Divergence Free Virtual Elements for the Stokes Problem on Polygonal Meshes

Seminar on Numerical Analysis

Virtual Element Methods (VEM)

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Outline

- Continuous Problem
- Virtual formulation
- Approximation and convergence properties
- Reduced spaces and reduced problem
- Numerical example

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The model Stokes problem

Find $(\mathbf{u}, p)$ such that (s.t.)

$$\begin{aligned} -\nu \boldsymbol{\Delta} \mathbf{u} - \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \Gamma = \partial\Omega \end{aligned}$$

The model Stokes problem

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q = [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ s.t.

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u},q) &= 0 && \forall q \in Q, \end{aligned}$$

with

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= \int_\Omega \nu \boldsymbol{\nabla} \mathbf{u} : \boldsymbol{\nabla} \mathbf{v} \, \mathrm{d}\Omega \\ b(\mathbf{v},q) &:= \int_\Omega \operatorname{div} \mathbf{v} \, q \, \mathrm{d}\Omega \end{aligned}$$

Existence and Uniqueness are well known (see [NumCM]).

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Preliminaries

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into general polygonal elements K with

$$h_K := \mathrm{diam}(K) \qquad \text{and} \qquad h := \sup_{K \in \mathcal{T}_h} h_K .$$

Assumptions on $\mathcal{T}_h$

(A1) K is star-shaped wrt a ball of radius $\geq \gamma h_K$,

(A2) the distance between any two vertices of K is $\geq c h_K$

The local virtual spaces

Let

- $\mathbb{P}_k(K)$ the set of polynomials on K of degree $\leq k$
- $\mathbb{B}_k(K) := \{v \in C^0(\partial K) : v|_e \in \mathbb{P}_k(e)\ \forall e \subset \partial K\}$
- $\mathcal{G}_k(K) := \nabla(\mathbb{P}_{k+1}(K)) \subseteq [\mathbb{P}_k(K)]^2$
- $\mathcal{G}_k(K)^\perp \subseteq [\mathbb{P}_k(K)]^2$ be the L^2-orth. complement to $\mathcal{G}_k(K)$

On each element, we define for $k \geq 2$ the local virtual spaces

$$\mathbf{V}_h^K := \left\{ \mathbf{v} \in [H^1(K)]^2 : \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2, \begin{cases} -\nu\,\boldsymbol{\Delta}\,\mathbf{v} - \nabla s \in \mathcal{G}_{k-2}(K)^\perp, \\ \operatorname{div} \mathbf{v} \in \mathbb{P}_{k-1}(K), \end{cases} \text{for some } s \in L^2(K) \right\}$$

and

$$Q_h^K := \mathbb{P}_{k-1}(K).$$

Degrees of freedom I

For $\mathbf{v} \in \mathbf{V}_h^K$ we have the linear operator $\mathbf{D_V}$ for the local degrees of freedom (DoFs)

- $\mathbf{D_V 1}$: the values of $\mathbf{v}$ at the vertices of K
- $\mathbf{D_V 2}$: the values of $\mathbf{v}$ at $k-1$ distinct points of every edge $e \subset \partial K$
- $\mathbf{D_V 3}$: the moments of $\mathbf{v}$

$$\int_K \mathbf{v} \cdot \mathbf{g}_{k-2}^{\perp} \, \mathrm{d}K \quad \text{for all } \mathbf{g}_{k-2}^{\perp} \in \mathcal{G}_{k-2}(K)^{\perp}$$

- $\mathbf{D_V 4}$: the moments up to order $k-1$ and greater than zero of $\operatorname{div} \mathbf{v}$ in K,

$$\int_K (\operatorname{div} \mathbf{v}) q_{k-1} \, \mathrm{d}K \quad \text{for all } q_{k-1} \in \mathbb{P}_{k-1}(K)/\mathbb{R}$$

Degrees of freedom II

For $q \in Q_h^K$ we have the local degrees of freedom

- $\mathbf{D_Q}$: the moments up to order $k-1$ of q in K,

$$\int_K q\, p_{k-1}\, \mathrm{d}K \quad \text{for all } p_{k-1} \in \mathbb{P}_{k-1}(K)$$

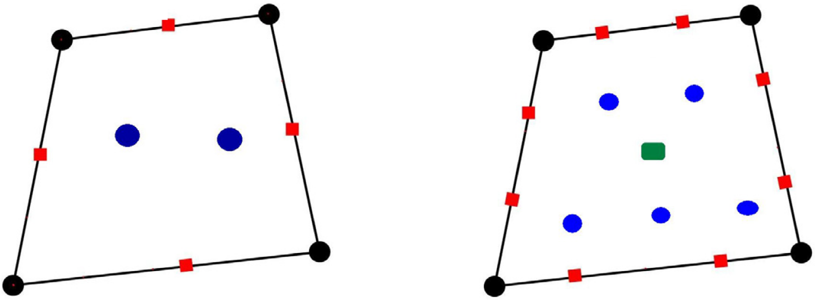
Proposition

The linear operators $\mathbf{D_V}$ and $\mathbf{D_Q}$ are a unisolvent set of DoFs for the virtual spaces $\mathbf{V}_h^K$ and Q_h^K, respectively.

Proof.

See [1]. □

Degrees of freedom III



DoFs for $k = 2, k = 3$. We denote $\mathbf{D_V 1}$ with the black dots, $\mathbf{D_V 2}$ with the red squares, $\mathbf{D_V 3}$ with the green rectangles, $\mathbf{D_V 4}$ with the blued dots inside the elements.

The global spaces

We define the global virtual element spaces as

$$\mathbf{V}_h := \{\mathbf{v} \in [H_0^1(\Omega)]^2 : \mathbf{v}|_K \in \mathbf{V}_h^K \text{ for all } K \in \mathcal{T}_h\},$$

and

$$Q_h := \{q \in L_0^2(\Omega) : q|_K \in Q_h^K \text{ for all } K \in \mathcal{T}_h\}.$$

Moreover, by construction

$$\operatorname{div} \mathbf{V}_h \subseteq Q_h.$$

The discrete bilinearform $b_h(.,.)$

We do not approximate the bilinearform b, i.e.,

$$b(\mathbf{v}_h, q_h) = \sum_{K \in \mathcal{T}_h} b^K(\mathbf{v}_h, q_h) \qquad \text{for all } \mathbf{v}_h \in \mathbf{V}_h, q_h \in Q_h.$$

The form above is **computable** from the degrees of freedom $\mathbf{D_V 1}$, $\mathbf{D_V 2}$ and $\mathbf{D_V 4}$.

Some important properties I

Observation

The quantity $a^K(\mathbf{q}_k, \mathbf{v})$ is exactly **computable** for all $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2$ and for all $\mathbf{v} \in \mathbf{V}_h^K$.
However, for any $(\mathbf{v}, \mathbf{w}) \in \mathbf{V}_h^K \times \mathbf{V}_h^K$, $a^K(\mathbf{v}, \mathbf{w})$ is **not computable**.

We want do define a computable discrete local bilinearform $a_h^K(\cdot, \cdot) : \mathbf{V}_h^K \times \mathbf{V}_h^K \to \mathbb{R}$.

Some important properties II

Properties of $a_h^K(.,.)$

- ***k*-consistency:** for all $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2$ and $\mathbf{v}_h \in \mathbf{V}_h^K$

$$a_h^K(\mathbf{q}_k, \mathbf{v}_h) = a^K(\mathbf{q}_k, \mathbf{v}_h)$$

- **stability:** there exist two positive constants α_*, α^*, independent of h and K, s.t., for all $\mathbf{v}_h \in \mathbf{V}_h^K$, it holds

$$\alpha_* \, a^K(\mathbf{v}_h, \mathbf{v}_h) \leq a_h^K(\mathbf{v}_h, \mathbf{v}_h) \leq \alpha^* \, a^K(\mathbf{v}_h, \mathbf{v}_h)$$

Some important properties III

Energy projection operator

For all $K \in \mathcal{T}_h$, we introduce the energy projection operator $\Pi_k^{\nabla,K} : \mathbf{V}_h^K \to [\mathbb{P}_k(K)]^2$, defined by

$$\begin{cases} a^K(\mathbf{q}_k, \mathbf{v}_h - \Pi_k^{\nabla,K}\mathbf{v}_h) = 0 & \text{for all } \mathbf{q}_k \in [\mathbb{P}_k(K)]^2 \\ P^{0,K}(\mathbf{v}_h - \Pi_k^{\nabla,K}\mathbf{v}_h) = \mathbf{0}, \end{cases}$$

where $P^{0,K}$ is the L^2-projection operator onto the constant functions on K.

Some important properties IV

Stabilizing

We introduce a symmetric stabilizing bilinear form $\mathcal{S}^K : \mathbf{V}_h^K \times \mathbf{V}_h^K \to \mathbb{R}$, satisfying

$$c_* a^K(\mathbf{v}_h, \mathbf{v}_h) \leq \mathcal{S}^K(\mathbf{v}_h, \mathbf{v}_h) \leq c^* a^K(\mathbf{v}_h, \mathbf{v}_h).$$

A possible choice is for instance

$$\mathcal{S}^K(\mathbf{v}_h, \mathbf{w}_h) = \alpha_K \, \overline{\mathbf{v}}_h^\top \overline{\mathbf{w}}_h,$$

with $\overline{\mathbf{v}}, \overline{\mathbf{w}} \in \mathbb{R}^{N_K}$ the vectors of local DoFs and α_K a suitably chosen constant.

The discrete bilinearform $a_h(.,.)$

For each polygon K we set

$$\begin{aligned} a_h^K(\mathbf{u}_h, \mathbf{v}_h) :=& a^K\left(\Pi_k^{\nabla,K}\mathbf{u}_h, \Pi_k^{\nabla,K}\mathbf{v}_h\right) \\ &+ \mathcal{S}^K\left((I - \Pi_k^{\nabla,K})\mathbf{u}_h, (I - \Pi_k^{\nabla,K})\mathbf{v}_h\right), \end{aligned}$$

which is k-consistent and stable.
The global approximated bilinearform is then

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} a_h^K(\mathbf{u}_h, \mathbf{v}_h)$$

Approximation of the right hand side

For $K \in \mathcal{T}_h$, let $\Pi_{k-2}^{0,K} : [L^2(K)]^2 \to [\mathbb{P}_{k-2}(K)]^2$ be the L^2-projection operator. Then

$$\mathbf{f}_h := \Pi_{k-2}^{0,K}\mathbf{f} \qquad \text{for all } K \in \mathcal{T}_h.$$

Then the right hand side is given by

$$(\mathbf{f}_h, \mathbf{v}_h) = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f}_h \cdot \mathbf{v}_h \, \mathrm{d}K,$$

which consists only of computable terms.

The discrete problem

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ s.t.

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}_h, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h, \end{aligned}$$

Exact divergence-freeness

Introducing the kernels

$$\begin{aligned} \mathbf{Z} &:= \{\mathbf{v} \in \mathbf{V} : b(\mathbf{v}, q) = 0 \quad \text{for all } q \in Q\}, \\ \mathbf{Z}_h &:= \{\mathbf{v}_h \in \mathbf{V}_h : b(\mathbf{v}_h, q_h) = 0 \quad \text{for all } q_h \in Q_h\}, \end{aligned}$$

there holds the inclusion

$$\mathbf{Z}_h \subseteq \mathbf{Z}.$$

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An approximation result

Proposition

Let $\mathbf{u} \in \mathbf{V} \cap [H^{s+1}(\Omega)]^2$ *with* $0 \leq s \leq k$. *Under the assumptions* $(\mathbf{A1})$ *and* $(\mathbf{A2})$ *on the decomposition* $\mathcal{T}_h$, *there exist* $\mathbf{u}_I \in \mathbf{V}_h$ *s.t.*

$$\|\mathbf{u} - \mathbf{u}_I\|_{0,K} + h_K\,|\mathbf{u} - \mathbf{u}_I|_{1,K} \leq C\,h_K^{s+1}|\mathbf{u}|_{s+1,D(K)}$$

where C *is a constant independent of* h, *and* $D(K)$ *denotes the neighbourhood ("diamond") of* K.

Proof.

See [1]. □

The discrete inf - sup condition

Proposition

Given the discrete spaces $\mathbf{V}_h$ *and* Q_h*, there exists a positive* $\overline{\beta}$*, independent of* h*, such that*

$$\sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \overline{\beta} \|q_h\|_Q \qquad \text{for all } q_h \in Q_h.$$

Proof.

Via a Fortin operator π_h. □

Existence

Theorem

The approximate problem has a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$, *verifying the estimate*

$$\|\mathbf{u}_h\|_1 + \|p_h\|_Q \leq C\|\mathbf{f}\|_0$$

Moreover, the $\inf$ - $\sup$ condition implies

$$\operatorname{div} \mathbf{V}_h = Q_h.$$

Convergence

An observation

If $\mathbf{u} \in \mathbf{V}$ is the velocity solution of the continuous problem, then it also solves:
Find $\mathbf{u} \in \mathbf{Z}$

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \text{for all } \mathbf{v} \in \mathbf{Z}.$$

Analogously, if $\mathbf{u}_h \in \mathbf{V}_h$ is the velocity solution of the approximate problem, then it also solves:
Find $\mathbf{u}_h \in \mathbf{Z}_h$

$$a_h(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}_h, \mathbf{v}_h) \qquad \text{for all } \mathbf{v}_h \in \mathbf{Z}_h.$$

Convergence

Theorem

Let $\mathbf{u} \in \mathbf{Z}$ *be the solution of the problem on the kernel of* $b(.,.)$ *and* $\mathbf{u}_h \in \mathbf{Z}_h$ *be the solution of the problem on the kernel of* $b_h(.,.)$*. Then it holds:*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \le C\, h^k \left(|\mathbf{f}|_{k-1} + |\mathbf{u}|_{k+1}\right).$$

Theorem

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the solution of the continuous problem and* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *be the solution of the approximate problem. Then it holds:*

$$\|p - p_h\|_Q \le C\, h^k \left(|\mathbf{f}|_{k-1} + |\mathbf{u}|_{k+1} + |p|_k\right).$$

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Reduced spaces

For $k \geq 2$, we define the local VEM spaces

$$\widehat{\mathbf{V}}_h^K := \left\{ \mathbf{v} \in [H^1(K)]^2 : \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2, \begin{cases} -\nu\,\boldsymbol{\Delta}\,\mathbf{v} - \nabla s \in \mathcal{G}_{k-2}(K)^{\perp}, \\ \operatorname{div}\mathbf{v} \in \mathbb{P}_0(K), \end{cases} \text{for some } s \in H^1(K) \right\}$$

and

$$\widehat{Q}_h^K := \mathbb{P}_0(K).$$

The global spaces are then

$$\widehat{\mathbf{V}}_h := \{\mathbf{v} \in [H_0^1(\Omega)]^2 : \mathbf{v}|_K \in \widehat{\mathbf{V}}_h^K \text{ for all } K \in \mathcal{T}_h\},$$

and

$$\widehat{Q}_h := \{q \in L_0^2(\Omega) : q|_K \in \widehat{Q}_h^K \text{ for all } K \in \mathcal{T}_h\}.$$

A reduced set of DoFs

For $\mathbf{v} \in \widehat{\mathbf{V}}_h^K$ we have the linear operator $\widehat{\mathbf{D}}_{\mathbf{V}}$ for the local degrees of freedom

- $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{1}$: the values of $\mathbf{v}$ at the vertices of K
- $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{2}$: the values of $\mathbf{v}$ at $k-1$ distinct points of every edge $e \subset \partial K$
- $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{3}$: the moments of $\mathbf{v}$

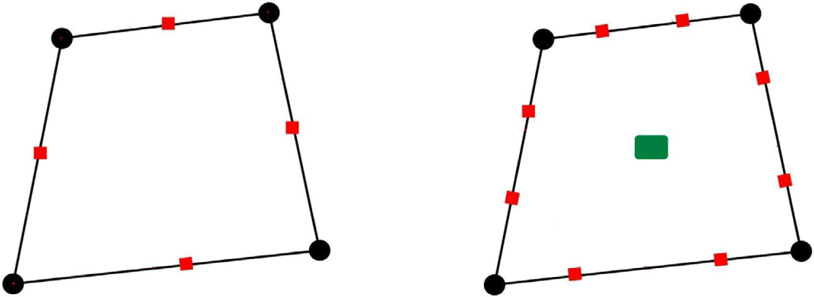
$$\int_K \mathbf{v} \cdot \mathbf{g}_{k-2}^{\perp} \, \mathrm{d}K \quad \text{for all } \mathbf{g}_{k-2}^{\perp} \in \mathcal{G}_{k-2}(K)^{\perp}$$

For $q \in \widehat{Q}_h^K$ we have the local degrees of freedom

- $\widehat{\mathbf{D}}_{\mathbf{Q}}$: the moment of q in K,

$$\int_K q \, \mathrm{d}K$$

A reduced set of DoFs II



DoFs for $k = 2, k = 3$. We denote $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{1}$ with the black dots, $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{2}$ with the red squares, $\widehat{\mathbf{D}}_{\mathbf{V}}\mathbf{3}$ with the green rectangles.

The reduced discrete problem

Find $(\widehat{\mathbf{u}}_h, p_h) \in \widehat{\mathbf{V}}_h \times \widehat{Q}_h$ s.t.

$$\begin{aligned} a_h(\widehat{\mathbf{u}}_h, \widehat{\mathbf{v}}_h) + b(\widehat{\mathbf{v}}_h, \widehat{p}_h) &= (\mathbf{f}_h, \widehat{\mathbf{v}}_h) && \forall \widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h, \\ b(\widehat{\mathbf{u}}_h, \widehat{q}_h) &= 0 && \forall \widehat{q}_h \in \widehat{Q}_h, \end{aligned}$$

All terms involved are computable wrt to the **reduced basis**!

Moreover, there exists a $\widehat{\beta} > 0$ such that

$$\sup_{0 \neq \widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h} \frac{b(\widehat{\mathbf{v}}_h, \widehat{q}_h)}{\|\widehat{\mathbf{v}}_h\|_1} \geq \widehat{\beta} \|\widehat{q}_h\|_Q \qquad \text{for all } \widehat{q}_h \in \widehat{Q}_h.$$

Proposition

Let $(\mathbf{u}_h, p_h)$ *be the solution of the full scheme and* $(\widehat{\mathbf{u}}_h, \widehat{p}_h)$ *be the solution of the reduced scheme. Then*

$$\widehat{\mathbf{u}}_h = \mathbf{u} \qquad \textit{and} \qquad \widehat{p}_h|_K = \Pi_0^{0,K} p_h \quad \textit{for all } K \in \mathcal{T}_h.$$

Proof.

See [1]. □

		$k=2$	$k=3$	$k=4$	$k=5$
$\mathcal{V}_h$	$h=1/4$	34.408%	43.715%	48.484%	51.494%
	$h=1/8$	30.260%	39.506%	44.547%	47.863%
	$h=1/16$	28.460%	37.624%	42.753%	46.185%
	$h=1/32$	27.634%	36.749%	41.911%	45.392%
$\mathcal{T}_h$	$h=1/2$	49.230%	56.737%	59.751%	61.369%
	$h=1/4$	47.761%	55.427%	58.616%	60.377%
	$h=1/8$	45.937%	53.889%	57.314%	59.253%
	$h=1/16$	45.171%	53.243%	56.767%	58.780%
$\mathcal{Q}_h$	$h=1/4$	43.835%	52.287%	56.031%	58.181%
	$h=1/8$	39.875%	48.706%	52.892%	55.411%
	$h=1/16$	38.066%	47.041%	51.417%	54.098%
	$h=1/32$	37.202%	46.238%	50.701%	53.458%

Percentage saving of DoFs in the reduced problem with respect the original one [1].

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Error computation I

We do not know the approximate solution $\mathbf{u}_h$ point-wise inside the elements. Hence we need a suitable (*computable*) polynomial projection of the VEM solution $\mathbf{u}_h$.

tensor-valued L^2-projection

For $K \in \mathcal{T}_h$ and $k \geq 2$, we introduce the L^2-projection operator $\boldsymbol{\Pi}_{k-1}^{0,K} : [L^2(K)]^{2\times 2} \to [\mathbb{P}_{k-1}(K)]^{2\times 2}$, defined by

$$\int_K \Big(\mathbf{A} - \boldsymbol{\Pi}_{k-1}^{0,K}\mathbf{A}\Big) : \mathbf{P}_{k-1} \, \mathrm{d}x = 0,$$

for all $\mathbf{A} \in [L^2(K)]^{2\times 2}$ and $\mathbf{P}_{k-1} \in [\mathbb{P}_{k-1}(K)]^{2\times 2}$.

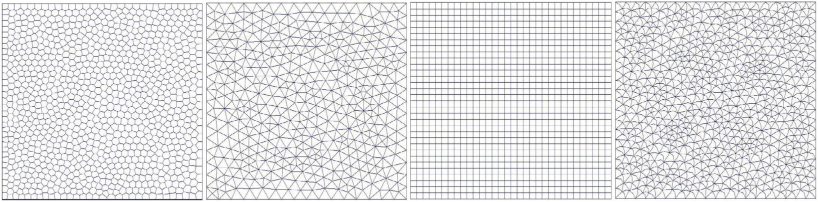
Error computation II

Error measure

$$\delta(\mathbf{u}) := \left(\sum_{K \in \mathcal{T}_h} \left\| \nabla \mathbf{u} - \boldsymbol{\Pi}^{0,K}_{k-1} (\nabla \mathbf{u}_h) \right\|^2 \right)^{1/2}$$

$$\delta(p) := \|p - p_h\|_0$$

Four meshes



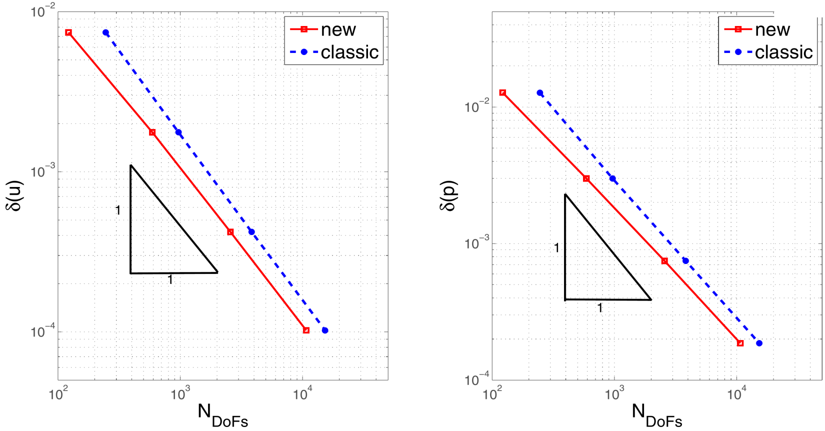
Example of polygonal meshes: $\mathcal{V}_{1/32}$, $\mathcal{T}_{1/16}$, $\mathcal{Q}_{1/32}$, $\mathcal{W}_{1/20}$.

Example 1

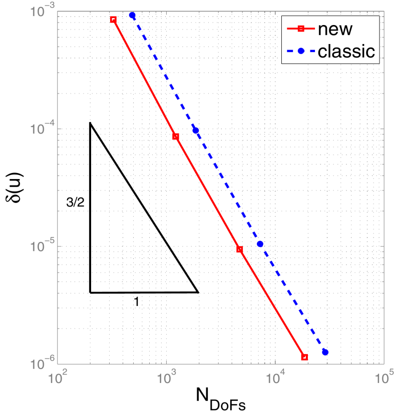
We consider the unit square $\Omega = [0,1]^2$. The functions

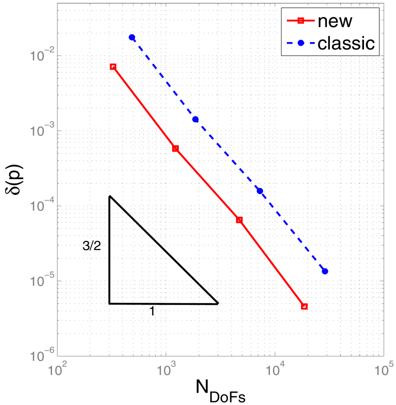
$$\mathbf{u}(x,y) = \begin{pmatrix} -\frac{1}{2}\cos^2(x)\cos(y)\sin(y) \\ \frac{1}{2}\cos^2(y)\cos(x)\sin(x) \end{pmatrix} \quad p(x,y) = \sin(x) - \sin(y)$$

are chosen as exact solutions, with the load vector $\mathbf{f}$ computed accordingly. Furthermore, we have homogeneous boundary conditions on the whole $\partial\Omega$.

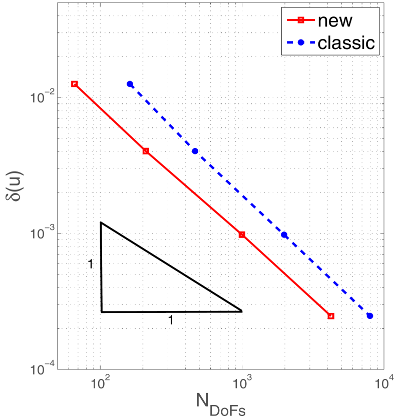


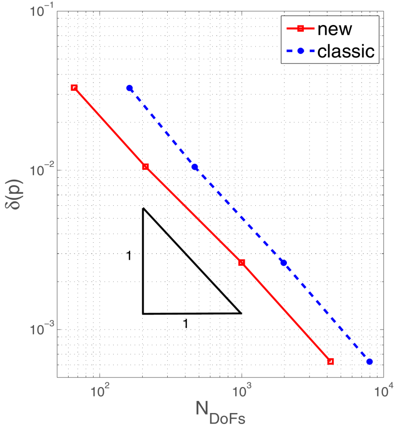
Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{V}_h$ with $k = 2$.



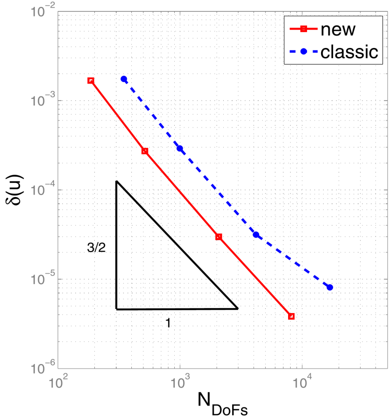


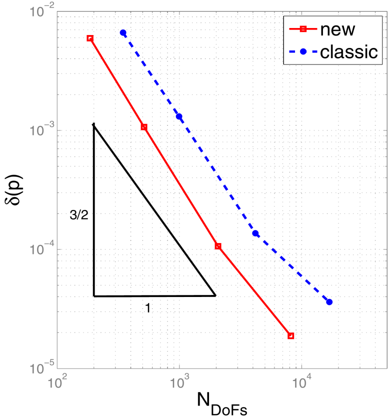
Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{V}_h$ with $k = 3$.



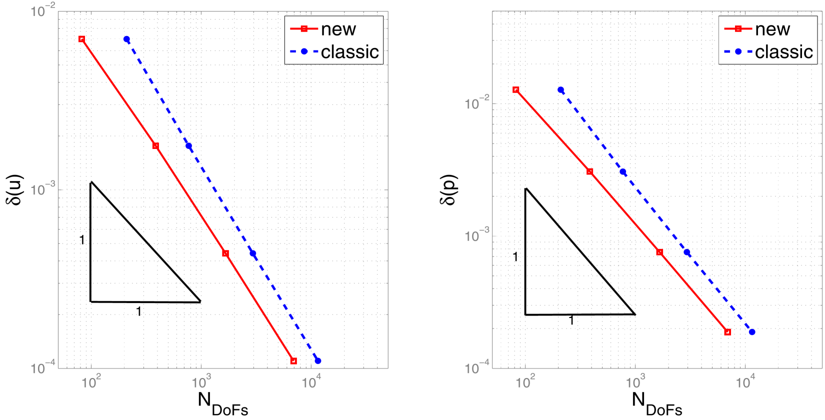


Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{T}_h$ with $k = 2$.

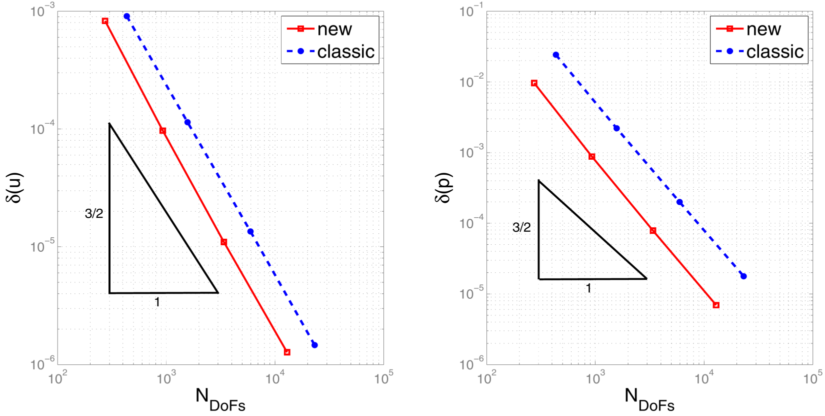




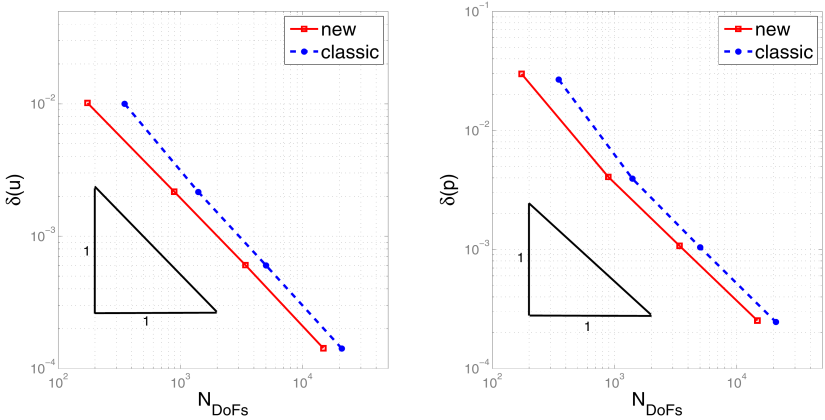
Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{T}_h$ with $k = 3$.



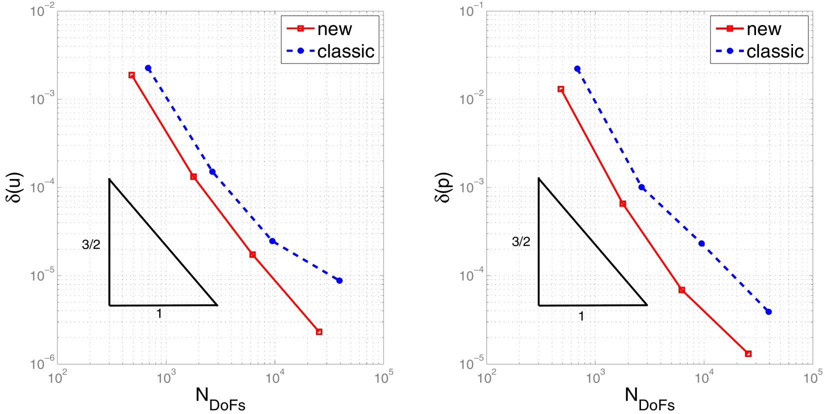
Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{Q}_h$ with $k = 2$.



Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{Q}_h$ with $k = 3$.



Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{W}_h$ with $k = 2$.



Behaviour of $\delta(\mathbf{u})$ and $\delta(p)$ for the sequence of meshes $\mathcal{W}_h$ with $k = 3$.

Example 2

Numerical check of equality

We consider the unit square $\Omega = [0,1]^2$. The polynomial functions

$$\mathbf{u}(x,y) = \begin{pmatrix} y^4 + 1 \\ x^4 + 2 \end{pmatrix} \qquad p(x,y) = x^3 - y^3$$

are chosen as exact solutions, with the load vector $\mathbf{f}$ computed accordingly.

Example 2

Numerical check of equality

Discrepancy measure

$$\varepsilon(\mathbf{u}) := \left(\sum_{K \in \mathcal{T}_h} \left\| \boldsymbol{\Pi}_{k-1}^{0,K} \boldsymbol{\nabla}(\mathbf{u}_h - \widehat{\mathbf{u}}_h) \right\|^2 \right)^{1/2}$$

$$\varepsilon(p) := \left(\sum_{K \in \mathcal{T}_h} \left\| \Pi_0^{0,K} p_h - \widehat{p}_h \right\|^2 \right)^{1/2}$$

Example 2

Numerical check of equality

		$k=2$		$k=3$	
		$\varepsilon(\mathbf{u})$	$\varepsilon(p)$	$\varepsilon(\mathbf{u})$	$\varepsilon(p)$
$\mathcal{V}_h$	$h=1/4$	1.8366063e−13	1.2397027e−13	9.9584405e−12	9.5750683e−13
	$h=1/8$	5.6291757e−13	1.7037760e−13	1.0257081e−11	6.4888535e−13
	$h=1/16$	1.5395183e−12	5.3823612e−13	1.2017070e−11	1.5761308e−12
	$h=1/32$	3.5162644e−12	5.2896229e−13	2.7289064e−11	9.7059278e−12
$\mathcal{T}_h$	$h=1/2$	1.6148091e−13	2.7158830e−14	5.3139688e−12	9.5716694e−13
	$h=1/4$	4.1349069e−13	6.9691214e−14	9.1204063e−11	4.8537564e−13
	$h=1/8$	1.3353440e−12	1.0109968e−13	2.7989324e−11	8.0028046e−12
	$h=1/16$	3.1038037e−12	2.3636051e−13	2.4258164e−11	1.4188633e−11
$\mathcal{Q}_h$	$h=1/4$	1.6747387e−13	8.0270859e−14	8.0510701e−12	2.1761282e−13
	$h=1/8$	4.3127288e−13	1.7217954e−13	2.9673107e−12	1.5735525e−13
	$h=1/16$	1.0294053e−12	2.2290502e−13	4.2024937e−12	7.9220732e−13
	$h=1/32$	2.4711285e−12	2.4074145e−13	7.6167571e−12	5.9492426e−13
$\mathcal{W}_h$	$h=4/10$	9.1587957e−13	7.5286364e−14	1.6072996e−11	1.0673512e−13
	$h=2/10$	1.3107628e−12	1.1154735e−13	1.2916868e−11	2.8184370e−13
	$h=1/10$	4.0885427e−12	3.8760675e−13	5.2532025e−11	7.5670394e−13
	$h=1/20$	7.2877771e−12	6.6196792e−13	8.4382876e−11	1.6915676e−12

$\varepsilon(\mathbf{u})$ and $\varepsilon(p)$ for the meshes $\mathcal{V}_h, \mathcal{T}_h, \mathcal{Q}_h, \mathcal{W}_h$ with $k=2,3$.

[1] Beirão da Veiga, L., Lovadina, C. and Vacca, G. Divergence free virtual elements for the stokes problem on polygonal meshes. *ESAIM: M2AN 51*, 2 (2017), 509-535.

Thank you!