Exercise sheet 2

1. (Lax–Milgram) Let X be a separable and reflexive Banach space. Let $A : X \to X^*$ be a linear, continuous and coercive operator, i.e., it holds

$$\begin{aligned} \langle Au, v\rangle_X &\leq C\|u\|_X\|v\|_X \quad &\forall u, v \in X, \\ \langle Au, u\rangle_X &\geq C\|u\|_X^2 \quad &\forall u \in X. \end{aligned}$$

   Prove that for every $b \in X^*$ there exists a unique solution $u \in X$ to the operator equation $Au = b$ by using the Browder–Minty theorem.

2. (Quasilinear PDE on intersection space) Let $s \geq 0$, $p > 1$, $f \in X^*$ for $X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Show that the quasilinear PDE $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + su = f$ with homogeneous Dirichlet boundary has a unique solution on X.

3. (Auxiliary inequality) Let $p > 1$. Show that there are constant $c(p), C(p)$ depending only on p such that it holds

$$c(p)(|\xi| + |\eta|)^{p-2} \leq \int_0^1 |\xi + \tau(\eta - \xi)|^{p-2} \,\mathrm{d}\tau \leq C(p)(|\xi| + |\eta|)^{p-2}$$

   for every ξ, $\eta \in \mathbb{R}^n$ fulfilling $|\xi| + |\eta| > 0$.

4. (Fundamental theorem of calculus for Sobolev functions)

   (a) Let $g \in L^1(a, b)$ and $C \in \mathbb{R}$. Consider the function f defined by

$$f(t) = C + \int_a^t g(s) \,\mathrm{d}s.$$

   Then f is continuous on $[a, b]$ and $f \in W^{1,1}(a, b)$ with $f' = g$ in weak sense.

   (b) Any function $f \in W^{1,1}(a, b)$ is equal a.e. to a continuous function $\tilde{f}$ on $[a, b]$ and we have for any $x, y \in [a, b]$

$$\tilde{f}(y) = \tilde{f}(x) + \int_x^y f'(s) \,\mathrm{d}s;$$

   in other words, we have for a.a. $x, y \in [a, b]$

$$f(y) = f(x) + \int_x^y f'(s) \,\mathrm{d}s.$$

   Hint: You may use that $f' = 0$ in weak sense implies $f = c$ a.e. for some constant c.