# The Stokes problem and divergence conforming discretization

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# Topic of this talk

- The Stokes problem

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- The four Brezzi's conditions

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- The Nitsche method

# Motivation: Stokes Equation

We study the Stokes Equation in one of its simplest forms:

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\begin{aligned}
-\Delta \mathbf{u} - \nabla p &= \mathbf{f}, \\
\nabla \cdot \mathbf{u} &= 0,
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where $\mathbf{u}$ is the velocity field, $p$ is the (negative) pressure and $\mathbf{f}$ is the volume forces.

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The Stokes problem is a mathematical model problem for saddle point problem and physical model equation for fluid dynamic.

We will consider the mathematical aspect of the problem.

# The Stokes problem

Strong formulation:
Given $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$, find $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$ such that the following holds,

$$\begin{aligned} -\Delta \mathbf{u} - \nabla p = \mathbf{f}, &\quad \forall \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{u} = 0, &\quad \forall \mathbf{x} \in \Omega, \\ \mathbf{u} = \mathbf{g}, &\quad \forall \mathbf{x} \in \Gamma. \end{aligned}$$

Where $\Omega$ is the domain and $\Gamma = \partial\Omega$ is the boundary.

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Where $\Omega$ is the domain and $\Gamma = \partial\Omega$ is the boundary.
Weak formulation:
Find $(\mathbf{u}, p) \in \mathbf{H}^1_{\mathbf{g}}(\Omega) \times L^2_0(\Omega)$, such that

$$
\begin{aligned}
(\nabla\mathbf{u}, \nabla\mathbf{v})_{L^2(\Omega)} + (p, \nabla\cdot\mathbf{v})_{L^2(\Omega)} &= (\mathbf{f}, \mathbf{v})_{L^2(\Omega)}\,, \\
(\nabla\cdot\mathbf{u}, q)_{L^2(\Omega)} &= 0
\end{aligned}
$$

$\forall\ (\mathbf{v}, q) \in \mathbf{H}^1_0 \times L^2_0(\Omega)$.

# The Stokes problem

More abstractly written we have:

Find $(\mathbf{u}, p) \in V \times Q$, such that

$$a\left(\mathbf{u}, \mathbf{v}\right) + b\left(p, \mathbf{v}\right) = \left(\mathbf{f}, \mathbf{v}\right)_{L^2(\Omega)}, \quad \forall \mathbf{v} \in V$$
$$b\left(q, \mathbf{u}\right) = 0 \quad \forall q \in Q,$$

where,

$$a\left(\mathbf{u}, \mathbf{v}\right) = \left(\nabla \mathbf{u}, \nabla \mathbf{v}\right)_{L^2(\Omega)} \text{ and } b\left(q, \mathbf{v}\right) = \left(q, \nabla \cdot \mathbf{v}\right)_{L^2(\Omega)}.$$

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We can also write it as

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$$

# Brezzi's conditions:

Boundedness of $a(\cdot,\cdot)$:

$$a\left(\mathbf{u},\mathbf{v}\right) \leq C_1 \|\mathbf{u}\|_V \|\mathbf{v}\|_V \quad \forall\, \mathbf{u},\mathbf{v} \in V. \tag{1}$$

Boundedness of $b(\cdot,\cdot)$:

$$b\left(q,\mathbf{v}\right) \leq C_2 \|q\|_Q \|\mathbf{v}\|_V \quad \forall\left(q,\mathbf{v}\right) \in Q \times V. \tag{2}$$

Coercivity of $a(\cdot,\cdot)$:

$$a\left(\mathbf{u},\mathbf{u}\right) \geq C_3 \|\mathbf{u}\|_V^2 \quad \forall\, \mathbf{u} \in V. \tag{3}$$

Coercivity of $b(\cdot,\cdot)$ (inf-sup):

$$\sup_{\mathbf{v}\in V} \frac{b\left(q,\mathbf{v}\right)}{\|\mathbf{v}\|_V} \geq C_4 \|q\|_Q \quad \forall\, q \in Q \tag{4}$$

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The weak problem has a unique solution $(\mathbf{u},p) \in \mathbf{H}^1_{\mathbf{g}}(\Omega) \times L^2_0(\Omega)$.

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If we use conforming discretization; that is, $V_h \subset V$ and $Q_h \subset Q$ then the three first conditions holds in the discrate case.

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If the discretization spaces are $V_h = S \times S \times S$ and $Q_h = S$, for some spline space $S$, then the discrate inf-sup condition does not hold:

$$\sup_{\mathbf{v}_h \in V_h} \frac{(q_n, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_{V_h}} \geq C_4 \|q_h\|_{Q_h} \quad \forall\, q_h \in Q_h.$$

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We need a discretization that is satisfy the discrete inf-sup condition!

# IGA discretization

Let $\mathcal{M}_h$ be the parametric mech, $Q$ an element in $\mathcal{M}_h$.

Tensor product B–spline basis function:

$$B^{k_1,k_2,k_3}_{i_1,i_2,i_3} := B^{k_1}_{i_1} \otimes B^{k_2}_{i_2} \otimes B^{k_3}_{i_4}.$$

Tensor product B–spline space

$$S^{k_1,k_2,k_3}_{\alpha_1,\alpha_2,\alpha_3}(\mathcal{M}_h) := \left\{ B^{k_1,k_2,k_3}_{i_1,i_2,i_3} \right\}^{n_1,n_2,n_3}_{i_1,i_2,i_3=1}$$

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Assume $\Omega$ can be exactly parametrized by a geometrical mapping:

$$\mathbf{F}(\widehat{\mathbf{x}}) : \widehat{\Omega} \to \Omega,$$

$$\mathcal{K}_h = \{K : K \in \mathbf{F}(Q), Q \in \mathcal{M}_h\}, \quad h_K := \|D\mathbf{F}\|_{L^\infty(Q)} h_Q.$$

# Discrete Spaces: TH and RT spaces

We define the Taylor–Hood spaces as

$$\widehat{\mathcal{V}}_h^{TH} := S_{\alpha_1,\alpha_2,\alpha_3}^{k_1+1,k_2+1,k_3+1} \times S_{\alpha_1,\alpha_2,\alpha_3}^{k_1+1,k_2+1,k_3+1} \times S_{\alpha_1,\alpha_2,\alpha_3}^{k_1+1,k_2+1,k_3+1},$$
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We constrain RT spaces as

$$\begin{aligned}\widehat{\mathcal{V}}_{0,h}^{RT} &:= \left\{\widehat{\mathbf{v}} \in \widehat{\mathcal{V}}_h^{RT} : \widehat{\mathbf{v}} \cdot \widehat{\mathbf{n}} = 0 \text{ on } \partial\widehat{\Omega}\right\},\\ \widehat{\mathcal{Q}}_{0,h}^{RT} &:= \left\{\widehat{q} \in \widehat{\mathcal{Q}}_h^{RT} : \int_{\widehat{\Omega}} \widehat{q}\, d\widehat{\mathbf{x}} = 0\right\}.\end{aligned}$$

The RT spaces and it's constraind spaces form a bounded discrete cochain complex with the divergence operator:

$$\widehat{\mathcal{V}}_h^{RT} \xrightarrow{\widehat{div}} \widehat{\mathcal{Q}}_h^{RT} \text{ and } \widehat{\mathcal{V}}_{0,h}^{RT} \xrightarrow{\widehat{div}} \widehat{\mathcal{Q}}_{0,h}^{RT}$$

# Properties of RT spaces

**Theorem**

*The discrete inf-sup condition holds for* $\widehat{\mathcal{V}}_{0,h}^{RT}$ *and* $\widehat{\mathcal{Q}}_{0,h}^{RT}$.

**Proof.**

Blackboard! Braess [2007] $\square$

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**Theorem**

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**Theorem**

*If $\widehat{\mathbf{v}}_h \in \widehat{\mathcal{V}}_{0,h}^{RT}$ satisfies*

$$(\nabla \cdot \widehat{\mathbf{v}}_h, \widehat{q}_h)_{L^2(\Omega)} = 0 \quad \forall \widehat{q}_h \in \widehat{\mathcal{Q}}_{0,h}^{RT}$$

*then $\nabla \cdot \widehat{\mathbf{v}}_h = 0$.*

That is, the discretization gives velocity fields which are pointwise divergence-free.

# Violation of the inf-sup condition

What happens if we strongly impose the full Dirichlet boudary condotions? That is, we set the constraind spaces to be

$$\widehat{\mathcal{V}}_{0,h}^{RT} := \left\{ \widehat{\mathbf{v}} \in \widehat{\mathcal{V}}_h^{RT} : \widehat{\mathbf{v}} = 0 \text{ on } \partial\widehat{\Omega} \right\},$$

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The mapping $\widehat{\mathcal{V}}_{0,h}^{RT} \xrightarrow{\widehat{div}} \widehat{\mathcal{Q}}_{0,h}^{RT}$ is not surjective. Hence there exists a $\widehat{q}_h \in \widehat{\mathcal{Q}}_{0,h}^{RT}$ such that $\widehat{q}_h \neq 0$ and

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We cannot strongly impose the full Dirichlet boudary conditions with this methodology.

# Divergence preserving transformation

For the pressure we use the standard IGA inverse composition

$$q = \mathcal{F}\left(\widehat{q}\right) = \widehat{q} \circ \widehat{\mathbf{F}}^{-1}, \quad \forall \widehat{q} \in L^2_0\left(\widehat{\Omega}\right).$$

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For the velocity field we use the divergence preserving transformation

$$\mathbf{v} = \mathcal{F}^{\mathrm{div}}\left(\widehat{\mathbf{v}}\right) = \frac{\mathbf{J}}{\det \mathbf{J}} \widehat{\mathbf{v}} \circ \widehat{\mathbf{F}}^{-1}, \quad \forall \widehat{\mathbf{v}} \in \mathbf{H}_0\left(\widehat{\mathrm{div}}; \widehat{\Omega}\right),$$

where $\mathbf{J}$ is the jacobian. This is sometimes called the contravariant Piola transformation. It has the important property that preserves the divergence.

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\mathbf{H}_0\left(\widehat{\mathrm{div}}; \widehat{\Omega}\right) \arrow[r, "\widehat{\mathrm{div}}"] \arrow[d, "\mathcal{F}^{\mathrm{div}}"'] & L_0^2\left(\widehat{\Omega}\right) \arrow[d, "\mathcal{F}"] \\
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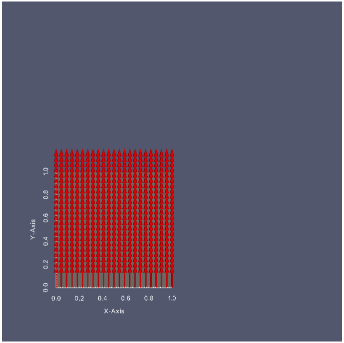
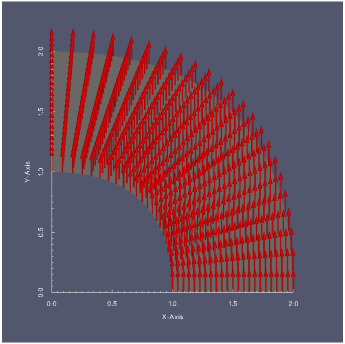
$$\mathcal{V}_{0,h} = \left\{\mathbf{v} \in \mathbf{H}_0\left(\mathrm{div}; \Omega\right) : \mathcal{F}^{\mathrm{div}}\left(\mathbf{v}\right)^{-1} \in \widehat{\mathcal{V}}_{0,h}\right\} \text{ and } \mathcal{Q}_{0,h} = \left\{q \in L^2_0\left(\Omega\right) : \mathcal{F}\left(q\right)^{-1} \in \widehat{\mathcal{Q}}_{0,h}\right\}$$

# Inverse composition transformation



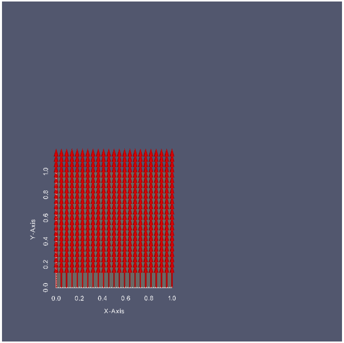

Parametric domain

$\xrightarrow{\mathcal{F}}$

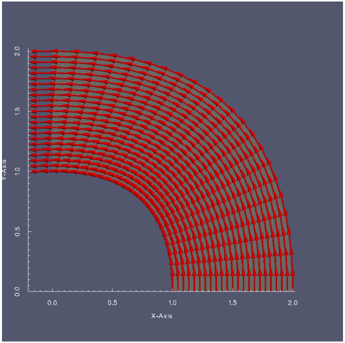



Physical domain

# Divergence preserving transformation




$$\xrightarrow{\mathcal{F}^{\text{div}}}$$

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Standard inverse composition:

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Divergence preserving transformation:

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$$\nabla \mathbf{v} = \frac{1}{\det \mathbf{J}} \mathbf{J}\, \widehat{\nabla}\, \widehat{\mathbf{v}}\, \mathbf{J}^{-T} + \frac{1}{\det \mathbf{J}} \sum_{i=1}^{d} \nabla \mathbf{J}\, \widehat{\mathbf{v}} - \frac{\nabla \det \mathbf{J}}{\det \mathbf{J}^2} \mathbf{J}\, \widehat{\mathbf{v}}$$

# Matrix properties

$$\begin{pmatrix} A_{11} & 0 & 0 & B_1^T \\ 0 & A_{22} & 0 & B_2^T \\ 0 & 0 & A_{33} & B_3^T \\ B_1 & B_2 & B_3 & 0 \end{pmatrix}$$

System matrix when using inverse composition transformation. The zero blocks comes from component preserving transformation.

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$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & B_1^T \\ A_{21} & A_{22} & A_{23} & B_2^T \\ A_{31} & A_{32} & A_{33} & B_3^T \\ B_1 & B_2 & B_3 & 0 \end{pmatrix}$$

System matrix when using divergence preserving transformation. This matrix is denser.

# The Nitsche method

We need to weakly impose the tangential components of the Dirichlet boundary conditions.

We use the Nitsche method to weakly impose the whole Dirichlet boundary conditions.

We redefine the bilinear forms to

$$a_h(\mathbf{u}_h, \mathbf{u}_h) = (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_\Omega - (\mathbf{n} \cdot \nabla \mathbf{u}_h, \mathbf{v}_h)_\Gamma - (\mathbf{n} \cdot \nabla \mathbf{v}_h, \mathbf{u}_h)_\Gamma + \sum_{F \in \Gamma_h} \int_F \frac{C_N}{h_F} \mathbf{u}_h \cdot \mathbf{v}_h dS,$$

$$b_h(p_h, \mathbf{v}_h) = (p_h, \nabla \cdot \mathbf{u}_h)_\Omega - (\mathbf{v}_h \cdot \mathbf{n}, p_h)_\Gamma$$

and add correct term on the right hand side.

# Nitsche penalty term

The pressure error asymptotically scales with the square root of $C_N$ [Evans and Hughes, 2013a]. So we need to put a good value for $C_N$. Evans suggests

$$C_N = 5\,(k+1)\,,$$

where $k = \min\{k_1, k_2, k_3\}$.

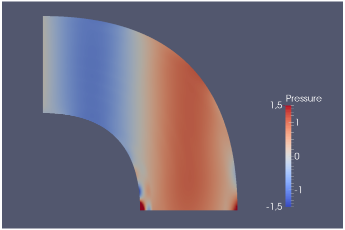
Evans and Hughes [2013d] gives an explicit choice for $C_N$ depending on shape, size, polynomial degree, and the NURBS weighting.

# Testing the Nitsche penalty with manufactured solutions

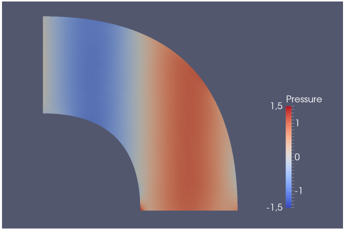
The test problem:

$$\begin{aligned}
&\Omega \text{ is an quarte annulus } \subset \left\{(x,y) \in \mathbb{R}^2 |\, x,y \in (0,2)\right\}, \\
&\mathbf{u} = \nabla \times \left(\sin^2(\pi x/2)\sin^2(\pi y/2)\right), \\
&p = -\sin(\pi x), \\
&\mathbf{f} = -\Delta \mathbf{u} - \nabla p,
\end{aligned}$$

# Nitsche penalty parameter

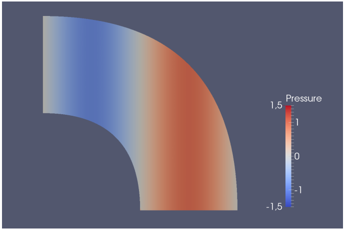



3 refinements, $k = 2$,
$C_N / h_F = 632$,
$L^2$-error velocity: 0.0119,
$L^2$-error pressure: 0.24123

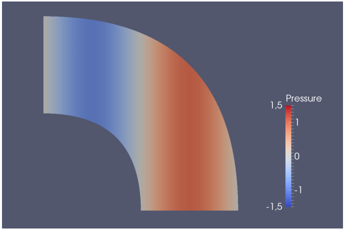



3 refinements, $k = 2$,
$C_N / h_F = 120$,
$L^2$-error velocity: 0.0118,
$L^2$-error pressure: 0.06309

# Nitsche penalty parameter




5 refinements, $k = 2$,
$C_N/h_F = 2297$,
$L^2$-error velocity: 0.0001214,
$L^2$-error pressure: 0.00315




5 refinements, $k = 2$,
$C_N/h_F = 480$,
$L^2$-error velocity: 0.0001236,
$L^2$-error pressure: 0.00080

# $L^2$-error: $k = 2$ (velocity and pressure)

| # Refined | 2 | 3 | 4 | 5 | rate avg | rate |
|---|---|---|---|---|---|---|
| Velocity | 5.01e-02 | 1.18e-02 | 1.08e-03 | 1.24e-04 | 3.27 | 3.29 |
| Pressure | 2.08e-01 | 6.31e-02 | 6.44e-03 | 8.00e-04 | 3.04 | 3.16 |

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