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The Ciarlet-Raviart Method for Biharmonic Problems on General Polygonal Domains: Mapping Properties and Preconditioning

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THE CIARLET-RAVIART METHOD FOR BIHARMONIC PROBLEMS ON GENERAL POLYGONAL DOMAINS: MAPPING PROPERTIES AND PRECONDITIONING *

WALTER ZULEHNER†

Abstract. For biharmonic boundary value problems, the Ciarlet-Raviart mixed method is considered on polygonal domains without additional convexity assumptions. Mapping properties of the involved operators on the continuous as well as on the discrete level are studied. Based on this, efficient preconditioners are constructed and numerical experiments are shown.

Key words. biharmonic equation, Ciarlet-Raviart method, mixed methods, mapping properties, preconditioning

AMS subject classifications. 65N22, 65F08, 65F10

1. Introduction. We consider the first biharmonic boundary value problem: Find y such that

$$\Delta^2 y = f \quad \text{in } \Omega, \qquad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma, \tag{1.1}$$

where Ω is an open and bounded set in $\mathbb{R}^2$ with a polygonal Lipschitz boundary Γ, Δ and $\partial/\partial n$ denote the Laplace operator and the derivative in the direction normal to the boundary, respectively, and $f \in H^{-1}(\Omega)$. Here and throughout the paper we use $L^2(\Omega)$, $H^m(\Omega)$, and $H^m_0(\Omega)$ with its dual space $H^{-m}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces with corresponding norms $\|.\|_0$, $\|.\|_m$, $|.|_m$, and $\|.\|_{-m}$ for positive integers m, see, e.g., [1]. Problems of this type occur, for example, in fluid mechanics, where y is the stream function of a two-dimensional Stokes flow, see, e.g., [10], and in elasticity, where y is the vertical deflection of a clamped Kirchhof plate, see, e.g., [6].

The standard (primal) variational formulation of (1.1) reads: Find $y \in H^2_0(\Omega)$ such that

$$\int_\Omega \Delta y \, \Delta z \, dx = \langle f, z\rangle \quad \text{for all } z \in H^2_0(\Omega), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product in $H^* \times H$ for a Hilbert space H with dual H^*, here for $H = H^1_0(\Omega)$. (If $H = \mathbb{R}^n$, we use $\langle \cdot, \cdot \rangle$ for the Euclidean inner product.) Existence and uniqueness of a solution to (1.2) is guaranteed by the theorem of Lax-Milgram, see, e.g., [21], [17].

Conforming finite element methods based on (1.2) require approximation spaces of continuously differentiable functions, which are not so easy to construct for unstructured meshes. Another challenging issue for (1.2) is the construction of efficient preconditioners for iterative methods for solving a discretized version of (1.2). Standard techniques which might help to resolve these difficulties are discontinuous Galerkin methods or mixed methods. We focus here on the well-known mixed method by Ciarlet-Raviart, see [8], for which an auxiliary variable

$$u = -\Delta y$$

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is introduced. For the Stokes problem u is the vorticity of the flow, for plate bending problems u can be interpreted as bending moment. With this auxiliary variable the fourth order differential equation in (1.1) can be rewritten as a system of two second-order equations

$$-\Delta y = u, \quad -\Delta u = f \quad \text{in } \Omega. \tag{1.3}$$

Finite element methods for (1.3) were studied on convex domains Ω by many authors, see, e.g, [8], [24], [9], [3], [10]. The equivalence of variational formulations for (1.1) and for (1.3) is a subtle issue, which, for the first biharmonic problem, was already addressed in the pioneering paper [8] for convex domains, and essentially settled in [4] for domains without convexity assumptions.

Strongly related to the Ciarlet-Raviart mixed method is a boundary operator formulation for another auxiliary variable

$$\lambda = u|_\Gamma$$

on the continuous as well as on the discrete level, see [7], [11]. On the discrete level, this approach can be seen as a reduction of the mixed problem to a Schur complement problem.

For convex domains and the more for non-convex domains, preconditioning the mixed method is still a challenging issue because the mapping properties of the involved linear operators are by far not trivial. One possible approach is the use of mesh-dependent norms for the mixed method, see [3]. However, the analysis was restricted to convex domains and, more severely, the resulting preconditioner requires a preconditioner for a matrix which can be interpreted as a discretization of a differential operator of order 4. In [25] preconditioners were studied which require only standard components, motivated by a reasonable trade-off between optimality (in the sense of mesh-independent convergence rates) and practicability. For the boundary operator formulation preconditioning was studied in [23] quite in the spirit of operator preconditioning, see, e.g., [15] and [18] for a general discussion of this concept. The preconditioner proposed in [23] leads to mesh-independent convergence rates for convex domains.

The aim of this paper is to fight for both optimality and practicability without convexity assumptions. We will extend results from [23] for the reduced problem in λ and show some preliminary results on a class of preconditioners for the original (non-reduced) mixed formulation in y and u.

The paper is organized as follows. In Sections 2 and 3 the mapping properties are analyzed for the mixed and the reduced formulation, respectively. After discussing the discretized problems quite in the spirit of the analysis of the corresponding continuous problems in Section 4, the main results on preconditioning are developed in Section 5. A few numerical experiments are presented in Section 6 for illustrating the theoretical results, followed by concluding remarks in Section 7. Some technical details on harmonic extension operators needed for the analysis in the previous sections are collected in an appendix.

2. The Ciarlet-Raviart method.

Here we shortly recall known results on the original mixed formulation and its modification in [4].

2.1. The original method.

We consider the following standard mixed variational formulation for (1.3): For $f \in H^{-1}(\Omega)$, find $u \in H^1(\Omega)$ and $y \in H_0^1(\Omega)$ such that

$$
(2.1)\qquad
\begin{aligned}
&\int_\Omega u\, v\; dx && -\int_\Omega \nabla v\cdot\nabla y\; dx = 0 && \text{for all } v\in H^1(\Omega),\\
-&\int_\Omega \nabla u\cdot\nabla z\; dx && \qquad\qquad\qquad = -\langle f, z\rangle && \text{for all } z\in H^1_0(\Omega).
\end{aligned}
$$

where ∇ denotes the gradient. This problem has the typical structure of a saddle point problem:

$$
\begin{aligned}
a(u,v)+b(v,y) &= 0 && \text{for all } v\in V,\\
b(u,z) &= -\langle f,z\rangle && \text{for all } z\in Q
\end{aligned}
$$

for the Hilbert spaces

$$
V = H^1(\Omega) \quad\text{and}\quad Q = H^1_0(\Omega)
$$

and the bilinear forms

$$
(2.2)\qquad a(u,v)=\int_\Omega u\, v\; dx \quad\text{and}\quad b(v,z) = -\int_\Omega \nabla v\cdot\nabla z\; dx.
$$

If the linear operator $\mathcal{A}\colon X \longrightarrow X^*$ with $X = V\times Q$ is introduced by

$$
\left\langle \mathcal{A}\begin{bmatrix} u\\ y\end{bmatrix}, \begin{bmatrix} v\\ z\end{bmatrix}\right\rangle = a(u,v)+b(v,y)+b(u,z),
$$

the mixed variational problem (2.1) can be rewritten as a linear operator equation

$$
\mathcal{A}\begin{bmatrix} u\\ y\end{bmatrix} = -\begin{bmatrix} 0\\ f\end{bmatrix}.
$$

Observe that the bilinear form a is symmetric, i.e., $a(u,v)=a(v,u)$, and non-negative, i.e., $a(v,v)\geq 0$. In this case it is well-known that $\mathcal{A}$ is an isomorphism from X onto X^*, if and only if the following conditions are satisfied, see, e.g., [5]:

1. a is bounded: There is a constant $\|a\|>0$ such that
$$
|a(u,v)| \leq \|a\|\, \|u\|_V\, \|v\|_V \quad \text{for all } u,\ v\in V.
$$
2. b is bounded: There is a constant $\|b\|>0$ such that
$$
|b(v,z)| \leq \|b\|\, \|v\|_V \|z\|_Q \quad \text{for all } v\in V,\ z\in Q.
$$
3. a is coercive on the kernel of b: There is a constant $\alpha>0$ such that
$$
a(v,v)\geq \alpha\, \|v\|_V^2 \quad \text{for all } v\in \ker B
$$
with $\ker B = \{w\in V\colon b(w,z)=0 \text{ for all } z\in Q\}$.
4. b satisfies the inf-sup condition: There is a constant $\beta>0$ such that
$$
\inf_{0\neq z\in Q}\ \sup_{0\neq v\in V} \frac{b(v,z)}{\|v\|_V\, \|z\|_Q} \geq \beta.
$$

Here $\|\cdot\|_V$ and $\|\cdot\|_Q$ denote the norms in V and Q, respect. We will refer to theses conditions as Brezzi's conditions with constants $\|a\|$, $\|b\|$, α, and β.

For (2.1) one of these conditions is not satisfied: the bilinear form a is not coercive on $\ker B$. Nevertheless, for convex domains Ω, existence of a unique solution and error estimates could be established, see, e.g. [8], [24], [9], [3], and many others. But even for convex domains, and the more for non-convex domains, not having an isomorphism makes it hard to develop efficient preconditioners.

2.2. The modified method. In [4] it was proposed to replace the space $H^1(\Omega)$ for the unknown u by the following Hilbert space of less regularity

$$H^{-1}(\Delta,\Omega) = \{v \in L^2(\Omega) \colon \Delta v \in H^{-1}(\Omega)\},$$

equipped with the norm

$$\|v\|_{-1,\Delta} = \left(\|v\|_0^2 + \|\Delta v\|_{-1}^2\right)^{1/2}.$$

Here Δv denotes the application of the Laplace operator to v in the distributional sense. The original space $H^1(\Omega)$ is a proper subset of the new space $H^{-1}(\Delta,\Omega)$. This requires to extend the definition of the bilinear form b accordingly by

$$b(v,z) = \langle \Delta v, z\rangle,$$

which, of course, coincides with the original definition for $v \in H^1(\Omega)$. Then the extended version of (2.1) for this larger primal space reads: For $f \in H^{-1}(\Omega)$, find $u \in H^{-1}(\Delta,\Omega)$ and $y \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega u\, v\; dx + \langle \Delta v, y\rangle = 0 && \text{for all } v \in H^{-1}(\Delta,\Omega),\\
&\langle \Delta u, z\rangle \qquad\qquad\quad = -\langle f, z\rangle && \text{for all } z \in H_0^1(\Omega).
\end{aligned} \tag{2.3}$$

We recall the following result from [4].

THEOREM 2.1. *The bilinear forms a and b, given by*

$$a(u,v) = \int_\Omega u\, v\; dx \quad \textit{and} \quad b(v,z) = \langle \Delta v, z\rangle$$

for $V = H^{-1}(\Delta,\Omega)$ and $Q = H_0^1(\Omega)$ with the norms $\|v\|_V = \|v\|_{-1,\Delta}$ and $\|z\|_Q = |q|_1$ satisfy Brezzi's conditions with the constants

$$\|a\| = \|b\| = \alpha = 1 \quad \textit{and} \quad \beta = (1 + c_F^2)^{-1/2},$$

where c_F denotes the constant in Friedrichs' inequality: $\|v\|_0 \le c_F\, |v|_1$ for all $v \in H_0^1(\Omega)$.

The problems (1.2) and (2.3) are fully equivalent for convex as well as for non-convex polygonal domains, since both problems are uniquely solvable and it is easy to see that (u,y) with $u = -\Delta y$ solves (2.3) if $y \in H_0^2(\Omega)$ solves (1.2). This has already been recognized in [4] in the context of the Stokes problem.

REMARK 1. *Analogous results follow for the second biharmonic boundary value problem, whose boundary conditions are given by $u = \Delta u = 0$ on Γ. Its primal variational formulation is identical with (1.2) with $H_0^2(\Omega) \cap H_0^1(\Omega)$ for y, z instead of $H_0^2(\Omega)$, its original mixed variational formulation is identical with (2.1) with $H_0^1(\Omega)$ for u, v instead of $H^1(\Omega)$. One obtains a mixed variational formulation which is fully equivalent to the primal variational problem by using $H_0^{-1}(\Delta,\Omega) = \{v \in H^{-1}(\Delta,\Omega) \colon \gamma^0 v = 0\}$ for u, v instead of $H_0^1(\Omega)$, see Section 4 for a discussion of the trace operator γ^0.*

3. Reduction to a boundary operator equation. Next we want to reduce the variational problem (2.3) for y and u to a variational problem for the trace λ of u only. For this we need two decomposition results. The first decomposition is closely related to results in [2]. The focus here is the formulation in the framework of space decompositions.

LEMMA 3.1. *$H^{-1}(\Delta,\Omega) = H_0^1(\Omega) \oplus \mathscr{H}(\Omega)$ with*

$$\mathscr{H}(\Omega) = \{v \in L^2(\Omega) \colon \Delta v = 0\},$$

where $\oplus$ denotes the direct sum of Hilbert spaces, whose canonical norm is given here by

$$\|(v_0, v_1)\|^2_{H_0^1(\Omega)\oplus\mathscr{H}(\Omega)} = |v_0|_1^2 + \|v_1\|_0^2.$$

In details, for each $v \in H^{-1}(\Delta,\Omega)$, there is a unique decomposition

$$v = v_0 + v_1 \quad \text{with} \quad v_0 \in H_0^1(\Omega) \quad \text{and} \quad v_1 \in \mathscr{H}(\Omega),$$

and there are positive constants $\underline{c}$ and $\overline{c}$ such that

$$\underline{c}\,\left(|v_0|_1^2 + \|v_1\|_0^2\right) \le \|v\|^2_{-1,\Delta} \le \overline{c}\,\left(|v_0|_1^2 + \|v_1\|_0^2\right) \quad \text{for all } v \in H^{-1}(\Delta,\Omega).$$

The constants $\underline{c}$ and $\overline{c}$ depend only on the constant c_F of Friedrichs' inequality.

Proof. For $v \in H^{-1}(\Delta,\Omega)$, let $v_0 \in H_0^1(\Omega)$ be the unique solution to the variational problem

$$\int_\Omega \nabla v_0 \cdot \nabla z \; dx = -\langle \Delta v, z\rangle \quad \text{for all } y \in H_0^1(\Omega). \tag{3.1}$$

By taking the supremum over all $z \in H_0^1(\Omega)$ we obtain $|v_0|_1 = \|\Delta v\|_{-1}$.

For $v_1 = v - v_0$, we have $\Delta v_1 = \Delta v - \Delta v_0 = 0$ in the distributional sense. Hence $v_1 \in \mathscr{H}(\Omega)$. On the other hand, if $v = v_0 + v_1$ with $v_0 \in H_0^1(\Omega)$ and $v_1 \in \mathscr{H}(\Omega)$, then $-\Delta v_0 = -\Delta v + \Delta v_1 = -\Delta v$, which is equivalent to the variational problem (3.1). So, v_0 is the unique solution of (3.1).

Furthermore, we have

$$\begin{aligned}\|v\|^2_{-1,\Delta} &= \|v\|_0^2 + \|\Delta v\|_{-1}^2 = \|v_0 + v_1\|_0^2 + |v_0|_1^2 \\ &\le 2\,\|v_0\|_0^2 + 2\,\|v_1\|_0^2 + |v_0|_1^2 \le (2c_F^2 + 1)\,|v_0|_1^2 + 2\,\|v_1\|_0^2\end{aligned}$$

and

$$\begin{aligned}|v_0|_1^2 + \|v_1\|_0^2 &= |v_0|_1^2 + \|v - v_0\|_0^2 \le |v_0|_1^2 + 2\,\|v\|_0^2 + 2\,\|v_0\|_0^2 \\ &\le 2\,\|v\|_0^2 + (2c_F^2 + 1)\,|v_0|_1^2 = 2\,\|v\|_0^2 + (2c_F^2 + 1)\,\|\Delta v\|_{-1}^2.\end{aligned}$$

Then the estimates immediately follow with $1/\underline{c} = \overline{c} = \max(2, 2c_F^2 + 1)$. ∎

NOTATION 1. *For estimates of the form*

$$\underline{c}\, f_2(x) \le f_1(x) \le \overline{c}\, f_2(x) \quad \text{for all } x \in H,$$

where f_1 and f_2 are non-negative functions, with some positive constants $\underline{c}$, $\overline{c}$ independent of $x \in H$ and, later on for discretized problems, also independent of the mesh size, we shortly write

$$f_1(x) \sim f_2(x) \quad \text{for all } x \in H.$$

If $f_1(x) = \langle M_1 x, x\rangle$, $f_2(x) = \langle M_2 x, x\rangle$ for symmetric and positive definite matrices $M_1, M_2 \in \mathbb{R}^{n\times n}$ with $H = \mathbb{R}^n$, we use the simplified notation $M_1 \sim M_2$.

With this notation the estimates in the last lemma can be written as

$$\|v\|_{-1,\Delta}^2 \sim |v_0|_1^2 + \|v_1\|_0^2 \quad \text{for all } v \in H^{-1}(\Delta,\Omega).$$

Corollary 3.2. *For all $v \in H^{-1}(\Delta,\Omega)$, we have $v\big|_{\Omega'} \in H^1(\Omega')$ for all open sets Ω' with $\overline{\Omega'} \subset \Omega$.*

Proof. We use Weyl's lemma to conclude that $\mathscr{H}(\Omega) \subset C^\infty(\Omega)$. Then the statement immediately follows from the decomposition $\varphi v = \varphi v_0 + \varphi v_1$ for a test function $\varphi \in C_0^\infty(\Omega)$ with φ identical to 1 on Ω'. □

For the description of a further decomposition of $\mathscr{H}(\Omega)$ we need trace and extension operators for $H^{-1}(\Delta,\Omega)$.

The properties for the trace operator are well-known and shortly summarized here, they easily follow from the results in [14], [13]. The boundary Γ of the polygonal domain Ω can be written as

$$\Gamma = \Gamma_C \cup \Gamma_E \quad \text{with} \quad \Gamma_E = \bigcup_{k=1}^K \Gamma_k,$$

where Γ_C denotes the set of all corners of Γ and Γ_k, $k = 1,2,\ldots,K$, are the edges of Γ, considered as open line segments. The trace operator γ^0, given by

$$\gamma^0 v = \left(\gamma_k^0 v\right)_{k=1,\ldots,K} \quad \text{with} \quad \gamma_k^0 v = v|_{\Gamma_k}$$

for smooth functions on $\overline{\Omega}$, has a unique continuous extension as an operator

$$\gamma^0 \colon H^{-1}(\Delta,\Omega) \longrightarrow H_{pw}^{-1/2}(\Gamma) = \Pi_{k=1}^K H^{-1/2}(\Gamma_k),$$

where $H^{-1/2}(\Gamma_k)$ is the dual of $\widetilde{H}^{1/2}(\Gamma_k)$, see [19] for details. (Another widely used notation for $\widetilde{H}^{1/2}(\Gamma_k)$ is $H_{00}^{1/2}(\Gamma_k)$, see [17].) The standard norm in $H^{-1/2}(\Gamma_k)$ is denoted by $\|.\|_{-1/2,\Gamma_k}$. The norm in $H_{pw}^{-1/2}(\Gamma)$, denoted by $\|\cdot\|_{-1/2,\Gamma}$, is the canonical product norm of its factor spaces, given by

$$\|\mu\|_{-1/2,\Gamma}^2 = \sum_{k=1}^K \|\mu_k\|_{-1/2,\Gamma_k}^2 \quad \text{for } \mu = (\mu_k)_{k=1,\ldots,K} \in H_{pw}^{-1/2}(\Gamma).$$

Notation 2. *For the notation of norms or duality products for functions on the boundary Γ or some edge Γ_k, we explicitly use Γ or Γ_k as subscripts. A subscript pw (piecewise) is used for spaces of functions on Γ which are products of spaces of functions defined on the edges Γ_k for $k \geq 1$. For simplicity we omit this subscript for the corresponding norm. A subscript h (mesh size) is used for mesh-dependent norms.*

The intersection of the kernel of γ^0 and $\mathscr{H}(\Omega)$, given by

$$N = \ker \gamma^0 \cap \mathscr{H}(\Omega) = \{v \in L^2(\Omega) \colon \Delta v = 0 \text{ and } \gamma^0 v = 0\}, \tag{3.2}$$

is known to be finite dimensional. The dimension of N is equal to the number of reentrant corners of Ω, see [14], [13].

The existence of an extension operator and its properties, which are well-known for convex or smooth domains, will be extended to general polygonal domains in the next theorem, see the appendix for the proof.

THEOREM 3.3. *There is a linear operator*

$$E^0\colon H^{-1/2}_{pw}(\Gamma) \longrightarrow H^{-1}(\Delta,\Omega)$$

which is a right inverse of γ^0 with the following properties:

1. $\operatorname{im} E^0 \subset \mathscr{H}(\Omega)$, *where* $\operatorname{im} L$ *denotes the image of a linear operator* L.
2. $\mathscr{H}(\Omega) = \operatorname{im} E^0 \perp N$, *where the symbol* $\perp$ *denotes the* L^2*-orthogonal decomposition.*
3. $\|E^0\mu\|_0^2 \sim \|\mu\|^2_{-1/2,\Gamma}$ *for all* $\mu \in H^{-1/2}_{pw}(\Gamma)$.

The first part means that E^0 can be viewed as a harmonic extension operator, the second part contains the required decomposition result, and the last part shows that E^0 is an isomorphism between the trace space and its image. The existence of the right inverse E^0 immediately implies that the trace operator γ^0 maps from $H^{-1}(\Delta,\Omega)$ onto $H^{-1/2}_{pw}(\Gamma)$.

Lemma 3.1 and Theorem 3.3 allow the following reduction of (2.3):

THEOREM 3.4. *Let $u \in H^{-1}(\Delta,\Omega)$ and $y \in H^1_0(\Omega)$ be the unique solution of (2.3). Then $\lambda = \gamma^0 u \in H^{-1/2}_{pw}(\Gamma)$ is the unique solution of the variational problem*

$$\int_\Omega E^0\lambda\, E^0\mu \; dx = -\int_\Omega u_0\, E^0\mu \; dx \quad \text{for all } \mu \in H^{-1/2}_{pw}(\Gamma), \tag{3.3}$$

where $u_0 \in H^1_0(\Omega)$ is the unique weak solution of the Dirichlet problem for the Laplace operator, i.e.

$$\int_\Omega \nabla u_0 \cdot \nabla z \; dx = \langle f, z\rangle \quad \text{for all } z \in H^1_0(\Omega). \tag{3.4}$$

Proof. From Lemma 3.1 and the second part of Theorem 3.3 it follows that there is a unique elements $u_0 \in H^1_0(\Omega)$ such that $u = u_0 + E^0\lambda + n \quad$ for some $n \in N$. The second line of (2.3) simplifies to (3.4), since $\Delta(E^0\lambda + n) = 0$ according to the first part of Theorem 3.3. (3.3) follows from the first line of (2.3) for test functions of the form $v = E^0\mu$ with $\mu \in H^{-1/2}_{pw}(\Gamma)$, since $E^0\mu$ is orthogonal to n according to the second part of Theorem 3.3. Using the third part of Theorem 3.3 the well-posedness of (3.3) follows from the theorem of Lax-Milgram. □

This generalizes the boundary operator equation, formulated in [11] for smooth domains Ω, to the case of general polygonal domains Ω.

4. Discretization. Let $\mathcal{T}_h$ be an admissible triangulation of the domain Ω. We proceed as usual to construct a conforming finite element space for approximating $H^{-1}(\Delta,\Omega)$ by choosing piecewise linear functions which lie in this space. From Corollary 3.2 it immediately follows that a piecewise smooth function lies in $H^{-1}(\Delta,\Omega)$ iff it is continuous. This leads to the standard finite element space

$$\mathcal{S}_h(\Omega) = \left\{ v_h \in C(\overline{\Omega})\colon v_h|_T \in P_1 \text{ for all } T \in \mathcal{T}_h \right\},$$

where P_1 denotes the set of linear polynomials. Additionally we introduce

$$\mathcal{S}_{h,0}(\Omega) = \mathcal{S}_h(\Omega) \cap H^1_0(\Omega).$$

Using $\mathcal{S}_h(\Omega)$ and $\mathcal{S}_{h,0}(\Omega)$ as approximation spaces for $H^{-1}(\Delta,\Omega)$ and $H^1_0(\Omega)$, respectively, we obtain the following conforming finite element method for (2.3): Find

$u_h \in \mathcal{S}_h(\Omega)$ and $y_h \in \mathcal{S}_{h,0}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega u_h\, v_h \, dx && - \int_\Omega \nabla v_h \cdot \nabla y_h \, dx = 0 && \text{for all } v_n \in \mathcal{S}_h(\Omega), \\ &- \int_\Omega \nabla u_h \cdot \nabla z_h \, dx && = -\langle f, z_h\rangle && \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega). \end{aligned} \tag{4.1}$$

Observe that $\mathcal{S}_h(\Omega) \subset H^1(\Omega)$. Therefore, the definition (2.2) of b can be used. This is exactly the original discrete problem studied in [8]. So, on the discrete level, there is no direct influence of the use of $H^{-1}(\Delta,\Omega)$ for u, v instead of $H^1(\Omega)$.

Analogously to Theorem 2.1 the well-posedness of the discrete problem can be shown.

THEOREM 4.1. *Brezzi's conditions are satisfied for (4.1) on the discrete spaces $V = \mathcal{S}_h(\Omega)$ and $Q = \mathcal{S}_{h,0}(\Omega)$ and the norms $\|v\|_V = \|v\|_{-1,\Delta,h}$ and $\|z_h\|_Q = |z_h|_1$, where*

$$\|v_h\|_{-1,\Delta,h} = \left(\|v_h\|_0^2 + \|\Delta v_h\|_{-1,h}^2\right)^{1/2} \quad \textit{with} \quad \|\ell\|_{-1,h} = \sup_{z_h \in \mathcal{S}_{h,0}(\Omega)} \frac{|\langle \ell, z_h\rangle|}{|z_h|_1},$$

with the same constants as in Theorem 2.1 for the continuous problem (2.3).

Proof. The proof follows the corresponding proof in [4] for the continuous problem.

1. Let $u_h, v_h \in \mathcal{S}_h(\Omega)$. Then
$$|a(u_h, v_h)| \le \|u_h\|_0 \, \|v_h\|_0 \le \|u_h\|_{-1,\Delta,h} \|v_h\|_{-1,\Delta,h}.$$
2. Let $v_h \in \mathcal{S}_h(\Omega)$, $z_h \in \mathcal{S}_{h,0}(\Omega)$. Then
$$|b(v_h, z_h)| \le \|\Delta v_h\|_{-1,h} \, |z_h|_1 \le \|v_h\|_{-1,\Delta,h} \, |z_h|_1.$$
3. Let $v_h \in \ker B_h = \{w_h \in \mathcal{S}_h(\Omega) \colon b(w_h, z_h) = 0 \text{ for all } z_h \in \mathcal{S}_{h,0}(\Omega)\}$. Then
$$a(v_h, v_h) = \|v_h\|_0^2 = \|v_h\|_{-1,\Delta,h}^2.$$
4. Let $0 \ne z_h \in \mathcal{S}_{h,0}(\Omega)$. Then
$$\sup_{0 \ne v_h \in \mathcal{S}_h(\Omega)} \frac{b(v_h, z_h)}{\|v_h\|_{-1,\Delta,h}} \ge \frac{b(-z_h, z_h)}{\|z_h\|_{-1,\Delta,h}} = \frac{|z_h|_1^2}{\|z_h\|_1} \ge \left(c_F^2 + 1\right)^{-1/2} |z_h|_1.$$

□

Observe that the norms introduced for the space $H^{-1}(\Delta,\Omega)$ in (2.1) and its discrete counterpart $\mathcal{S}_h(\Omega)$ in (4.1) are similar but different. For the discrete problem the norm is mesh-dependent.

The actual computations will be performed in matrix-vector notation. We will now rewrite (4.1) in this way. Let $\underline{v}_h$ and $\underline{z}_h$ be the coefficient vectors of $v_h \in \mathcal{S}_h(\Omega)$ and $z_h \in \mathcal{S}_{h,0}(\Omega)$ with respect to the nodal bases in these spaces, respectively. The splitting into interior nodes and nodes on the boundary Γ induces a corresponding block structure of $\underline{v}_h$:

$$\underline{v}_h = \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix}.$$

The mass matrix M_h and the stiffness matrix K_h representing $\|\cdot\|_0$ and $|\cdot|_1$ on $\mathcal{S}_h(\Omega)$, respectively, can be partitioned accordingly:

$$M_h = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \quad \text{and} \quad K_h = \begin{bmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{bmatrix}.$$

Then the variational problem (4.1) reads in matrix-vector notation

$$\begin{bmatrix} M_{00} & M_{01} & -K_{00} \\ M_{10} & M_{11} & -K_{10} \\ -K_{00} & -K_{01} & 0 \end{bmatrix} \begin{bmatrix} \underline{u}_{h,0} \\ \underline{\lambda}_h \\ \underline{y}_h \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\underline{f}_h \end{bmatrix}. \tag{4.2}$$

A reduction of this block system to a single system for $\underline{\lambda}_h$ can be easily achieved by eliminating $\underline{u}_{h,0}$ and $\underline{y}_h$ using the third and the first block line, respectively. This leads to

$$S_h \, \underline{\lambda}_h = \underline{g}_h \tag{4.3}$$

with

$$S_h = M_{11} - M_{10} K_{00}^{-1} K_{01} - K_{10} K_{00}^{-1} M_{01} + K_{10} K_{00}^{-1} M_{00} K_{00}^{-1} K_{01}$$

and the right-hand side

$$\underline{g}_h = \left(K_{10} K_{00}^{-1} M_{00} - M_{10} \right) K_{00}^{-1} \underline{f}_h.$$

The matrix S_h is known as a Schur complement of the block system.

As in the continuous case the reduction to the boundary can also be done by a decomposition result for the finite element space $\mathcal{S}_h(\Omega)$, which reveals some extra structural information of the Schur complement matrix.

We start with the following discrete version of Lemma 3.1.

LEMMA 4.2. *$\mathcal{S}_h(\Omega) = \mathcal{S}_{h,0}(\Omega) \oplus \mathscr{H}_h(\Omega)$ with*

$$\mathscr{H}_h(\Omega) = \{ v_h \in \mathcal{S}_h(\Omega) \colon \int_\Omega \nabla v_h \cdot \nabla z_h \, dx = 0 \textit{ for all } z_h \in \mathcal{S}_{h,0}(\Omega) \}.$$

In details, for each $v_h \in \mathcal{S}_h(\Omega)$, we have the following unique decomposition:

$$v_h = \hat{v}_{h,0} + \hat{v}_{h,1} \quad \textit{with} \quad \hat{v}_{h,0} \in \mathcal{S}_{h,0}(\Omega) \quad \textit{and} \quad \hat{v}_{h,1} \in \mathscr{H}_h(\Omega)$$

and

$$\|v_h\|_{-1,\Delta,h}^2 \sim |\hat{v}_{h,0}|_1^2 + \|\hat{v}_{h,1}\|_0^2 \quad \textit{for all } v_h \in \mathcal{S}_h(\Omega).$$

with the same constants as in Lemma 3.1.

The proof of this lemma is a complete copy of the proof in the continuous case and is, therefore, omitted.

Observe that, for $v_h \in \mathcal{S}_h(\Omega)$, the decompositions in Lemma 3.1 and Lemma 4.2 are different, in general. The space $\mathscr{H}_h(\Omega)$ is known as the space of discrete harmonic functions.

Next we introduce the trace space of functions from $\mathcal{S}_h(\Omega)$ by

$$\mathcal{S}_h(\Gamma) = \{ \mu_h = v_h|_\Gamma \colon v_h \in \mathcal{S}_h(\Omega) \}.$$

For each $\mu_h \in \mathcal{S}_h(\Gamma)$, there is a unique element $v_h \in \mathcal{S}_h(\Omega)$ with $v_h\big|_\Gamma = \mu_h$ and

$$\int_\Omega \nabla v_h \cdot \nabla z_h \, dx = 0 \quad \text{for all } z_h \in \mathcal{S}_{h,0}(\Omega).$$

The associated mapping $E_h \colon \mathcal{S}_h(\Gamma) \longrightarrow \mathscr{H}_h(\Omega)$, $\mu_h \mapsto v_h$ is the well-known discrete harmonic extension. In matrix-vector notation, this mapping reads

$$\underline{\mu}_h \mapsto \underline{v}_h = \begin{bmatrix} E_{01} \\ I \end{bmatrix} \underline{\mu}_h \quad \text{with} \quad E_{01} = -K_{00}^{-1} K_{01}.$$

Here $\underline{\mu}_h$ denotes the coefficient vector of μ_h with respect to the nodal basis of $\mathcal{S}_h(\Gamma)$. It is easy to see that E_h is bijective.

Analogously to Theorem 3.4 we now obtain

THEOREM 4.3. *Let $u_h \in \mathcal{S}_h(\Omega)$ and $y_h \in \mathcal{S}_{h,0}(\Omega)$ be the unique solution of (4.1). Then $\lambda_h = u_h\big|_\Gamma \in \mathcal{S}_h(\Gamma)$ is the unique solution of the variational problem*

$$\int_\Omega E_h\lambda_h \, E_h\mu_h \; dx = -\int_\Omega \hat{u}_{h,0} \, E_h\mu \; dx \quad \textit{for all } \mu \in \mathcal{S}_h(\Gamma), \tag{4.4}$$

where $\hat{u}_{h,0} \in \mathcal{S}_{h,0}(\Omega)$ is the unique solution of the discrete variational problem

$$\int_\Omega \nabla \hat{u}_{h,0} \cdot \nabla z_h \; dx = \langle f, z_h \rangle \quad \textit{for all } z_h \in \mathcal{S}_{h,0}(\Omega).$$

The proof, which is completely analogous to the continuous case, is omitted.

Moreover, as already observed in [23], it is easy to show that the matrix representation of the bilinear form on the left hand side in (4.4) is the Schur complement S_h:

$$\int_\Omega E_h\lambda_h \, E_h\mu_h \; dx = \langle S_h \, \underline{\lambda}_h, \underline{\mu}_h \rangle \quad \text{for all } \lambda_h, \; \mu_h \in \mathcal{S}_h(\Gamma). \tag{4.5}$$

5. Preconditioning. Starting point for the construction of a preconditioner for (4.2) is Theorem 4.1, which shows the well-posedness of the discrete problem with respect to the norms $\|\cdot\|_{-1,\Delta,h}$ and $|\cdot|_1$, whose matrix representations are given by

$$\|v_h\|_{-1,\Delta,h} = \|\underline{v}_h\|_{P_h} \text{ with } P_h = M_h + \begin{bmatrix} K_{00} \\ K_{10} \end{bmatrix} K_{00}^{-1} \begin{bmatrix} K_{00} & K_{01} \end{bmatrix} \text{ and } |z_h|_1 = \|\underline{z}_h\|_{K_{00}}.$$

Here the following notation is used:

NOTATION 3. *For a positive definite matrix $M \in \mathbb{R}^{n\times n}$ the associated inner product is given by $\langle x, y\rangle_M = \langle Mx, y\rangle$. Both the vector norm and the matrix norm associated with the inner product $\langle\cdot,\cdot\rangle_M$ are denoted by $\|\cdot\|_M$.*

Therefore, as a consequence of Theorem 4.1 the spectrum of the preconditioned matrix $\mathcal{P}_h^{-1}\mathcal{A}_h$ with

$$\mathcal{P}_h = \begin{bmatrix} P_h & 0 \\ 0 & K_{00} \end{bmatrix} \quad \text{and} \quad \mathcal{A}_h = \begin{bmatrix} M_{00} & M_{01} & -K_{00} \\ M_{10} & M_{11} & -K_{10} \\ -K_{00} & -K_{01} & 0 \end{bmatrix}$$

is bounded away from 0 and ∞ uniformly with respect to the mesh size h. The application of this preconditioner requires an efficient method for multiplying $\mathcal{P}_h^{-1}$ with a vector, which in general is too costly. In practice, the blocks P_h and K_{00} are replaced by efficient preconditioners $\hat{P}_h$ and $\hat{K}_{00}$ leading to a practical preconditioner

$$\hat{\mathcal{P}}_h = \begin{bmatrix} \hat{P}_h & 0 \\ 0 & \hat{K}_{00} \end{bmatrix}. \tag{5.1}$$

Standard multilevel or multigrid methods are available for $\hat{K}_{00}$. Therefore, we will concentrate on the construction of an efficient preconditioner $\hat{P}_h$ for P_h.

Lemma 4.2 gives a first hint for preconditioning P_h. In matrix-vector notation it states that

$$\|\underline{v}_h\|_{P_h}^2 \sim \|\underline{\hat{v}}_{h,0}\|_{K_{00}}^2 + \|\underline{\hat{v}}_{h,1}\|_{M_h}^2 \quad \text{for all } \underline{v}_h = \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix}$$

with

$$\underline{\hat{v}}_{h,0} = \begin{bmatrix} I & K_{00}^{-1} K_{01} \end{bmatrix} \begin{bmatrix} \underline{v}_{h,0} \\ \underline{\mu}_h \end{bmatrix} \quad \text{and} \quad \underline{\hat{v}}_{h,1} = \begin{bmatrix} -K_{00}^{-1} K_{01} \\ I \end{bmatrix} \underline{\mu}_h,$$

which, by elementary calculations, leads to

$$P_h \sim \begin{bmatrix} I & 0 \\ K_{10} K_{00}^{-1} & I \end{bmatrix} \begin{bmatrix} K_{00} & 0 \\ 0 & S_h \end{bmatrix} \begin{bmatrix} I & K_{00}^{-1} K_{01} \\ 0 & I \end{bmatrix}.$$

This motivates the use of preconditioners of the following form

$$\hat{P}_h = \begin{bmatrix} I & 0 \\ -\hat{E}_{01}^T & I \end{bmatrix} \begin{bmatrix} \hat{K}_{00} & 0 \\ 0 & \hat{S}_h \end{bmatrix} \begin{bmatrix} I & -\hat{E}_{01} \\ 0 & I \end{bmatrix}$$

with three essential components $\hat{K}_{00}$, $\hat{S}_h$, and $\hat{E}_{01}$. It is reasonable to choose the same preconditioner for $\hat{K}_{00}$ in $\hat{P}_h$ as in $\hat{\mathcal{P}}_h$. Candidates for $\hat{S}_h$ are preconditioners for S_h. The third component $\hat{E}_{01}$ is considered as an approximation of $E_{01} = -K_{00}^{-1} K_{01}$. The associated mapping $\hat{E}_h \colon \mathcal{S}_h(\Gamma) \longrightarrow \mathcal{S}_h(\Omega)$, given by

$$\underline{\mu}_h \mapsto \begin{bmatrix} \hat{E}_{01} \\ I \end{bmatrix} \underline{\mu}_h,$$

can be seen as an approximation to the discrete harmonic extension E_h.

Preconditioners of this type have been intensively studied in the context of domain decomposition methods. A typical result reads, see [20] for the proof.

THEOREM 5.1. *Assume that $\hat{K}_{00} \sim K_{00}$, $\hat{S}_h \sim S_h$, and that there is a positive constant such that*

$$\left\| \begin{bmatrix} \hat{E}_{01} \\ I \end{bmatrix} \underline{\mu}_h \right\|_{P_h}^2 \leq c \, \|\underline{\mu}_h\|_{S_h}^2 \quad \textit{for all } \underline{\mu}_h.$$

Then $\hat{P}_h \sim P_h$.

Observe that the last condition translates to

$$\|\hat{E}_h \, \mu_h\|_{-1,\Delta,h}^2 \leq c \, \|E_h \, \mu_h\|_0^2 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma), \tag{5.2}$$

i.e., the approximate harmonic extension has to be bounded with respect to the given norms.

Next we discuss the choice of the two remaining components $\hat{S}_h$ and $\hat{E}_{01}$.

5.1. Schur complement preconditioning. The mapping property of S_h is contained in the following theorem, which does not rely on any convexity assumption. This generalizes a result in [23], where the convex case was considered.

THEOREM 5.2. *For the norm* $\|\cdot\|_{-1/2,\Gamma,h}$ *in* $\mathcal{S}_h(\Omega)$*, given by*

$$\|\mu_h\|_{-1/2,\Gamma,h}^2 = \|\mu_h\|_{-1/2,\Gamma}^2 + h\,\|\mu_h\|_{0,\Gamma}^2,$$

we have

$$\langle S_h\,\underline{\mu}_h, \underline{\mu}_h\rangle = \|E_h\mu_h\|_0^2 \sim \|\mu_h\|_{-1/2,\Gamma,h}^2 + \|\Pi_N E_h\,\mu_h\|_0^2 \quad \text{for all } \mu_h \in \mathcal{S}_h(\Gamma), \tag{5.3}$$

where Π_N *is the* L^2*-orthogonal projection onto* N*, see (3.2).*

Proof. We closely follow the proof in [23] and denote the classical harmonic extension operator as a mapping from $H^{1/2}(\Gamma)$ onto $H^1(\Omega)$ by E^1, see the appendix for details. The symbol c is used as a generic constant, which might change its value at each appearance.

For $\mu_h \in \mathcal{S}_h(\Omega) \subset H^{1/2}(\Gamma)$, let $v = E^1\mu_h \in H^1(\Omega)$ and $v_h = E_h\mu_h \in \mathcal{S}_h(\Omega)$ be its harmonic and the discrete harmonic extension, respectively.

For $v^* = (I-\Pi_N)\,v$ and $v_h^* = (I-\Pi_N)\,v_h$, there exists $z \in H^2(\Omega)\cap H_0^1(\Omega)$ with $v^*-v_h^* = \Delta z$, since $\mathrm{im}(I-\Pi_N) = \mathrm{im}\,\Delta$ for the Laplace operator $\Delta\colon H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow L^2(\Omega)$, see (7.3) in the appendix. Then we have

$$\begin{aligned}\|v_h^* - v^*\|_0^2 &= -\int_\Omega (v_h^* - v^*)\,\Delta z\;dx = -\int_\Omega (v_h - v)\,\Delta z\;dx\\ &= \int_\Omega \nabla(v_h - v)\cdot\nabla z\;dx = \int_\Omega \nabla(v_h - v)\cdot\nabla(z - z_h)\;dx\end{aligned}$$

for an arbitrary element $z_h \in \mathcal{S}_{h,0}(\Omega)$. The last identity follows from the Galerkin orthogonality. Therefore, for the pointwise interpolant z_h of z, we obtain

$$\|v_h^* - v^*\|_0^2 \le |v_h - v|_1\,|z - z_h|_1 \le c\,h\,|v|_1\,\|z\|_2 \le c\,h\,\|\mu_h\|_{1/2,\Gamma}\,\|v_h^* - v^*\|_0,$$

using the approximation property of $\mathcal{S}_h(\Omega)$ for functions in $H^2(\Omega)$ and the mapping properties of E^1 and Δ. This implies

$$\|v_h^* - v^*\|_0 \le c\,h\,\|\mu_h\|_{1/2,\Gamma} \le c\,h^{1/2}\,\|\mu_h\|_{0,\Gamma}$$

by using an inverse inequality for the second estimate. Furthermore,

$$\|v^*\|_0 = \|(I-\Pi_N)E^1\mu_h\|_0 = \|E^0\mu_h\|_0 \le c\,\|\mu_h\|_{-1/2,\Gamma},$$

see Theorem 7.2 in the appendix, and Theorem 3.3, part 3. Therefore, we obtain

$$\|(I-\Pi_N)E_h\mu_h\|_0 = \|v_h^*\|_0 \le \|v^*\|_0 + \|v_h^* - v^*\|_0 \le \bar{c}\,\left(\|\mu_h\|_{-1/2,\Gamma} + h^{1/2}\,\|\mu_h\|_{0,\Gamma}\right).$$

With $\|E_h\mu_h\|_0^2 = \|(I-\Pi_N)E_h\mu_h\|_0^2 + \|\Pi_N E_h\mu_h\|_0^2$ the second estimate easily follows.

For the first estimate we start with

$$\|E_h\mu_h\|_0 = \|v_h\|_0 \ge \|v_h^*\|_0 \ge \|v^*\|_0 - \|v_h^* - v^*\|_0 \ge c\,\left(\|\mu_h\|_{-1/2,\Gamma} - h^{1/2}\,\|\mu_h\|_{0,\Gamma}\right)$$

Using the inverse inequality $\|v_h\|_0 \ge c\,h^{1/2}\,\|\mu_h\|_{0,\Gamma}$ and $\|E_h\mu_h\|_0 \ge \|\Pi_N E_h\mu_h\|_0$, the first inequality easily follows. ∎

In order to construct a preconditioner for S_h we start by first considering the term $\|\mu_h\|_{-1/2,\Gamma,h}$ in (5.3) only. A preconditioner for this norm, i.e. an easy to invert approximation to the matrix representing this norm, was already proposed in [23] based on preconditioners for $\|\cdot\|_{-1/2,\Gamma_k}$, $k \geq 1$. We follow this idea but replace the preconditioner for $\|\cdot\|_{-1/2,\Gamma_k}$, for which FFT (fast Fourier transform) was used in [23], by a simpler standard multilevel preconditioner of the type as analyzed in [16].

For this, let $\mathcal{T}_\ell$, $\ell = 0, 1, 2, \ldots, L$ be a hierarchy of uniformly refined subdivisions of Ω of mesh size h_ℓ with $\mathcal{T}_L = \mathcal{T}_h$ with associated finite element spaces $\mathcal{S}_\ell(\Omega)$ of continuous and piecewise linear functions and their trace spaces $\mathcal{S}_\ell(\Gamma)$. Furthermore, let $\mathcal{S}_\ell(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$ be the linear span of all nodal basis functions from $\mathcal{S}_\ell(\Gamma)$ associated with nodes from Γ_C and from Γ_E, respectively.

Then the proposed preconditioner is of additive Schwarz type, given by

$$\left(\hat{S}_h^{(0)}\right)^{-1} = h_L\, R_{C,L}\, A_{C,L}\, R_{C,L}^T + \sum_{\ell=0}^{L} h_\ell\, R_{E,\ell}\, A_{E,\ell}\, R_{E,\ell}^T.$$

Here $R_{C,L}$ and $R_{E,\ell}$ denote the matrix representations of the canonical embeddings of $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$ into $\mathcal{S}_L(\Gamma)$, respectively. The matrices $A_{C,L}$ and $A_{E,\ell}$ are given by

$$A_{C,L} = \bar{M}_{C,L}^{-1}\, K_{C,L}\, \bar{M}_{C,L}^{-1}, \quad A_{E,\ell} = \bar{M}_{E,\ell}^{-1}\, K_{E,\ell}\, \bar{M}_{E,\ell}^{-1},$$

where $\bar{M}_{C,L}$ and $\bar{M}_{E,\ell}$ are the matrix representations of the discrete version of the norm $\|\cdot\|_{0,\Gamma}$ which results from the elementwise use of the trapezoidal rule on $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$, respectively. $\bar{K}_{C,L}$ and $\bar{K}_{E,\ell}$ are the matrix representations of the norm $|\cdot|_{1,\Gamma}$ on $\mathcal{S}_L(\Gamma_C)$ and $\mathcal{S}_\ell(\Gamma_E)$, respectively, where this norm is given by

$$|\mu_h|_{1,\Gamma}^2 = \sum_{k=1}^{K} \int_{\Gamma_k} |\nabla_{\Gamma_k}\, \mu_h|^2 \, ds \quad \text{with the tangential gradient } \nabla_{\Gamma_k}.$$

Observe that the boundary mass matrices $\bar{M}_{C,L}$ and $\bar{M}_{E,\ell}$ are diagonal. So, the application of the preconditioner requires only the multiplication by boundary stiffness matrices $\bar{K}_{C,L}$ and $\bar{K}_{E,\ell}$ and some componentwise scaling on each refinement level.

Now we have

Theorem 5.3. *$\|\mu_h\|_{-1/2,\Gamma,h}^2 \sim \left\langle \hat{S}_h^{(0)}\, \underline{\mu}_h, \underline{\mu}_h \right\rangle$ for all $\mu_h \in \mathcal{S}_h(\Gamma)$.*

Proof. Each part of the proof is based on fairly standard arguments from [23], [22], [16], and [12]. We just have to put known things together.

First of all, for the decomposition $\mathcal{S}_L(\Gamma) = \mathcal{S}_L(\Gamma_C) \oplus \mathcal{S}_L(\Gamma_E)$ we obtain

$$\|\mu_L\|_{-1/2,\Gamma,h}^2 \sim h_L\, \|\mu_C\|_{0,\Gamma}^2 + \|\mu_E\|_{-1/2,\Gamma}^2$$

for all $\mu_L \in \mathcal{S}_L(\Gamma)$ with $\mu_L = \mu_C + \mu_E$, $\mu_C \in \mathcal{S}_L(\Gamma_C)$, $\mu_E \in \mathcal{S}_L(\Gamma_E)$, see [23], Proposition 7.5.

Next, using the multiscale representation of the norm $\|\cdot\|_{-1/2,\Gamma_k}$ from Theorem 4 in [22], following the main idea from [16] and replacing the norm $\|\cdot\|_{0,\Gamma_k}$ in this representation by the norm $h_\ell^{-1}\, \|\cdot\|_{-1,\Gamma_k}$, and applying Proposition 3 from [12] one obtains

$$\|\mu_E\|_{-1/2,\Gamma}^2 \sim \inf \left\{ \sum_{\ell=0}^{L} h_\ell^{-1}\, \|\mu_\ell\|_{-1,\Gamma}^2 \colon \mu_E = \sum_{\ell=0}^{L} \mu_\ell,\ \mu_\ell \in \mathcal{S}_\ell(\Gamma_E) \right\}$$

for all $\mu_E \in \mathcal{S}_L(\Gamma_E)$, where $\|\cdot\|_{-1,\Gamma}$ denotes the canonical norm in $H^{-1}_{pw}(\Gamma)$.

Finally, using

$$\|\mu_\ell\|^2_{-1,\Gamma} \sim \langle A^{-1}_{E,\ell}\, \underline{\mu_\ell}, \underline{\mu_\ell} \rangle \quad \text{for all } \mu_\ell \in \mathcal{S}_\ell(\Gamma_E)$$

from [16] and

$$h_L\, \|\mu_C\|^2_{0,\Gamma} \sim h_L^{-1}\, \langle A^{-1}_{C,L}\, \underline{\mu}_C, \underline{\mu}_C \rangle \quad \text{for all } \mu_C \in \mathcal{S}_L(\Gamma_C),$$

which follows from a simple scaling argument, we obtain a stable space decomposition, whose associated additive Schwarz operator is $\hat{S}^{(0)}_h$. □

For studying the term $\|\Pi_N E_h\, \mu_h\|^2_0$ in (5.3) we assume that a basis $\{s_1, \dots, s_J\}$ of N is known. Then we have

$$\|\Pi_N v_h\|^2_0 = \langle M^T_{Nh} M^{-1}_N\, M_{Nh}\, \underline{v}_h, \underline{v}_h \rangle$$

with the mass matrices

$$M_N = \left(\int_\Omega s_i\, s_j\; dx \right) \quad \text{and} \quad M_{Nh} = \left(\int_\Omega s_i\, \varphi_j\; dx \right),$$

where $\{\varphi_1, \varphi_2, \dots, \varphi_I\}$ denotes the nodal basis of $\mathcal{S}_h(\Omega)$. Hence

$$\|\Pi_N E_h\, \mu_h\|^2_0 = \left\langle M^T_{Nh} M^{-1}_N\, M_{Nh} \begin{bmatrix} E_{01} \\ I \end{bmatrix} \underline{\mu}_h, \begin{bmatrix} E_{01} \\ I \end{bmatrix} \underline{\mu}_h \right\rangle = \left\langle U_h M^{-1}_N\, U^T_h\, \underline{\mu}_h, \underline{\mu}_h \right\rangle$$

with $U_h = \begin{bmatrix} E^T_{01} & I \end{bmatrix} M^T_{Nh}$. Observe that the rank of $U_h M^{-1}_N U^T_h$ is equal to the dimension of N, i.e., the (fixed) number of reentrant corners.

So, in summary, we obtain

$$\langle S_h\, \underline{\mu}_h, \underline{\mu}_h \rangle = \|E_h\, \mu_h\|^2_0 \sim \left\langle \left[\hat{S}^{(0)}_h + U_h M^{-1}_N U^T_h \right] \underline{\mu}_h, \underline{\mu}_h \right\rangle \quad \text{for all } \mu \in \mathcal{S}_h(\Gamma),$$

which completes the proof of

THEOREM 5.4. *$S_h \sim \hat{S}^{(1)}_h$ with $\hat{S}^{(1)}_h = \hat{S}^{(0)}_h + U_h M^{-1}_N U^T_h$.*

Furthermore, we have

$$\left(\hat{S}^{(1)}_h\right)^{-1} \underline{\mu}_h = \left[I - V_h (M_N + U^T_h V_h)^{-1} U^T_h \right] \left(\hat{S}^{(0)}_h\right)^{-1} \underline{\mu}_h \quad \text{with} \quad V_h = \left(\hat{S}^{(0)}_h\right)^{-1} U_h,$$

by the Sherman-Morrison-Woodbury formula. The matrix $I - V_h (M_N + U^T_h V_h)^{-1} U^T_h$ has to be computed only once and the computational costs are rather low for domains with a small number of reentrant corners. This makes $\hat{S}^{(1)}_h$ an efficient preconditioner for S_h.

REMARK 2. *Instead of using a basis $\{s_1, \dots, s_J\}$ of N, which is hardly known in practice, it suffices to use a basis $\{\hat{s}_1, \dots, \hat{s}_J\}$ of a space $\hat{N}$ as long as $\operatorname{im} \Delta \oplus \hat{N}$ is an L^2-stable decomposition. Such a basis is known and consists of functions of the form*

$$\hat{s}_j(r_j, \theta_j) = \eta(r_j)\, r_j^{\pi/\omega_j}\, \sin((\omega_j/\pi)\, \theta_j).$$

Here (r_j, θ_j) denotes the polar coordinates centered at a reentrant corner with internal angle ω_j spanned by $\theta_j = 0$ and $\theta_j = \omega_j$, and $\eta(r_j)$ is a cutoff function which is identical to 1 in a neighborhood of the corner, see [14], [13].

5.2. Approximate discrete harmonic extensions. The evaluation of $v_h = E_h\,\mu_h$, where E_h is the discrete harmonic extension requires the exact solve of the linear system

$$K_{00}\,\underline{v}_{h,0} = -K_{01}\,\underline{\mu}_h.$$

If instead we use an inner iteration by performing r steps of the Richardson method with preconditioner $\hat{K}_{00}$ and initial guess 0, then we end up with an approximate harmonic extension $\hat{E}_h^{(r)}$, given by

$$\underline{\mu}_h \mapsto \begin{bmatrix} \hat{E}_{01}^{(r)} \\ I \end{bmatrix} \underline{\mu}_h \quad \text{with} \quad \hat{E}_{01}^{(r)} = \left[I - (I - \hat{K}_{00}^{-1} K_{00})^r \right] E_{01}.$$

(For $r = 1$ we simply get $\hat{E}_{01}^{(1)} = -\hat{K}_{00}^{-1} K_{01}$.) We will now show that this approximate discrete harmonic extension satisfies the third condition of Theorem 5.1 under reasonable assumptions.

LEMMA 5.5. *Assume that the inner iteration converges in the corresponding energy norm with a convergence rate $q < 1$ which independent of h, i.e.*

$$\|I - \hat{K}_{00}^{-1} K_{00}\|_{K_{00}} \le q.$$

Then, for $r = \mathcal{O}(|\ln h|)$, there is a constant c such that

$$\|\hat{E}_h^{(r)}\,\mu_h\|_{-1,\Delta,h} \le c\,\|E_h\,\mu_h\|_0 \quad \textit{for all } \mu_h \in \mathcal{S}_h(\Gamma).$$

Proof. For all $\mu_h \in \mathcal{S}_h(\Gamma)$, we have

$$\begin{aligned} \|\hat{E}_h^{(r)}\,\mu_h\|_{-1,\Delta,h} &\le \|E_h\,\mu_h\|_{-1,\Delta,h} + \|(\hat{E}_h^{(r)} - E_h)\,\mu_h\|_{-1,\Delta,h} \\ &= \|E_h\,\mu_h\|_0 + \|(\hat{E}_h^{(r)} - E_h)\,\mu_h\|_1 \\ &\le \|E_h\,\mu_h\|_0 + (c_F^2 + 1)^{1/2}\,|(\hat{E}_h^{(r)} - E_h)\,\mu_h|_1. \end{aligned}$$

Now it easily follows that

$$\begin{aligned} |(\hat{E}_h^{(r)} - E_h)\,\mu_h|_1 = \|(\hat{E}_{01}^{(r)} - E_{01})\,\underline{\mu}_h\|_{K_{00}} &= \|(I - \hat{K}_{00}^{-1} K_{00})^r E_{01}\,\underline{\mu}_h\|_{K_{00}} \\ \le q^r\,\|E_{01}\,\underline{\mu}_h\|_{K_{00}} \le q^r\,|E_h\,\mu_h|_1 &\le c\,q^r h^{-1}\,\|E_h\,\mu_h\|_0. \end{aligned}$$

If $r = \mathcal{O}(|\ln h|)$, the factor $q^r h^{-1}$ is uniformly bounded, which completes the proof. □

6. Numerical experiments. We consider the following simple biharmonic test problem:

$$\Delta^2 y = f \quad \text{in } \Omega, \qquad y = \frac{\partial y}{\partial n} = 0 \quad \text{on } \Gamma$$

on two domains, the square $\Omega = \Omega_S = (-1,1)^2$ (representing the convex case) and the L-shaped domain $\Omega = \Omega_L$ depicted in figures 6.1 and 6.2, where also the initial mesh (level $\ell = 0$) is shown. The right-hand side $f(x)$ is chosen such that

$$y(x) = \left[1 - \cos(2\pi x_1)\right]\left[1 - \cos(4\pi x_2)\right]$$

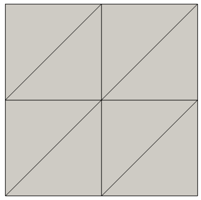

FIG. 6.1. $\Omega = \Omega_S$

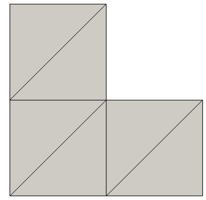

FIG. 6.2. $\Omega = \Omega_L$

is the exact solution to the problem. The initial meshes are uniformly refined until the final level $\ell = L$.

We will present numerical results demonstrating the quality of the preconditioners $\hat{S}_h^{(0)}$ and (in the non-convex case) $\hat{S}_h^{(1)}$ for S_h and the preconditioner $\hat{\mathcal{P}}_h$ for $\mathcal{A}_h$. The Schur complement preconditioners were tested by applying the preconditioned gradient (PG), the conjugate gradient method (CG) and its preconditioned variant (PCG) to (4.3), the method of choice for (4.2) was the preconditioned minimal residual method (PMINRES). In all experiments a reduction of the Euclidean norm of the initial residual by a factor of 10^{-8} was used as stopping criterion for the iterative methods, where the initial guess was chosen randomly out of the range spanned by the corresponding exact quantities.

For the preconditioner $\hat{K}_{00}$, which is used as first diagonal block in $\hat{P}_h$, as the last diagonal block in $\hat{\mathcal{P}}_h$, and as preconditioner in the inner iteration for the approximate discrete harmonic extension, we always choose one multigrid V-cycle with one step of forward and backward Gauss-Seidel smoothing. The action of the exact inverse K_{00}, as needed for S_h and for the exact discrete harmonic extension E_h, was realized by applying an inner iteration with 10 V-cycles. The preconditioner $\hat{S}_h^{(1)}$ was constructed with the modification as described in Remark 2.

Table 6.1 shows the observed number of iterations for (4.3) in the convex case $\Omega = \Omega_S$. The first column contains the level L of refinement. The next three columns show the results for CG, for PG and PCG both with the preconditioner $\hat{S}_h^{(0)}$.

TABLE 6.1
Number of iterations for (4.3), $\Omega = \Omega_S$ (square)

L	CG	PG	PCG
6	66	77	20
7	91	77	21
8	116	75	21
9	157	71	20

As expected the number of iterations grows for CG without preconditioning if the mesh size decreases. The second column shows that the preconditioner alone with PG already leads to convergence rates which are uniformly bounded in h. Of course, the use of this preconditioner in PCG results in a further reduction of the number of iterations.

Table 6.2 shows the results for the L-shaped domain $\Omega = \Omega_L$ representing a non-convex case. The second and third columns contain the numbers of iterations for PG with the preconditioners $\hat{S}_h^{(0)}$ and $\hat{S}_h^{(1)}$, respectively, while in the next two columns the corresponding results for PCG are shown.

TABLE 6.2
Number of iterations for (4.3), $\Omega = \Omega_L$

L	PG($\hat{S}_h^{(0)}$)	PG($\hat{S}_h^{(1)}$)	PCG($\hat{S}_h^{(0)}$)	PCG($\hat{S}_h^{(1)}$)
6	337	104	24	22
7	427	114	25	23
8	385	114	26	23
9	457	110	25	23

By comparing the second and the third column one sees that $\hat{S}_h^{(1)}$ performs significantly better as a preconditioner in PG than $\hat{S}_h^{(0)}$, as expected from the analysis. Nevertheless, as seen from the fourth column, PCG works well for the non-optimal preconditioner $\hat{S}_h^{(0)}$. A relevant further improvement by using the better preconditioner $\hat{S}_h^{(0)}$ in PCG was not observed. This important feature was observed here experimentally and is not yet supported by analysis. If confirmed, this would considerably contribute to practicability, in particular, for a possible extension to three-dimensional problems, see the concluding remarks.

Finally, Table 6.3 shows some preliminary results for preconditioning the non-reduced system (4.2), whose total numbers n of unknowns are shown in the second column. PMINRES was applied to the L-shaped domain $\Omega = \Omega_L$, for preconditioning the Schur complement the non-optimal preconditioner $\hat{S}_h^{(0)}$ was used. The third and the second columns contain the results if using the costly exact discrete harmonic extension E_h and the less costly approximate version $\hat{E}_h^{(r)}$ with r inner iterations, respectively. The chosen value of r is shown in parenthesis in the fourth column.

TABLE 6.3
Number of iterations for (4.2), $\Omega = \Omega_L$

L	n	PMINRES(E_h)	PMINRES($\hat{E}_h^{(r)}$)
4	1 538	46	43 (3)
5	6 146	45	48 (3)
6	24 578	43	47 (4)
7	98 306	43	48 (4)
8	393 218	44	46 (5)
9	1 572 866	44	56 (5)

It can be seen that a modest increase of the number r of inner iterations keeps the number of iterations in the range of the observed number of iterations if using the costly exact discrete harmonic extension. This is in accordance with Lemma 5.5.

7. Concluding remarks. Efficient Schur complement preconditioners were derived and analyzed for convex and non-convex polygonal domains, respectively. There is experimental evidence that $\hat{S}_h^{(0)}$ works also fine for non-convex polygonal domains in combination with a Krylov subspace method (PCG for (4.3) or PMINRES for (4.2)). This is specially advantageous for a possible extension to three-dimensional problems, where $\hat{S}_h^{(1)}$ would be much harder to construct. Preconditioning the reduced system (4.3) has, therefore, reached a satisfactory state. Observe, however, that the computational costs for evaluating one residual for (4.3) is relatively high, since it requires the application of (an accurate approximation of) the discrete harmonic extension E_h

twice.

The situation is less clear for the non-reduced problem (4.2). Here the evaluation of the residual is computationally inexpensive. The computational costs for applying $\hat{\mathcal{P}}_h$ depend mainly on the choice for $\hat{E}_h$. If $\hat{E}_h = E_h$, the computational costs of one step of PCG for (4.3) and one step of PMINRES for (4.2) are roughly the same. The number of iterations differ by a factor of about 2, see Tables 6.1, third column and 6.3, last column, and so do the observed computing times, as expected. Possible improvements are to use symmetric indefinite preconditioners, see [25], based on the same components as the proposed symmetric and positive definite block diagonal preconditioner, in particular with the same extension $\hat{E}_h = E_h$, in combination of Krylov subspace methods such as GMRES. Another possible improvement is the replacement of E_h by more efficient approximate discrete harmonic extensions. The few numerical experiments with an inner iteration as shown in Table 6.3 already lead to an improvement in computing time by almost a factor of 2. Efficient approximate harmonic extensions are well-developed and understood as bounded operators from $\mathcal{S}_h(\Gamma) \subset H^{1/2}(\Gamma)$ to $H^1(\Omega)$, see, e.g., [20]. Here the challenge of future work is the construction of efficient approximate harmonic extensions which satisfy (5.2).

Appendix: Harmonic extension operators. The trace space of functions from $H^1(\Omega)$ is $H^{1/2}(\Gamma)$. The well-known harmonic extension operator

$$E^1 \colon H^{1/2}(\Omega) \longrightarrow H^1(\Omega)$$

is given by the following variational problem: For $\mu \in H^{1/2}(\Gamma)$, find $v = E^1\lambda \in H^1(\Omega)$ such that $v\big|_\Gamma = \mu$ and

$$\int_\Omega \nabla v \cdot \nabla z \, dx = 0 \quad \text{for all } z \in H^1_0(\Omega). \tag{7.1}$$

An essential property of E^1 is

$$|E^1\mu|_1^2 \sim \|\mu\|^2_{1/2,\Gamma} \quad \text{for all } \mu \in H^{1/2}(\Gamma),$$

where $\|\cdot\|_{1/2,\Gamma}$ denotes the standard norm in $H^{1/2}(\Gamma)$.

A harmonic extension operator

$$E^0 \colon H^{-1/2}_{pw}(\Gamma) \longrightarrow H^{-1}(\Delta, \Omega)$$

which is more appropriate in the context of this paper is given by the following variational problem: For $\mu \in H^{-1/2}_{pw}(\Gamma)$, find $v = E^0\mu \in \operatorname{im}\Delta$ such that $\gamma^0 v = \mu$ and

$$\int_\Omega v \, \Delta z \, dx = \langle \mu, \gamma^1 z\rangle_\Gamma \quad \text{for all } z \in H^2(\Omega) \cap H^1_0(\Omega), \tag{7.2}$$

with

$$\langle \mu, g\rangle_\Gamma = \sum_{k=1}^K \langle \mu_k, g_k\rangle_{\Gamma_k} \quad \text{for } \mu = (\mu_k)_{k=1,\dots,K} \in H^{-1/2}_{pw}(\Gamma),\ g = (g_k)_{k=1,\dots,K} \in H^{1/2}_{pw}(\Gamma)$$

where $\langle \cdot, \cdot\rangle_{\Gamma_k}$ denotes the duality product in $H^{-1/2}_{pw}(\Gamma) \times H^{1/2}_{pw}(\Gamma)$ and γ^1 is the trace operator, given by

$$\gamma^1 z = (\gamma^1_k z)_{k=1,\dots,K} \quad \text{with} \quad \gamma^1_k z = \frac{\partial z}{\partial n}\Big|_{\Gamma_k} \quad \text{for} \quad z \in H^2(\Omega) \cap H^1_0(\Omega).$$

Here, $H^{1/2}_{pw}(\Gamma)$ denotes the product space $\prod_{k=1}^{K} H^{1/2}(\Gamma_k)$ and $\operatorname{im}\Delta$ is the image of the Laplace operator $\Delta\colon H^2(\Omega)\cap H^1_0(\Omega) \longrightarrow L^2(\Omega)$.

For this definition of Δ, it is known that

$$L^2(\Omega) = \operatorname{im}\Delta \perp N, \tag{7.3}$$

which immediately implies that $\operatorname{im}\Delta = N^\perp = \operatorname{im}(I-\Pi_N)$, [14], [13].

REMARK 3. *For smooth functions on $\overline{\Omega}$ the right-hand side in (7.2) can be rewritten in a more traditional fashion as*

$$\langle \mu, \gamma^1 z\rangle_\Gamma = \int_\Gamma \mu \frac{\partial z}{\partial n}\, dS.$$

Variants of (7.2) are often called very weak formulations of the corresponding Dirichlet problem for the Laplace operator.

In [10], page 184, this harmonic extension operator was studied for smooth and for convex polygonal domains Ω, however on a not yet appropriate trace space in the case of convex polygonal domains. We will show now that E^0 is well-defined on general polygonal domains.

THEOREM 7.1. *For each $\mu \in H^{-1/2}_{pw}(\Gamma)$, there is a unique solution $v = E^0\mu \in \operatorname{im}(\Delta)$ to (7.2) and*

$$\|E^0\mu\|_0^2 \sim \|\mu\|^2_{-1/2,\Gamma} \quad \textit{for all } \mu \in H^{-1/2}_{pw}(\Gamma).$$

Proof. For $v = \Delta w$ with $w \in W = H^2(\Omega)\cap H^1_0(\Omega)$, problem (7.2) coincides with the second biharmonic boundary value problem for w, which is known to be well-posed. For given $\mu \in H^{-1/2}_{pw}(\Gamma)$, let w_μ be its solution. Then

$$\|w_\mu\|_2 \sim \|\ell_\mu\|_{W^*} \quad \text{for all } \mu \in H^{-1/2}_{pw}(\Gamma),$$

where ℓ_μ denotes the linear functional on the right-hand side of (7.2), given by $z \mapsto \langle \mu, \gamma^1 z\rangle_\Gamma$.

The trace operator $\gamma^1\colon W \longrightarrow \widetilde{H}^{1/2}_{pw}(\Gamma)$ is well-defined, bounded and surjective, see [14]. Therefore, the bilinear form $(z,\mu) \mapsto \langle \mu, \gamma^1 z\rangle_\Gamma$ is well-defined and bounded on $W \times H^{-1/2}_{pw}(\Gamma)$, and it satisfies an inf-sup condition. Therefore,

$$\|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H^{-1/2}_{pw}(\Gamma).$$

Using $\|\Delta w\|_0 \sim \|w\|_2$ for all $w \in H^2(\Omega)\cap H^1_0(\Omega)$, we finally obtain

$$\|E^0\mu\|_0 = \|\Delta w_\mu\|_0 \sim \|\ell_\mu\|_{W^*} \sim \|\mu\|_{-1/2,\Gamma} \quad \text{for all } \mu \in H^{-1/2}_{pw}(\Gamma).$$

□

We have the following relation between E^1 and E^0 on the domain $H^{1/2}(\Gamma)$, where both extension operators exist.

THEOREM 7.2. $E^0\mu = (I-\Pi_N)E^1\mu$ *for all* $\mu \in H^{1/2}(\Gamma)$.

Proof. Both $E^0\mu$ and $E^1\lambda$ are harmonic and have the same trace μ. Therefore, $E^1\mu - E^0\mu \in N$, i.e., there is an $n \in N$ with $E^1\mu = E^0\mu + n$. Moreover, $E^0\mu \in \operatorname{im}\Delta$ by definition. From (7.3) it follows that $n = \Pi_N E^1\mu$, which implies $E^0\mu = E^1\mu - n = E^1\mu - \Pi_N E^1\mu$. □

Theorem 3.3 is a simple consequence of the last two theorems.

REMARK 4. *For convex domains, N is trivial. Only in this case the two harmonic extension operators coincide.*

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