High order FEM vs. IgA

by J.A.Evans, T.J.Hughes and A.Reali, 2014

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Jan 20, 2017

Overview

Introduction

Introduction

- Spectral approximation properties of FE and B-splines
- Investigation of eigenvalue approximation in former papers
  $\rightarrow$ Good approximation quality of B-spline
  $\rightarrow$ FE approximation diverged with p

Introduction

What are the effects of this results to BVP and IVP?

- This question will be answered throughout this presentation

Introduction

- Need approximations from a global perspective
- "Pythagorean eigenvalue error theorem" pertains to all modes

Introduction

- Only the Laplace operator is considered
- Domain $\Omega \subset \mathbb{R}^d$ is bounded and connected
- $\partial\Omega$ is Lipschitz
- $H^m(\Omega) := \{f \in L^2 | D^\alpha f \in L^2, \forall |\alpha| \leq m\}$
- $V \subseteq (H^m(\Omega))^n$ closed
- Functions in V satisfy appropriate boundary conditions
- $d, m, n \in \mathbb{N}$

- $(\cdot, \cdot)$ and $a(\cdot, \cdot)$ are symmetric, bilinear with

  1. $a(v, w) \leq \|v\|_E \|w\|_E$
  2. $\|w\|_E^2 = a(w, w)$
  3. $(v, w) \leq \|v\| \|w\|$
  4. $\|w\|^2 = (w, w)$

  where $\|\cdot\|_E$ is the energy norm which is equivalent to the $(H^m(\Omega))^n$ norm on V, $\|\cdot\|$ is the $L^2(\Omega)^n$ norm, v, $w \in V$

The Elliptic eigenvalue problem

Continuous eigenvalue problem formulation

Find eigenvalues $\lambda_\ell \in \mathbb{R}^+$ and eigenfunctions $u_\ell \in V$ for $\ell \in \mathbb{N}$, s.t., for all $w \in V$

$$\lambda_\ell(w, u_\ell) = a(w, u_\ell)$$

$\rightarrow 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $(u_k, u_\ell) = \delta_{k\ell}$

$$\Rightarrow \|u_\ell\|_E^2 = a(u_\ell, u_\ell) = \lambda_\ell \text{ and } a(u_k, u_\ell) = 0, \ell \neq k$$

Discrete eigenvalue problem formulation

Find eigenvalues $\lambda_\ell^h \in \mathbb{R}^+$ and eigenfunctions $u_\ell^h \in V^h$, s.t., for all $w^h \in V^h$

$$\lambda_\ell^h (w^h, u_\ell^h) = a(w^h, u_\ell^h)$$

$\rightarrow 0 < \lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_N^h$ where $dim(V^h) = N$ and $(u_k^h, u_\ell^h) = \delta_{k\ell}$

$$\Rightarrow \|u_\ell^h\|_E^2 = a(u_\ell^h, u_\ell^h) = \lambda_\ell^h \text{ and } a(u_k^h, u_\ell^h) = 0, \ell \neq k$$

- Comparison of $\{\lambda_\ell^h, u_\ell^h\}$ to $\{\lambda_\ell, u_\ell\}$ for $\ell = 1, \ldots, N$ is important
- Based on "Pythagorean eigenvalue error theorem" as introduced by G. Strang and G. Fix

$$\frac{\lambda_\ell^h - \lambda_\ell}{\lambda_\ell} + \|u_\ell^h - u_\ell\|^2 = \frac{\|u_\ell^h - u_\ell\|_E^2}{\lambda_\ell}, \qquad \forall \ell = 1, \ldots, N$$

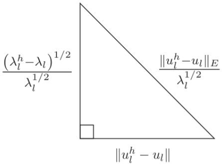


Figure: Graphical representation of the Pythagorean eigenvalue error theorem

The Elliptic eigenvalue problem

Variational model problem

$$\lambda_\ell (w, u_\ell) = a(w, u_\ell)$$

where

$$a(w, u_\ell) = \int_0^1 \frac{\partial w}{\partial x} \frac{\partial u_\ell}{\partial x} dx$$

$$(w, u_\ell) = \int_0^1 w u_\ell dx$$

and with homogeneous Dirichlet boundary conditions and $\Omega = (0, 1)$

The Elliptic eigenvalue problem

For $\ell \in \mathbb{N}$,

- Eigenvalues are $\lambda_\ell = \pi^2 \ell^2$
- Eigenfunctions are $u_\ell = \sqrt{2} sin(\ell \pi x)$

The Elliptic eigenvalue problem

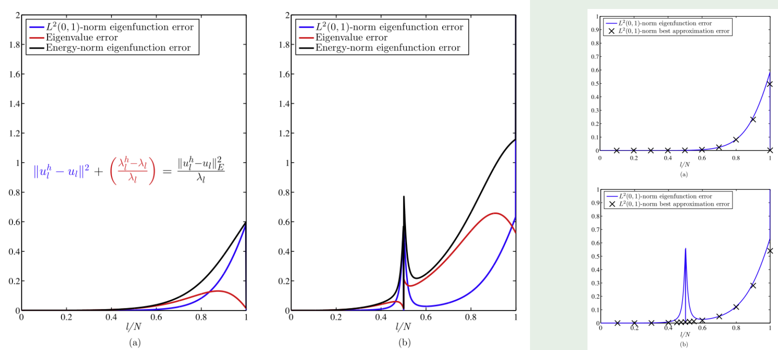
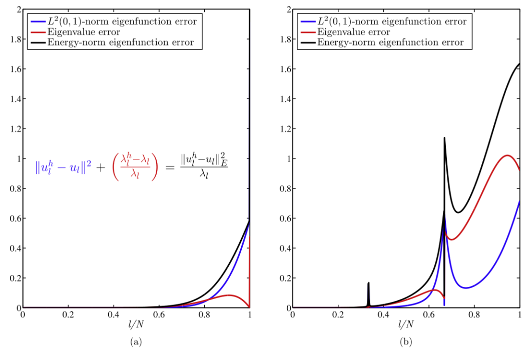


Figure: Pythagorean eigenvalue error theorem budget and L^2-eigenfunction error for quadratic elements. (a) C^1-continuous B-splines; (b) C^0 continuous finite elements, N = 99

The Elliptic eigenvalue problem



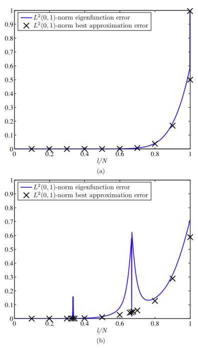
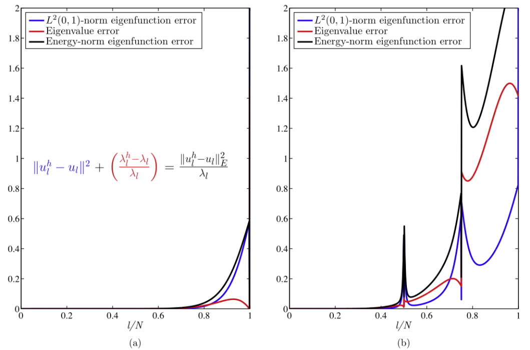


Figure: Pythagorean eigenvalue error theorem budget and L^2-eigenfunction error for cubic elements. (a) C^2-continuous B-splines; (b) C^0 continuous finite elements, N = 99

The Elliptic eigenvalue problem



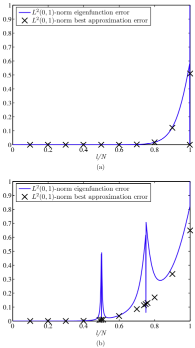


Figure: Pythagorean eigenvalue error theorem budget for quartic elements. (a) C^3-continuous B-splines; (b) C^0 continuous finite elements, N $= 99$

Results of this section

- An accurate eigenvalue does not imply an accurate eigenfunction
- The higher the eigenvalue the greater the eigenfunction error is false
- B-splines yield better approximations
- "outlier" modes at the end of the spectrum
- "outlier" modes do not spoil the accuracy in the interior

The Elliptic boundary value problem

The Elliptic boundary value problem

- Let $f \in (L^2(\Omega))^n$

Continuous variational problem

Find $u \in V$ such that for all $w \in V$

$$a(w, u) = (w, f)$$

Discrete variational problem

Find $u^h \in V^h$ such that for all $w^h \in V^h$

$$a(w^h, u^h) = (w^h, f)$$

The Elliptic boundary value problem

- Approximation of the continuous problem by the discrete one
- Eigenfunction expansion

$$u = \sum_{\ell=1}^{\infty} d_\ell u_\ell \text{ and } u^h = \sum_{\ell=1}^{N} d_\ell^h u_\ell^h$$

where d_ℓ and d_ℓ^h denote the Fourier-coefficients of the continuous and the discrete solutions, respectively

The Elliptic boundary value problem

- $\lambda_\ell d_\ell = f_\ell \overset{def.}{=} (u_\ell, f)$
- $\lambda_\ell^h d_\ell^h = f_\ell^h \overset{def.}{=} (u_\ell^h, f)$

$\Rightarrow u(x) = \sum_{\ell=1}^{\infty} \frac{f_\ell}{\lambda_\ell} u_\ell(x)$ and $u^h(x) = \sum_{\ell=1}^{N} \frac{f_\ell^h}{\lambda_\ell^h} u_\ell^h(x)$

The Elliptic boundary value problem

- The error in the solution is

$$e(x) = u^h(x) - u(x) = \bar{e}(x) + e'(x)$$

with

$$\bar{e}(x) = \sum_{\ell=1}^{N} \bar{e}_\ell(x) = \sum_{\ell=1}^{N} \left(\frac{f_\ell^h}{\lambda_\ell^h} u_\ell^h(x) - \frac{f_\ell}{\lambda_\ell} u_\ell(x) \right)$$

$$e'(x) = \sum_{\ell=1}^{N} e'_\ell(x) = \sum_{\ell=N+1}^{\infty} \left(-\frac{f_\ell}{\lambda_\ell} u_\ell(x) \right)$$

The Elliptic boundary value problem

- For a specific $\ell \in 1, \ldots, N$

$$\|\bar{e}_\ell\| \leq 2\frac{\|f\|}{\lambda_\ell}\left(\frac{\|e_\ell\|_E}{\lambda_\ell^{1/2}}\left(1+\frac{1}{2}\frac{\|e_\ell\|_E}{\lambda_\ell^{1/2}}\right)\left(1+\frac{\|e_\ell\|_E^2}{\lambda_\ell}\right)+\frac{1}{2}\frac{\|e_\ell\|_E^2}{\lambda_\ell}\right)$$

$$\|\bar{e}_\ell\|_E \leq 2\frac{\|f\|\|e_\ell\|_E}{\lambda_\ell}\left(1+\frac{\|e_\ell\|_E}{\lambda_\ell^{1/2}}\left(1+\frac{\|e_\ell\|_E}{\lambda_\ell^{1/2}}+\frac{1}{2}\frac{\|e_\ell\|_E^2}{\lambda_\ell}\right)\right)$$

where $e_\ell = u_\ell^h - u_\ell$

The Elliptic boundary value problem

- Discrete solution can be large in error
- Elliptic boundary-value problems are usually forgiving

The Parabolic initial-value problem

The Parabolic initial-value problem

- Let $f \in L^2((0,T);(L^2(\Omega))^n)$ and $U \in (L^2(\Omega))^n)$ and $T \in \mathbb{R}^+$

Continuous variational problem

Find $u \in V_T$ such that for all $w \in V$ and almost all $t \in (0,T)$

$$\langle w, \frac{\partial u}{\partial t}(t)\rangle + a(w,u(t)) = (w, f(t))$$
$$(w, u(0)) = (w, U)$$

where $V_T := \{v \in L^2((0,T);V) : \frac{\partial v}{\partial t} \in L^2((0,T);V^*)\}$
and $\langle \cdot, \cdot \rangle$ is the duality pairing

The Parabolic initial-value problem

Semi-discrete variational problem

Find $u^h \in V_T^h$ such that for all $w^h \in V^h$ and almost all $t \in (0, T)$

$$\begin{aligned} \langle w^h, \frac{\partial u^h}{\partial t}(t)\rangle + a(w^h, u^h(t)) &= (w^h, f(t)) \\ (w^h, u^h(0)) &= (w^h, U) \end{aligned}$$

where $V_T^h := \{v \in L^2((0,T); V^h) : \frac{\partial v}{\partial t} \in L^2((0,T); V^{h*})\}$
and $\langle \cdot, \cdot \rangle$ is the duality pairing

The Parabolic initial-value problem

$$u(t) = \sum_{\ell=1}^{\infty} d_\ell(t) u_\ell \text{ and } u^h(t) = \sum_{\ell=1}^{N} d_\ell^h(t) u_\ell^h$$

yield

$$\begin{aligned} d_\ell'(t) + \lambda_\ell d_\ell(t) &= f_\ell(t) \stackrel{def}{=} (u_\ell, f(t)) \\ d_\ell(0) &= U_\ell \stackrel{def}{=} (u_\ell, U) \end{aligned}$$

and

$$\begin{aligned} d_\ell'^h(t) + \lambda_\ell^h d_\ell^h(t) &= f_\ell^h(t) \stackrel{def}{=} (u_\ell^h, f(t)) \\ d_\ell^h(0) &= U_\ell^h \stackrel{def}{=} (u_\ell^h, U) \end{aligned}$$

The Parabolic initial-value problem

- Solving the ordinary differential equations yield

$$d_\ell(t) = U_\ell exp(-\lambda_\ell t) + \int_0^t exp(-\lambda_\ell(t-\tau))f_\ell(\tau)d\tau$$

and

$$d_\ell^h(t) = U_\ell^h exp(-\lambda_\ell^h t) + \int_0^t exp(-\lambda_\ell^h(t-\tau))f_\ell^h(\tau)d\tau$$

The Parabolic initial-value problem

- From the Fourier-coefficients, we obtain

$$u(x,t) = \sum_{\ell=1}^{\infty} \left(U_\ell exp(-\lambda_\ell t) + \int_0^t exp(-\lambda_\ell (t-\tau)) f_\ell(\tau) d\tau \right) u_\ell(x)$$

and

$$u^h(x,t) = \sum_{\ell=1}^{N} \left(U_\ell^h exp(-\lambda_\ell^h t) + \int_0^t exp(-\lambda_\ell^h (t-\tau)) f_\ell^h(\tau) d\tau \right) u_\ell^h(x)$$

- The error is $e(x,t) = u^h(x,t) - u(x,t) = \overline{e}(x,t) + e'(x,t)$

The Parabolic initial-value problem

- $\bar{e}(x,t)$ is caused by eigenvalue and eigenfunction errors
- Errors in decay rates are only due to eigenvalue errors
- Initial error is due to projection error
- Important are times up to $t = \mathcal{O}(\lambda_\ell^{-1})$
- Similar for f
- $\|\bar{e}_\ell\| \to 0$ as $-\lambda_\ell t \to -\infty$
- $\|\bar{e}_\ell\|_E \to 0$ as $-\lambda_\ell t \to -\infty$

The Hyperbolic initial-value problem

The Hyperbolic initial-value problem

- Let $f \in L^2((0,T);(L^2(\Omega))^n)$, $U_0 \in (L^2(\Omega))^n)$, $U_1 \in V^*$ and $T \in \mathbb{R}^+$

Continuous variational problem

Find $u \in V_T$ such that for all $w \in V$ and almost all $t \in (0,T)$

$$\begin{aligned}
\langle w, \frac{\partial^2 u}{\partial t^2}(t)\rangle + a(w, u(t)) &= (w, f(t)) \\
(w, u(0)) &= (w, U_0) \\
\langle w, \frac{\partial u}{\partial t}(0)\rangle &= \langle w, U_1\rangle
\end{aligned}$$

where $V_T := \{v \in L^2((0,T);V) : \frac{\partial v}{\partial t} \in L^2\left((0,T);(L^2(\Omega))^n\right)$ and $\frac{\partial^2 v}{\partial t^2} \in L^2((0,T);V^*)\}$ and $\langle \cdot, \cdot \rangle$ is the duality pairing

The Hyperbolic initial-value problem

Semi-discrete variational problem

Find $u^h \in V_T^h$ such that for all $w^h \in V^h$ and almost all $t \in (0, T)$

$$\begin{aligned}
\langle w^h, \frac{\partial^2 u^h}{\partial t^2}(t)\rangle + a(w^h, u^h(t)) &= (w^h, f(t)) \\
(w^h, u^h(0)) &= (w^h, U_0) \\
\langle w^h, \frac{\partial u^h}{\partial t}(0)\rangle &= \langle w^h, U_1\rangle
\end{aligned}$$

where $V_T^h := \{v \in L^2((0,T); V^h) : \frac{\partial v}{\partial t} \in L^2\left((0,T); V^h\right)$ and $\frac{\partial^2 v}{\partial t^2} \in L^2((0,T); V^{h*})\}$ and $\langle\cdot,\cdot\rangle$ is the duality pairing

The Hyperbolic initial-value problem

- Proceeding as in the previous chapter yield

$$
\begin{aligned}
d_\ell''(t) + \lambda_\ell d_\ell(t) = f_\ell(t) &\overset{def}{=} (u_\ell, f(t)) \\
d_\ell(0) = U_{0\ell} &\overset{def}{=} (u_\ell, U_0) \\
d_\ell'(0) = U_{1\ell} &\overset{def}{=} \langle u_\ell, U_1 \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
d_\ell''^h(t) + \lambda_\ell^h d_\ell^h(t) = f_\ell^h(t) &\overset{def}{=} (u_\ell^h, f(t)) \\
d_\ell^h(0) = U_{0\ell}^h &\overset{def}{=} (u_\ell^h, U_0) \\
d_\ell'^h(0) = U_{1\ell}^h &\overset{def}{=} \langle u_\ell^h, U_1 \rangle
\end{aligned}
$$

The Hyperbolic initial-value problem

- The solutions of these ordinary differential equations are

$$d_\ell(t) = U_{0\ell} cos(\omega_\ell t) + \frac{U_{1\ell}}{\omega_\ell} sin(\omega_\ell t) + \frac{1}{\omega_\ell} \int_0^t sin(\omega_\ell (t-\tau)) f_\ell(\tau) d\tau$$

and

$$d_\ell^h(t) = U_{0\ell}^h cos(\omega_\ell^h t) + \frac{U_{1\ell}^h}{\omega_\ell^h} sin(\omega_\ell^h t) + \frac{1}{\omega_\ell^h} \int_0^t sin(\omega_\ell^h (t-\tau)) f_\ell^h(\tau) d\tau$$

with $\omega_\ell = (\lambda_\ell)^{1/2}$ and $\omega_\ell^h = (\lambda_\ell^h)^{1/2}$

The Hyperbolic initial-value problem

- Plug the Fourier-coefficients into eigenfunction expansion
- $e(x, t) = u^h(x, t) - u(x, t) = \overline{e}(x, t) + e'(x, t)$
- Modal error can be bounded by eigenfunctions and eigenvalue errors
- Modal errors oscillate in time

The Hyperbolic initial-value problem

Numerical investigation of hyperbolic approximations

For $U_0 = sin(51\pi x)$, $U_1 = 0$ and $f = 0$ the exact solution to the hyperbolic initial-value problem is

$$u(x,t) = sin(51\pi x)cos(51\pi t)$$

with solution coefficients

$$U_{0\ell} = \begin{cases} 1/\sqrt{2} & \text{if } \ \ell = 51 \\ 0 & \text{otherwise} \end{cases}, U_{1\ell} = 0 \text{ and } f_\ell = 0$$

on $\Omega = (0,1)$

The Hyperbolic initial-value problem

- Approximation is given by

$$u^h(x,t) = \sum_{\ell=1}^{N} \{ U_{0\ell}^h cos(\omega_\ell^h t) + \frac{U_{1\ell}^h}{\omega_\ell^h} sin(\omega_\ell^h t)$$
$$+ \frac{1}{\omega_\ell^h} \int_0^t sin(\omega_\ell^h (t-\tau)) f_\ell^h(\tau) d\tau \} u_\ell^h(x)$$

- C^3-continuous quartic B-splines and C^0-continuous quartic finite elements

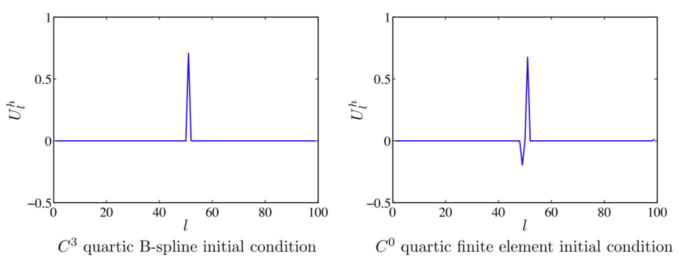


Figure: Initial condition coefficients $U^h_{0\ell}$ for C^3-cont. quartic B-splines(left) and C^0-cont. quartic FE-solutions(right), N = 99

- Single wave and composition of two different waves

The Hyperbolic initial-value problem

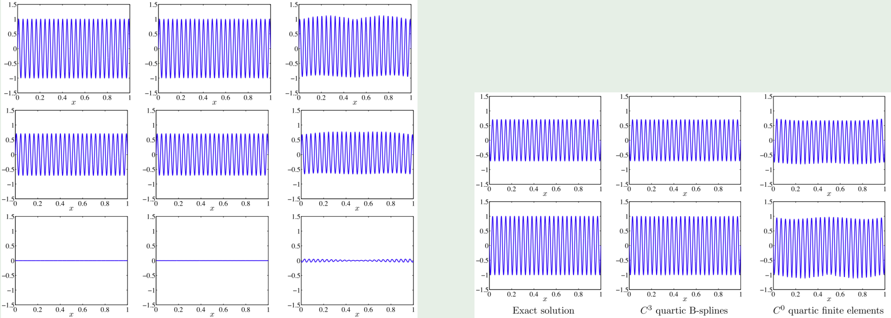


Figure: Exact and numerical solutions for $t = 0, 0.25/51, 0.5/51, 0.75/51, 1/51$, N = 99

The Hyperbolic initial-value problem

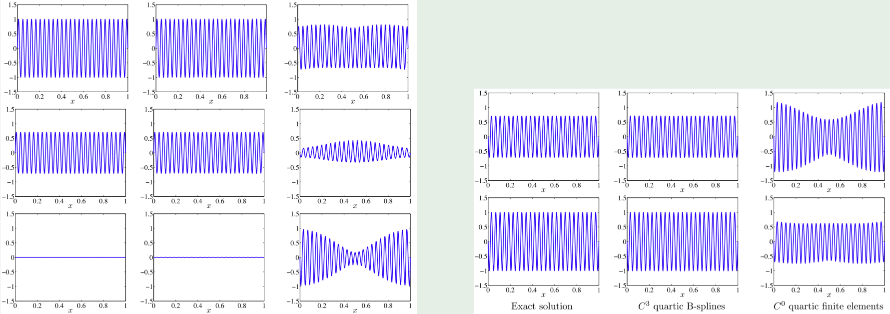
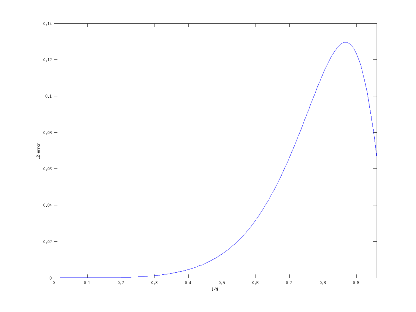


Figure: Exact and numerical solutions for $t = 10, 10.25/51, 10.5/51, 10.75/51, 11/51$, N = 99

Own results

- Own tests with Laplace-eigenvalue problem



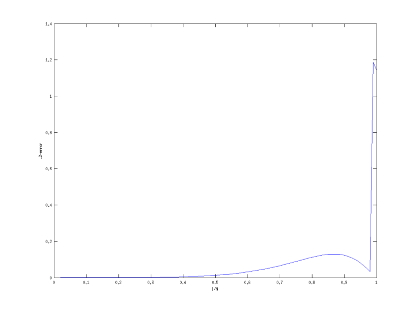
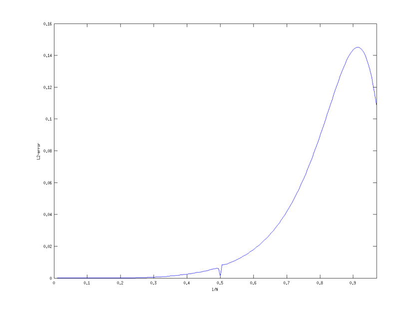


Figure: Eigenvalue error for polynomial degree 2 and with C^1-continuity, N $= 100$

Own results



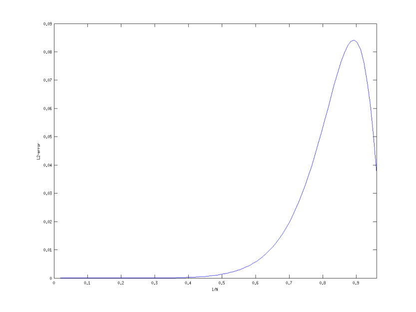
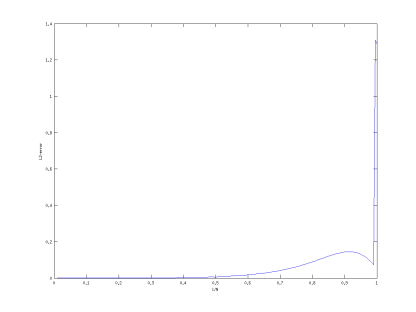


Figure: Eigenvalue error for polynomial degree 3 and with C^1 and C^2-continuity, respectively, N = 100

Own results



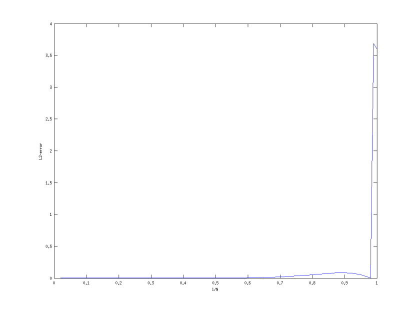
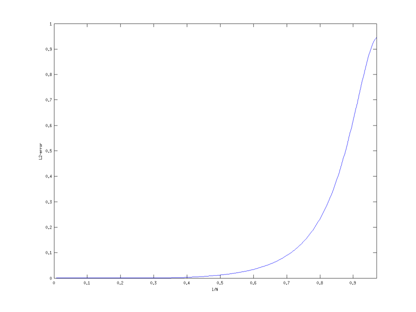


Figure: Eigenvalue error for polynomial degree 3 and with C^1 and C^2-continuity, respectively, N = 100

Own results



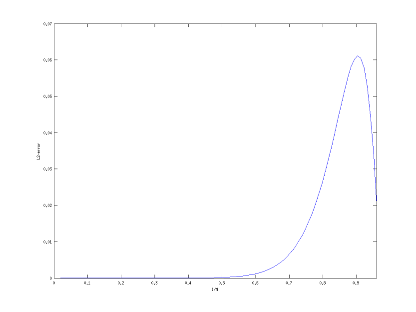


Figure: Eigenvalue error for polynomial degree 4 and with C^1 and C^3-continuity, respectively, N $= 100$

Own results

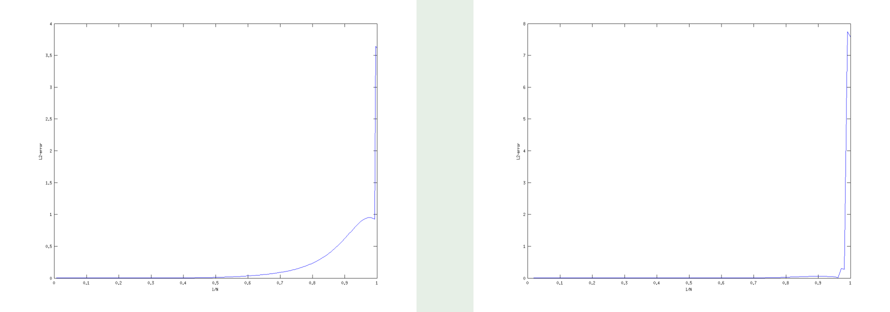


Figure: Eigenvalue error for polynomial degree 4 and with C^1 and C^3-continuity, respectively, $\mathrm{N} = 100$

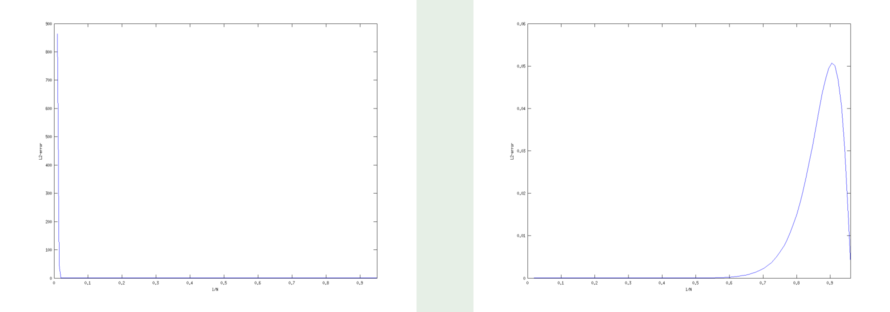


Figure: Eigenvalue error for polynomial degree 5 and with C^1 and C^4-continuity, respectively N = 50, 100

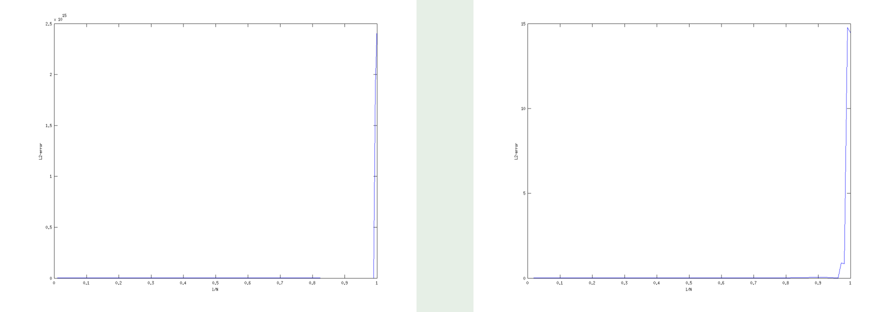


Figure: Eigenvalue error for polynomial degree 5 and with C^1 and C^4-continuity, respectively, N = 50, 100

Own results

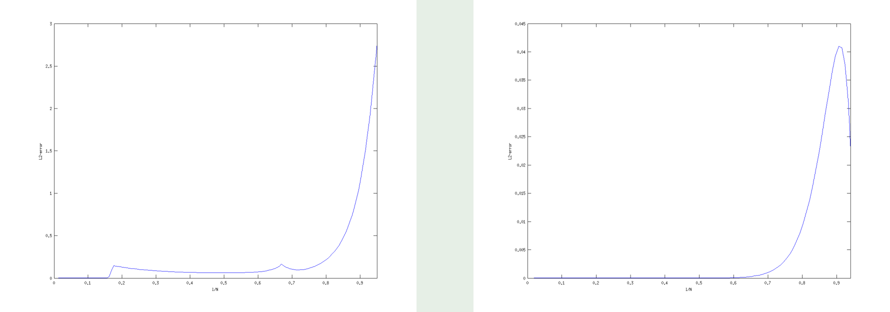


Figure: Eigenvalue error for polynomial degree 6 and with C^3 and C^5-continuity, respectively N $= 50$, 100

Own results

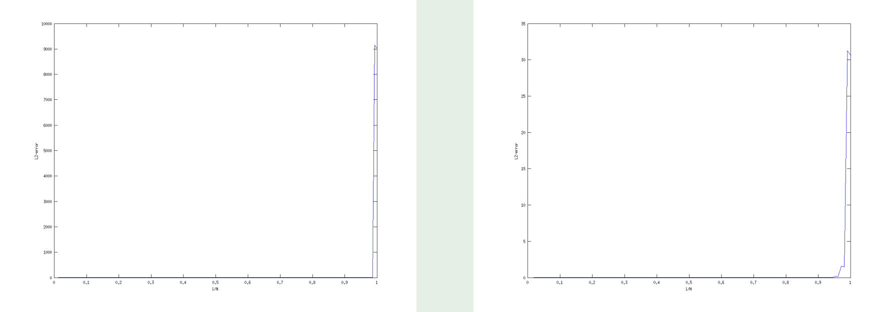


Figure: Eigenvalue error for polynomial degree 6 and with C^3 and C^5-continuity, respectively, N $=$ 50, 100

Own results

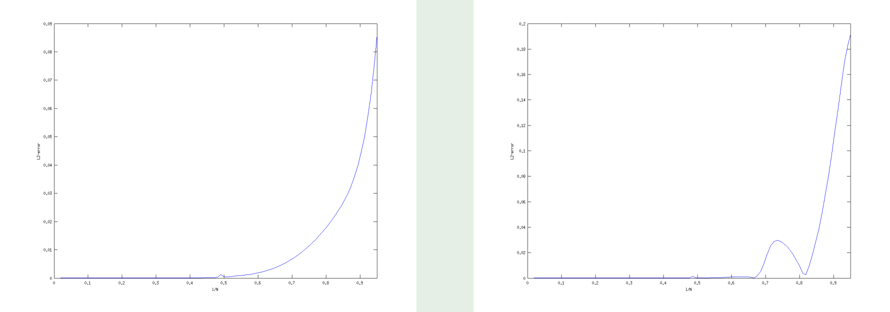


Figure: Eigenvalue error for polynomial degree 5 and 6 with C^3 and C^4-continuity, respectively, N = 50

Own results

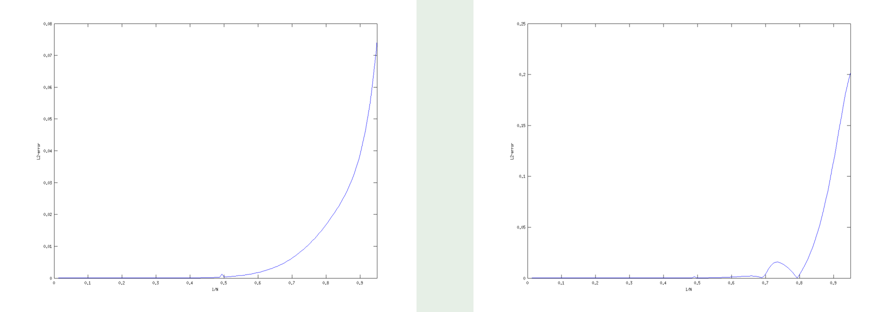
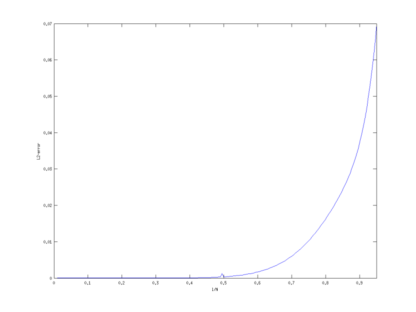


Figure: Eigenvalue error for polynomial degree 5 and 6 with C^3 and C^4-continuity, respectively, N = 75

Own results



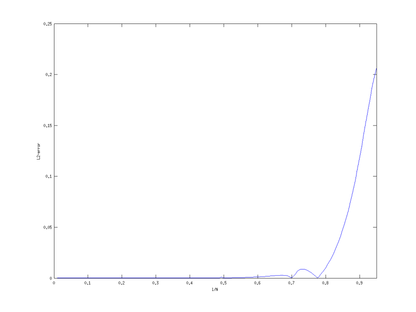


Figure: Eigenvalue error for polynomial degree 5 and 6 with C^3 and C^4-continuity, respectively, N = 100

Summary

- First considered the elliptic eigenvalue problem
- Then looked at corresponding BVP, parabolic IVP and hyperbolic IVP
- Solution errors can be expressed in therms of eigenvalue and eigenfunction errors
- B-spline eigenvalue spectrum does not have optical branches
- B-spline eigenfunction error is indistinguishable in L^2-norm
- B-spline approximations are much more accurate
- Own results for eigenvalue error for different p and r

References

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