# Analysis-suitable adaptive T-mesh refinement with linear complexity (by [Morgenstern & Peterseim, 2015])

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# References

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# Overview

1. Adaptive mesh-refinement
2. Analysis-suitability
3. The overlay
4. Nestedness
5. Linear complexity

# Overview

T-splines can be used for local refinement, but in general the T-splines are not linearly independent. To have this property, we need *analysis-suitable* T-meshes.

The proposed refinement algorithm provides the following

1. the preservation of analysis-suitablity and nestedness of the generated T-spline spaces,
2. a bounded cardinality of the overlay,
3. linear computational complexity of the refinement procedure.

# Adaptive mesh refinement

We consider only a 2D-index domain, as the physical domain can be obtained via a suitable mapping.

**Definition 1 (Initial mesh, element).**

Given positive numbers $M, N \in \mathbb{N}$, the initial mesh $\mathcal{G}_0$ is a tensor product mesh consisting of closed squares (also denoted elements) with side length 1, i.e.

$$\mathcal{G}_0 := \Big\{[m-1, m] \times [n-1, n] : m \in \{1, \ldots, M\}, n \in \{1, \ldots, N\}\Big\}.$$

**Definition 2.**

The level of an element $K$ is defined by

$$\ell(K) := -\log_2 |K|$$

# (p,q)-patches

**Definition 3.**

Given an element K and polynomial degrees p and q, the (p,q)-patch is defined by

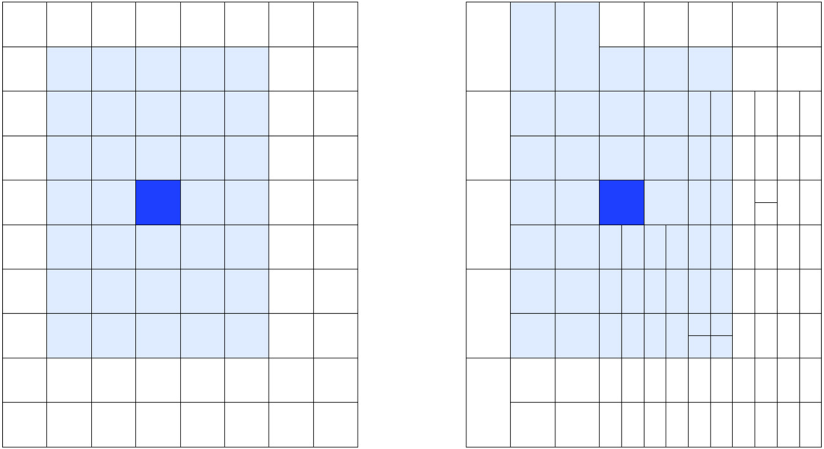
$$\mathcal{G}^{p,q}(K) := \{K \in \mathcal{G} : \mathrm{Dist}(K', K) \leq \boldsymbol{D}^{p,q}(l(K))\}$$

with

$$\boldsymbol{D}^{p,q}(k) := \begin{cases} 2^{-k/2}(\lfloor \frac{p}{2} \rfloor + \frac{1}{2}, \lceil \frac{q}{2} \rceil + \frac{1}{2}) \text{if } k \text{ is even} \\ 2^{-(k+1)/2}(\lceil \frac{p}{2} \rceil + \frac{1}{2}, 2\lfloor \frac{q}{2} \rfloor + 1) \text{if } k \text{ is odd} \end{cases}$$

Note that $\mathrm{Dist}(K', K)$ is the vector-valued distance between the midpoints of $K'$ and $K$.

# Example for a (p,q)-patch



# Refining an element

From now on, we assume $p, q \geq 2$. This ensures that neighbouring elements of $K$ are always in $\mathcal{G}^{p,q}(K)$ and nested elements $K \subseteq \hat{K}$ have nested (p,q)-patches, i.e. $\mathcal{G}^{p,q}(K) \subseteq \mathcal{G}^{p,q}(\hat{K})$.

**Definition 4.**

Given an arbitrary element $K = [\mu, \mu + \tilde{\mu}] \times [\nu, \nu + \tilde{\nu}]$, we define the operators

$$\mathrm{bisect}_x(K) := [\mu, \mu + \frac{\tilde{\mu}}{2}] \times [\nu, \nu + \tilde{\nu}], [\mu + \frac{\tilde{\mu}}{2}, \mu + \tilde{\mu}] \times [\nu, \nu + \tilde{\nu}]$$

and

$$\mathrm{bisect}_y(K) := [\mu, \mu + \tilde{\mu}] \times [\nu, \nu + \frac{\tilde{\nu}}{2}], [\mu, \mu + \tilde{\mu}] \times [\nu + \frac{\tilde{\nu}}{2}, \nu + \tilde{\nu}].$$

**Definition 5 (Bisection, Multiple bisections).**

Given a mesh $\mathcal{G}$ and an element $K \in \mathcal{G}$, we denote by $\mathrm{bisect}(\mathcal{G}, K)$ the mesh that results from a level dependent bisection of $K$,

$$\mathrm{bisect}(\mathcal{G}, K) := \mathcal{G} \setminus K \cup \mathsf{child}(K),$$

with

$$\mathsf{child}(K) := \begin{cases} \mathrm{bisect}_x(K), & \text{if } \ell(K) \text{ is even} \\ \mathrm{bisect}_y(K), & \text{if } \ell(K) \text{ is odd} \end{cases}.$$

The bisection of multiple elements $\mathcal{M} \subseteq \mathcal{G}$ is defined by successive bisections in an arbitrary order, i.e.

$$\mathrm{bisect}(\mathcal{G}, \mathcal{M}) := \mathrm{bisect}(\mathrm{bisect}(\ldots \mathrm{bisect}(\mathcal{G}, K_1)), K_J).$$

# A superset

Now we can define our refinement algorithm but first need some superset of $\mathcal{M}$.

**Algorithm** (*Closure*).

Given a mesh $\mathcal{G}$ and a set of marked elements $\mathcal{M} \subseteq \mathcal{G}$ to be bisected, the closure $\mathrm{clos}_{\mathcal{G}}^{p,q}(\mathcal{M})$ of $\mathcal{M}$ is computed as follows

$\tilde{\mathcal{M}} := \mathcal{M}$

**repeat**

**for all** $K \in \tilde{\mathcal{M}}$ **do**

$\widetilde{\mathcal{M}} := \tilde{\mathcal{M}} \cup \{K' \in \mathcal{G}^{p,q}(K) : l(K') < l(K)\}$

**end for**

**until** $\widetilde{\mathcal{M}}$ stops growing

**return** $\mathrm{clos}_{\mathcal{G}}^{p,q}(\mathcal{M}) = \widetilde{\mathcal{M}}$

# The refinement algorithm

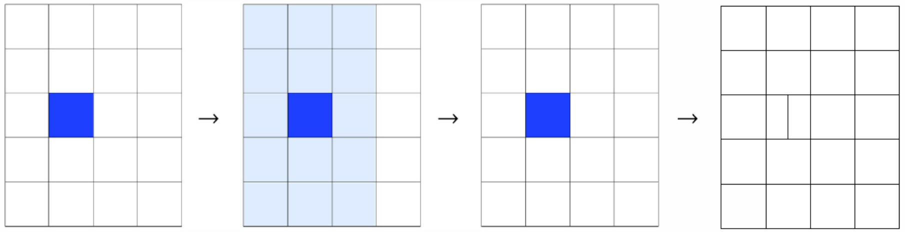
**Algorithm** (*Refinement*).

Given a mesh $\mathcal{G}$ and a set of marked elements $\mathcal{M} \subseteq \mathcal{G}$ to be bisected, $\mathrm{ref}^{p,q}(\mathcal{G}, \mathcal{M})$ is defined by

$$\mathrm{ref}^{p,q}(\mathcal{G}, \mathcal{M}) := \mathrm{bisect}(\mathcal{G}, \mathrm{clos}_{\mathcal{G}}^{p,q}(\mathcal{M})).$$

In the following examples, the polynomial degrees are $p = q = 3$ and $M = 4, N = 5$.

# Examples



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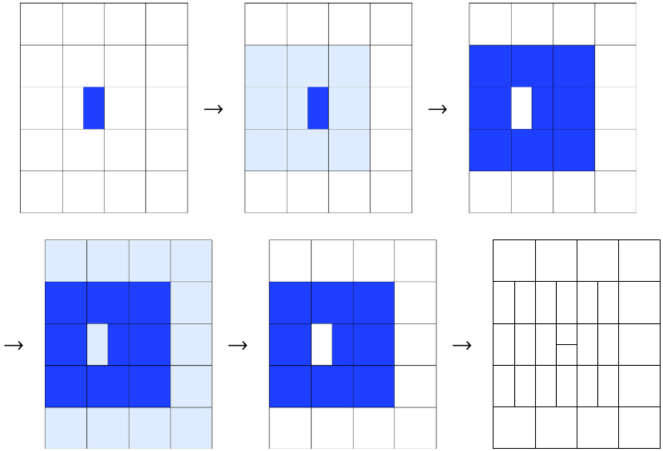

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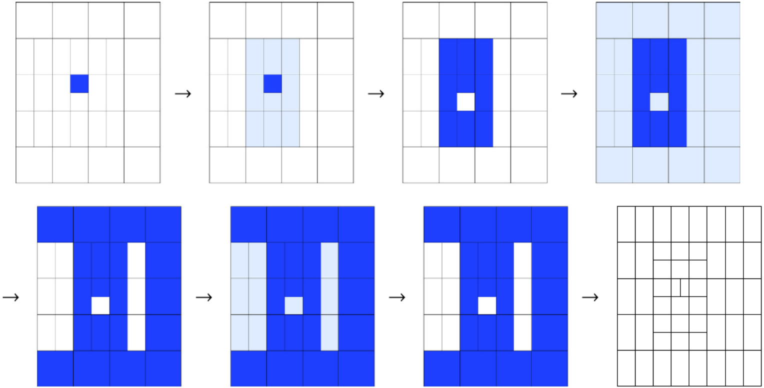

# The concept of admissibility

**Definition 6 ($(p,q)$-admissible bisections).**

Given a mesh $\mathcal{G}$ and an element $K \in \mathcal{G}$, the bisection of $K$ is called (p,q)-admissible if all $K' \in \mathcal{G}^{p,q}$ satisfy $l(K') \geq l(K)$.
In the case of several elements $\mathcal{M} = \{K_1, \dots, K_J\} \subseteq \mathcal{G}$ , the bisection $\mathrm{bisect}(\mathcal{G}, \mathcal{M})$ is (p,q)-admissible if there is an order $(\sigma(1), \dots, \sigma(J))$ (this is, if there is a permutation $\sigma$ of $1, \dots, J)$ such that

$$\mathrm{bisect}(G, M) = \mathrm{bisect}(\mathrm{bisect}(\dots \mathrm{bisect}(\mathcal{G}, K_{\sigma(1)}), \dots), K_{\sigma(J)})$$

is a concatenation of (p,q)-admissible bisections.

**Definition 7 (Admissible refinement).**

A refinement $\mathcal{G}$ of $\mathcal{G}_0$ is (p,q)-admissible if there is a sequence of meshes $\mathcal{G}_1, \ldots, \mathcal{G}_J = \mathcal{G}$ and markings $\mathcal{M}_j \subseteq \mathcal{G}_j$ for $j = 0, \ldots, J-1$, such that $\mathcal{G}_{j+1} = \mathrm{bisect}(\mathcal{G}_j, \mathcal{M}_j)$ is a (p,q)-admissible bisection for all $j = 0, \ldots, J-1$. The set of all (p,q)-admissible meshes, which is the initial mesh and all its admissible refinements, is denoted by $\mathbb{A}^{p,q}$.

# Preservance of admissibility

**Lemma 8 (Local quasi-uniformity).**

*Given* $K \in \mathcal{G} \in \mathbb{A}^{p,q}$, *any* $K' \in \mathcal{G}^{p,q}(K)$ *satisfies* $\ell(K') \geq \ell(K) - 1$.

Proof.

See [5]. □

**Proposition 9.**

*Any admissible mesh* $\mathcal{G}$ *and any set of marked elements* $\mathcal{M} \subseteq \mathcal{G}$ *satisfy* $\mathrm{ref}^{p,q}(\mathcal{G}, \mathcal{M}) \in \mathbb{A}^{p,q}$.

Proof.

By induction and with Lemma 8. For details, see [5]. □

# Analysis-suitability

To ensure that the T-spline blending functions of a refined mesh are still linearly independent, we need the concept of analysis-suitable meshes.

**Definition 10.**

Consider an admissible mesh $\mathcal{G} \in \mathbb{A}^{p,q}$. The set of vertices of $\mathcal{G}$ is denoted by $\mathcal{N}$. We define the active region

$$\mathcal{AR} := [\lceil\frac{p}{2}\rceil, M - \lceil\frac{p}{2}\rceil] \times [\lceil\frac{q}{2}\rceil, N - \lceil\frac{q}{2}\rceil]$$

and the set of active nodes $\mathcal{N}_A := \mathcal{N} \cap \mathcal{AR}$

**Definition 11.**

We denote by $hSk$ (resp. $vSk$) the horizontal (resp. vertical) skeleton, which is the union of all horizontal (resp. vertical) edges. Note that $hSk \cap vSk = \mathcal{N}$ .

### Definition 12 (T-junction extension).

Denote by $\mathcal{T} \subset \mathcal{N}_A$ the set of all active nodes with valence three and refer to them as T-junctions. Consider a T-junction $T = (t_1, t_2) \in \mathcal{T}$ of type $\dashv$. Clearly, $t_1$ is one of the entries of $\boldsymbol{X}(t_2)$. Then extract from $\boldsymbol{X}(t_2)$ the $p+1$ consecutive indices $i_{-\lfloor p/2 \rfloor}, \dots, i_{\lceil p/2 \rceil}$ such that $i_0 = t_1$. We denote

$$\mathrm{ext}_e^{p,q}(T) := [i_{-\lfloor p/2 \rfloor}, i_0] \times \{t_2\}, \qquad \mathrm{ext}_f^{p,q}(T) := (i_0, i_{\lceil p/2 \rceil}] \times \{t_2\},$$

$$\mathrm{ext}^{p,q}(T) := \mathrm{ext}_f^{p,q}(T) \cup \mathrm{ext}_e^{p,q}(T).$$

Here $\boldsymbol{X}(y) := \{z \in [0, M] : (z, y) \in vSk\}$ is a global index set (analogous definition for the x-direction).

### Definition 13 (Analysis-suitability).

A mesh is analysis-suitable if horizontal T-junction extensions do not intersect vertical T-junction extensions.

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A mesh is analysis-suitable if horizontal T-junction extensions do not intersect vertical T-junction extensions.

**Theorem 14.**

*All admissible meshes are analysis suitable.*

Proof.

By induction over admissible bisections. For details, see [5]. □

**Corollary 15.**

*All admissible meshes provide T-spline blending functions that are non-negative, linearly independent, and form a partition of unity. Moreover, on each element $K \in \mathcal{G} \in \mathbb{A}^{p,q}$, there are not more than $2(p+1)(q+1)$ T-spline basis functions that have a support on $K$.*

Proof.

See [3, 1] □

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Proof.

See [3, 1] □

# Overlay

To use this algorithm for a posteriori error-driven refinement, we need some theoretical properties on the *overlay*, which is the common coarsest refinement.

**Definition 16 (Overlay).**

We define the operator $\mathrm{Min}_\subseteq$ which yields all minimal elements of a set that is partially ordered by " $\subseteq$ ",

$$\mathrm{Min}_\subseteq(\mathcal{M}) := \{K \in \mathcal{M} : \forall K' \in \mathcal{M} : K' \subseteq K \to K' = K\}$$

The overlay of $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ is defined by

$$\mathcal{G}_1 \otimes \mathcal{G}_2 := \mathrm{Min}_\subseteq(\mathcal{G}_1 \cup \mathcal{G}_2)$$

**Proposition 17.**

*$\mathcal{G}_1 \otimes \mathcal{G}_2$ is the coarsest refinement of $\mathcal{G}_1$ and $\mathcal{G}_2$ in the sense that for any $\hat{\mathcal{G}}$ being a refinement of $\mathcal{G}_1$ and $\mathcal{G}_2$, and $\mathcal{G}_1 \otimes \mathcal{G}_2$ being a refinement of $\hat{\mathcal{G}}$, it follows that $\hat{\mathcal{G}} = \mathcal{G}_1 \otimes \mathcal{G}_2$.*

Proof.

Blackboard, see [5]. □

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Proof.

Blackboard, see [5]. □

# Two theoretical properties

**Proposition 18.**

*For any admissible meshes $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$, the overlay $\mathcal{G}_1 \otimes \mathcal{G}_2$ is also admissible.*

**Lemma 19.**

*For all $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ holds*

$$\#(\mathcal{G}_1 \otimes \mathcal{G}_2) + \#\mathcal{G}_0 \leq \#\mathcal{G}_1 + \#\mathcal{G}_2.$$

The second property is a assumption in [2].

# Nestedness

Now the nesting behaviour of the T-spline spaces corresponding to admissible meshes is investigated (for details, see [5, 4]).

**Definition 20.**

For any partitions $\mathcal{G}_1, \mathcal{G}_2$ of $\overline{\Omega}$ we introduce the refinement relation " $\preceq$ ", which is defined using the overlay

$$\mathcal{G}_1 \preceq \mathcal{G}_2 \Leftrightarrow \mathcal{G}_1 \otimes \mathcal{G}_2 = \mathcal{G}_2$$

**Corollary 21.**

*Denote the skeleton of a mesh $\mathcal{G}$ by $Sk(G) := hSk(G) \cup vSk(G)$ . Then for rectangular partitions $\mathcal{G}_1, \mathcal{G}_2$ of $\overline{\Omega}$ holds the equivalence*

$$\mathcal{G}_1 \preceq \mathcal{G}_2 \Leftrightarrow Sk(\mathcal{G}_1) \subseteq Sk(\mathcal{G}_2)$$

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# Mesh extension

**Definition 22.**

Given a rectangular partition $\mathcal{G}$ of $\overline{\Omega}$ , denote by $\mathrm{ext}^{p,q}(G)$ the union of all T-junction extensions in the mesh $\mathcal{G}$. Then the extended mesh $\mathcal{G}^{\mathrm{ext}}$ is defined as the unique rectangular partition of $\overline{\Omega}$ such that

$$Sk(\mathcal{G}^{\mathrm{ext}}) = Sk(\mathcal{G}) \cup \mathrm{ext}^{p,q}(\mathcal{G}).$$

Sketch of an extended mesh.

# Mesh perturbation

**Definition 23.**

Given a partition G of $\overline{\Omega}$ into axis-aligned rectangles, we define by $Ptb(\mathcal{G})$ the set of all continuous and invertible mappings $\delta : \overline{\Omega} \to \overline{\Omega}$ such that the corners $(0,0), (M,0), (M,N), (0,N)$ are fixed points of $\delta$ and

$$\delta(\mathcal{G}) = \{\delta(K) : K \in \mathcal{G}\}$$

is also a partition of $\overline{\Omega}$ into axis-aligned rectangles.

Note for $\delta \in Ptb(\mathcal{G})$, the corresponding skeleton satisfies $Sk(\delta(\mathcal{G})) = \delta(Sk(\mathcal{G}))$. In general, such a perturbation $\delta$ does not map T-junction extensions to the corresponding extensions in the perturbed mesh, i.e.

$$\mathrm{ext}^{p,q}_{\delta(\mathcal{G})}(\delta(T)) \neq \delta(\mathrm{ext}^{p,q}_{\mathcal{G}}(T)).$$

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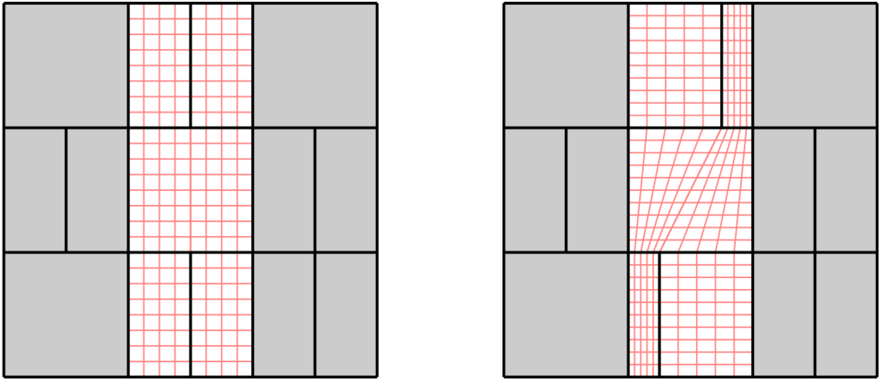

Example for a perturbation.

**Theorem 24.**

*Given two analysis-suitable meshes $\mathcal{G}_1$ and $G_2$, if for all $\delta \in Ptb(\mathcal{G}_2)$ holds*

$$(\delta(\mathcal{G}_1))^{\text{ext}} \preceq (\delta(\mathcal{G}_2))^{\text{ext}}$$

*then the T-spline spaces corresponding to $G_1$ and $G_2$ are nested.*

Proof.

See [4]. □

**Theorem 24.**

*Given two analysis-suitable meshes* $\mathcal{G}_1$ *and* $G_2$*, if for all* $\delta \in Ptb(\mathcal{G}_2)$ *holds*

$$(\delta(\mathcal{G}_1))^{\text{ext}} \preceq (\delta(\mathcal{G}_2))^{\text{ext}}$$

*then the T-spline spaces corresponding to* $G_1$ *and* $G_2$ *are nested.*

Proof.

See [4]. □

**Theorem 25.**

*Any two meshes $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ that are nested in the sense $\mathcal{G}_1 \preceq \mathcal{G}_2$ satisfy for all $\delta \in Ptb(\mathcal{G}_2)$*

$$(\delta(\mathcal{G}_1))^{\text{ext}} \preceq (\delta(\mathcal{G}_2))^{\text{ext}}.$$

Proof.

It is sufficient to show

$$\text{ext}^{p,q}(\delta(\mathcal{G}_1)) \cup Sk(\delta(\mathcal{G}_1)) \subseteq \text{ext}^{p,q}(\delta(\mathcal{G}_2)) \cup Sk(\delta(\mathcal{G}_2)).$$

First, let $K \in \mathcal{G}_1 \in \mathbb{A}^{p,q}$ and $\mathcal{G}_2 := \text{bisect}(\mathcal{G}_1)$, then $\mathcal{G}_1 \preceq \mathcal{G}_2 \Rightarrow Sk(\delta(\mathcal{G}_1)) \subseteq Sk(\delta(\mathcal{G}_2))$.
The second part includes comparison of different cases. For further details, see [5]. □

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*Any two meshes $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ that are nested in the sense $\mathcal{G}_1 \preceq \mathcal{G}_2$ satisfy for all $\delta \in Ptb(\mathcal{G}_2)$*

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Combination of these two results gives us:

**Corollary 26.**

*For any two meshes $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{A}^{p,q}$ that are nested in the sense $\mathcal{G}_1 \preceq \mathcal{G}_2$ , the corresponding T-spline spaces are also nested.*

# Linear complexity

The following estimate shows that the number of refined elements depends at most linearly on the number of marked elements.

**Theorem 27.**

*Any sequence of admissible meshes* $\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_J$ *with*

$$\mathcal{G}_j = \mathrm{ref}^{p,q}(\mathcal{G}_{j-1}, \mathcal{M}_{j-1}), \quad \mathcal{M}_{j-1} \subseteq \mathcal{G}_{j-1} \text{ for } j \in \{1, \dots, J\}$$

*satisfies*

$$|\mathcal{G}_J \setminus \mathcal{G}_0| \leq C_{p,q} \sum_{j=0}^{J-1} |\mathcal{M}_j|$$

*with* $C_{p,q} = (3 + \sqrt{2})(4d_p + 1)(4d_q + \sqrt{2})$ *and*

$$d_p = (1 + 2^{-1/2})p + 1 + \frac{5}{4}\sqrt{2}, \quad d_q = (1 + \sqrt{2})q + \frac{3}{2} + \sqrt{2}.$$

# Sketch of proof

One can show the following:

- for $K \in \bigcup \mathbb{A}^{p,q}$ and $\tilde{K} \in \mathcal{M}$, define $\lambda(K, \tilde{K})$ by

$$\lambda(K,\tilde{K}) := \begin{cases} 2^{\ell(K)-\ell(\tilde{K})/2}, & \text{if } \ell(K) \leq \ell(\tilde{K}) + 1 \text{ and } \mathrm{Dist}(K,\tilde{K}) \leq 2^{1-\ell(K)/2}(d_p, d_q) \\ 0 \text{ otherwise} \end{cases}$$

- for all $j \in \{0, \ldots, J-1\}$ and $\tilde{K} \in \mathcal{M}_j$ holds

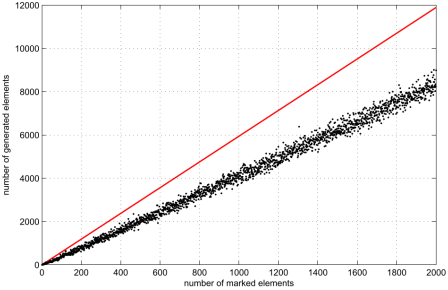
$$\sum_{K \in \mathcal{G}_J \setminus \mathcal{G}_0} \lambda(K, \tilde{K}) \leq (3 + \sqrt{2})(4d_p + 1)(4d_q + \sqrt{2}) = C_{p,q},$$

- each $K \in \mathcal{G}_J \setminus \mathcal{G}_0$ satisfies

$$\sum_{\tilde{K} \in \mathcal{M}} \lambda(K, \tilde{K}) \geq 1.$$

# Remarks and Examples

- The result of the theorem is not trivial, as there is no uniform bound for the number of generated elements.
- The large constant $C_{p,q}$ was not observed in the numerical experiments by the authors.




Generated and marked elements for randomly refined (3,3)-admissible meshes.

# Observed bounds for higher degrees of (p,q)

| p \ q | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 5 | 7 | 7 | 7 | 7 | 8 | 8 |
| 3 | 6 | 6 | 7 | 7 | 8 | 8 | 9 | 11 |
| 4 | 7 | 8 | 8 | 8 | 11 | 10 | 10 | 12 |
| 5 | 7 | 7 | 9 | 10 | 10 | 12 | 11 | 13 |
| 6 | 7 | 8 | 10 | 10 | 11 | 12 | 12 | 16 |
| 7 | 8 | 11 | 10 | 13 | 12 | 12 | 16 | 14 |
| 8 | 9 | 10 | 11 | 17 | 13 | 13 | 15 | 15 |
| 9 | 9 | 11 | 12 | 14 | 14 | 16 | 16 | 23 |

Maximal observed ratios for random refinement.

# Observed bounds for higher degrees of (p,q)

| $p$ \ $q$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 2 | 24 | 33 | 46 | 56 | 69 | 78 | 91 | 100 |
| 3 | 33 | 46 | 65 | 78 | 97 | 109 | 128 | 140 |
| 4 | 46 | 65 | 91 | 110 | 136 | 154 | 179 | 198 |
| 5 | 56 | 78 | 110 | 132 | 163 | 186 | 216 | 238 |
| 6 | 69 | 97 | 136 | 164 | 202 | 229 | 268 | 295 |
| 7 | 78 | 110 | 154 | 186 | 229 | 260 | 304 | 335 |
| 8 | 91 | 128 | 180 | 217 | 268 | 304 | 355 | 391 |
| 9 | 100 | 141 | 198 | 239 | 295 | 335 | 391 | 431 |

Maximal observed ratios when refining the lower left.

Thank you!