Matrix assembly by low rank tensor approximation

Felix Scholz

13.02.2017

References

Angelos Mantzaflaris, Bert Juettler, Boris Khoromskij, and Ulrich Langer.
Matrix generation in isogeometric analysis by low rank tensor approximation.
In *Curves and Surfaces*, volume 9213 of *LNCS*, pages 321–340. Springer, 2015.

Angelos Mantzaflaris, Bert Juettler, Boris Khoromskij, and Ulrich Langer.
Low rank tensor methods in galerkin-based isogeometric analysis.
Technical report, Radon Institute for Computational and Applied Mathematics, 2016.

W. Hackbusch.
Tensor spaces and numerical tensor calculus.
Springer, Berlin, 2012.

M. Griebel and H. Harbrecht.
Approximation of two-variate functions: Singular value decomposition versus sparse grids.
IMA Journal of Numerical Analysis, 2011.

Bert Jüttler and Dominik Mokriš.
Low rank spline interpolation of boundary value curves.
G+S Report No. 47, 2016.

Overview

- Motivation
- Singular Value Decomposition (SVD) of functions
- Discrete Singular Value Decomposition
- Numerical examples
- Generalisation to arbitrary dimensions

Motivation

Given a parametrisation of the physical domain Ω by a regular tensor product B-Spline (or NURBS) function

$$F : \hat{\Omega} \longrightarrow \Omega$$

we consider the weak formulation an elliptic equation as a model problem:
Find $u \in H_0^1(\Omega)$, such that

$$a(u, v) = (f, v)_{0,\Omega} \; \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_{\Omega} \nabla_x u(x) \cdot (A(x) \nabla_x v(x)) + c u(x) v(x) \mathrm{d}x.$$

Motivation

We consider the 2D-case with $\hat{\Omega} = [0,1]^2$, $A \equiv I$ and $c \equiv 1$.
As the discrete space of functions on the parametric domain we choose a tensor spline space

$$S_p(\Xi) = S_{p_1}(\Xi_1) \otimes S_{p_2}(\Xi_2)$$

with the B-spline basis

$$\hat{B}_{i,p}(\xi) = \hat{B}_{i_1,p_1}(\xi_1)\hat{B}_{i_2,p_2}(\xi_2)$$

and set the discrete space of functions on the physical domain to be (up to the boundary conditions)

$$V_h := \mathrm{span}\{\hat{B}_i \circ F^{-1}\} = \mathrm{span}\{B_i\}.$$

Motivation

Computing the L^2-product of the basis elements we get the entries of the mass matrix:

$$\begin{aligned} M_{ij} &= \int_{\Omega} B_i(x) B_j(x) \mathrm{d}x \\ &= \int_{\hat{\Omega}} |\det \nabla_\xi F(\xi)| \hat{B}_i(\xi) \hat{B}_j(\xi) \mathrm{d}\xi \\ &= \int_0^1 \int_0^1 \omega(\xi) \hat{B}_{i_1}(\xi_1) \hat{B}_{j_1}(\xi_1) \hat{B}_{i_2}(\xi_2) \hat{B}_{j_2}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2, \end{aligned}$$

where $\omega(\xi) = \det \nabla_\xi F(\xi)$. Analogously we get for the stiffness matrix

$$\begin{aligned} S_{ij} &= \int_{\Omega} \nabla_x B_i \cdot \nabla_x B_j \mathrm{d}x \\ &= \sum_{p,q=1}^{2} \int_0^1 \int_0^1 K_{pq}(\xi) \frac{\partial}{\partial \xi_p} \hat{B}_i \frac{\partial}{\partial \xi_q} \hat{B}_j \mathrm{d}\xi, \end{aligned}$$

where $K(\xi) = (\det \nabla_\xi F)(\nabla_\xi F)^{-1}(\nabla_\xi F)^{-T}$.

Motivation

The complexity of computing the bivariate integrals in S_{ij} and M_{ij} by Gauss quadrature is in $O(n^2 p^6)$ (if $n = n_1 = n_2$ and $p_1 = p_2$). We want to decompose the functions $\omega(\xi)$ and $K_{pq}(\xi)$ into products of univariate functions so we only need to compute univariate integrals where the complexity is $O(np^3)$.

Singular value decomposition of a function

Any bivariate continuous function $f \in C([0,1] \times [0,1])$ has a *singular value decomposition*

$$f(\xi_1, \xi_2) = \sum_{r=1}^{\infty} \sigma_r u_r(\xi_1) v_r(\xi_2),$$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$ and $\{u_r\}_{r \geq 1}$ and $\{v_r\}_{r \geq 1}$ are $L^2((0,1))$-orthonormal systems of continuous functions. The sum converges in $L^2((0,1) \times (0,1))$.

The *rank R-approximation*

$$f_R(\xi_1, \xi_2) = \sum_{r=1}^{R} \sigma_r u_r(\xi_1) v_r(\xi_2)$$

is the best approximation of f by a rank R function in the L^2-norm.

Singular value decomposition of a function

Lemma

If $f \in H^s((0,1) \times (0,1))$, *then*

(i) *the singular values decay like*

$$\sigma_r \lesssim r^{-s}.$$

(ii) *the approximation error fulfils*

$$\|f - f_R\|_{L^2} = \sqrt{\sum_{r=R+1}^{\infty} \sigma_r^2} \lesssim R^{\frac{1}{2}-s}$$

Proof.

Griebel, 2011. [4] □

Discrete singular value decomposition

A low rank approximation of the functions ω and K_{pq} is computed as follows:

1. Project the function ω or K_{pq} into a suitable spline space.
2. Decompose the coefficient tensor using matrix SVD.
3. Choose the rank R such that the overall approximation error is lower than a given constant ϵ, for example the discretisation error.

Projection into a spline space

The function $\omega = \det \nabla F$ is a tensor product spline function of higher polynomial degrees $q_d = 2p_d - 1$ and lower smoothness than F. We can thus represent it exactly with respect to a B-Spline basis $\{\bar{B}_i\}_{i \leq (m_1, m_2)}$.

$$\omega(\xi) = \Pi\omega(\xi) = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} \omega_{i_1 i_2} \bar{B}_{i_1}(\xi_1) \bar{B}_{i_2}(\xi_2).$$

For the functions K_{pq} we choose a sufficiently refined spline space $\mathrm{span}\{\bar{B}_i\}$ such that

$$||K_{pq} - \Pi K_{pq}||_{L^2(\hat{\Omega})} = ||K_{pq} - \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} (K_{pq})_{i_1 i_2} \bar{B}_{i_1} \bar{B}_{i_2}||_{L^2(\hat{\Omega})} \leq \epsilon_\Pi.$$

Decomposition of the coefficient tensor

We compute the SVD of the $m_1 \times m_2$ matrix $W = (\omega_{i_1 i_2})$, i.e.

$$W = U \Sigma V^T = \sum_{r=1}^{\min(m_1,m_2)} \sigma_r u_r v_r^T,$$

where U is an orthogonal $m_1 \times m_1$-matrix with columns u_r, V is an orthogonal $m_2 \times m_2$-matrix with columns v_r and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{\min(m_1,m_2)})$. We assume $\sigma_1 \geq \ldots \geq \sigma_{\min(m_1,m_2)}$.

Decomposition of the coefficient tensor

The rank-R approximation

$$W_R = \sum_{r=1}^{R} \sigma_r u_r v_r^T$$

is the best approximation of W by a matrix of rank R in the Frobenius norm and fulfils

$$||W - W_R||_F = \sqrt{\sum_{r=R+1}^{\min(m_1,m_2)} \sigma_r^2}.$$

Decomposition of ω

We multiply the decomposed coefficient tensor with the basis of the projection space to get the decomposition

$$\begin{aligned}\Pi\omega(\xi) &= \sum_{r=1}^{\min(m_1,m_2)} \sigma_r \left(\sum_{i_1=1}^{m_1}(u_r)_{i_1}\bar{B}_{i_1}(\xi_1)\right)\left(\sum_{i_2=1}^{m_2}(v_r)_{i_2}\bar{B}_{i_2}(\xi_2)\right) \\ &= \sum_{r=1}^{\min(m_1,m_2)} \mathcal{U}_r(\xi_1)\mathcal{V}_r(\xi_2)\end{aligned}$$

Error estimate for the low rank approximation

Lemma

The rank R-approximation

$$\Lambda_R \omega(\xi) = \sum_{r=1}^{R} \mathcal{U}_r(\xi_1)\mathcal{V}_r(\xi_2)$$

fulfils

$$||\Pi\omega - \Lambda_R\omega||_{L^\infty(\hat{\Omega})} \leq ||W - W_R||_F = \sqrt{\sum_{r=R+1}^{\min(m_1,m_2)} \sigma_r^2}.$$

Thus for a given accuracy ϵ_Λ we can choose the smallest rank R such that the approximation error is below ϵ_Λ.

Assembly of the matrices

The entries of approximated mass matrix are

$$M_{ij} \approx \bar{M}_{ij} = \sum_{r=1}^{R} \int_0^1 \mathcal{U}_r(\xi_1)\hat{B}_{i_1}(\xi_1)\hat{B}_{j_1}(\xi_1)\mathrm{d}\xi_1 \cdot \int_0^1 \mathcal{V}_r(\xi_2)\hat{B}_{i_2}(\xi_2)\hat{B}_{j_2}(\xi_2)\mathrm{d}\xi_2$$

and thus $\bar{M}$ can be written in the *Kronecker format*

$$\bar{M} = \sum_{r=1}^{R} X_r \otimes Y_r$$

where each X_r is a $n_1 \times n_1$ and Y_r a $n_2 \times n_2$-matrix containing the univariate integrals.
For the stiffness matrix we can proceed in the same way.

Computational complexity

We assume $n_1 = n_2 = n$, $m_1 = m_2 = m$, $p_1 = p_2 = p$ and $q_1 = q_2 = q$.

- The complexity is bounded from below by the number of non-zeros in the matrix, which is $O(n^2 p^2)$.
- The complexity of computing the matrix SVD up to rank R is $O(Rm^2)$.
- For assembling the matrices X_r and Y_r using univariate element-wise Gauss quadrature the complexity is $O(Rnp^3)$
- The complexity for computing the Kronecker sum $\sum_{r=1}^{R} X_r \otimes Y_r$ is $O(Rn^2 p^2)$.

Since generally $n \gg p$, the overall complexity is dominated by the last step and is thus $O(Rn^2 p^2)$.

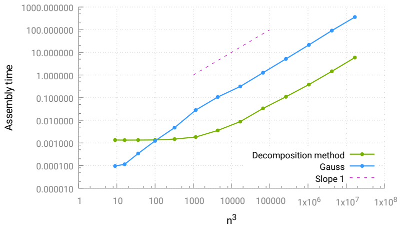
Numerical examples

For the method to be efficient we need to be able to choose the rank low.

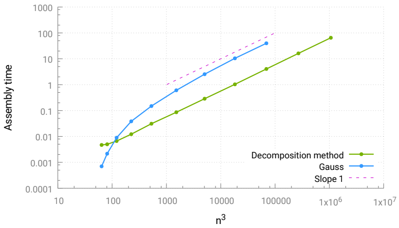
Table: Rank values for accuracy $\epsilon_\Lambda = \epsilon_\Pi = 10^{-8}$

				
n	2×3	2×3	5×5	8×8
p	$(1,2)$	$(1,2)$	$(4,4)$	$(7,7)$
$\mathrm{rank}(\omega)$	1	1	7	8
$\mathrm{rank}(K)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 17 & 17 \\ 17 & 17 \end{pmatrix}$	$\begin{pmatrix} 14 & 18 \\ 18 & 15 \end{pmatrix}$

Numerical examples



(a) Quarter annulus, $p = 2$



(b) 2nd Coons surface (star), $p = 7$

Figure: Comparison of computation times for the stiffness matrix using the decomposition method and an element-wise Gauss method.

Numerical examples

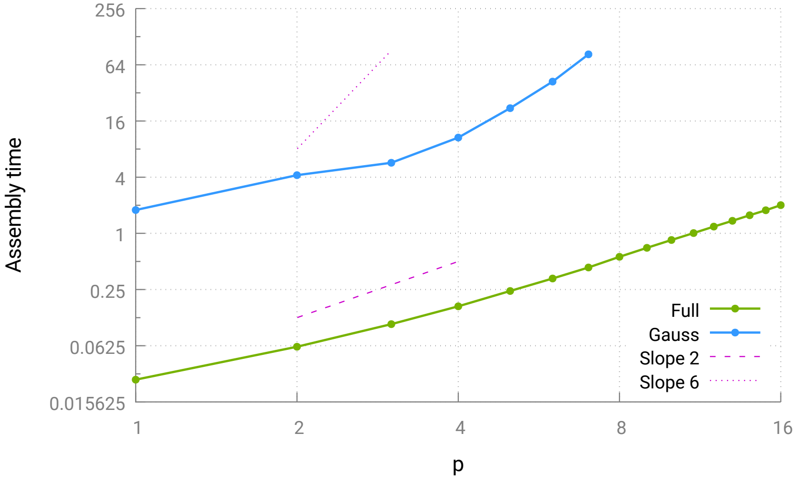


Figure: Comparison of the p-Dependence of the computation times for the stiffness matrix using the decomposition method and an element-wise Gauss method. Computed on the quarter annulus with $400 \times 400 = 160000$ DOF.

Generalisation to arbitrary dimensions

- The tensor decomposition method can be generalised to arbitrary dimensions since any $d-$tensor $T \in \mathbb{W}_{(n_1,\ldots,n_d)}$ possesses a canonical representation

$$T = \sum_{r=1}^{R} v_1^r \otimes v_2^r \otimes \ldots \otimes v_d^r,$$

  where $v_k^r \in \mathbb{R}^{n_k}$.

- However, the truncation operator

$$\Lambda_R T = \underset{\mathrm{rank}(U) \leq R}{\mathrm{argmin}} \, ||T - U||_2$$

  leads to a non-linear optimisation problem.