Monotone operators in nonlinear PDEs

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Tu 12-13:30
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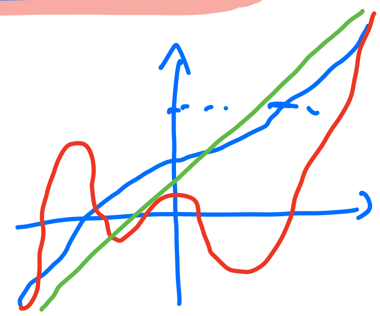
0. Motivation



We want to generalize a result like:

A function $F:\mathbb{R}\to\mathbb{R}$ fulfilling:
- F monotonically increasing
- F continuous
- F coercive, $F(u)\to\pm\infty$ if $x\to\pm\infty$

then the equation $F(u)=b$ has a solution $u\in\mathbb{R}$ $\forall b\in\mathbb{R}$ ie F is surjective

If F is strictly monotone, then the solution u is unique.

The theory on monotone operators wants to generalize this result to equations of the form $\boxed{Au=b}$ in a reflexive Banach space.

$X\cong X''$, $\langle Jx,x'\rangle_{X'}=\langle x',x\rangle_X$ $\forall x'\in X'$

Theorem 0.1 Let X be a separable & reflexive Banach space, and let the operator $A: X\to X'$ fulfill:

- monotone i.e. $\langle \underset{\in X^*}{Au}-Av,\,u-v\rangle_X\geq 0$ $\forall u,v\in X$
- hemicontinuous ie. $t\mapsto \underbrace{\langle A(u+tv),w\rangle_X}_{\in\mathbb{R}}$ is continuous in $[0,1]$ for any $u,v,w\in X$ $\xrightarrow{t\to 0}\langle Au,w\rangle_X$
- coercive ie. $\lim\limits_{\|u\|_X\to\infty}\dfrac{\langle Au,u\rangle_X}{\|u\|_X}=\infty$

$\Longrightarrow$ A is surjective ie. $\forall b\in X^*\ \exists u\in X:\ Au=b$

Sketch of a proof:

1) Galerkin approximation: Since X is separable, there is a basis $(w_i)_{i\in\mathbb{N}}$ of X s.t. for $X_n=\operatorname{span}\{w_1,\dots,w_n\}$ it holds

$$X=\overline{\bigcup_{n=1}^{\infty}X_n}=\overline{\operatorname{span}\{w_1,\dots\}}$$

We want to approximate $Au=b$ in the space X_n

$\rightsquigarrow u_n\in X_n$ (show this by Brouwer fixed pt. theorem)

every cont. mapping in a closed ball in $\mathbb{R}^d$ has a fixed pt.

2) A priori estimate: We want to show that u_n is bounded.

Since $A: X\to X'$ is coercive, there exists a $R_0>0$ s.t. for any $\|u\|_X>R_0$ it holds:

$$\frac{\langle Au,u\rangle_X}{\|u\|_X}\geq 1+\|b\|_{X^*}$$

$\langle b,u\rangle\leq\|b\|\cdot\|u\|$

$$\Rightarrow \langle Au, u\rangle_X \geq (1+\|b\|_{X^*})\cdot\|u\|_X$$

$$\Rightarrow \underbrace{\langle Au, u\rangle_X - \langle b, u\rangle_X}_{=0} \geq (1+\|b\|_{X^*})\|u\|_X - \langle b, u\rangle_X$$

$$\geq (1+\|b\|_{X^*})\|u\|_X - \|b\|_{X^*}\|u\|_X$$

$$\geq \|u\|_X > R_0$$

Now, if $u\in X$ with $\|u\|_X > R_0$ is a solution of $Au=b$, it would mean $0 \geq R_0 > 0$ ↯

$\Rightarrow$ $u \in X$ the solution of $Au=b$ fulfills $\|u\|_X \leq R_0$

3) Weak convergence Since X is reflexive, it follows by the Eberlein-Smulian theorem that there exists a weakly converging subsequence $(u_{n_k})_{k\in\mathbb{N}}$ & a limit pt. $\underline{u}$ s.t.

$$u_{n_k} \rightharpoonup \underline{u} \text{ in } X \qquad (f(u_n)\to f(u) \ \forall f\in X^*)$$

4) Existence We need to show that this $\underline{u}$ is a solution of $Au=b$ (Minty trick)

$$\langle Au_{n_k}, w\rangle = \langle b, w\rangle \quad \forall w\in X_{n_k}$$

we need to prove

$$\langle Au, w\rangle = \langle b, w\rangle \quad \forall w\in X$$

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Lemma 0.2 (Minty) Let X be a Banach space & $A: X\to X^*$ a hemicontinuous & monotone operator. Then:

(i) If A is maximal monotone ie. for any fixed $u\in X$, $b\in X^*$ it holds

$$\langle b - Av, u-v\rangle_X \geq 0 \quad \forall v\in X$$

$$\Longrightarrow \quad Au = b$$

$\langle Au - Av, u-v\rangle \geq 0$ take $b = Au$

(ii) If A type M ie.

$$u_n \rightharpoonup u \text{ in } X$$

$$Au_n \rightharpoonup b \ (=Au) \text{ in } X^*$$

$$\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \leq \langle b, u\rangle_X$$

$$\Rightarrow Au = b$$

(iii) If $u_n \to u$ in X & $Au_n \rightharpoonup b$ in X^*
or $u_n \rightharpoonup u$ in X & $Au_n \to b$ in X^*

$\Rightarrow Au = b$

Proof (i) Let $u \in X$, $b \in X^*$ be given s.t. A is max. monotone. $w \in X$ arbitrary but fixed.

Then we set $v := u + tw$, $t > 0$, $w \in X$ arbitrary but fixed.

Then we have

$$\langle b - Av, u - v\rangle \geq 0 \Rightarrow \langle b - A(u + t\cdot(w)), -w\rangle \geq 0$$

A is hemicontinuous $\Rightarrow \langle b - Au, w\rangle_X \geq 0$

We can do the same for $v = u - tw$ $\Rightarrow \langle b - Au, w\rangle_X \leq 0$

$\Rightarrow \langle b - Au, w\rangle_X = 0$ $\forall w \in X$ $\Rightarrow b = Au$

(ii) $$0 \overset{A \text{ monotone}}{\leq} \langle Au_n - Av, u_n - v\rangle_X = \langle Au_n, u_n\rangle_X - \langle Av, u_n\rangle_X - \langle Au_n - Av, v\rangle_X$$

Take $\limsup\limits_{n\to\infty}$

$\langle Au_n, v\rangle \to \langle b, v\rangle$

$$0 \leq \underbrace{\limsup \langle Au_n, u_n\rangle_X}_{\leq \langle b, u\rangle_X} - \underbrace{\limsup \langle Av, u_n\rangle_X}_{= \langle Av, u\rangle_X \text{ since } u_n \rightharpoonup u} - \underbrace{\limsup \langle Au_n - Av, v\rangle_X}_{= \langle b - Av, v\rangle_X}$$

$$= \langle b - Av, u - v\rangle_X$$

$\Rightarrow A$ max. monotone

$\overset{(i)}{\Rightarrow} Au = b$

(iii) follows from Lemma 0.3

Lemma 0.3 Let X be a Banach space.

(i) If $x_n \rightharpoonup x$ in X, then $\|x_n\|_X \leq C$ $\forall n \in \mathbb{N}$

(ii) If $x_n \to x$ in X, $f_n \overset{*}{\rightharpoonup} f$ in X^* then $\langle f_n, x_n\rangle_X \to \langle f, x\rangle_X$

(iii) If $x_n \rightharpoonup x$ in X, $f_n \to f$ in X^* then $\langle f_n, x_n\rangle_X \to \langle f, x\rangle_X$

(iv) Let X be reflexive. Let $(x_n)_n$ be bounded. If all weakly convergent subsequences of (x_n) converge to the same limit point x, then $x_n \rightharpoonup x$ in X

Proof: Functional analysis & see Exercise class

1. Monotone operators

Definition 1.1 Let X be a Banach space, let $A: X \to X^*$. Then A is called:

(i) monotone iff. $\langle Au - Av, u-v \rangle_X \geq 0 \quad \forall u, v \in X$

(ii) strictly monotone iff $\langle Au - Av, u-v \rangle_X > 0 \quad \forall u, v \in X, u \neq v$

(iii) strongly monotone iff $\exists c > 0$ s.t. $\langle Au - Av, u-v \rangle_X \geq c \cdot \|u-v\|_X^2 \quad \forall u, v \in X$
>0 if $u \neq v$

(iv) coercive iff $\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle_X}{\|u\|_X} = \infty$

It holds: (i) A strongly monotone $\Rightarrow$ A strictly mon. $\Rightarrow$ A mon.

(ii) A strongly monotone, then A is coercive since

$$\langle Au, u \rangle_X = \langle Au + A(0), u \rangle_X - \langle A(0), u \rangle_X$$
$$\geq c\, \|u-0\|_X^2 - \|A(0)\|_{X^*} \|u\|_X$$

divide by $\|u\|_X$

$$\frac{\langle Au, u \rangle_X}{\|u\|_X} \geq \underbrace{c\,\|u\|_X}_{\to \infty} - \|A(0)\| \longrightarrow \infty \quad \text{for } \|u\|_X \to \infty$$

Example: 1) let $f: \mathbb{R} \to \mathbb{R}$, $X = \mathbb{R}$, $X^* = \mathbb{R}$,

$$\langle f(u) - f(v), u-v \rangle_X = \underline{(f(u) - f(v))} \cdot (\overset{\geq 0}{\overbrace{u-v}}) \geq 0$$

(i). f (strictly) monotone $\Leftrightarrow$ f (strictly) monotonically increasing

(ii) f coercive $\Leftrightarrow$ $f(u) \to \pm\infty$ if $u \to \pm\infty$

2) $g: \mathbb{R} \to \mathbb{R}$, $g(u) = \begin{cases} |u|^{p-2} u, & u \neq 0 \\ 0 & u = 0 \end{cases}$

Exercise: (i) g is strictly monotone for $p > 1$

(ii) $\langle g(u) - g(v), u-v \rangle \geq c \cdot |u-v|^p$ for $p \geq 2$

(iii) g strongly mon. for $p = 2$

Definition 1.2 Let X, Y be Banach spaces, $A: X \to Y$. Then A is called

Ex: $Id: H^1(\Omega) \to L^2(\Omega)$
$u_n \rightharpoonup u$ in $H^1(\Omega)$
$\Rightarrow u_n \to u$ in $L^2(\Omega)$

(i) completely continuous iff

$$u_n \rightharpoonup u \text{ in } X \Rightarrow Au_n \to Au \text{ in } Y$$

(A is weak-strong continuous)

(ii) demicontinuous iff

$$u_n \to u \text{ in } X \Rightarrow Au_n \rightharpoonup Au \text{ in } Y$$

(A is strong-weak convergent)

(iii) hemicontinuous if $Y = X^*$ and $t \mapsto \langle A(u+tv), w\rangle_X$ is continuous in $[0,1]$ for any $u, v, w \in X$

(A weakly continuous)

(iv) bounded iff A maps bounded sets in X to bounded sets in Y

(v) locally bounded iff $\forall u \in X\ \exists\, \varepsilon(u) > 0, K(u) : \|Av\|_Y \le K$ for any $v \in X$ with $\|u - v\|_X < \varepsilon$

strong-strong,
weak-weak conv.

A is monotone

A compl. cont. $\Rightarrow$ A cont. $\Rightarrow$ A demicontinuous $\Leftarrow$ A hemicontinuous

A bounded $\Rightarrow$ A locally bounded.

Lemma 1.3 X be a reflexive Banach space, $A: X \to X^*$

(i) If A completely continuous, then A is compact.

(ii) If A demicontinuous, then A is locally bounded.

(iii) If A monotone, then A is locally bounded.

(iv) If A monotone & hemicontinuous, then A is demicontinuous.

Proof (i) compact $\hat{=}$ relatively compact wrt sequence

We want to show that for every bdd subset $M \subseteq X$ the image $A(M)$ is relatively compact wrt sequences.

Let $(Au_n)_n \subseteq A(M)$. Since M is bounded, we know $(u_n)_n$ is bounded.

By Eberlein-Šmulian theorem (bc. X refl. BS) $\exists (u_{n_k})_k, u \in X$ s.t. $u_{n_k} \rightharpoonup u$ in X $\Rightarrow$ $Au_{n_k} \to Au$ in X $\Rightarrow$ $A(M)$ is relatively compact.

(ii) Proof by contradiction: Let A not be locally bounded, i.e. $\exists u \in X$ & a sequence $(u_n) \subseteq X$ s.t. $u_n \to u$ but $\|Au_n\|_{X^*} \to \infty$.

However, A is demicont., it holds $Au_n \to Au$ in X^*. Thus, $\|Au_n\| \le C$
$\lightning$

(iii) Proof by contradiction: Let A not be locally bounded, then there is a $u \in X$ & a sequence $(u_n)_n \subseteq X$ s.t. $u_n \to u$ in X but $\|Au_n\|_{X^*} \to \infty$

We define: $$a_n = (\underset{\uparrow}{1} + \|Au_n\|_{X^*} \cdot \|u_n - u\|_X)^{-1} > 0 \qquad \frac{1}{1+\dots}$$
$$\le 1$$

Since A is monotone, we know $\forall v \in X$
$$0 \le \langle Au_n - Av, u_n - v\rangle_X$$
$$= \langle Au_n - Av, (u_n - u) + (u - v)\rangle_X$$

$$\Rightarrow a_n \cdot \langle Au_n, \underbrace{v-u}_{u-v}\rangle \le a_n (\langle Au_n, u_n - u\rangle_X - \langle Av, u_n - v\rangle)$$
$$\le \underbrace{a_n \|Au_n\|_{X^*} \cdot \|u_n - u\|_X}_{\le 1} + \underbrace{a_n}_{\le 1} \underbrace{\|Av\|_{X^*}}_{=C_1} (\underbrace{\|u_n\|_X}_{\le C_2} + \underbrace{\|v\|_X}_{=C_3})$$
$$\le 1 + c(v,u)$$

Since this holds for any $v \in X$, we can select $v = 2u - v$
$$\Rightarrow -a_n \langle Au_n, v - u\rangle_X \le 1 + c(v,u) \qquad \forall n$$
$$\Rightarrow \sup_n |a_n \langle Au_n, \underset{\uparrow}{w}\rangle_X| \le \tilde{c}(w,u) < \infty$$

Uniform boundedness principle (linear & cont. operator $a_n Au_n : X \to \mathbb{R}$ are pointwise bounded) tells us that
$$\sup_n \|a_n Au_n\|_{X^*} \le c(u) < \infty$$

$$\Rightarrow \|Au_n\|_{X^*} \le \frac{c(u)}{a_n} = c(u) \cdot (1 + \|Au_n\|_{X^*} \cdot \underbrace{\|u_n - u\|_X}_{\to 0}) \le c(u) + \frac{1}{2}\|Au_n\|_{X^*}$$

Since $\|u_n - u\|_X \to 0$, there is a $n_0 \in \mathbb{N}$ s.t. $c(u) \cdot \|u_n - u\|_X < \frac{1}{2} \quad \forall n \ge n_0$
$$\Rightarrow \|Au_n\|_{X^*} \le 2c(u) \quad \lightning$$

(iv) Let $(u_n)_n \subseteq X$ be a sequence with $u_n \to u \in X$.

A is monotone $\overset{(iii)}{\Rightarrow}$ A locally bounded $\Rightarrow \|Au_n\|_{X^*} \le C$

$\overset{\text{Eberlein}}{\underset{\text{Smulian}}{\Rightarrow}} \exists (u_{n_k})$ s.t. $Au_{n_k} \rightharpoonup b$ in X^*

1) Minty's trick: $b = Au$
(Lemma 0.2 (iii))

2) Every subsequence of $(Au)_n$ is converging weakly to Au

since otherwise there would be a subsequence $(Au_{n_\ell})_\ell \subseteq X^*$ s.t.

$Au_{n_\ell} \longrightarrow c \neq b$ in X^*

$\overset{\text{Minty}}{\Longrightarrow} c = Au$ ↯ $Au = b$

$\Rightarrow$ every subsequence is converging weakly to $b = Au$

1.1.1. Theorem of Browder – Minty

Theorem 1.5 (Browder-Minty) Let X be a separable & reflexive BS. Further, let $A: X \to X^*$ be monotone, coercive, hemicontinuous. Then for any $b \in X^*$ there is a solution $u \in X$ of

$$Au = b. \qquad \langle Au - b, w \rangle_X = 0 \quad \forall w \in X$$

The solution set is closed, bounded & convex. If A is strictly monotone, then the solution of $Au = b$ is unique.

Proof Since X is separable, there is a basis $(w_i)_{i\in\mathbb{N}}$ of X.
We set $X_n = \operatorname{span}(w_1, \dots, w_n)$ & we look for an approximate solution $u_n \in X_n$ of the form $u_n = \sum_{j=1}^{n} c_n^j w_j$ to the Galerkin system

$$\langle Au_n - b, w_k \rangle_X = 0 \quad \forall k = 1, \dots, n \qquad (G)$$

$$Au_n - \Pi_n b = 0, \qquad \Pi_n : X, X^* \to X_n \text{ orth. projection}$$

1. (G) has a solution

Define $c_n = \begin{pmatrix} c_n^1 \\ \vdots \\ c_n^n \end{pmatrix} \in \mathbb{R}^n$, $|c_n| := \left\| \sum_{j=1}^{n} c_n^j w_j \right\|_X$

$$g_n^k(c_n) = \left\langle A\left(\sum_{j=1}^{n} c_n^j w_j\right) - b, w_k \right\rangle_X = 0 \quad \forall k = 1, \dots, n$$

$g_n^k : \mathbb{R}^n \to \mathbb{R}$

We look for a vector c_n s.t. $g_n^k(c_n) = 0 \quad \forall k = 1, \dots, n$

$$\longrightarrow \underbrace{\begin{pmatrix} g_n^1(c_n) \\ \vdots \\ g_n^n(c_n) \end{pmatrix}}_{= g_n(c_n)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

A is monotone & hemicontinuous $\Rightarrow$ A demicontinuous by Lemma 1.3(iv)

$\Rightarrow g_n : \mathbb{R}^n \to \mathbb{R}^n$ is continuous since if $c_\ell \to c$ as $\ell \to \infty$ w.r.t. $|\cdot|$ in $\mathbb{R}^n$ then $\sum_{j=1}^n c_\ell^j w_j \longrightarrow \sum_{j=1}^n c^j w_j$ in X

$\Rightarrow g_n(c_\ell) \to g_n(c) \Rightarrow g_n$ is continuous

Brouwer: Every continuous mapping of a closed ball in $\mathbb{R}^n$ into itself has a fixed pt.

Corollary: Let $g = (g^1, \dots, g^n) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, that fulfills $\exists R > 0 : \sum_{i=1}^n g^i(x)\, x_i = (g(x), x) \geq 0 \quad \forall x$ with $|x| = R$.

Then there exists a solution x_0 of $g(x_0) = 0$ with $|x_0| \leq R$.

<u>Proof:</u> Proof by contradiction. Let $g(x_0) = 0$ have no solution in $\overline{B_R(0)} \subseteq \mathbb{R}^n$. We define $f^i(x) = -R \cdot \frac{g^i(x)}{|g(x)|}$, $i = 1, \dots, n$

Since $g(x_0) = 0$ has no solution, it holds $|g(x)| > 0 \quad \forall x \in \overline{B_R(0)}$

$\Rightarrow f = (f^1, \dots, f^n)^T$ is well-defined, continuous, maps the closed ball $\overline{B_R(0)}$ into itself.

$\overset{\text{Brouwer}}{\Longrightarrow} \exists x^* \in \overline{B_R(0)}$ s.t. $x^* = f(x^*)$

$\quad \& \quad |x^*| = |f(x^*)| = \left| -R \frac{g(x^*)}{|g(x^*)|} \right| = R$

$$\Rightarrow 0 \leq \sum_{i=1}^n g^i(x^*) \cdot x_i^* = -\sum_{i=1}^n \underbrace{f^i(x^*)}_{= x_i^*} \cdot x_i^* \frac{|g(x^*)|}{R}$$

$$= -\underbrace{|x^*|^2}_{= R^2} \cdot \frac{|g(x^*)|}{R} = -R\,|g(x^*)| < 0 \quad ↯$$

Let $c = (c^1, \dots, c^n)^T$, $v = \sum_{j=1}^n c^j w_j$

$$\sum_{k=1}^n g_n^k(c) \cdot c^k \underset{\text{Def. of } g_n^k}{=} \langle Av, v \rangle_X - \langle b, v \rangle_X$$

$\langle b, v \rangle_X \leq \|b\|_{X^*} \cdot \|v\|_X$

$-\langle b, v \rangle_X \geq -\|b\|_{X^*} \cdot \|v\|_X$

Since A is coercive i.e. $\frac{\langle Aw, w \rangle_X}{\|w\|_X} \to \infty$ as $\|w\|_X \to \infty$

$\Rightarrow \exists R_0 > 0$ s.t. $\forall \|w\|_X \geq R_0$ it holds $\langle Aw, w \rangle_X \geq \|b\|_{X^*} \cdot \|w\|_X$

Hence, for any c with $|c| = \|v\|_X = R_0$ it holds

$$\langle Av, v \rangle_X \geq \|b\|_{X^*} \|v\|_X$$

$$\Rightarrow \sum_{k=1}^n g_n^k(c) \cdot c^k \geq \|b\|_{X^*} \|v\|_X - \|b\|_{X^*} \|v\|_X = 0$$

$\xRightarrow[\text{of Brouwer}]{\text{Corollary}}$ $\exists u_n$ of (6) with $\|u_n\|_X \leq R_0$

a priori estimate

independent of n

② Boundedness of $(Au_n)_n$