Isogeometric Tearing and Interconnecting

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27-1-2017

Overview

Motivation

- design solvers for large-scale problems
- handle complicated computational geometries/meshes (NURBS)
  - numerical simulation problem subdivided into independent sub-problems
  - then coupled in appropriate ways
- NURBS-based IGA particularly suited for FETI methods $\rightarrow$IETI-methods
- IsogEometric Tearing and Interconnecting (IETI) method combines
  - advanced solver design of FETI
  - exact geometry representation of IGA

Big picture

- computational domain is subdivided into non-overlapping subdomains
- each subdomain gets its own set of equations derived from global equation
- to ensure the equivalence to global equation:
  →introduce continuity conditions at subdomain interfaces (Lagrange multipliers)
- FETI-inspired method also provides a powerful solver design:
  - eliminating the original variables from the resulting saddle point problem
  - obtain a system in the Lagrange multipliers (i.e., only on the interface)
  - solution of original problem computed from solution of this interface problem

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Model problems

$\Omega \subset \mathbb{R}^2$ open, bounded and connected Lipschitz domain
boundary $\partial\Omega = \Gamma_D \cup \Gamma_N, \Gamma_D \cap \Gamma_N = \emptyset$

Model problem 1 (scalar diffusion)

$$
\begin{aligned}
-\operatorname{div}(\alpha \nabla u) &= f \text{ in } \Omega \\
u &= g_D \text{ on } \Gamma_D \\
\alpha \frac{\partial}{\partial \boldsymbol{n}} u &= g_N \text{ on } \Gamma_N
\end{aligned}
$$

Model problem 2 (linearized elasticity)

$$
\begin{aligned}
-\textbf{div}(\boldsymbol{\Sigma}(\boldsymbol{u})) &= \boldsymbol{f} \text{ in } \Omega \\
\boldsymbol{u} &= \boldsymbol{g}_D \text{ on } \Gamma_D \\
\boldsymbol{\Sigma}(\boldsymbol{u})\boldsymbol{n} &= \boldsymbol{f}_N \text{ on } \Gamma_N
\end{aligned}
$$

Model problems

Variational form (scalar diffusion)

Find $u \in V_g$

$$a_1(u, v) = l_1(v) \text{ for } \forall v \in V_0 \qquad \text{(mod1var)}$$

where

$$V_0 = \{v \in \mathcal{H}^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$$
$$V_g = \{v \in \mathcal{H}^1(\Omega) : v = g_D \text{ on } \Gamma_D\}$$

Model problems

Variational form (linearized elasticity)

Find $\boldsymbol{u} \in \boldsymbol{V}_g$

$$a_2(\boldsymbol{u}, \boldsymbol{v}) = l_2(\boldsymbol{v}) \text{ for } \forall \boldsymbol{v} \in \boldsymbol{V}_0 \qquad \text{(mod2var)}$$

where

$$\boldsymbol{V}_0 = \{v \in \mathcal{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$$
$$\boldsymbol{V}_g = \{v \in \mathcal{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{g}_D \text{ on } \Gamma_D\}$$

Equivalence to minimization

data such that

- $a_i(.,..)$ bounded, symmetric, positive definite
- $l_i(.)$ bounded linear functional

Then (mod1var) and (mod2var) equivalent to minimization problems

$$u = \operatorname*{argmin}_{v \in V_g} \frac{1}{2} a_1(v, v) - l_1(v) \qquad \text{(mod1min)}$$

$$\boldsymbol{u} = \operatorname*{argmin}_{\boldsymbol{v} \in \boldsymbol{V}_g} \frac{1}{2} a_2(\boldsymbol{v}, \boldsymbol{v}) - l_2(\boldsymbol{v}) \qquad \text{(mod2min)}$$

NURBS geometry mappings

From the lecture

- $p \in \mathbb{N}$ degree
- $\mathbf{s} = (s_i)_{i=1}^{n_s+p+1}$, $s_1 = 0$, $s_{n_s+p+1} = 1$ open knot vector
- n_s number of univariate B-spline functions

Bivariate NURBS

- $(B_{i,p}^s)_{i=1}^{n_s}, (B_{j,q}^t)_{j=1}^{n_t}$ B-spline families
- $\mathcal{R} := \{(i,j) : i = 1, \cdots, n_s, j = 1, \cdots, n_t\}$ bivariate B-spline coordinates
- Bivariate NURBS $R_{(i,j)} := \dfrac{B_{i,p}^s(\xi_1)B_{j,q}^t(\xi_2)w_{(i,j)}}{\sum_{(k,l)\in\mathcal{R}} B_{k,p}^s(\xi_1)B_{l,q}^t(\xi_2)w_{(k,l)}}$
- $w_{(i,j)} \in \mathbb{R}^+$ weights

NURBS geometry mappings

NURBS surface

- control points $C_{(i,j)} \in \mathbb{R}^2$

$$F(\xi_1, \xi_2) := \sum_{(i,j)\in\mathcal{R}} R_{(i,j)} C_{(i,j)}$$

- parameter domain $\hat{\Omega} = (0,1)^2$, physical domain $\Omega = F(\hat{\Omega})$
- assume geometry mapping is continuous and bijective (i.e. not self-penetrating)

Single-patch NURBS discretization

wlog. consider Model problem 1 (scalar diffusion), set

- $V_h := \operatorname{span}\{\hat{R}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathcal{R}} \subset \mathcal{H}^1(\Omega), \hat{R}_{\boldsymbol{k}} = R_{\boldsymbol{k}} \circ F^{-1}$
- $V_{0h} := \{v \in V_h : v\big|_{\Gamma_D} = 0\}$
- $V_{gh} := \{v \in V_h : v\big|_{\Gamma_D} = g_D\} = \tilde{g} + V_{0h}$
- function $u_h \in V_h$ and DOFs $u_{\boldsymbol{k}} \in \mathbb{R}$

$$u_h(x) = \sum_{\boldsymbol{k}\in\mathcal{R}} u_{\boldsymbol{k}} \hat{R}_{\boldsymbol{k}}(x)$$

Multi-patch NURBS discretization

- physical domain Ω represented by $N \in \mathbb{N}$ single-patch NURBS
- geometry mappings

$$F^{(i)} : \hat{\Omega} \to \Omega^{(i)} = F^{(i)}(\hat{\Omega}) \subset \Omega, i = 1, \cdots, N$$

- domain decomposition

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}^{(i)} \text{ and } \Omega^{(i)} \cap \Omega^{(j)} = \emptyset \text{ for } i \neq j$$

- $V_h^{(i)} := \text{span}\,\{\hat{R}_{\boldsymbol{k}}^{(i)}\}_{\boldsymbol{k}\in\mathcal{R}} \subset \mathcal{H}^1(\Omega^{(i)})$,
- $\hat{R}_{\boldsymbol{k}}^{(i)} = R_{\boldsymbol{k}}^{(i)} \circ (F^{(i)})^{-1}$
- $\cdots$

Multi-patch NURBS discretization

Product space

include patches

$$\prod V_h = \{v \in \mathcal{L}^2(\Omega) : v\big|_{\Omega^{(i)}} \in V_h^{(i)}, i = 1, \cdots, N\} \equiv \prod_{i=1}^{N} V_h^{(i)}$$

Discretization

- $V_h \longleftarrow \prod V_h \cap \mathcal{C}^0(\Omega)$
- $V_{0h} := \{v \in V_h : v\big|_{\Gamma_D} = 0\}$
- $V_{gh} := \{v \in V_h : v\big|_{\Gamma_D} = g_D\} = \tilde{g} + V_{0h}$
- $u_h\big|_{\Omega^{(i)}} = \sum_{\boldsymbol{k} \in \mathcal{R}} u_{\boldsymbol{k}}^{(i)} \hat{R}_{\boldsymbol{k}}^{(i)}$

Multi-patch NURBS discretization

Interfaces

- Interfaces

$$\Gamma^{(i,j)} := \partial\Omega^{(i)} \cap \partial\Omega^{(j)}$$

- Indices of nonempty interfaces (interface tuples)

$$C_\Gamma := \{(i,j) : \Gamma^{(i,j)} \neq \emptyset\}$$

- Basis indices on interface

$$\mathcal{B}(i,j) := \{\boldsymbol{k} \in \mathcal{R}^{(i)} : \text{supp}\, \hat{R}_{\boldsymbol{k}}^{(i)} \cap \Gamma^{(i,j)} \neq \emptyset\}, (i,j) \in C_\Gamma$$

Def.: Fully-matching subdomains

Definition

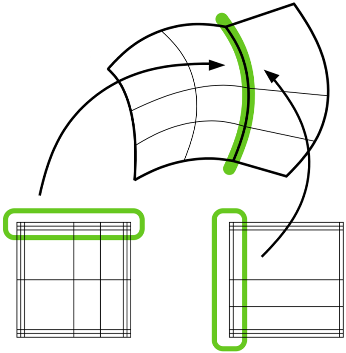
Let $\Gamma^{(i,j)}$ be an edge. Subdomains $\Omega^{(i)}, \Omega^{(j)}$ are fully matching IFF

1. the interface $\Gamma^{(i,j)}$ is the image of an entire edge of the respective parameter domains
2. for each index $\boldsymbol{k} \in \mathcal{B}(i,j)$, there must be a unique index $\boldsymbol{l} \in \mathcal{B}(j,i)$, such that

$$\hat{R}_{\boldsymbol{k}}^{(i)} = \hat{R}_{\boldsymbol{l}}^{(j)}$$

Exmp.: Fully-matching subdomains

Fully matching subdomains $\Omega^{(i)}, \Omega^{(j)}$



All weights equal to 1.

$$p^{(i)} = 2$$
$$s^{(i)} = \{0, 0, 0, 0.5, 0.75, 1, 1, 1\}$$
$$q^{(i)} = 2,$$
$$t^{(i)} = \{0, 0, 0, 0.5, 1, 1, 1\}$$

$$p^{(j)} = 1$$
$$s^{(j)} = \{0, 0, 1, 1\}$$
$$q^{(j)} = 2$$
$$t^{(j)} = \{0, 0, 0, 0.25, 0.5, 1, 1, 1\}$$

Variational forms

- assume $l_1(.)$ assembled from contributions $l_1^{(i)}(.)$ on $\Omega^{(i)}$
- $a_1^{(i)}(.,..)$ restriction of $a_1(.,..)$ on $\Omega^{(i)}$
- minimization problems on subdomains (according to (mod1min))

$$u_h = \underset{v_h \in V_g}{\operatorname{argmin}} \sum_{i=1}^{N} \left(\frac{1}{2} a_1^{(i)}(v_h, v_h) - l_1^{(i)}(v_h) \right) \qquad \text{(mod1mini)}$$

- condition 2 implies: each DOF

$$u_{\boldsymbol{k}}^{(i)} \cong u_{\boldsymbol{l}}^{(j)}, \boldsymbol{k} \in \mathcal{B}(i,j), \boldsymbol{l} \in \mathcal{B}(j,i)$$

- Identify fully matching DOFs and solve system $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$?

Why IETI?

- $\boldsymbol{K u} = \boldsymbol{f}$ could be solved by standard sparse LU-factorization
- for large problem size memory/runtime complexity of direct solvers inefficient
- $\rightarrow$efficient iterative solvers
  (PCG: conjugate gradient + preconditioner)
- standard FEM preconditioners well studied
  (multigrid-/domain decomposition methods)
- assembling $\boldsymbol{K}$ structural subdomain properties lost
- One way out (inspired by the FETI methods):
  IETI uses local structure of (mod1mini)
  - parallelization ✓
  - exploit tensor product structure ✓

IETI basics - Continuity constraints

$$\prod V_h = \{v \in \mathcal{L}^2(\Omega) : v\big|_{\Omega^{(i)}} \in V_h^{(i)}, i = 1, \cdots, N\} \equiv \prod_{i=1}^N V_h^{(i)}$$

- $v^{(i)} \in V_h^{(i)}$ i-th comp. of $v \in \prod V_h$
- ig. discontinuous across subdomain interfaces
  $\rightarrow$impose continuity conditions
- Unique interfaces: $C := \{(i,j) \in C_\Gamma : i < j\}$
  $\rightarrow$master $\Omega^{(i)}$ and slave $\Omega^{(j)}$
- IETI refinement and definition inspires rewriting for $\boldsymbol{k} \in \mathcal{B}(i,j)$

$$\hat{R}_{\boldsymbol{k}}^{(i)}\Big|_{\Gamma^{(i,j)}} = \sum_{\boldsymbol{l} \in \mathcal{B}(j,i)} a_{\boldsymbol{k},\boldsymbol{l}}^{(i,j)} \hat{R}_{\boldsymbol{l}}^{(j)}\Big|_{\Gamma^{(i,j)}}, \; a_{\boldsymbol{k},\boldsymbol{l}}^{(i,j)} = 0 \text{ except for one } = 1$$

- impose continuity by imposing condition for $(i,j) \in C$

$$u_{\boldsymbol{k}}^{(i)} - \sum_{\boldsymbol{l} \in \mathcal{B}(j,i)} a_{\boldsymbol{k},\boldsymbol{l}}^{(i,j)} u_{\boldsymbol{l}}^{(j)} = 0 \text{ for } \forall \boldsymbol{k} \in \mathcal{B}(i,j)$$

IETI basics - Essential boundary conditions

- let $\Gamma^{(i,D)} := \Gamma_D \cap \partial\Omega^{(i)}$ and
  $\mathcal{D}(i) := \{\boldsymbol{k} \in \mathcal{R}^{(i)} : \text{supp}\, \hat{R}_{\boldsymbol{k}}^{(i)} \cap \Gamma^{(i,D)} \neq \emptyset\}$
- assumed $V_{gh} = \tilde{g} + V_{0h}$ $(\tilde{g}|_{\Gamma_D} = g_D|_{\Gamma_D})$

$$g_D|_{\Gamma^{(i,D)}} = \sum_{\boldsymbol{k} \in \mathcal{D}(i)} g_{\boldsymbol{k}}^{(i)} \hat{R}_{\boldsymbol{k}}^{(i)}\Big|_{\Gamma^{(i,D)}}, g_{\boldsymbol{k}}^{(i)} \in \mathbb{R}$$

- impose incorporate essential boundary conditions for $\Gamma^{(i,D)} \neq \emptyset$

$$u_{\boldsymbol{k}}^{(i)} = g_{\boldsymbol{k}}^{(i)} \text{ for } \forall \boldsymbol{k} \in \mathcal{D}(i)$$

- continuity between physical subdomains and "virtual" neighbour subdomain

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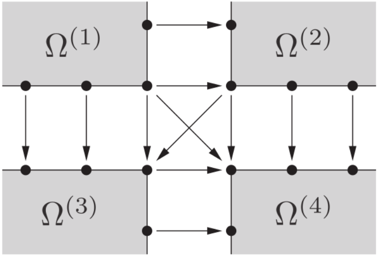
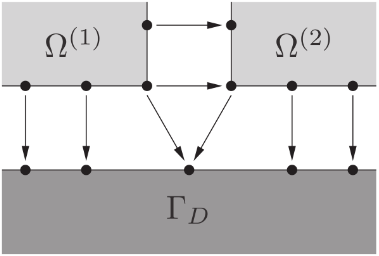


(a) Fully redundant coupling at a subdomain vertex.



(b) Incorporation of essential boundary conditions by coupling to virtual neighbour subdomains.

Figure: Illustration of fully redundant coupling and all floating setting. Arrows indicate coupling conditions and point from master subdomain to slave subdomain.

IETI basics - Jump operator

- total number of constraints $J := \sum_{(i,j)\in C} |\mathcal{B}(i,j)| + \sum_{i=1}^{N} |\mathcal{D}(i)|$
- fix numbering of constraints
- set for $\boldsymbol{y} \in \mathbb{R}^J$ corresponding components $\boldsymbol{y}_{\boldsymbol{k}}^{(i,j)}, \boldsymbol{y}_{\boldsymbol{k}}^{(i,D)}$
- introduce Jump Operator

$$B : \prod V_h \to \mathbb{R}^J$$

$$(Bu)_{\boldsymbol{k}}^{(i,j)} = u_{\boldsymbol{k}}^{(i)} - \sum_{\boldsymbol{l}\in\mathcal{B}(j,i)} a_{\boldsymbol{k},\boldsymbol{l}}^{(i,j)} u_{\boldsymbol{l}}^{(j)} \text{ for } \forall \boldsymbol{k} \in \mathcal{B}(i,j)$$

$$(Bu)_{\boldsymbol{k}}^{(i,D)} = u_{\boldsymbol{k}}^{(i)}$$

IETI basics - Jump operator

- write $\mathcal{C}^0$-continuity/essential BC as

$$Bu = \boldsymbol{b}$$

where $\boldsymbol{b} = (b_m)_{m=1}^J$ with $b_m = 0$ for cont. $= g_{\boldsymbol{k}}^{(i)}$ for essent.

- $\rightarrow$for $\tilde{g}\big|_{\Gamma_D} = g_D\big|_{\Gamma_D}$ have $B\tilde{g} = \boldsymbol{b}$
- new minimization problem from (mod1mini)

$$u_h = \operatorname*{argmin}_{v \in \prod V_h, Bv = \boldsymbol{b}} \sum_{i=1}^{N} \left(\frac{1}{2} a_1^{(i)}(v^{(i)}, v^{(i)}) - l_1^{(i)}(v^{(i)}) \right) \tag{mod1minB}$$

Saddle point formulation

Corresponding representations

$$v^{(i)} \in V_h^{(i)} \leftrightarrow \boldsymbol{v}^{(i)} \qquad v \in \prod V_h \leftrightarrow \boldsymbol{v} = (\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(N)})$$

$$a^{(i)}(.,..) \leftrightarrow \boldsymbol{K}^{(i)} \qquad f^{(i)}(.) \leftrightarrow \boldsymbol{f}^{(i)}, \boldsymbol{f} = (\boldsymbol{f}^{(1)}, \cdots, \boldsymbol{f}^{(n)})$$

- set global stiffness matrix

$$\boldsymbol{K} := \begin{bmatrix} \boldsymbol{K}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \boldsymbol{K}^{(N)} \end{bmatrix}$$

- set signed boolean jump matrix $\boldsymbol{B} \leftrightarrow B$

Saddle point formulation

- represent (mod1minB) by saddle point problem

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^\top \\ \boldsymbol{B} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{b} \end{bmatrix}$$

- **Beware:** $\boldsymbol{\lambda}$ only unique up to $\ker B^\top$!
- Classical FETI →equation solely of $\boldsymbol{\lambda}$
- not straight forward: $\boldsymbol{K}^{(i)}$ not invertible

  Problem 1 kernel spanned by constant functions

  Problem 2 kernel spanned by rigid body motions
- →follow dual-primal FETI →IETI!

Dual-primal formulation

- NURBS geometry mapping F (open knot vectors)
- therefore vertex of $\hat{\Omega}$ has unique $\boldsymbol{k}_0 \in \mathcal{R}^{(i)}$ with $R^{(i)}_{\boldsymbol{k}_0} = 1$ and $R^{(i)}_{\boldsymbol{k}} = 0, \boldsymbol{k} \neq \boldsymbol{k}_0$ there
- $\rightarrow$DOF associated with vertices of $\hat{\Omega}$ primal DOF
- others non-primal or remaining DOF
- define $\tilde{\boldsymbol{W}}_h := \{v \in \prod V_h : v \text{ continuous in all vertices}\} \subset \prod V_h$
- for continuity identify primal DOF with common point in $\hat{\Omega}$
- fix a global numbering of these primal DOF $\rightarrow$representation $(\boldsymbol{v} \leftrightarrow \tilde{\boldsymbol{v}})$ and set $\tilde{\boldsymbol{v}} = (\tilde{\boldsymbol{v}}_P, \tilde{\boldsymbol{v}}_R)^\top = (\tilde{\boldsymbol{v}}_P, \tilde{\boldsymbol{v}}_R^{(1)}, \cdots, \tilde{\boldsymbol{v}}_R^{(N)})^\top$
- Jump operator on $\tilde{\boldsymbol{W}}_h$ given by $\tilde{\boldsymbol{B}}\tilde{\boldsymbol{u}} = Bu$ where $\tilde{\boldsymbol{B}} = (\tilde{\boldsymbol{B}}_P, \tilde{\boldsymbol{B}}_R) = (\tilde{\boldsymbol{B}}_P, \tilde{\boldsymbol{B}}_R^{(1)}, \cdots, \tilde{\boldsymbol{B}}_R^{(N)})$

Setting up the global system

- we know $u \in \tilde{\boldsymbol{W}}_h$ $\rightarrow$reformulate SPP: replace $\prod V_h$ by $\tilde{\boldsymbol{W}}_h$
- rearrange local contributions $\boldsymbol{u}^{(i)}, \boldsymbol{f}^{(i)}, \boldsymbol{K}^{(i)}$

$$\tilde{\boldsymbol{K}}^{(i)} := \begin{bmatrix} \tilde{\boldsymbol{K}}_{PP}^{(i)} & \tilde{\boldsymbol{K}}_{PR}^{(i)} \\ \tilde{\boldsymbol{K}}_{RP}^{(i)} & \tilde{\boldsymbol{K}}_{RR}^{(i)} \end{bmatrix}, \tilde{\boldsymbol{f}}^{(i)} = \begin{bmatrix} \tilde{\boldsymbol{f}}_P^{(i)} \\ \tilde{\boldsymbol{f}}_R^{(i)} \end{bmatrix}, \tilde{\boldsymbol{u}}^{(i)} = \begin{bmatrix} \tilde{\boldsymbol{u}}_P^{(i)} & \tilde{\boldsymbol{u}}_R^{(i)} \end{bmatrix}$$

- rearrange global contributions $\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{K}$

$$\tilde{\boldsymbol{K}} := \begin{bmatrix} \tilde{\boldsymbol{K}}_{PP} & \tilde{\boldsymbol{K}}_{PR} \\ \tilde{\boldsymbol{K}}_{RP} & \tilde{\boldsymbol{K}}_{RR} \end{bmatrix}, \tilde{\boldsymbol{f}} = \begin{bmatrix} \tilde{\boldsymbol{f}}_P \\ \tilde{\boldsymbol{f}}_R \end{bmatrix}, \tilde{\boldsymbol{u}} = \begin{bmatrix} \tilde{\boldsymbol{u}}_P & \tilde{\boldsymbol{u}}_R \end{bmatrix}$$

- **Beware:** $\tilde{\boldsymbol{K}}_{PP}$ ist NOT block diagonal!

Setting up the global system

- block diagonal $\tilde{\boldsymbol{K}}_{RR}$

$$\tilde{\boldsymbol{K}} := \begin{bmatrix} \tilde{\boldsymbol{K}}_{RR}^{(1)} & & 0 \\ & \ddots & \\ 0 & & \tilde{\boldsymbol{K}}_{RR}^{(N)} \end{bmatrix}, \tilde{\boldsymbol{f}}_R = \begin{bmatrix} \tilde{\boldsymbol{f}}_R^{(1)} \\ \cdots \\ \tilde{\boldsymbol{f}}_R^{(N)} \end{bmatrix}$$

New global system (primal/residual DOFs)

$$\begin{bmatrix} \tilde{\boldsymbol{K}} & \tilde{\boldsymbol{B}}^\top \\ \tilde{\boldsymbol{B}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{f}} \\ \tilde{\boldsymbol{b}} \end{bmatrix}$$

Essential boundary conditions

- so far $\tilde{\boldsymbol{K}}$ is singular since essential BC not in $\tilde{\boldsymbol{W}}_h$
- essential primal DOF associated with Γ_D
- floating primal DOF in interior of Ω or associated with Γ_N

$$\tilde{\boldsymbol{u}} = (\tilde{\boldsymbol{u}}_d, \tilde{\boldsymbol{u}}_f, \tilde{\boldsymbol{u}}_R)^\top = (\tilde{\boldsymbol{u}}_d, \tilde{\boldsymbol{u}}_f, \tilde{\boldsymbol{u}}_R^{(1)}, \cdots, \tilde{\boldsymbol{u}}_R^{(N)})^\top$$

$$\tilde{\boldsymbol{B}} = \begin{bmatrix} \tilde{\boldsymbol{B}}_d & \tilde{\boldsymbol{B}}_f & \tilde{\boldsymbol{B}}_R \end{bmatrix}$$

$$\tilde{\boldsymbol{K}} = \begin{bmatrix} \tilde{\boldsymbol{K}}_{dd} & \tilde{\boldsymbol{K}}_{df} & \tilde{\boldsymbol{K}}_{dR} \\ \tilde{\boldsymbol{K}}_{fd} & \tilde{\boldsymbol{K}}_{ff} & \tilde{\boldsymbol{K}}_{fR} \\ \tilde{\boldsymbol{K}}_{Rd} & \tilde{\boldsymbol{K}}_{Rf} & \tilde{\boldsymbol{K}}_{RR} \end{bmatrix}, \tilde{\boldsymbol{f}} = \begin{bmatrix} \tilde{\boldsymbol{f}}_d \\ \tilde{\boldsymbol{f}}_f \\ \tilde{\boldsymbol{f}}_R \end{bmatrix}$$

- let $\tilde{\boldsymbol{g}}_d$ with entries values of g_D at essential primal DOF
- want $\tilde{\boldsymbol{u}}_d = \tilde{\boldsymbol{g}}_d$

Global system including essential boundary conditions

Primal problem

Find $u \in \tilde{\boldsymbol{W}}_h$ represented as $\tilde{\boldsymbol{u}}$, $\boldsymbol{\lambda} \in \mathbb{R}^J$ such that

$$\begin{bmatrix} \bar{\boldsymbol{K}} & \bar{\boldsymbol{B}}^\top \\ \bar{\boldsymbol{B}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}} \\ \bar{\boldsymbol{b}} \end{bmatrix}$$

$$\bar{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{I} & 0 & 0 \\ 0 & \tilde{\boldsymbol{K}}_{ff} & \tilde{\boldsymbol{K}}_{fR} \\ 0 & \tilde{\boldsymbol{K}}_{Rf} & \tilde{\boldsymbol{K}}_{RR} \end{bmatrix} \qquad \bar{\boldsymbol{f}} = \begin{bmatrix} \tilde{\boldsymbol{g}}_d \\ \tilde{\boldsymbol{f}}_f - \tilde{\boldsymbol{K}}_{fd}\tilde{\boldsymbol{g}}_d \\ \tilde{\boldsymbol{f}}_R - \tilde{\boldsymbol{K}}_{Rd}\tilde{\boldsymbol{g}}_d \end{bmatrix}$$

$$\bar{\boldsymbol{B}} = \begin{bmatrix} 0 & \tilde{\boldsymbol{B}}_f & \tilde{\boldsymbol{B}}_R \end{bmatrix} \qquad \bar{\boldsymbol{b}} = \tilde{\boldsymbol{b}} - \tilde{\boldsymbol{B}}_d\tilde{\boldsymbol{g}}_d$$

Notation primal-/remaining DOFs

- notation to identify primal and remaining DOF

$$\bar{\boldsymbol{K}}_{PP} = \begin{bmatrix} \boldsymbol{I} & 0 \\ 0 & \tilde{\boldsymbol{K}}_{ff} \end{bmatrix} \qquad \bar{\boldsymbol{K}}_{PR} = \begin{bmatrix} 0 \\ \tilde{\boldsymbol{K}}_{fR} \end{bmatrix}$$

$$\bar{\boldsymbol{K}}_{RP} = \begin{bmatrix} \boldsymbol{I} & \\ 0 & \tilde{\boldsymbol{K}}_{Rf} \end{bmatrix} \qquad \bar{\boldsymbol{K}}_{RR} = \begin{bmatrix} \tilde{\boldsymbol{K}}_{RR} \end{bmatrix}$$

$$\bar{\boldsymbol{f}}_{P} = \begin{bmatrix} \tilde{\boldsymbol{g}}_{d} \\ \tilde{\boldsymbol{f}}_{f} - \tilde{\boldsymbol{K}}_{fd}\tilde{\boldsymbol{g}}_{d} \end{bmatrix} \qquad \bar{\boldsymbol{f}}_{R} = \bar{\boldsymbol{f}} = \begin{bmatrix} \tilde{\boldsymbol{f}}_{R} - \tilde{\boldsymbol{K}}_{Rd}\tilde{\boldsymbol{g}}_{d} \end{bmatrix}$$

$$\bar{\boldsymbol{B}}_{P} = \begin{bmatrix} 0 & \tilde{\boldsymbol{B}}_{f} \end{bmatrix} \qquad \bar{\boldsymbol{B}}_{R} = \begin{bmatrix} \tilde{\boldsymbol{B}}_{R} \end{bmatrix}$$

Dual problem

- for both model problems $\bar{\boldsymbol{K}}$ regular

$$\begin{bmatrix} \bar{\boldsymbol{K}} & \bar{\boldsymbol{B}}^\top \\ \bar{\boldsymbol{B}} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{u}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{f}} \\ \bar{\boldsymbol{b}} \end{bmatrix}$$

- first line reads as $\tilde{\boldsymbol{u}} = \bar{\boldsymbol{K}}^{-1}(\bar{\boldsymbol{f}} - \bar{\boldsymbol{B}}^\top \boldsymbol{\lambda})$
- insert in second line

$$\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{B}}^\top \boldsymbol{\lambda} = \bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{f}} - \bar{\boldsymbol{b}} \qquad (*)$$

On the (*) system

- to realize $\bar{\boldsymbol{K}}^{-1}$ use block factorization

$$\bar{\boldsymbol{K}}^{-1} = \begin{bmatrix} \boldsymbol{I} & 0 \\ -\bar{\boldsymbol{K}}_{RR}^{-1}\bar{\boldsymbol{K}}_{RP} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{S}}_{PP}^{-1} & 0 \\ 0 & \bar{\boldsymbol{K}}_{RR}^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\bar{\boldsymbol{K}}_{RR}^{-1}\bar{\boldsymbol{K}}_{RP} \\ 0 & \boldsymbol{I} \end{bmatrix}$$

  where $\bar{\boldsymbol{S}}_{PP} = \bar{\boldsymbol{K}}_{PP} - \bar{\boldsymbol{K}}_{RR}^{-1}\bar{\boldsymbol{K}}_{RP}$
- **Note:** $\bar{\boldsymbol{K}}_{RR}$ is block diagonal (solve local problems independently on each subdomain i.e. sparse LU fact.)
- $\bar{\boldsymbol{S}}_{PP}$ can be assembled from local contributions $\bar{\boldsymbol{S}}_{PP}^{(i)} = \bar{\boldsymbol{K}}_{PP}^{(i)} - (\bar{\boldsymbol{K}}_{RR}^{(i)})^{-1}\bar{\boldsymbol{K}}_{RP}^{(i)}$ and by [Davis 2006] it is
  - sparse
  - can be factorized using standard sparse LU factorization

On the (*) system

- solve symmetric and positive definite system (*)

$$\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{B}}^{\top}\boldsymbol{\lambda} = \bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{f}} - \bar{\boldsymbol{b}} \qquad (*)$$

  for $\boldsymbol{\lambda}$ by CG algorithm

- calculate

$$\tilde{\boldsymbol{u}}_P = \bar{\boldsymbol{S}}_{PP}^{-1}\left(\bar{\boldsymbol{f}}_P - \bar{\boldsymbol{B}}_P^{\top}\boldsymbol{\lambda} - \bar{\boldsymbol{K}}_{PR}\bar{\boldsymbol{K}}_{RR}^{-1}(\bar{\boldsymbol{f}}_R - \bar{\boldsymbol{B}}_R^{\top}\boldsymbol{\lambda})\right)$$

$$\tilde{\boldsymbol{u}}_R = \bar{\boldsymbol{K}}_{RR}^{-1}\left(\bar{\boldsymbol{f}}_P - \bar{\boldsymbol{B}}_P^{\top}\boldsymbol{\lambda} - \bar{\boldsymbol{K}}_{RP}\tilde{\boldsymbol{u}}_P\right)$$

- FETI [Farhat et al. 1994] $\rightarrow \kappa(\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{B}}^{\top}) = \mathcal{O}(H/h)$ where H, h denote subdomain size/FE mesh size
- [Kleiss et al. 2012] $\rightarrow$IETI behaves similarly

Preconditioner (scaled Dirichlet dual-primal)

- construction in [Kleiss et al. 2012] follows scaled Dirichlet preconditioner – extended to dual-primal formulation
- again separation of DOFs $\boldsymbol{K}^{(i)} = \begin{bmatrix} \boldsymbol{K}_{II}^{(i)} & \boldsymbol{K}_{IB}^{(i)} \\ \boldsymbol{K}_{BI}^{(i)} & \boldsymbol{K}_{BB}^{(i)} \end{bmatrix}$

  interior DOF subscript I on $\{\Omega^{(i)}\}^\circ$ DOFs
  boundary DOF subscript B ond $\Omega^{(i)}$ DOFs
- dual-primal Dirichlet preconditioner defined by $\boldsymbol{S}_{BB}^{(i)} = \boldsymbol{K}_{BB}^{(i)} - \boldsymbol{K}_{BI}^{(i)}(\boldsymbol{K}_{II}^{(i)})^{-1}\boldsymbol{K}_{IB}^{(i)}$

$$\boldsymbol{M}^{-1} := \sum_{i=1}^{N} \boldsymbol{D}^{(i)}\boldsymbol{B}^{(i)} \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{S}_{BB}^{(i)} \end{bmatrix} (\boldsymbol{B}^{(i)})^\top \boldsymbol{D}^{(i)}$$

Preconditioner (scaled Dirichlet dual-primal)

- here $\boldsymbol{B}^{(i)}$ restriction of $\boldsymbol{B}$ to the interface conditions associated with $\Omega^{(i)}$
- scaled diagonal matrix $\boldsymbol{D}^{(i)} \in \mathbb{R}^{J \times J}$ (Model problem I $\leftrightarrow$ Model Problem 2)
- **Remark:** $\boldsymbol{K}_{II}^{(i)}$ can be factorized as easily and cheaply as $\boldsymbol{K}_{RR}^{(i)}$
- **Preconditioned FETI**

$$\kappa(\boldsymbol{M}^{-1}\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{B}}^{\top}) = \mathcal{O}(1 + \log(H/h)^2)$$

- numerics show similar behaviour for IETI

Isogeometric Tearing and Interconnecting ALGORITHM

Step 1 for each $i = 1, \cdots, N$ locally on each subdomain $\Omega^{(i)}$ (in parallel)

1. assemble local $\boldsymbol{K}^{(i)}, \boldsymbol{f}^{(i)}$ using a fully local numbering of the DOF
2. partition $\boldsymbol{K}^{(i)}, \boldsymbol{f}^{(i)}$ in primal-/remaining DOFs to $\tilde{\boldsymbol{K}}, \tilde{\boldsymbol{f}}$
3. factorize $\boldsymbol{K}_{RR}^{(i)}$ and calculate $\bar{\boldsymbol{S}}_{PP}^{(i)}$
4. partition $\boldsymbol{K}^{(i)}, \boldsymbol{f}^{(i)}$ into interior/boundary DOFs
5. factorize $\boldsymbol{K}_{II}^{(i)}$

Isogeometric Tearing and Interconnecting ALGORITHM

Step 2 assemble and factorize $\bar{\boldsymbol{S}}_{PP}$ and calculate $\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{f}} - \bar{\boldsymbol{b}}$

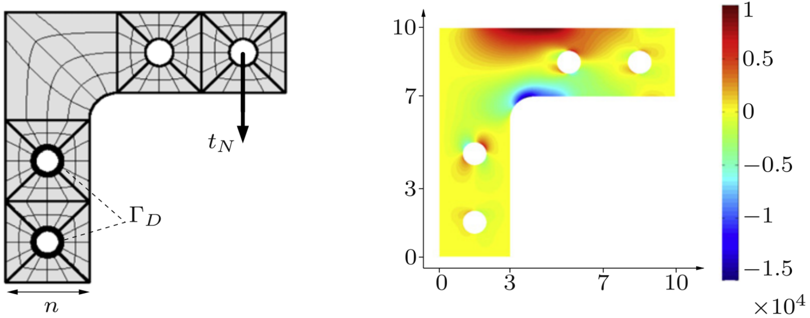
Step 3 solve $\bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{B}}^{\top}\boldsymbol{\lambda} = \bar{\boldsymbol{B}}\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{f}} - \bar{\boldsymbol{b}}$ by PCG with preconditioner $\boldsymbol{M}^{-1}$

Step 4 calculate

$$\tilde{\boldsymbol{u}}_P = \bar{\boldsymbol{S}}_{PP}^{-1}\left(\bar{\boldsymbol{f}}_P - \bar{\boldsymbol{B}}_P^{\top}\boldsymbol{\lambda} - \bar{\boldsymbol{K}}_{PR}\bar{\boldsymbol{K}}_{RR}^{-1}(\bar{\boldsymbol{f}}_R - \bar{\boldsymbol{B}}_R^{\top}\boldsymbol{\lambda})\right)$$

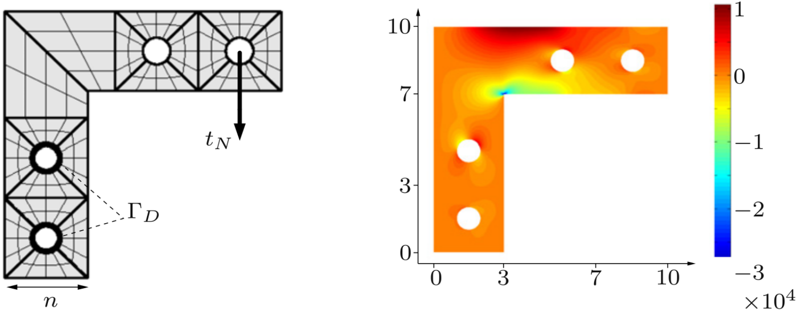
Step 5 for each $i = 1, \cdots, N$ obtain $\tilde{\boldsymbol{u}}_R^{(i)}$ (in parallel)

$$\tilde{\boldsymbol{u}}_R^{(i)} = (\bar{\boldsymbol{K}}_{RR}^{(i)})^{-1}\left(\bar{\boldsymbol{f}}_P^{(i)} - (\bar{\boldsymbol{B}}_P^{(i)})^{\top}\boldsymbol{\lambda} - \bar{\boldsymbol{K}}_{RP}^{(i)}\tilde{\boldsymbol{u}}_P\right)$$



(a) Setting in case (A). (b) Stress component σ_{11}.

Figure: Case (A), bracket with rounded reentrant corner.



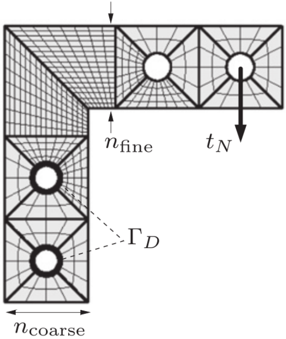
(a) Setting in case (B).

(b) Stress component σ_{11}.

Figure: Case (B), bracket with sharp reentrant corner.

Case	n	$\#\boldsymbol{\lambda}$	condition numbers $\mathbf{F}$	condition numbers $\mathbf{M}^{-1}\mathbf{F}$	(P)CG iterations $\mathbf{F}$	(P)CG iterations $\mathbf{M}^{-1}\mathbf{F}$
(A)	8	464	173.95	39.60	69	42
	16	784	470.09	67.98	103	49
	32	1424	1230.11	105.87	149	58
	64	2704	3074.62	154.10	> 200	67
(B)	8	484	174.17	51.72	70	43
	16	820	509.39	85.71	106	50
	32	1492	1388.58	130.24	150	60
	64	2836	3543.44	185.42	> 200	67

Figure: Condition numbers and (P)CG iterations for cases (A) and (B) of a bracket under load.



(a) Setting in case (C).

Figure: Case (C), bracket with sharp reentrant corner and local h-refinement near the corner. The ratio $n_{fine}/n_{coarse} = 4$ is the same for all chosen meshes.

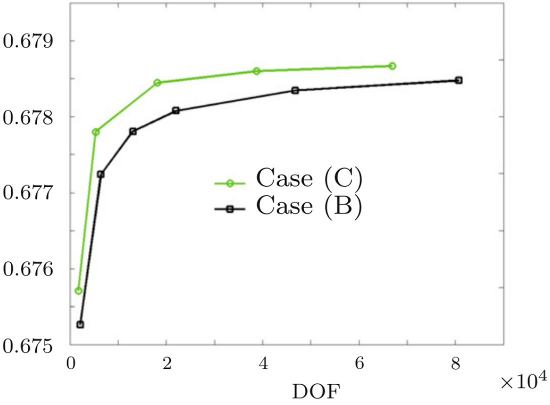
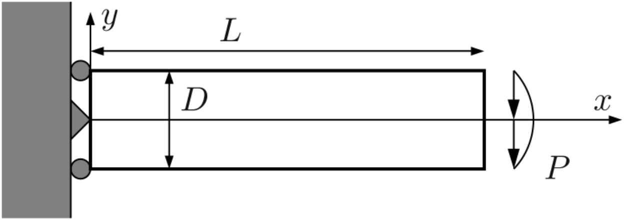


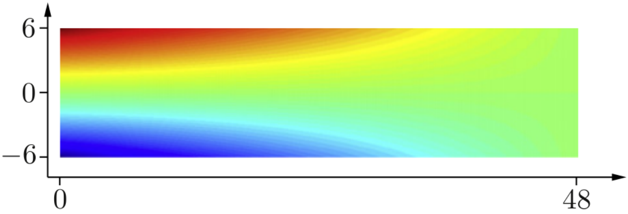
Figure: Comparison of the energy norms of the discrete solutions in cases (B) and (C).

n_{coarse}	n_{fine}	$\#\boldsymbol{\lambda}$	conditon numbers $\mathbf{F}$	$\mathbf{M}^{-1}\mathbf{F}$	$\mathbf{M}_A^{-1}\mathbf{F}$	(P)CG iterations $\mathbf{F}$	$\mathbf{M}^{-1}\mathbf{F}$	$\mathbf{M}_A^{-1}\mathbf{F}$
4	16	428	469.89	344.72	86.40	71	69	45
8	32	708	1324.01	542.19	130.75	105	89	57
16	64	1268	3419.50	820.16	185.40	148	103	66
32	128	2388	8448.24	1170.29	258.89	> 200	120	75

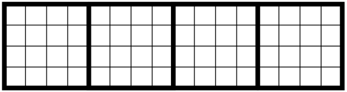
(b) Condition numbers and (P)CG iterations for case (C).



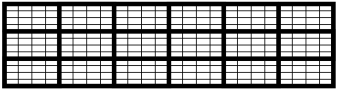
(a) Problem setting.



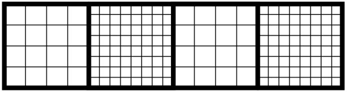
(b) Stress component σ_{11}.



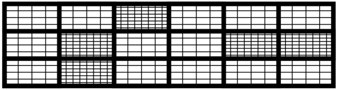
Setting (A), 4 subdomains, fully matching.



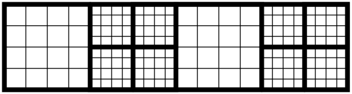
Setting (B), 18 subdomains, fully matching.



Setting (C), 4 subdomains, 2 of which are h-refined.



Setting (D), 18 subdomains, 5 of which are h-refined.



Setting (E), 10 subdomains, with hanging vertices.

Setting	n_{coarse}	n, n_{fine}	$\#\boldsymbol{\lambda}$	condition numbers $\mathbf{F}$	condition numbers $\mathbf{M}^{-1}\mathbf{F}$	condition numbers $\mathbf{M}_A^{-1}\mathbf{F}$	(P)CG iterations $\mathbf{F}$	(P)CG iterations $\mathbf{M}^{-1}\mathbf{F}$	(P)CG iterations $\mathbf{M}_A^{-1}\mathbf{F}$
(A)	*	8	88	33.80	11.81	†	24	15	†
	*	16	152	77.02	14.99	†	33	16	†
	*	32	280	170.81	18.38	†	40	18	†
	*	64	536	378.38	23.49	†	50	20	†
(B)	*	8	700	53.87	13.50	†	48	24	†
	*	16	1100	116.68	16.79	†	67	27	†
	*	32	2140	264.72	20.38	†	85	30	†
	*	64	4060	595.88	24.21	†	120	34	†
(C)	8	16	136	66.03	109.10	28.43	23	31	21
	16	32	248	157.41	146.07	35.53	37	38	22
	32	64	472	358.85	184.14	43.02	48	41	24
	64	128	920	795.88	224.55	51.11	62	47	28
(D)	4	8	580	46.38	74.78	31.02	50	53	35
	8	16	940	102.90	109.54	38.59	68	63	39
	16	32	1660	231.74	142.50	47.01	93	71	43
	32	64	3100	530.70	175.41	55.94	123	77	46
(E)	*	8	338	184.11	351.08	98.71	53	58	37
	*	16	578	387.44	433.00	141.50	69	63	41
	*	32	1058	813.77	532.37	194.07	90	69	46
	*	64	2018	1733.71	647.72	257.28	119	73	52

(d) Condition numbers and (P)CG iterations of the unpreconditioned and the preconditioned interface problem.
*In cases (A), (B), and (E), the number of knot spans is the same on all subdomains.
†In fully matching settings, we have $\mathbf{M}_A^{-1} = \mathbf{M}^{-1}$.

The End