Multipatch Discontinuous Galerkin IGA

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References:

U. Langer and I. Toulopoulos, "Analysis of multipatch discontinuous galerkin iga approximations to elliptic boundary value problems", ,

U. Langer, A. Mantzaflaris, S. Moore, and I. Toulopoulos, "Multipatch discontinuous galerkin isogeometric analysis", ,

Multipatch Domain

- $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$
- Interior faces: $F_{ij} := \bar{\Omega}_i \cap \bar{\Omega}_j$
- Boundary faces $F_i \in \mathcal{F}_B$
- All faces $\mathcal{F} := \mathcal{F}_I \cup \mathcal{F}_B$

Multipatch DG IGA Discretization

- Reference domain $\hat{\Omega} := (0,1)^d$
- $\mathcal{T}_H(\Omega) := \{\Omega_i\}_{i=1}^N$
- d knot-vectors on each subdomain: $\Xi_n^i, n = 1, \ldots, d$
- Parametric mesh $T_{\hat{\Omega}}^{(i)} := \Xi_1^i \times \ldots \times \Xi_d^i := \{\hat{E}_m\}_{m=1}^{M_i}$
- $T_{\Omega_i}^{(i)} := \{E_m^{(i)}\}_{m=1}^{M_i}$
- Assumption: quasi-uniform mesh

Multipatch Spline Spaces

- For each parametric mesh $T_{\hat{\Omega}}^{(i)}$ construct $\hat{\mathbb{B}}^{(i)} := \text{span}\{\hat{B}_j^{(i)}(\hat{x})\}$
- with tensor-product splines $\hat{B}_j^{(i)}(\hat{x}) := \hat{B}_{j_1}^{(i)}(\hat{x}) \dots \hat{B}_{j_d}^{(i)}(\hat{x})$
- Physical spline space $\mathbb{B}(\mathcal{T}_H(\Omega)) := \mathbb{B}^{(1)} \times \dots \times \mathbb{B}^{(N)}$
- with $\mathbb{B}^{(i)} := (\hat{\mathbb{B}}^{(i)} \circ \Phi_i^{-1})\big|_{\Omega_i}$

Model Problem

$$-\nabla \cdot (\alpha \nabla u) = f \qquad \text{in } \Omega$$

$$u = u_D \qquad \text{on } \partial\Omega$$

With $\alpha\big|_{\Omega_i} = \text{const} =: \alpha_i$

DG Notations

$v^i := v\big|_{\Omega_i}$

On common interfaces $F_{ij} \in \mathcal{F}_I$:

$$[\![v]\!] := v^i - v^j$$

$$\{v\} := \frac{1}{2}(v^i + v^j)$$

On boundary faces $F_i \in \mathcal{F}_B$:

$$[\![v]\!] := v^i$$

$$\{v\} := v^i$$

DG Variational Problem

$$a_h(u_h, v_h) = l(v_h) + p_D(u_D, v_h), \;\; \forall v_h \in \mathbb{B}(\mathcal{T}_H(\Omega))$$

$$a_h(u_h, v_h) := \sum_{i=1}^{N} \left(\int_{\Omega_i} \alpha_i \nabla u_h \nabla v_h dx - s_i(u_h, v_h) + p_i(u_h, v_h) \right)$$

$$s_i(u_h, v_h) := \sum_{\mathcal{F}} \int_F \{\alpha \nabla u_h\} \cdot n_{ij} [\![v_h]\!] + \{\alpha \nabla v_h\} \cdot n_{ij} [\![u_h]\!] ds$$

$$p_i(u_h, v_h) := \sum_{\mathcal{F}} \int_F \mu(\frac{\alpha_j}{h_j} + \frac{\alpha_i}{h_i}) [\![u_h]\!] [\![v_h]\!] ds$$

$$p_D(u_D, v_h) := \sum_{F_i} \int_{F_i} \left(\mu \frac{\alpha}{h} u_D v_h - \alpha u_D \nabla v n \right) ds$$

$$a(u,v) = -\sum_{i=1}^{N} \int_{\Omega_i} \nabla \cdot (\alpha \nabla u) v dx$$

$$
\begin{aligned}
a(u,v) &= -\sum_{i=1}^{N} \int_{\Omega_i} \nabla \cdot (\alpha \nabla u) v dx \\
&= \sum_{i=1}^{N} \int_{\Omega_i} \alpha \nabla u \nabla v dx - \sum_{i=1}^{N} \int_{\partial\Omega_i} \alpha \nabla u n v ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \int_{\Omega_i} \alpha \nabla u \nabla v dx - \sum_{i=1}^{N} \int_{\partial\Omega_i} \alpha \nabla u n v ds \\
&= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \alpha_i \nabla u_i v_i n_{ij} + \alpha_j \nabla u_j v_j n_{ji} ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \alpha_i \nabla u_i v_i n_{ij} + \alpha_j \nabla u_j v_j n_{ji} ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds \\
&= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} [\![\alpha \nabla u v]\!] ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds
\end{aligned}
$$

$$= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} [\![\alpha \nabla u v]\!] ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds$$

$$= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\}[\![v]\!] + [\![\alpha \nabla u]\!]\{v\} ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\} [\![v]\!] + [\![\alpha \nabla u]\!] \{v\} ds - \sum_{F_i} \int_{F_i} \alpha \nabla u v n ds \\
&= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\} [\![v]\!] + \{\alpha \nabla v\} [\![u]\!] - \frac{\mu}{h} [\![u]\!] [\![v]\!] ds \\
&- \sum_{F_i} \int_{F_i} \alpha \nabla u v n + \alpha (u - u_D) \nabla v n - \frac{\mu}{h} (u - u_D) v ds
\end{aligned}
$$

$$= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\}[\![v]\!] + \{\alpha \nabla v\}[\![u]\!] - \frac{\mu}{h}[\![u]\!][\![v]\!] ds$$

$$- \sum_{F_i} \int_{F_i} \alpha \nabla u v n + \alpha (u - u_D) \nabla v n - \frac{\mu}{h}(u - u_D) v ds$$

$$= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\}[\![v]\!] + \{\alpha \nabla v\}[\![u]\!] - \frac{\mu}{h}[\![u]\!][\![v]\!] ds$$

$$- \sum_{F_i} \int_{F_i} \alpha \nabla u v n + \alpha u \nabla v n - \frac{\mu}{h} u v ds$$

$$= \sum_{i=1}^{N} a_i(u,v) - \sum_{F_{ij}} \int_{F_{ij}} \{\alpha \nabla u\}[\![v]\!] + \{\alpha \nabla v\}[\![u]\!] - \frac{\mu}{h}[\![u]\!][\![v]\!] ds$$

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$$- \sum_{F_i} \int_{F_i} \{\alpha \nabla u\}[\![v]\!] + \{\alpha \nabla v\}[\![u]\!] - \frac{\mu}{h}[\![u]\!][\![v]\!] ds$$

DG Variational Problem

$$a_h(u_h, v_h) = l(v_h) + p_D(u_D, v_h), \quad \forall v_h \in \mathbb{B}(\mathcal{T}_H(\Omega))$$

$$a_h(u_h, v_h) := \sum_{i=1}^{N} \left(\int_{\Omega_i} \alpha_i \nabla u_h \nabla v_h dx - s_i(u_h, v_h) + p_i(u_h, v_h) \right)$$

$$s_i(u_h, v_h) := \sum_{\mathcal{F}} \int_F \{\alpha \nabla u_h\} \cdot n_{ij} [\![v_h]\!] + \{\alpha \nabla v_h\} \cdot n_{ij} [\![u_h]\!] ds$$

$$p_i(u_h, v_h) := \sum_{\mathcal{F}} \int_F \mu (\frac{\alpha_j}{h_j} + \frac{\alpha_i}{h_i}) [\![u_h]\!] [\![v_h]\!] ds$$

$$p_D(u_D, v_h) := \sum_{\mathcal{F}_B} \int_{F_i} \left(\mu \frac{\alpha}{h} u_D v_h - \alpha u_D \nabla v n \right) ds$$

Inverse Inequalities

Lemma 1 (Inverse Inequality)

For $u_h \in \mathbb{B}(\mathcal{T}_H(\Omega))$, *we have*

$$\|\nabla v_h\|_{L^2(F_{ij})} \leq c_I h^{-\frac{1}{2}} \|\nabla v_h\|_{L^2(\Omega_i)}$$

Boundedness and Ellipticity

- dG-norm for $u \in H_h^l := H^1(\Omega) \cap H^l(\mathcal{T}_H(\Omega)) + \mathbb{B}(\mathcal{T}_H(\Omega))$

$$\|u\|_{dG}^2 := \sum_{i=1}^{N} \left(\alpha_i \|\nabla u^{(i)}\|_{L^2(\Omega_i)}^2 + p_i(u^{(i)}, u^{(i)}) \right)$$

$\forall u \in H_h^l, u_h, v_h \in \mathbb{B}(\mathcal{T}_H(\Omega))$ we get the following results:

Lemma 2

$a_h(u_h, u_h) \geq C \|u_h\|_{dG}^2$

Lemma 3

$a_h(u, v_h) \leq C \left(\|u\|_{dG}^2 + \sum_{F_{ij}} h \|\{\alpha \nabla u\}\|_{L^2(F_{ij})} \right)^{\frac{1}{2}} \|v_h\|_{dG}$

Error Estimates (Outlook)

Theorem 4

For the exact solution $u \in H_h^l, l \geq 2$ *and the discrete solution* $u_h \in \mathbb{B}(\mathcal{T}_H(\Omega))$, *the following estimate can be shown:*

$$\|u - u_h\|_{dG} \leq \sum_{i=1}^{N} C_i \left(h_i^{l-1} + \sum_{F_{ij} \subset \partial\Omega_i} \alpha_i \frac{h_i}{h_j} h_i^{l-1} \right) \|u\|_{H^l(\Omega_i)}$$

Thank you for your attention!