dG IgA: Error Estimation and Outlook

Agnes Seiler

Johannes Kepler University Linz

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

Institut für Angewandte Geometrie

Agenda

1. Error Estimates in a 2D Setting
2. Outlook: Non-matching interface parameterizations

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2. Outlook: Non-matching interface parameterizations

Recall: Notation

$F_{ij} = \Omega_i \cap \Omega_j$	edges in physical domain

Spaces:

$\mathrm{H}^{\ell}_{\mathcal{T}} = H^1(\Omega) \cap H^{\ell}(\mathcal{T}(\Omega))$ $\mathrm{H}^{\ell}_{\mathrm{h}} = \mathrm{H}^{\ell}_{\mathcal{T}} + \mathbb{B}_{\mathrm{h}}(\mathcal{T}(\Omega))$	$u \in \mathrm{H}^{\ell}_{\mathcal{T}}, \ell \geqslant 2$
$\Pi_h : H^{\ell}(\Omega) \mapsto \mathbb{B}_h(\mathcal{T}(\Omega))$	quasi interpolant
p	degree of B-Splines

Approximation estimates

Lemma

Let $u \in H^\ell(\Omega_i)$ with $\ell \geqslant 2$. Then for $0 \leqslant \ell \leqslant p+1$, there exist constants

$$C_i := C_i\big(\max_{\ell_0 \leqslant \ell} \|D^{\ell_0}\Phi_i\|_{L^\infty(\Omega_i)}, \|u\|_{H^\ell(\Omega_i)}\big)$$

such that

- $h_i^\beta \|\nabla u^i - \nabla \Pi_h u^i\|^2_{L^2(F_{ij})} \leqslant C_i h_i^{2\ell-3+\beta}$
- $\|u - \Pi_h u\|^2_{dG} \leqslant \sum_{i=1}^N C_i h_i^{2\ell-2} \|u\|^2_{H^\ell(\Omega_i)} + \sum_{F_{ij}} C_i \alpha^j \frac{h_i}{h_j} h_i^{2\ell-2} \|u\|^2_{H^\ell(\Omega_i)}$

Recall: Boundedness of the bilinear form

Lemma

There is a constant C independent of h_i such that for all $u \in \mathrm{H}_{\mathrm{h}}^{\ell}$ and for all $v \in \mathbb{B}_{\mathrm{h}}(\mathcal{T}(\Omega))$ we have

$$a_h(u,v) \leqslant C \left(\|u\|_{dG}^2 + \sum_{F_{ij}} h_i \alpha^i \|\nabla u^i\|_{L^2(F_{ij})}^2 + h_j \alpha^j \|\nabla u^j\|_{L^2(F_{ij})}^2 \right)^{\frac{1}{2}} \|v\|_{dG}.$$

Main error estimation

Theorem

Let $u \in \mathrm{H}^{\ell}_{\mathcal{T}}$, $\ell \geqslant 2$, be the solution of

$$a(u, v) = F(v) \quad \forall v \in H^1_0(\Omega)$$

and let $u_h \in \mathbb{B}_\mathrm{h}(\mathcal{T}(\Omega))$ be the solution of

$$a_h(u_h, v_h) = F(v_h) + p_D(u_d, v_h) \quad \forall v_h \in \mathbb{B}_\mathrm{h}(\mathcal{T}(\Omega)).$$

Then

$$\|u - u_h\|_{dG} \leqslant \sum_{i=1}^{N} C_i \left(h_i^{\ell-1} + \sum_{F_{ij}} \alpha^j \frac{h_i}{h_j} h_i^{\ell-1} \right) \|u\|_{H^\ell(\Omega_i)}.$$

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Introduction and Problem Set

Change of notation: $F_{ij} \rightarrow e$

$$a(u,v) = \sum_{i=1}^{2} \int_{\Omega_i} \alpha_i \nabla u \cdot \nabla v d\Omega$$
$$-\frac{1}{2} \sum_{e \in F_I} \int_e \{\nabla u \cdot n\}[v] + \{\nabla v \cdot n\}[u] de + \sum_{e \in F_I} \frac{\delta}{h_e} \int_e [u][v] de$$

Recall: find u_i^k such that

$$\sum_{k=1}^{n} \sum_{i \in \mathcal{R}^k} u_i^k a(b_i^k, b_j^\ell) = F(b_j^\ell) \quad \forall j \in \mathcal{R}^\ell, \ell = 1, \dots, n$$

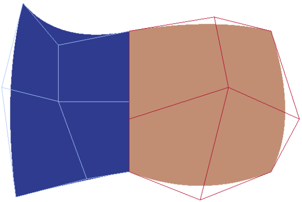
Introduction and Problem Set

Problems caused by non-matching interface parameterizations (not considered previously)

↓

Challenge 1: Find reparameterizations
Challenge 2: Find suitable elements for numerical quadrature rule

- Experimental results

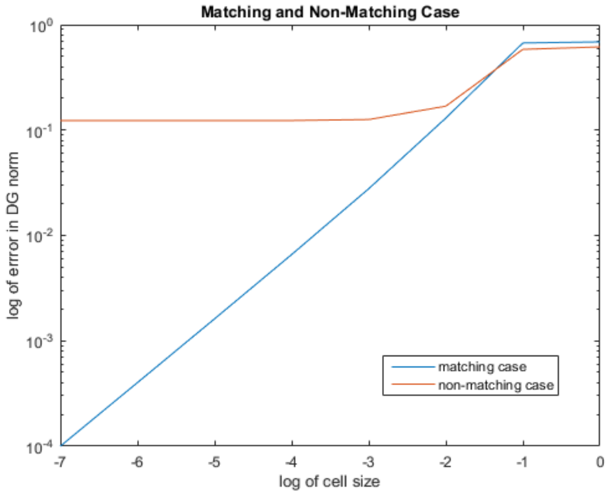


↑

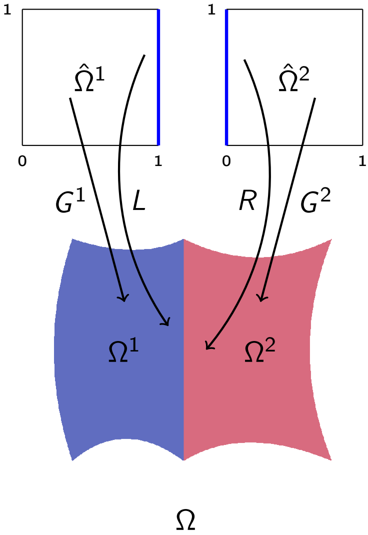
same curve, potentially different parameterizations

Why do we consider non-matching parameterizations?

If we don't treat non-matching parameterizations differently:



Example Domain and Simplifications



$$L = G^1|_{(G^1)^{-1}(e)}$$
$$R = G^2|_{(G^2)^{-1}(e)}$$

Computing the Integral Terms

$$\int_e [u][v]de = \int_e (u|_{\Omega^1} - u|_{\Omega^2})(v|_{\Omega^1} - v|_{\Omega^2})de$$
$$= \int_e u|_{\Omega^1} v|_{\Omega^1} - u|_{\Omega^2} v|_{\Omega^1} - u|_{\Omega^1} v|_{\Omega^2} + u|_{\Omega^2} v|_{\Omega^2} de$$

$$\Big\downarrow \quad \text{plug in } b_i^k, b_j^\ell,\ k, \ell = 1, 2$$

term of interest: $(-1)^{k+\ell} \int_e b_i^k(x)|_{\Omega^k} b_j^\ell(x)|_{\Omega^\ell} dx$

Compute one Integral Value Exemplarily

For $k = 1$, $\ell = 2$ we get

$$-\int_e b_i^1(\boldsymbol{x})|_{\Omega^1} b_j^2(\boldsymbol{x})|_{\Omega^2} d\boldsymbol{x}$$

Reparameterizations

Let

$$\lambda : [0,1] \to \{1\} \times [0,1],$$
$$\varrho : [0,1] \to \{0\} \times [0,1]$$

such that $L \circ \lambda = R \circ \varrho$.

$$\Rightarrow e = (L \circ \lambda)([0,1]) = (R \circ \varrho)([0,1])$$

Set $P(t) := L(\lambda(t)) = R(\varrho(t))$, $t \in [0,1]$.

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For $k = 1$, $\ell = 2$ we get

$$-\int_e b_i^1(\boldsymbol{x})|_{\Omega^1} b_j^2(\boldsymbol{x})|_{\Omega^2} d\boldsymbol{x}$$
$$= -\int_e \left(\beta_i^1 \circ (G^1)^{-1}\right)(\boldsymbol{x}) \left(\beta_j^2 \circ (G^2)^{-1}\right)(\boldsymbol{x}) d\boldsymbol{x}$$

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\begin{aligned}
& - \int_e b_i^1(\boldsymbol{x})|_{\Omega^1} b_j^2(\boldsymbol{x})|_{\Omega^2} d\boldsymbol{x} \\
&= - \int_e \left(\beta_i^1 \circ (G^1)^{-1}\right)(\boldsymbol{x}) \left(\beta_j^2 \circ (G^2)^{-1}\right)(\boldsymbol{x}) d\boldsymbol{x} \\
&= - \int_e \left(\beta_i^1 \circ L^{-1}\right)(\boldsymbol{x}) \left(\beta_j^2 \circ R^{-1}\right)(\boldsymbol{x}) d\boldsymbol{x}
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&= - \int_0^1 \left(\beta_i^1 \circ L^{-1}\right)(P(t)) \cdot \\
&\qquad \left(\beta_j^2 \circ R^{-1}\right)(P(t)) \|\dot{P}(t)\| dt
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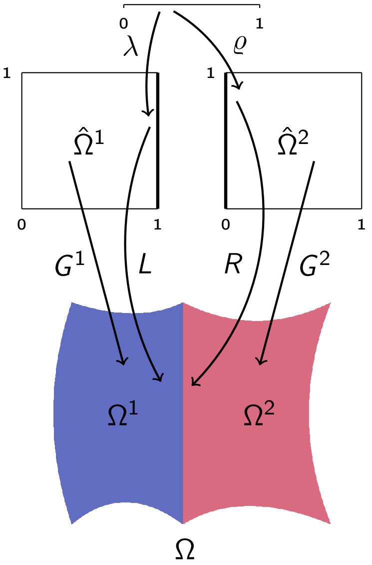
Two Challenges



Tasks

1. Find λ and ϱ.
   Cases of interest:
   $L \neq R \Rightarrow \lambda \neq id$ or $\varrho \neq id$
2. Apply a suitable quadrature to

$$\int_0^1 \beta_i^1(\lambda(t))\beta_j^2(\varrho(t))\|\dot{P}(t)\|dt.$$

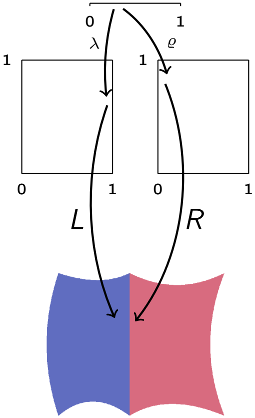
Challenge 1: How to find the Reparameterizations

- Compute discrete version of $\varrho = R^{-1} \circ L \circ \lambda$, i.e.

$$\varrho(t_i) = \varrho_i$$

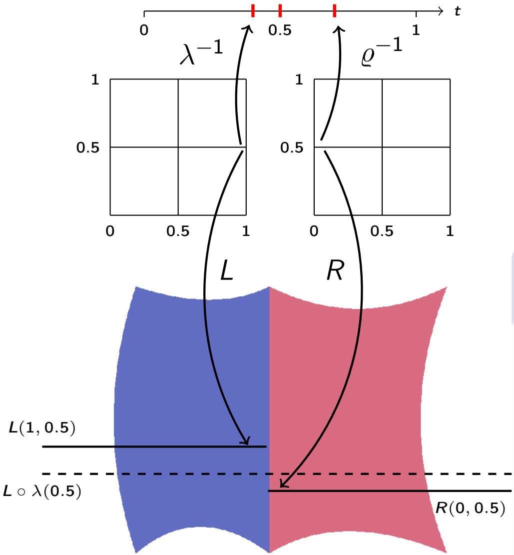
  by solving a least squares problem for some samples t_i.
- Solve a spline fitting problem of $(\varrho_i)_i$ with one dimensional splines.

$$\Rightarrow L \circ \lambda \approx R \circ \varrho$$



$$\varrho = R^{-1} \circ L \circ \lambda$$

Challenge 2: How to find the Quadrature Knots



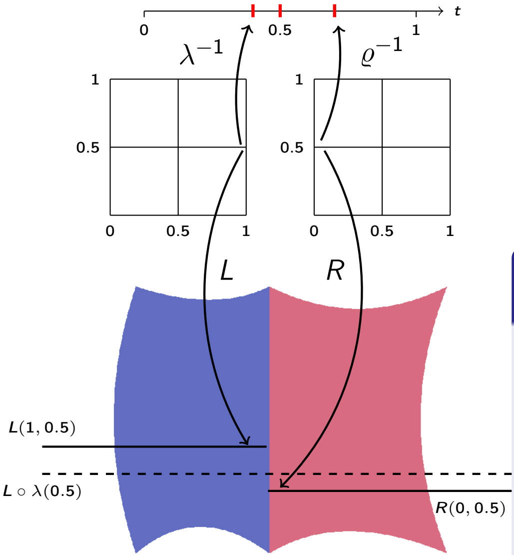
Find quadrature knots to integrate

$$\int_0^1 \beta_i^1(\lambda(t))\beta_j^2(\varrho(t))\|\dot{P}(t)\|dt.$$

Finding Interior Knots: Strategies

- Find the exact positions by inverting λ and ϱ.
- Split the t-knotspans into uniform segments.
- Use adaptive quadrature on whole t-knot spans.

Challenge 2: How to find the Quadrature Knots



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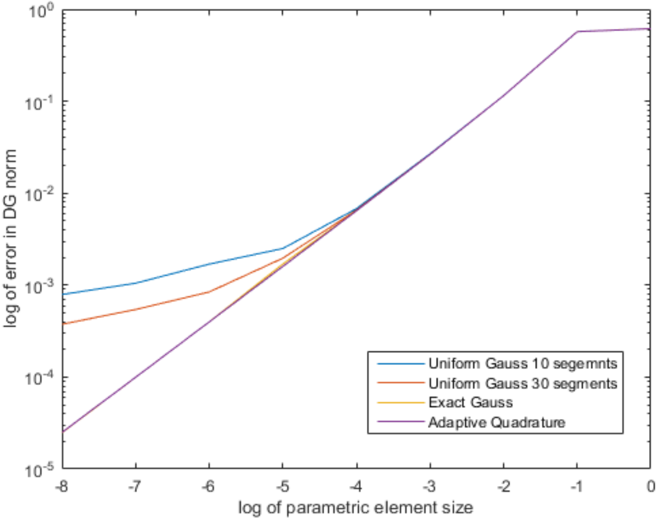
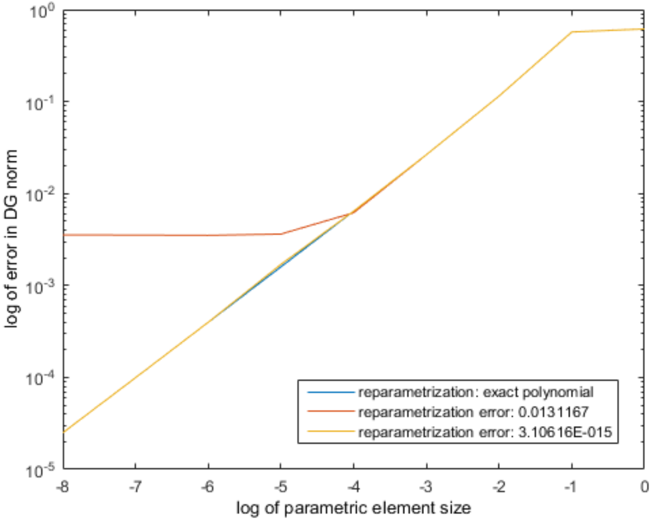
Influence of Quadrature Method



Influence of the Reparameterization



Literature

U. Langer, I. Toulopoulos
Analysis of Multipatch Discontinuous Galerkin IGA Approximatons to Elliptic Boundary Value Problems.
2014

U. Langer, A. Mantzaflaris, S.E. Moore, I. Toulopoulos
Multipatch Discontinuous Galerkin Isogeometric Analysis.
2014

Thank you for your attention!