# Optimality of a standard AFEM

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December 10, 2019

# Motivation

- Uniform refinement: optimal $\#P \sim \varepsilon^{-2}$ for $2d$ linear elements

Basic adaptive algorithm:

$$\text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine}$$

- Computational complexity?
- Want: similar optimal complexity for more functions

# Notation

- $P^c$ ... conforming partition
- $V_{P^c}$, $E_{P^c}$ ... interior vertices, edges
- $P^c_e$ ... two $\Delta$'s with edge $e$
- $[v]_e$ ... jump along $n_e$

# Error Estimator

From previous talks:

$$\eta_e(P^c, f, w_{P^c}) := diam(e)\|[\nabla w_{P^c}]\cdot n_e\|^2_{L_2(e)} + \sum_{\Delta\in P^c_e} diam(\Delta)^2\|f\|^2_{L_2(\Delta)} \quad (1)$$

$$\mathcal{E}(P^c, f, w_{P^c}) := \left[\sum_{e\in E_{P^c}} \eta_e(P^c, f, w_{P^c})\right]^{\frac{1}{2}} \quad (2)$$

# Error Estimator

## Theorem (4.1: Refinement Error)

*Let* $f \in L_2(\Omega)$ *and* $\tilde{P}$ *a refinement of* $P^c$.

*Define*

$$\bar{F} = \bar{F}(P^c, \tilde{P}) := \{e \in E_{P^c} : \exists \Delta' \in P^c \text{ s.t. } \Delta' \notin \tilde{P}, \Delta' \cap \bigcup_{\Delta \in P_e^c} \Delta \neq \emptyset\}.$$

*Then*

$$|u_{\tilde{P}} - u_{P^c}|_{H^1(\Omega)} \leq C_1 \left[\sum_{e \in \bar{F}} \eta_e(P^c, f, u_{P^c})\right]^{\frac{1}{2}}. \qquad (3)$$

# Error Estimator

$$\bar{F} = \bar{F}(P^c, \tilde{P}) := \{e \in E_{P^c} : \exists \Delta' \in P^c \text{ s.t. } \Delta' \notin \tilde{P}, \Delta' \cap \bigcup_{\Delta \in P_e^c} \Delta \neq \emptyset\}.$$

- $\Delta'$ has been refined
- $\bar{F}$ ... edges of refined $\Delta$ in $P^c$
- $\#\bar{F} \lesssim \#\tilde{P} - \#P^c$

# Error Estimator

Well-known result:

**Theorem (4.2: Reliability)**

$$|u - u_{P^c}|_{H^1(\Omega)} \leq C_1 \mathcal{E}(P^c, f, u_{P^c})$$

**Proof.**

$H_0^1(\Omega) \simeq S_{\tilde{P}}$ with infinite uniform refinement. Or: Verfürth □

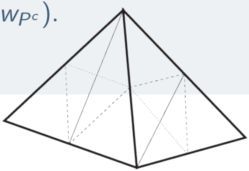
# Error Estimator[1]

## Theorem (4.3: Efficiency, local)

*Let* $P^c$ *be conforming,* $e \in E_{P^c}$, $\tilde{P}$ *have fully refined* $P^c_e$, $f_{P^c} \in S^0_{P^c}$,
$u_{\tilde{P}} := L^{-1}_{\tilde{P}} f_{P^c}$, *and* $w_{P^c} \in S_{P^c}$.

*Then:*

$$\sum_{\Delta \in P^c_e} |u_{\tilde{P}} - w_{P^c}|^2_{H^1(\Delta)} \gtrsim \eta_e(P^c, f_{P^c}, w_{P^c}).$$



- Not valid for general right-hand sides!

[1] Morin, Nochetto, and Siebert, "Data Oscillation and Convergence of Adaptive FEM", 2000.

# Error Estimator[2]

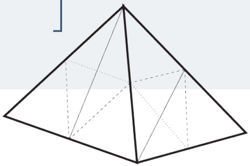
## Corollary (4.4: Efficiency, bulk)

*Same as before, and* $\underline{F} \in E_{P^c}$, $\tilde{P}$ *have fully refined* $P_e^c$ *for all* $e \in \underline{F}$.

*Then:*

$$|u_{\tilde{P}} - w_{P^c}|_{H^1(\Omega)} \geq c_2 \left[ \sum_{e \in \underline{F}} \eta_e(P^c, f_{P^c}, w_{P^c}) \right]^{\frac{1}{2}}$$

*and* $\#\tilde{P} - \#P^c \lesssim \#\underline{F}$.



- Not valid for general right-hand sides!

[2] Morin, Nochetto, and Siebert, "Data Oscillation and Convergence of Adaptive FEM", 2000.

# Error Estimator

Holds for further refinements, in particular:

**Corollary (4.5: Efficiency, global)**

$$|u - w_{P^c}|_{H^1(\Omega)} \geq c_2 \mathcal{E}(P^c, f_{P^c}, w_{P^c}).$$

# Optimality of

# an Idealized Adaptive Finite Element Method

# Optimality

## Definition

$$|u|_{\mathcal{A}^s} := \sup_{\varepsilon>0} \varepsilon \inf_{\{P:\inf_{u_P\in S_P} |u-u_P|_1\leq\varepsilon\}} [\#P - \#P_0]^s$$

## Definition

$$|u|_{\mathcal{A}^s} := \sup_{n\in\mathbb{N}} n^s \inf_{\#P-\#P_0\leq n} \inf_{u_P\in S_P} |u - u_P|_1$$

- $\mathcal{A}^s$ set of functions that can be approximated within $\varepsilon > 0$ with $\#P - \#P_0 \lesssim \varepsilon^{-1/s}$
- Contains $S_P$, $H^{1+2s}(\Omega) \cap H_0^1(\Omega) \subset A^s$ for $s \leq \frac{1}{2}$
- Contains many more functions (c.f. Besov spaces)

# Idealized AFEM

SOLVE$[f, \varepsilon] \to [P_k^c, u_{P_k^c}]$

$P_0^c := P_0, u_{P_0^c} := L_{P_0^c}^{-1} f$

while $C_1 \mathcal{E}(P_k^c, f, u_{P_k^c}) \geq \varepsilon$ do

$\tilde{P}_{k+1} :=$ REFINE$[P_k^c, f, u_{P_k^c}]$

$P_{k+1}^c :=$ MAKECONFORM$[\tilde{P}_{k+1}]$

$u_{P_{k+1}^c} := L_{P_{k+1}^c}^{-1} f$

done

END

- piecewise constant right-hand side
- ignore cost of linear solver

# Refinement Procedure

REFINE$[P^c, f, u_{P^c}] \to \tilde{P}$

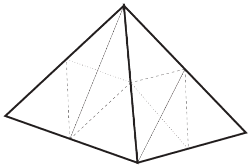
$\theta \in (0,1]$ fixed

Select $\underline{F} \subset E_{P^c}$ with minimal cardinality s.t.

$$\sum_{e \in \underline{F}} \eta_e(P^c, f_{P^c}, w_{P^c}) \geq \theta^2 \mathcal{E}(P^c, f_{P^c}, w_{P^c})^2. \quad (4)$$

$\tilde{P} :=$ full refinement of all $\Delta \in P^c_e \quad \forall e \in \underline{F}$

END



- Select percentage of largest errors
- Requirement: $\mathcal{O}(\#P^c)$ operations

✓ C++: `std::nth_element`

# Refinement Procedure

**Lemma (5.2)**

*Let* $f \in S^0_{P^c}$, $u := L^{-1} f \in \mathcal{A}^s$. *Then for* $\tilde{P} := REFINE[P^c, f, u_{P^c}]$, *we have*

$$\#\tilde{P} - \#P^c \lesssim |u - u_{P^c}|_1^{-\frac{1}{s}} |u|_{\mathcal{A}^s}^{\frac{1}{s}}.$$

- REFINE gives us optimal number of triangles (up to constant)
- Need: $\theta \in (0, c_2/C_1)$

# Refinement Procedure

From previous talk[3]:

## Theorem (3.2)

*$P_i$ a refinement of $P^c_{i-1}$, $P^c_i := MAKECONFORM[P_i]$. Then*

$$\#P^c_n - \#P^c_0 \lesssim \sum_{i=1}^{n} \#P_i - \#P^c_{i-1}.$$

- Removing hanging nodes does not give us many more triangles

[3] Binev, Dahmen, and DeVore, "Adaptive finite element methods with convergence rates", 2004.

## Theorem (5.3: Optimal Complexity)

*Let* $f \in S^0_{P_0}$, *then* $[P^c, u_{P^c}] = SOLVE[f, \varepsilon]$ *terminates with* $|u - u_{P^c}|_1 \leq \varepsilon$.
*If* $u \in \mathcal{A}^s$, *then* $\#P^c - \#P_0 \lesssim \varepsilon^{-\frac{1}{s}} |u|^{\frac{1}{s}}_{\mathcal{A}^s}$.

## Remark

*Bound on* $\#P^c - \#P_0$ *is the best one can achieve for* $u \in \mathcal{A}^s$.

## Proof.

error reduction, $\#P$ of REFINE, $\#P$ of MAKECONFORM

(4.4, (4), 4.2, 4.5, 5.2, 3.2) □

# Extensions

- $f \in L^2$: approximate $|f - f_{P^c}|_{H^{-1}} \leq \delta$
  (may induce additional refinements)
- Inexact solves: assume $|u_{P^c} - \tilde{u}_{P^c}|_1 \leq \delta$
  with $\lesssim \max\{1, \log(\delta^{-1}|u_{P^c} - u_{P^c}^{(0)}|_1)\}\#P^c$ operations
  E.g.: Multigrid
- Proof: same ideas, but more technical

# Thank you.

Binev, Peter, Wolfgang Dahmen, and Ron DeVore. "Adaptive finite element methods with convergence rates". In: *Numer. Math.* 97.2 (2004), pp. 219–268.

Morin, Pedro, Ricardo H Nochetto, and Kunibert G Siebert. "Data Oscillation and Convergence of Adaptive FEM". In: *SIAM J. Numer. Anal.* 38.2 (2000), pp. 466–488.

Stevenson, Rob. "Optimality of a standard adaptive finite element method". In: *Found. Comput. Math.* 7.2 (2007), pp. 245–269.