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The Timoshenko beam model

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Sworn Declaration

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Abstract

In this bachelor thesis, a model for the Timoshenko beam is derived. Starting with the equilibrium conditions and the Saint-Venant-Kirchhoff material law, a 3D beam model is derived, both in variational form and as minimization problem. This 3D model is then reduced to a 2D model, because of kinematical assumptions. The material law is modified, so we get a new minimization problem and therefore also a new variational problem. This new variational problem is then discretized with the Courant element and then the solution is calculated by the preconditioned CG-method.

For a fixed thickness and finer discretizations, the numerical solution gets always closer to the analytical solution, as expected. But if we have a fixed discretization and we then look at beams, that are always thinner, we can observe, that the numerical solution gets always smaller, and that the approximations for the analytical solution get worse. So the beam appears to be stiffer than it actually is.

Kurzfassung

In dieser Bachelorarbeit wird ein Modell für den Timoshenko Balken hergeleitet. Aus den Gleichgewichtsbedingungen und dem Saint-Venent-Kirchhoff Materialgesetz wird zuerst ein 3D Balkenmodell hergeleitet, sowohl in variationeller Form, als auch als Minimierungsproblem. Aufgrund von kinematischen Annahmen wird dieses 3D Modell dann auf ein 2D Modell reduziert. Das Materialgesetz wird dann modifiziert, und es ergibt sich dadurch dann ein neues Minimierungsproblem und damit auch ein neues Variationsproblem. Dieses Variationsproblem wird dann diskretisiert, wobei dazu das Courant Element verwendet wird. Anschließend wird die Lösung mit dem präkonditionierten CG-Verfahren berechnet.

Für fixe Balkendicke und feinere Diskretisierungen nähert sich die numerische Lösung, wie erwartet, immer mehr der analytischen Lösung an. Betrachtet man jedoch bei gleicher Diskretisierung immer dünner werdende Balken, so stellt man fest, dass die numerische Lösung immer kleiner wird und eine immer schlechtere Approximation für die analytische Lösung liefert. Der Balken wirkt also steifer, als er tatsächlich ist.

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Chapter 1

Introduction

Beam theory deals with the deformation of a beam under a load. In this bachelor thesis, a model for the Timoshenko beam is derived.

First a 3D model beam model is derived, by starting with the equilibrium conditions and then using the Saint-Venant-Kirchhoff material law.

In the third chapter, this 3D model is reduced by some kinematical assumptions for the beam, and we get the representation for the deformation of the Timoshenko beam. For a modified material law, we then derive a variational problem and also two second order differential equations

In chapter four, the variational problem is then discretized by using the Courant Element and we get a linear system of equations. This system is then solved for concrete values for the beam parameters by the preconditioned CG-method.

The results are represented in the fifth chapter. We first look at the results for different stepsizes h in the discretization. Then we also look at the results if we use a fixed number of intervals for the discretization, but we change the thickness of the beam and let it get smaller and smaller.

Chapter 2

The Elasticity Problem in $\mathbb{R}^3$

The content of this chapter is based on [3] and [2].

2.1 Variational form for σ

We want to deduce the variational form for

$$\begin{aligned} -\operatorname{Div} \boldsymbol{\sigma}(x) &= f(x) && \forall x \in \Omega, \\ u(x) &= 0 && \forall x \in \Gamma_D, \\ \boldsymbol{\sigma}(x) n(x) &= g_N(x) && \forall x \in \Gamma_N, \end{aligned}$$

where Γ_D and Γ_N are disjoint sets with $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\operatorname{Div} \boldsymbol{F}(x) = \left(\sum_{j=1}^{3} \frac{\partial F_{ij}}{\partial x_j}(x)\right)_{i=1,2,3}$.

Therefore the inner product with a test function $v(x)$ is built on both sides of the equation and then both sides are integrated over the integration domain Ω:

$$\int_\Omega -\operatorname{Div} \boldsymbol{\sigma}(x) \cdot v(x) \mathrm{d}x = \int_\Omega f(x) \cdot v(x) \mathrm{d}x.$$

Now the left hand side is transformed by using the definition of $\operatorname{Div} \boldsymbol{\sigma}(x)$:

$$\begin{aligned} \int_\Omega -\operatorname{Div} \boldsymbol{\sigma}(x) \cdot v(x) \mathrm{d}x &= -\int_\Omega \left(\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(x)\right)_i \cdot v(x) \mathrm{d}x \\ &= -\int_\Omega \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}(x)\right) v_i(x) \mathrm{d}x \\ &= -\sum_{i=1}^{3} \sum_{j=1}^{3} \int_\Omega \frac{\partial \sigma_{ij}}{\partial x_j}(x) v_i(x) \mathrm{d}x. \end{aligned}$$

After partial integration one gets:

$$\begin{aligned}\int_\Omega f(x)\cdot v(x)\mathrm{d}x &= \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\int_\Omega \sigma_{ij}(x)\frac{\partial v_i}{\partial x_j}(x)\mathrm{d}x - \int_{\partial\Omega}\sigma_{ij}(x)v_i(x)n_j(x)\mathrm{d}s\right)\\ &= \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}(x)\frac{\partial v_i}{\partial x_j}(x)\mathrm{d}x - \int_{\partial\Omega}\sum_{i=1}^{3}(\boldsymbol{\sigma}(x)n(x))_i\, v_i(x)\mathrm{d}s\\ &= \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}(x)\frac{\partial v_i}{\partial x_j}(x)\mathrm{d}x - \int_{\partial\Omega}(\boldsymbol{\sigma}(x)n(x))\cdot v(x)\mathrm{d}s.\end{aligned}$$

Next, the Neumann boundary condition is put in and $v(x) = 0\ \ \forall x \in \Gamma_D$ is used:

$$\int_\Omega f(x)\cdot v(x)\mathrm{d}x = \int_\Omega \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}(x)\frac{\partial v_i}{\partial x_j}(x)\mathrm{d}x - \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s.$$

With the help of the Frobenius inner product of matrices $A : B = \sum_{i,j=1}^{d} A_{ij}B_{ij}$ for $A, B \in \mathbb{R}^{d\times d}$ we can also write this as

$$\int_\Omega f(x)\cdot v(x)\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}(x) : \nabla v(x)\mathrm{d}x - \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s,$$

where $\nabla v(x)$ is the Jacobian matrix of v in x.
Now we can use the calculation rule for the Frobenius inner product

$$A : B = A : (\operatorname{sym} B) \quad \text{for } A, B \in \mathbb{R}^{d\times d}, A \text{ symmetric}$$

for $A = \boldsymbol{\sigma}(x)$ and $B = \nabla v(x)$ and use the strain-displacement relation

$$\boldsymbol{\varepsilon}(v(x)) = \operatorname{sym}\nabla v(x)$$

to get the following equation:

$$\int_\Omega f(x)\cdot v(x)\mathrm{d}x = \int_\Omega \boldsymbol{\sigma}(x) : \boldsymbol{\varepsilon}(v(x))\mathrm{d}x - \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s.$$

Then the required function sets are defined:

$$V = (H^1(\Omega))^3, \qquad\qquad V_0 = \{v \in V \mid v(x) = 0\ \ \forall x \in \Gamma_D\}.$$

Then we get the variational formulation: We look for $\boldsymbol{\sigma}(x) \in V^3$ such that for every $v \in V_0$ the following holds:

$$\int_\Omega \boldsymbol{\sigma}(x) : \boldsymbol{\varepsilon}(v(x))\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x + \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s.$$

2.2 Material law

As the next step we want to include the material law. For the elasticity problem in $\mathbb{R}^3$ we use the Saint-Venant-Kirchhoff material law:

$$\boldsymbol{\sigma}(x) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(u(x))$$

where the parameters λ and μ are the so-called Lamé constants. We get:

$$\begin{aligned}\boldsymbol{\sigma}(x) : \boldsymbol{\varepsilon}(v(x)) &= (\lambda\,(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))\,\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(u(x))) : \boldsymbol{\varepsilon}(v(x))\\ &= \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))(\mathbf{I} : \boldsymbol{\varepsilon}(v(x))) + 2\mu(\boldsymbol{\varepsilon}(u(x)) : \boldsymbol{\varepsilon}(v(x)))\\ &= \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))(\operatorname{tr}\boldsymbol{\varepsilon}(v(x))) + 2\mu(\boldsymbol{\varepsilon}(u(x)) : \boldsymbol{\varepsilon}(v(x))).\end{aligned}$$

As the next step we use the strain-displacement relation to calculate the trace

$$\operatorname{tr}\boldsymbol{\varepsilon}(u(x)) = \tfrac{1}{2}\left(\operatorname{tr}\nabla u(x) + \operatorname{tr}\nabla u(x)^{\mathsf{T}}\right) = \operatorname{tr}\nabla u(x) = \operatorname{div} u(x)$$

so we get:

$$\boldsymbol{\sigma}(x) : \boldsymbol{\varepsilon}(v(x)) = \lambda \operatorname{div} u(x) \operatorname{div} v(x) + 2\mu(\boldsymbol{\varepsilon}(u(x)) : \boldsymbol{\varepsilon}(v(x))).$$

From this and the Dirichlet boundary condition of the boundary value problem at the beginning we can get the following variational problem: We want to find $u \in V_0$ such that

$$a(u, v) = \langle f, v\rangle \quad \forall v \in V_0,$$

with

$$\begin{aligned}a(w, v) &= \int_\Omega \lambda \operatorname{div} w(x) \operatorname{div} v(x) + 2\mu(\boldsymbol{\varepsilon}(w(x)) : \boldsymbol{\varepsilon}(v(x)))\mathrm{d}x,\\ \langle f, v\rangle &= \int_\Omega f(x)\cdot v(x)\mathrm{d}x + \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s\end{aligned}$$

for all $w, v \in V$ and

$$V = (H^1(\Omega))^3, \qquad\qquad V_0 = \{v \in V \mid v(x) = 0 \;\; \forall x \in \Gamma_D\}.$$

Now we want to study the bilinear form $a(w, v)$: We can see that the bilinear form is symmetric, as $a : V \times V \to \mathbb{R}$ and $a(w, v) = a(v, w)$. For many materials we may also assume, that $\lambda > 0$ and $\mu > 0$, so in this case the bilinear form is also positive semi-definite:

$$\begin{aligned} a(v,v) &= \int_\Omega \boldsymbol{\sigma}(v(x)) : \boldsymbol{\varepsilon}(v(x))\mathrm{d}x \\ &= \int_\Omega \lambda \operatorname{div} v(x) \operatorname{div} v(x) + 2\mu(\boldsymbol{\varepsilon}(v(x)) : \boldsymbol{\varepsilon}(v(x)))\mathrm{d}x \\ &= \int_\Omega \lambda \left(\operatorname{div} v(x)\right)^2 + 2\mu \sum_{i,j=1}^{3} \boldsymbol{\varepsilon}(v(x))_{ij}^2 \,\mathrm{d}x \geq 0, \end{aligned}$$

with $\boldsymbol{\sigma}(v(x)) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(v(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(v(x))$.

As we know now that the bilinear form is symmetric and not negative, we can also write our variational problem as minimization problem of the energy functional J: We look for $u \in V_0$ such that

$$J(u) = \min_{v \in V_0} J(v), \quad J(v) = \tfrac{1}{2}a(v,v) - \langle f, v\rangle$$

because

$$\begin{aligned} & J(u) = \min_{v\in V_0} J(v) \\ \Leftrightarrow\ & J(u) \leq J(u+tw) \quad \forall w \in V_0, t \in [0,1] \\ \Leftrightarrow\ & J(u) \leq J(u) + t[a(u,w) - \langle f, w\rangle] + \tfrac{t^2}{2}a(w,w) \quad \forall w \in V_0, t \in [0,1] \\ \Leftrightarrow\ & 0 \leq t[a(u,w) - \langle f, w\rangle] + \tfrac{t^2}{2}a(w,w) \quad \forall w \in V_0, t \in [0,1] \\ \Leftrightarrow\ & 0 \leq a(u,w) - \langle f, w\rangle + \tfrac{t}{2}a(w,w) \quad \forall w \in V_0, t \in [0,1] \\ \Leftrightarrow\ & 0 \leq a(u,w) - \langle f, w\rangle \quad \forall w \in V_0. \end{aligned}$$

If we now chose $w = v$ and $w = -v$ we get $a(u,v) = \langle f, v\rangle$.

So in our case the minimization problem looks as follows: We want to find $u \in V_0$ such that

$$J(u) = \min_{v\in V_0} J(v)$$

with

$$J(v) = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}(v(x)) : \boldsymbol{\varepsilon}(v(x))\,\mathrm{d}x - \int_\Omega f(x)\cdot v(x)\mathrm{d}x - \int_{\Gamma_N} g_N(x)\cdot v(x)\mathrm{d}s$$

and

$$\boldsymbol{\sigma}(v(x)) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(v(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(v(x)).$$

2.3 3D beam model

As the next step the geometry is set: The beam is a cuboid, parallel to the axes with length L, width B and height t. We set $B = 1$ and we assume that $L >> t$. So the beam can be described by the following set:

$$\Omega = (0, L) \times (0, 1) \times \left(-\tfrac{t}{2}, \tfrac{t}{2}\right).$$

We assume that the beam is fixed on the two faces $x_1 = 0, x_1 = L$. On upper and lower face $x_3 = -\frac{t}{2}, x_3 = \frac{t}{2}$ we assume that we have a Neumann boundary condition given, and also on the other faces we assume a Neumann boundary condition.

So we can now split the boundary $\partial\Omega = \Gamma$ into three disjoint sets Γ_D, Γ_{N1}, Γ_{N2} such that $\Gamma = \Gamma_D \cup \Gamma_{N1} \cup \Gamma_{N2}$:

$$\begin{aligned}
\Gamma_D &= \left(\{0\} \times [0, 1] \times [-\tfrac{t}{2}, \tfrac{t}{2}]\right) \cup \left(\{L\} \times [0, 1] \times [-\tfrac{t}{2}, \tfrac{t}{2}]\right), \\
\Gamma_{N1} &= \left((0, L) \times (0, 1) \times \{-\tfrac{t}{2}\}\right) \cup \left((0, L) \times (0, 1) \times \{\tfrac{t}{2}\}\right), \\
\Gamma_{N2} &= \left((0, L) \times \{0\} \times [-\tfrac{t}{2}, \tfrac{t}{2}]\right) \cup \left((0, L) \times \{1\} \times [-\tfrac{t}{2}, \tfrac{t}{2}]\right).
\end{aligned}$$

Our assumptions now lead to the following boundary conditions: As the beam is fixed on Γ_D we get $u = 0$ on Γ_D. For Γ_{N1} we assume that we have a Neumann boundary condition g_{N1} given and for Γ_{N2} we assume $g_{N2} = 0$.

For the data f, g_{N1} we assume that they only have effect in the direction of x_3, so $f(x) = (0, 0, f_3(x))^\mathsf{T}$ and $g_{N1}(x) = (0, 0, p(x))^\mathsf{T}$, as we only look at a pure bending of the beam.

This leads to the variational problem: We look for $u \in V_0$ such that for every $v \in V_0$ the following holds:

$$\int_\Omega \boldsymbol{\sigma}(u(x)) : \boldsymbol{\varepsilon}(v(x))\mathrm{d}x = \int_\Omega f_3(x)v_3(x)\mathrm{d}x + \int_{\Gamma_{N1}} p(x)v_3(x)\mathrm{d}s$$

with

$$\boldsymbol{\sigma}(u(x)) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(u(x))$$

and

$$V = (H^1(\Omega))^3, \qquad V_0 = \{v \in V \mid v(x) = 0 \;\; \forall x \in \Gamma_D\}.$$

Now this problem can be formulated also as minimization problem: We want to find $u \in V_0$ such that

$$J(u) = \min_{v \in V_0} J(v)$$

with

$$J(v) = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}(v(x)) : \boldsymbol{\varepsilon}(v(x))\,\mathrm{d}x - \int_{\Omega} f_3(x)v_3(x)\mathrm{d}x - \int_{\Gamma_{N1}} p(x)v_3(x)\mathrm{d}s$$

and

$$\boldsymbol{\sigma}(v(x)) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(v(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(v(x)).$$

Chapter 3

Dimension reduction

This chapter is based on [3] and [2].

3.1 Kinematical assumptions

For the undeformed beam, described by Ω, the midsurface S is given by

$$S = (0, L) \times (0, 1) \times \{0\}.$$

We make the following kinematical assumptions:

1. The Reissner-Mindlin kinematical assumption: Straight lines which are orthogonal to the midsurface S of the undeformed beam remain straight lines after the deformation.
2. Pure bending: The in-plane displacements $u_1(x)$ and $u_2(x)$ at points of the midsurface S vanish, so for $x \in S$ we get $u_1(x) = 0$ and $u_2(x) = 0$.
3. Beam model: The displacement in the direction of x_2, $u_2(x)$ vanishes and $u_1(x)$ and $u_3(x)$ do not depend on x_2, this means $u_2(x) = 0$, $u_1(x) = u_1(x_1, x_3)$ and $u_3(x) = u_3(x_1, x_3)$.
4. The vertical displacement $u_3(x)$ does not depend on x_3, therefore we get $u_3(x) = u_3(x_1, x_2)$.

First, we take a closer look on assumption (1): For fixed $\underline{x} = (x_1, x_2) \in (0, L) \times (0, 1)$ the straight line orthogonal to the midsurface looks as follows:

$$l_{\underline{x}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : x_3 \in (-\tfrac{t}{2}, \tfrac{t}{2}) \right\}.$$

We assume that after the deformation the material points of $l_{\underline{x}}$ form the set $\varphi(l_{\underline{x}}) = \{\varphi(x) : x \in l_{\underline{x}}\}$ with $\varphi(x) = x + u(x)$ and that it is a straight line:

$$\varphi(l_{\underline{x}}) = \left\{ a_{\underline{x}} + x_3\, b_{\underline{x}} : x_3 \in (-\tfrac{t}{2}, \tfrac{t}{2}) \right\}.$$

So for any $x \in \Omega$, $\varphi(x)$ can be represented as $\varphi(x) = x + u(x)$, but also as $\varphi(x) = a_{\underline{x}} + x_3 b_{\underline{x}}$, for some $a_{\underline{x}}$ and $b_{\underline{x}}$. If we look at this component-wise, we get

$$\begin{aligned}
a_1(x_1,x_2) + x_3 b_1(x_1,x_2) &= x_1 + u_1(x_1,x_2,x_3),\\
a_2(x_1,x_2) + x_3 b_2(x_1,x_2) &= x_2 + u_2(x_1,x_2,x_3),\\
a_3(x_1,x_2) + x_3 b_3(x_1,x_2) &= x_3 + u_3(x_1,x_2,x_3).
\end{aligned}$$

For $x = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \in S$ we then get:

$$\begin{aligned}
a_1(x_1,x_2) &= x_1 + u_1(x_1,x_2,0),\\
a_2(x_1,x_2) &= x_2 + u_2(x_1,x_2,0),\\
a_3(x_1,x_2) &= u_3(x_1,x_2,0).
\end{aligned}$$

So for $b_{\underline{x}} = \frac{1}{x_3}\left(x + u(x) - a_{\underline{x}}\right)$ we then get:

$$\begin{aligned}
b_1(x_1,x_2) &= \tfrac{1}{x_3}\left(u_1(x_1,x_2,x_3) - u_1(x_1,x_2,0)\right),\\
b_2(x_1,x_2) &= \tfrac{1}{x_3}\left(u_2(x_1,x_2,x_3) - u_2(x_1,x_2,0)\right),\\
b_3(x_1,x_2) &= \tfrac{1}{x_3}\left(x_3 + u_3(x_1,x_2,x_3) - u_3(x_1,x_2,0)\right).
\end{aligned}$$

As the next step, assumption (2) is used: As $(x_1,x_2,0) \in S$ we get, that $u_1(x_1,x_2,0) = 0$ and $u_2(x_1,x_2,0) = 0$. Inserted in the formulas for $a_{\underline{x}}$ and $b_{\underline{x}}$ we get:

$$\begin{aligned}
a_{\underline{x}} &= \begin{pmatrix} x_1 \\ x_2 \\ u_3(x_1,x_2,0) \end{pmatrix},\\
b_{\underline{x}} &= \frac{1}{x_3}\begin{pmatrix} u_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) \\ x_3 + u_3(x_1,x_2,x_3) - u_3(x_1,x_2,0) \end{pmatrix}.
\end{aligned}$$

Next, we use assumption (3) : $u_2(x) = 0$, $u_1(x) = u_1(x_1,x_3)$ and $u_3(x) = u_3(x_1,x_3)$. This gives us

$$\begin{aligned}
a_{\underline{x}} &= \begin{pmatrix} x_1 \\ x_2 \\ u_3(x_1,0) \end{pmatrix},\\
b_{\underline{x}} &= \frac{1}{x_3}\begin{pmatrix} u_1(x_1,x_3) \\ 0 \\ x_3 + u_3(x_1,x_3) - u_3(x_1,0) \end{pmatrix}.
\end{aligned}$$

Now assumption (4), which says that $u_3(x) = u_3(x_1, x_3)$ does not depend on x_3, gives us $u_3(x) = u_3(x_1)$, and we get:

$$a_{\underline{x}} = \begin{pmatrix} x_1 \\ x_2 \\ u_3(x_1) \end{pmatrix},$$

$$b_{\underline{x}} = \frac{1}{x_3} \begin{pmatrix} u_1(x_1, x_3) \\ 0 \\ x_3 + u_3(x_1) - u_3(x_1) \end{pmatrix} = \begin{pmatrix} \frac{1}{x_3} u_1(x_1, x_3) \\ 0 \\ 1 \end{pmatrix}.$$

So we get:

$$a_{\underline{x}} = \begin{pmatrix} a_1(x_1, x_2) \\ a_2(x_1, x_2) \\ a_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} a_1(x_1) \\ a_2(x_2) \\ a_3(x_1) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ u_3(x_1) \end{pmatrix},$$

$$b_{\underline{x}} = \begin{pmatrix} b_1(x_1, x_2) \\ b_2(x_1, x_2) \\ b_3(x_1, x_2) \end{pmatrix} = \begin{pmatrix} b_1(x_1) \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{x_3} u_1(x_1, x_3) \\ 0 \\ 1 \end{pmatrix}.$$

Then we can now express $u(x)$ in terms of $a_{\underline{x}}$ and $b_{\underline{x}}$ as:

$$u(x) = \varphi(x) - x = a_{\underline{x}} + x_3 b_{\underline{x}} - x.$$

So we get:

$$\begin{aligned} u_1(x_1, x_2, x_3) &= u_1(x_1, x_3) = x_3 b_1(x_1), \\ u_2(x_1, x_2, x_3) &= 0, \\ u_3(x_1, x_2, x_3) &= u_3(x_1) = a_3(x_1). \end{aligned}$$

Now we look at the geometry of the problem, pictured in figure 3.1:

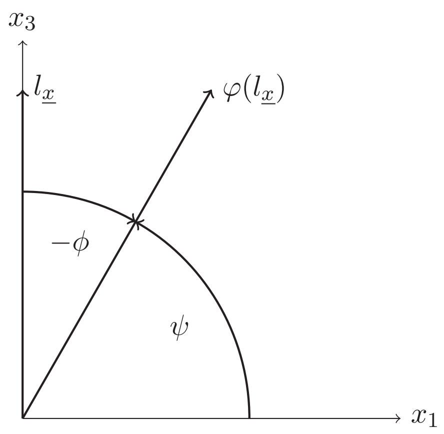

Figure 3.1: Geometry of the problem

For the angle $-\phi$ between $l_{\underline{x}}$ and $\varphi(l_{\underline{x}})$ and the angle ψ from the x_1 axis to $\varphi(l_{\underline{x}})$ we get:

$$\begin{aligned}
-\phi + \psi &= \frac{\pi}{2},\\
\tan(\phi) &= -\tan(-\phi) = -\frac{b_1(x_1)}{b_3} = -b_1(x_1),\\
\tan(\psi) &= \tan\left(\phi + \frac{\pi}{2}\right) = -\frac{1}{\tan(\phi)} = \frac{1}{b_1(x_1)}.
\end{aligned}$$

Now we rename $a_3(x_1)$ to $w(x_1)$ and $\tan(\phi)$ to θ, so $b_1(x_1) = -\theta(x_1)$. For $u(x)$ we then get:

$$\begin{pmatrix} u_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) \\ u_3(x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} -x_3\theta(x_1) \\ 0 \\ w(x_1) \end{pmatrix}.$$

Next we calculate $\nabla u(x)$ as:

$$\nabla u(x) = \begin{pmatrix} \frac{\partial(-x_3\theta(x_1))}{\partial x_1} & \frac{\partial(-x_3\theta(x_1))}{\partial x_2} & \frac{\partial(-x_3\theta(x_1))}{\partial x_3} \\ 0 & 0 & 0 \\ \frac{\partial w(x_1)}{\partial x_1} & \frac{\partial w(x_1)}{\partial x_2} & \frac{\partial w(x_1)}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -x_3\theta'(x_1) & 0 & -\theta(x_1) \\ 0 & 0 & 0 \\ w'(x_1) & 0 & 0 \end{pmatrix}.$$

Inserting this result into $\boldsymbol{\varepsilon}(u(x))$ we get:

$$\begin{aligned}
\boldsymbol{\varepsilon}(u(x)) &= \tfrac{1}{2}(\nabla u(x) + \nabla u(x)^{\mathsf{T}})\\
&= \frac{1}{2}\left(\begin{pmatrix} -x_3\theta'(x_1) & 0 & -\theta(x_1) \\ 0 & 0 & 0 \\ w'(x_1) & 0 & 0 \end{pmatrix} + \begin{pmatrix} -x_3\theta'(x_1) & 0 & w'(x_1) \\ 0 & 0 & 0 \\ -\theta(x_1) & 0 & 0 \end{pmatrix}\right)\\
&= \begin{pmatrix} -x_3\theta'(x_1) & 0 & \frac{1}{2}\left(w'(x_1) - \theta(x_1)\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(w'(x_1) - \theta(x_1)\right) & 0 & 0 \end{pmatrix}.
\end{aligned}$$

For $\boldsymbol{\sigma}(x) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(u(x))$ we then get:

$$\begin{aligned}
\boldsymbol{\sigma}(x) &= \lambda\left(-x_3\theta'(x_1)\right)\mathbf{I} + 2\mu\begin{pmatrix} -x_3\theta'(x_1) & 0 & \frac{1}{2}\left(w'(x_1) - \theta(x_1)\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(w'(x_1) - \theta(x_1)\right) & 0 & 0 \end{pmatrix}\\
&= \begin{pmatrix} -(\lambda + 2\mu)x_3\theta'(x_1) & 0 & \mu\left(w'(x_1) - \theta(x_1)\right) \\ 0 & -\lambda x_3\theta'(x_1) & 0 \\ \mu\left(w'(x_1) - \theta(x_1)\right) & 0 & -\lambda x_3\theta'(x_1) \end{pmatrix}.
\end{aligned}$$

3.2 Modification of the Material law

We can calculate $\boldsymbol{\sigma}$ from $\boldsymbol{\varepsilon}$ by the material law $\boldsymbol{\sigma}(x) = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(u(x)))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(u(x))$. By transforming this formula, we can also compute $\boldsymbol{\varepsilon}$ from $\boldsymbol{\sigma}$ by

$$\boldsymbol{\varepsilon}(u(x)) = \tfrac{1}{2\mu}\left(\boldsymbol{\sigma}(x) - \tfrac{\lambda}{3\lambda+2\mu}(\operatorname{tr}\boldsymbol{\sigma}(x))\mathbf{I}\right).$$

Starting with the calculation of $\boldsymbol{\varepsilon}$, we can split the material law in two parts: For $i \neq j$ we get

$$\varepsilon_{ij} = \tfrac{1}{2\mu}\sigma_{ij}$$

and for $i = j$ we get:

$$\varepsilon_{ii} = \tfrac{1}{2\mu}\left(\sigma_{ii} - \tfrac{\lambda}{3\lambda+2\mu}\operatorname{tr}\boldsymbol{\sigma}\right).$$

We can see above, that σ_{22} and σ_{33} are not zero. For physical reasons we set $\sigma_{22} = 0$ and $\sigma_{33} = 0$. Then we get:

$$\operatorname{tr}\boldsymbol{\sigma} = \sigma_{11} = -(\lambda + 2\mu)x_3\theta'(x_1),$$

so for $i = j$ we calculate:

$$\begin{aligned}
\varepsilon_{11} &= \tfrac{1}{2\mu}\left(\sigma_{11} - \tfrac{\lambda}{3\lambda+2\mu}\sigma_{11}\right) = \tfrac{1}{2\mu}\left(\tfrac{2\lambda+2\mu}{3\lambda+2\mu}\sigma_{11}\right) = \tfrac{\lambda+\mu}{\mu(3\lambda+2\mu)}\sigma_{11},\\
\varepsilon_{22} &= \tfrac{1}{2\mu}\left(\sigma_{22} - \tfrac{\lambda}{3\lambda+2\mu}\sigma_{11}\right) = -\tfrac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{11},\\
\varepsilon_{33} &= \tfrac{1}{2\mu}\left(\sigma_{33} - \tfrac{\lambda}{3\lambda+2\mu}\sigma_{11}\right) = -\tfrac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{11}.
\end{aligned}$$

With the help of the modulus of elasticity $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ and the shear modulus $G = \mu$ we can write the expressions above also as:

$$\begin{aligned}
\varepsilon_{11} &= \tfrac{1}{E}\sigma_{11},\\
\varepsilon_{22} &= \tfrac{2G-E}{2EG}\sigma_{11} = \tfrac{2G-E}{2EG}E\varepsilon_{11} = \tfrac{2G-E}{2G}\varepsilon_{11},\\
\varepsilon_{33} &= \tfrac{2G-E}{2EG}\sigma_{11} = \tfrac{2G-E}{2EG}E\varepsilon_{11} = \tfrac{2G-E}{2G}\varepsilon_{11},\\
\varepsilon_{ij} &= \tfrac{1}{2G}\sigma_{ij} \quad \text{for } i \neq j.
\end{aligned}$$

Then we can also calculate $\boldsymbol{\sigma}$ in relation to $\boldsymbol{\varepsilon}$ as:

$$\begin{aligned}
\sigma_{11} &= E\varepsilon_{11},\\
\sigma_{22} &= 0,\\
\sigma_{33} &= 0,\\
\sigma_{ij} &= 2G\varepsilon_{ij} \quad \text{for } i \neq j.
\end{aligned}$$

If we assume that ε_{11} does not change, we get the following matrices for $\boldsymbol{\varepsilon}(u)$ and $\boldsymbol{\sigma}(u)$:

$$\begin{aligned}
\boldsymbol{\varepsilon}(u(x)) &= \begin{pmatrix} \varepsilon_{11} & 0 & \varepsilon_{13} \\ 0 & \varepsilon_{22} & 0 \\ \varepsilon_{31} & 0 & \varepsilon_{33} \end{pmatrix} \\
&= \begin{pmatrix} -x_3\theta'(x_1) & 0 & \frac{1}{2}\left(w'(x_1)-\theta(x_1)\right) \\ 0 & -\frac{2G-E}{2G}x_3\theta'(x_1) & 0 \\ \frac{1}{2}\left(w'(x_1)-\theta(x_1)\right) & 0 & -\frac{2G-E}{2G}x_3\theta'(x_1) \end{pmatrix}, \\
\boldsymbol{\sigma}(u(x)) &= \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & 0 & 0 \\ \sigma_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} E\varepsilon_{11} & 0 & 2G\varepsilon_{13} \\ 0 & 0 & 0 \\ 2G\varepsilon_{31} & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -Ex_3\theta'(x_1) & 0 & G\left(w'(x_1)-\theta(x_1)\right) \\ 0 & 0 & 0 \\ G\left(w'(x_1)-\theta(x_1)\right) & 0 & 0 \end{pmatrix}.
\end{aligned}$$

With this new material law for $\boldsymbol{\sigma}(u)$ we can now calculate $\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(u)\,\mathrm{d}x$ as:

$$\begin{aligned}
\int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\varepsilon}(u)\,\mathrm{d}x &= \int_\Omega Ex_3^2\left(\theta'(x_1)\right)^2 + G\left(w'(x_1)-\theta(x_1)\right)^2\,\mathrm{d}x \\
&= E\int_\Omega x_3^2\left(\theta'(x_1)\right)^2\,\mathrm{d}x + G\int_\Omega \left(w'(x_1)-\theta(x_1)\right)^2\,\mathrm{d}x \\
&= E\int_0^L\int_0^1\int_{-\frac{t}{2}}^{\frac{t}{2}} x_3^2\left(\theta'(x_1)\right)^2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&+ G\int_0^L\int_0^1\int_{-\frac{t}{2}}^{\frac{t}{2}} \left(w'(x_1)-\theta(x_1)\right)^2\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= E\int_0^L \left(\theta'(x_1)\right)^2\,\mathrm{d}x_1 \int_{-\frac{t}{2}}^{\frac{t}{2}} x_3^2\,\mathrm{d}x_3 + Gt\int_0^L \left(w'(x_1)-\theta(x_1)\right)^2\,\mathrm{d}x_1 \\
&= E\frac{t^3}{12}\int_0^L \left(\theta'(x_1)\right)^2\,\mathrm{d}x_1 + Gt\int_0^L \left(w'(x_1)-\theta(x_1)\right)^2\,\mathrm{d}x_1.
\end{aligned}$$

Analogous to $u(x)$ we also can write $v(x)$ as:

$$v(x) = \begin{pmatrix} x_3\phi(x_1) \\ 0 \\ v(x_1) \end{pmatrix}.$$

Then we can write down the energy functional $J(v)$, depending on $\phi(x_1)$ and $v(x_1)$ as:

$$\begin{aligned}
J(\phi, v) &= \frac{1}{2}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}\,\mathrm{d}x - \int_\Omega f_3(x)v_3(x)\mathrm{d}x - \int_{\Gamma_{N1}} p(x)v_3(x)\mathrm{d}s \\
&= \frac{Et^3}{24}\int_0^L \left(\phi'(x_1)\right)^2\,\mathrm{d}x_1 + \frac{Gt}{2}\int_0^L \left(v'(x_1) - \phi(x_1)\right)^2\,\mathrm{d}x_1 \\
&- \int_\Omega f_3(x)v(x_1)\mathrm{d}x - \int_{\Gamma_{N1}} p(x)v(x_1)\mathrm{d}s.
\end{aligned}$$

We now use the shear correction factor κ for the correction of the part with the shear stress and get our final energy functional $J(\phi, v)$ as:

$$\begin{aligned}
J(\theta, w) &= \frac{Et^3}{24}\int_0^L \left(\phi'(x_1)\right)^2\,\mathrm{d}x_1 + \frac{\kappa Gt}{2}\int_0^L \left(v'(x_1) - \phi(x_1)\right)^2\,\mathrm{d}x_1 \\
&- \int_\Omega f_3(x)v(x_1)\mathrm{d}x - \int_{\Gamma_{N1}} p(x)v(x_1)\mathrm{d}s \\
&= \frac{1}{2}a((\phi, v), (\phi, v)) - \langle f, (\phi, v)\rangle.
\end{aligned}$$

From this we can now get the bilinear form $a((\theta, w), (\phi, v))$ and the linear form $\langle f, (\phi, v)\rangle$ as:

$$\begin{aligned}
a((\theta, w), (\phi, v)) &= \frac{Et^3}{12}\int_0^L \theta'(x_1)\phi'(x_1)\,\mathrm{d}x_1 \\
&\quad + \kappa Gt\int_0^L \left(w'(x_1) - \theta(x_1)\right)\left(v'(x_1) - \phi(x_1)\right)\,\mathrm{d}x_1, \\
\langle f, (\phi, v)\rangle &= \int_0^L \bar{f}(x_1)v(x_1)\,\mathrm{d}x_1,
\end{aligned}$$

with

$$\bar{f}(x_1) = \int_0^1\int_{-\frac{t}{2}}^{\frac{t}{2}} f_3(x)\,\mathrm{d}x_3\,\mathrm{d}x_2 + \int_0^1 p(x_1, x_2, -\tfrac{t}{2}) + p(x_1, x_2, \tfrac{t}{2})\,\mathrm{d}x_2.$$

We now look at our boundary conditions: The Dirichlet boundary condition for $u(x)$ leads to the following boundary conditions for $\theta(x_1)$ and $w(x_1)$:

$$\begin{aligned}
-x_3\theta(x_1) &= 0 && \forall x \in \Gamma_D, \\
w(x_1) &= 0 && \forall x \in \Gamma_D.
\end{aligned}$$

As we know that $\Gamma_D = \left(\{0\} \times [0,1] \times [-\frac{t}{2}, \frac{t}{2}]\right) \cup \left(\{L\} \times [0,1] \times [-\frac{t}{2}, \frac{t}{2}]\right)$ we can now fix $x_1 = 0$ or $x_1 = L$ and get:

$$\begin{aligned}
\theta(0) &= 0, & w(0) &= 0, \\
\theta(L) &= 0, & w(L) &= 0.
\end{aligned}$$

So we get: $\theta, w \in H_0^1(0,L) = \{v \in H^1(0,L) \mid v(0) = v(L) = 0\}$. So for $x = (\theta, w)$ we get: $x \in H_0^1(0,L) \times H_0^1(0,L) = \mathcal{V}$. So for $y = (\phi, v) \in \mathcal{V}$ we then get the following variational problem: We want to find $x \in \mathcal{V}$ such that

$$a(x,y) = \langle f, y \rangle \quad \forall y \in \mathcal{V},$$

with

$$\begin{aligned} a(x,y) = a((\theta,w),(\phi,v)) &= \frac{Et^3}{12} \int_0^L \theta'(x_1)\phi'(x_1)\,\mathrm{d}x_1 \\ &+ \kappa G t \int_0^L \left(w'(x_1) - \theta(x_1)\right)\left(v'(x_1) - \phi(x_1)\right)\,\mathrm{d}x_1, \\ \langle f, y \rangle = \langle f, (\phi, v) \rangle &= \int_0^L \bar{f}(x_1) v(x_1)\,\mathrm{d}x_1 \end{aligned}$$

for all x, y in $\mathcal{V}$ and

$$\mathcal{V} = H_0^1(0,L) \times H_0^1(0,L).$$

We can also split up the equation. For $v = 0$ we get:

$$\frac{Et^3}{12} \int_0^L \theta'(x_1)\phi'(x_1)\,\mathrm{d}x_1 - \kappa G t \int_0^L \left(w'(x_1) - \theta(x_1)\right)\phi(x_1)\,\mathrm{d}x_1 = 0$$

and for $\phi = 0$ we get:

$$\kappa G t \int_0^L \left(w'(x_1) - \theta(x_1)\right) v'(x_1)\,\mathrm{d}x_1 = \int_0^L \bar{f}(x_1) v(x_1)\,\mathrm{d}x_1.$$

3.3 Differential equation

We now want to derive the differential equations to this variational form. With partial integration we get the following terms:

$$\begin{aligned} \int_0^L \theta'(x_1)\phi'(x_1)\,\mathrm{d}x_1 &= -\int_0^L \theta''(x_1)\phi(x_1)\,\mathrm{d}x_1 + \left[\theta'(L)\phi(L) - \theta'(0)\phi(0)\right] \\ &= -\int_0^L \theta''(x_1)\phi(x_1)\,\mathrm{d}x_1, \\ \int_0^L \left(w'(x_1) - \theta(x_1)\right) v'(x_1)\,\mathrm{d}x_1 &= -\int_0^L \left(w''(x_1) - \theta'(x_1)\right) v(x_1)\,\mathrm{d}x_1 \\ &+ \left[\left(w'(L) - \theta(L)\right) v(L) - \left(w'(0) - \theta(0)\right) v(0)\right] \\ &= -\int_0^L \left(w''(x_1) - \theta'(x_1)\right) v(x_1)\,\mathrm{d}x_1. \end{aligned}$$

So if we use these results of the partial integration in our equation $a(x, y) = \langle f, y \rangle$ we get:

$$-\frac{Et^3}{12} \int_0^L \theta''(x_1)\phi(x_1)\,\mathrm{d}x_1 - \kappa Gt \int_0^L \left(w''(x_1) - \theta'(x_1)\right) v(x_1)\,\mathrm{d}x_1$$
$$-\kappa Gt \int_0^L \left(w'(x_1) - \theta(x_1)\right) \phi(x_1)\,\mathrm{d}x_1 = \int_0^L \bar{f}(x_1)v(x_1)\,\mathrm{d}x_1.$$

As this equation holds for all $\phi, v \in V_0$ it especially holds for $v = 0$, and in this case we get:

$$-\frac{Et^3}{12} \int_0^L \theta''(x_1)\phi(x_1)\,\mathrm{d}x_1 - \kappa Gt \int_0^L \left(w'(x_1) - \theta(x_1)\right) \phi(x_1)\,\mathrm{d}x_1 = 0 \quad \forall \phi \in V_0.$$

This leads to:

$$\tfrac{Et^3}{12}\theta''(x_1) + \kappa Gt \left(w'(x_1) - \theta(x_1)\right) = 0.$$

We could also write this as a differential equation for w:

$$w'(x_1) = \theta(x_1) - \tfrac{Et^3}{12\kappa Gt}\theta''(x_1).$$

But as the equation $a(x, y) = \langle f, y \rangle$ holds for all $\phi, v \in V_0$, it also holds for $\phi = 0$, and we get:

$$-\kappa Gt \int_0^L \left(w''(x_1) - \theta'(x_1)\right) v(x_1)\,\mathrm{d}x_1 = \int_0^L \bar{f}(x_1)v(x_1)\,\mathrm{d}x_1,$$

with $\bar{f}(x_1) = \int_0^1 \int_{-\frac{t}{2}}^{\frac{t}{2}} f_3(x)\,\mathrm{d}x_3\,\mathrm{d}x_2 + \int_0^1 p(x_1, x_2, -\frac{t}{2}) + p(x_1, x_2, \frac{t}{2})\,\mathrm{d}x_2$.
This leads to:

$$-\kappa Gt \left(w''(x_1) - \theta'(x_1)\right) = \bar{f}(x_1).$$

We could now also use the differential equation for w (differentiated on both sides) from above and get:

$$-\kappa Gt \left(-\frac{Et^3}{12\kappa Gt}\theta'''(x_1)\right) = \bar{f}(x_1)$$

and so in this case we get the equation for θ as:

$$\frac{Et^3}{12}\theta'''(x_1) = \bar{f}(x_1).$$

Chapter 4

Discretization

The next step is the discretization of the variational form of the last chapter, see also [2] and [1].

We are starting the discretization from the variational problem: We want to find $(\theta, w) \in \mathcal{V}$ such that for all $(\phi, v) \in \mathcal{V}$:

$$\begin{aligned} \frac{Et^3}{12} \int_0^L \theta'(x_1)\phi'(x_1)\,\mathrm{d}x_1 - \kappa G t \int_0^L \left(w'(x_1) - \theta(x_1)\right)\phi(x_1)\,\mathrm{d}x_1 = 0, \\ \kappa G t \int_0^L \left(w'(x_1) - \theta(x_1)\right) v'(x_1)\,\mathrm{d}x_1 = \int_0^L \bar{f}(x_1) v(x_1)\,\mathrm{d}x_1, \end{aligned}$$

with

$$\mathcal{V} = H_0^1(0, L) \times H_0^1(0, L).$$

We now rename x_1 to x, x_2 to y and x_3 to z, as we need x_1, x_2, x_3 for the nodes. So we get the variational equations as:

$$\begin{aligned} \frac{Et^3}{12} \int_0^L \theta'(x)\phi'(x)\,\mathrm{d}x - \kappa G t \int_0^L \left(w'(x) - \theta(x)\right)\phi(x)\,\mathrm{d}x = 0 \qquad \forall \phi \in V_0, \\ \kappa G t \int_0^L \left(w'(x) - \theta(x)\right) v'(x)\,\mathrm{d}x = \int_0^L \bar{f}(x) v(x)\,\mathrm{d}x \qquad \forall v \in V_0. \end{aligned}$$

4.1 Discrete Problem

For the discretization we use the Courant-Element. Therefore we choose nodes x_i, $i = 0, 1, \ldots, n$, with

$$0 = x_0 < x_1 < \cdots < x_n = L,$$

and $x_j = jh$, $h = \frac{L}{n}$, to get equally distributed intervals. Then we get our elements $T_k = (x_{k-1}, x_k)$, $k = 1, \dots, n$, and our mesh $\mathcal{T}_h = \{T_1, \dots T_n\}$. Then we define our space V_h as the space of all continuous and piecewise affine linear functions on $[0, L]$:

$$V_h = \{v \in C[0, L] \mid v|_T \in P_1 \text{ for all } T \in \mathcal{T}_h\},$$

where P_1 is the set of all polynomials of degree ≤ 1. Then we set:

$$V_{0h} = V_0 \cap V_h = \{v_h \in V_h \mid v_h(0) = v_h(L) = 0\}.$$

We now need to construct a basis for V_h. Therefore we use the nodal basis: To every node x_i, $i = 0, \dots n$ in $\mathcal{T}_h$ we assign a function $\varphi_i \in V_h$, which is uniquely defined by

$$\varphi_i(x_j) = \delta_{ij} \quad \text{for all } i, j = 0, \dots n.$$

These basis functions are called the hat functions. They have only local support and they are a basis of V_h, as they are linearly independent and we can write every function $v_h \in V_h$ as:

$$v_h(x) = \sum_{i=0}^{n} v_i \varphi_i(x) \quad \text{with } v_i = v_h(x_i).$$

Then we can write V_{0h} as:

$$V_{0h} = \{v_h \in V_h \mid v_h = \sum_{i=1}^{n-1} v_i \varphi_i(x)\}.$$

From this we get a discrete problem: We look for $\theta_h, w_h \in V_{0h}$ such that for all $\phi_h, v_h \in V_{0h}$ the following holds:

$$\begin{aligned} a_1((\theta_h, w_h), (\phi_h, v_h)) &= 0, \\ a_2((\theta_h, w_h), (\phi_h, v_h)) &= \langle f, v_h \rangle, \end{aligned}$$

with

$$\begin{aligned} a_1((\theta_h, w_h), (\phi_h, v_h)) &= \frac{Et^3}{12} \int_0^L \theta_h'(x)\phi_h'(x)\,\mathrm{d}x - \kappa G t \int_0^L w_h'(x)\phi_h(x)\,\mathrm{d}x \\ &\quad + \kappa G t \int_0^L \theta_h(x)\phi_h(x)\,\mathrm{d}x, \\ a_2((\theta_h, w_h), (\phi_h, v_h)) &= \kappa G t \int_0^L w_h'(x) v_h'(x)\,\mathrm{d}x - \kappa G t \int_0^L \theta_h(x) v_h'(x)\,\mathrm{d}x, \\ \langle f, v_h \rangle &= \int_0^L \bar{f}(x) v_h(x)\,\mathrm{d}x. \end{aligned}$$

We first look at the equation $a_1(\theta_h, w_h, \phi_h, v_h) = 0$. By using $\varphi_i, i = 0, \dots, n$ as test functions for ϕ_h, one after another, we get:

$$\frac{Et^3}{12}\int_0^L \left(\sum_{j=1}^{n-1}\theta_j\varphi_j(x)\right)' \varphi_i'(x)\,\mathrm{d}x - \kappa Gt\int_0^L \left(\sum_{j=1}^{n-1} w_j\varphi_j(x)\right)' \varphi_i(x)\,\mathrm{d}x$$
$$+\kappa Gt\int_0^L \left(\sum_{j=1}^{n-1}\theta_j\varphi_j(x)\right)\varphi_i(x)\,\mathrm{d}x = 0,$$

which we can simplify to:

$$\sum_{j=1}^{n-1}\frac{Et^3}{12}\theta_j\int_0^L \varphi_j'(x)\varphi_i'(x)\,\mathrm{d}x - \sum_{j=1}^{n-1}\kappa Gt w_j\int_0^L \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x$$
$$+\sum_{j=1}^{n-1}\kappa Gt\theta_j\int_0^L \varphi_j(x)\varphi_i(x)\,\mathrm{d}x = 0.$$

For the second equation we again use $\varphi_i, i = 0, \dots, n$ as test functions, this time for v_h and get:

$$\kappa Gt\int_0^L \left(\sum_{j=1}^{n-1} w_j\varphi_j(x)\right)' \varphi_i'(x)\,\mathrm{d}x - \kappa Gt\int_0^L \left(\sum_{j=1}^{n-1}\theta_j\varphi_j(x)\right)\varphi_i'(x)\,\mathrm{d}x$$
$$= \int_0^L \bar{f}(x)\varphi_i(x)\,\mathrm{d}x.$$

We can write this also as equation system for $\underline{\theta}_h = (\theta_j)_{j=1,\dots,n-1}$ and $\underline{w}_h = (w_j)_{j=1,\dots,n-1}$:

$$\begin{aligned}\tfrac{Et^3}{12}K_h\underline{\theta}_h - \kappa Gt N_h\underline{w}_h + \kappa Gt M_h\underline{\theta}_h &= 0,\\ \kappa Gt K_h\underline{w}_h - \kappa Gt N_h^{\mathsf{T}}\underline{\theta}_h &= \underline{f}_h,\end{aligned}$$

with

$$\begin{aligned}
K_h &= (K_{ij})_{i,j=1,\dots,n-1}, & K_{ij} &= \int_0^L \varphi_j'(x)\varphi_i'(x)dx,\\
M_h &= (M_{ij})_{i,j=1,\dots,n-1}, & M_{ij} &= \int_0^L \varphi_j(x)\varphi_i(x)dx,\\
N_h &= (N_{ij})_{i,j=1,\dots,n-1}, & N_{ij} &= \int_0^L \varphi_j'(x)\varphi_i(x)dx,\\
\underline{f}_h &= (f_i)_{i=1,\dots,n-1}, & f_i &= \int_0^L \bar{f}(x)\varphi_i(x)\,\mathrm{d}x.
\end{aligned}$$

By using the Euclidean inner product $(.\,,.)_{\ell_2}$, we get the following relations:

$$\begin{aligned} a_1((\theta_h, w_h),(\phi_h, v_h)) &= \left(\tfrac{Et^3}{12} K_h \underline{\theta}_h - \kappa G t N_h \underline{w}_h + \kappa G t M_h \underline{\theta}_h, \underline{\phi_h}\right)_{\ell_2}, \\ a_2((\theta_h, w_h),(\phi_h, v_h)) &= (\kappa G t K_h \underline{w}_h - \kappa G t N_h^{\mathsf{T}} \underline{\theta}_h, \underline{v_h})_{\ell_2}, \\ \langle f, v_h\rangle &= (\underline{f}_h, \underline{v}_h)_{\ell_2}, \end{aligned}$$

where we use the following notation: For any function $v_h \in V_{0h}$, $\underline{v}_h$ is the related vector of the coefficients in basis representation:

$$v_h = \sum_{i=1}^{n-1} v_i \varphi_i(x), \qquad\qquad \underline{v}_h = (v_i)_{i=1,\dots,n-1}.$$

4.2 Calculation of the matrices

We now split up the matrix N_h into its element matrices:

$$(N_h \underline{w}_h, \underline{\phi}_h)_{\ell_2} = \sum_{T\in\mathcal{T}_h} \int_T w_h'(x)\phi_h(x)\,\mathrm{d}x = \sum_{k=1}^{n} \sum_{i,j=1}^{n-1} \phi_i w_j \int_{T_k} \varphi_j'(x)\varphi_i(x)\,\mathrm{d}x.$$

For every element T_k, the integral is only non-zero for $i = j = 1$, if $k = 1$, for (i,j) with $i,j \in \{k-1,k\}$ if $2 \le k \le n-1$, and for $i = j = n-1$ if $k = n$. Therefore we can write:

$$(N_h \underline{w}_h, \underline{\phi}_h)_{\ell_2} = N_h^{(1)} w_1 \phi_1 + \sum_{k=2}^{n-1} \left(N_h^{(k)} \begin{pmatrix} w_{k-1} \\ w_k \end{pmatrix}, \begin{pmatrix} \phi_{k-1} \\ \phi_k \end{pmatrix} \right)_{\ell_2} + N_h^{(n)} w_{n-1}\phi_{n-1},$$

with

$$\begin{aligned} N_h^{(1)} &= \int_{T_1} \varphi_1'(x)\varphi_1(x)\,\mathrm{d}x, \\ N_h^{(k)} &= \begin{pmatrix} \int_{T_k} \varphi_{k-1}'(x)\varphi_{k-1}(x)\,\mathrm{d}x & \int_{T_k} \varphi_k'(x)\varphi_{k-1}(x)\,\mathrm{d}x \\ \int_{T_k} \varphi_{k-1}'(x)\varphi_k(x)\,\mathrm{d}x & \int_{T_k} \varphi_k'(x)\varphi_k(x)\,\mathrm{d}x \end{pmatrix}, \\ N_h^{(n)} &= \int_{T_n} \varphi_{n-1}'(x)\varphi_{n-1}(x)\,\mathrm{d}x. \end{aligned}$$

To calculate these integrals, we start by writing down the basis functions φ_i and its derivatives:

$$\varphi_j(x) = \begin{cases} 0 & 0 \leq x \leq x_{j-1} \\ \frac{x-x_{j-1}}{h} & x_{j-1} \leq x \leq x_j \\ \frac{x_{j+1}-x}{h} & x_j \leq x \leq x_{j+1} \\ 0 & x_{j+1} \leq x \leq L \end{cases},$$

$$\varphi_j'(x) = \begin{cases} 0 & 0 \leq x \leq x_{j-1} \\ \frac{1}{h} & x_{j-1} \leq x \leq x_j \\ -\frac{1}{h} & x_j \leq x \leq x_{j+1} \\ 0 & x_{j+1} \leq x \leq L \end{cases}.$$

From this we can now calculate the matrices:

$$\begin{aligned} N_h^{(1)} &= \int_{x_0}^{x_1} \frac{1}{h}\frac{x-x_0}{h}\,\mathrm{d}x = \frac{1}{h^2}\left[\frac{x^2}{2} - x_0 x\right]_{x_0}^{x_1} = \frac{1}{h^2}\left(\frac{x_1^2}{2} - x_0 x_1 - \frac{x_0^2}{2} + x_0^2\right) \\ &= \frac{1}{2h^2}(x_1 - x_0)^2 = \frac{1}{2}, \\ N_h^{(k)} &= \begin{pmatrix} \int_{x_{k-1}}^{x_k} -\frac{1}{h}\frac{x_k-x}{h}\,\mathrm{d}x & \int_{x_{k-1}}^{x_k} \frac{1}{h}\frac{x_k-x}{h}\,\mathrm{d}x \\ \int_{x_{k-1}}^{x_k} -\frac{1}{h}\frac{x-x_{k-1}}{h}\,\mathrm{d} & \int_{x_{k-1}}^{x_k} \frac{1}{h}\frac{x-x_{k-1}}{h}\,\mathrm{d}x \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \\ N_h^{(n)} &= \int_{x_{n-1}}^{x_n} -\frac{1}{h}\frac{x_n-x}{h}\,\mathrm{d}x = -\frac{1}{h^2}\left[x_n x - \frac{x^2}{2}\right]_{x_0}^{x_1} = -\frac{1}{h^2}\left(\frac{x_n^2}{2} - x_n x_{n-1} + \frac{x_{n-1}^2}{2}\right) \\ &= -\frac{1}{2h^2}(x_n - x_{n-1})^2 = -\frac{1}{2}. \end{aligned}$$

With these results, we can then assembly the matrix N_h as:

$$\begin{aligned} N_h &= \begin{pmatrix} \frac{1}{2} - \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & \frac{1}{2} - \frac{1}{2} & \frac{1}{2} & \ddots & & \vdots \\ 0 & -\frac{1}{2} & \frac{1}{2} - \frac{1}{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2} & 0 \\ \vdots & & \ddots & -\frac{1}{2} & \frac{1}{2} - \frac{1}{2} & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2} & \frac{1}{2} - \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} & 0 & \cdots & \cdots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & & \vdots \\ 0 & -\frac{1}{2} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2} & 0 \\ \vdots & & \ddots & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \cdots & \cdots & 0 & -\frac{1}{2} & 0 \end{pmatrix}. \end{aligned}$$

We can do the same procedure also for K_h and M_h and we get:

$$K_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix},$$

$$M_h = \frac{h}{6}\begin{pmatrix} 4 & 1 & 0 & \dots & \dots & 0 \\ 1 & 4 & 1 & \ddots & & \vdots \\ 0 & 1 & 4 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 4 & 1 \\ 0 & \dots & \dots & 0 & 1 & 4 \end{pmatrix}.$$

Then we calculate the loadvector:

$$\int_0^L \bar{f}(x)\varphi_i(x)\,\mathrm{d}x = \sum_{T\in\mathcal{T}_h}\int_T \bar{f}(x)\varphi_i(x)\,\mathrm{d}x = \sum_{k=1}^{n}\int_{T_k} \bar{f}(x)\varphi_i(x)\,\mathrm{d}x,$$

with $\bar{f}(x) = \int_0^1 \int_{-\frac{t}{2}}^{\frac{t}{2}} f_3(x,y,z)\,\mathrm{d}z\,\mathrm{d}y + \int_0^1 p\left(x,y,-\frac{t}{2}\right) + p\left(x,y,\frac{t}{2}\right)\,\mathrm{d}y$.
For any element T_k, the integral is only non-zero for $i=1$, if $k=1$, for $i=k-1$ and $i=k$, if $2 \le k \le n-1$ and for $i = n-1$, if $k=n$. So only the following terms are different from zero:

$$\int_{T_1} \bar{f}(x)\varphi_1(x)\,\mathrm{d}x,$$
$$\int_{T_k} \bar{f}(x)\varphi_{k-1}(x)\,\mathrm{d}x,$$
$$\int_{T_k} \bar{f}(x)\varphi_k(x)\,\mathrm{d}x,$$
$$\int_{T_n} \bar{f}(x)\varphi_{n-1}(x)\,\mathrm{d}x.$$

We approximate these integrals by using the trapezoidal rule:

$$\int_a^b g(\xi)\,\mathrm{d}\xi \approx (b-a)\frac{g(a)+g(b)}{2}.$$

So we get:

$$
\begin{aligned}
&\int_{T_1} \bar{f}(x)\varphi_1(x)\,\mathrm{d}x \approx \tfrac{h}{2}\Big(\bar{f}(x_0)\varphi_1(x_0) + \bar{f}(x_1)\varphi_1(x_1)\Big) = \tfrac{h}{2}\bar{f}(x_1) = f_1^{(1)},\\
&\int_{T_k} \bar{f}(x)\varphi_{k-1}(x)\,\mathrm{d}x \approx \tfrac{h}{2}\Big(\bar{f}(x_{k-1})\varphi_{k-1}(x_{k-1}) + \bar{f}(x_k)\varphi_{k-1}(x_k)\Big) = \tfrac{h}{2}\bar{f}(x_{k-1}) = f_0^{(k)},\\
&\int_{T_k} \bar{f}(x)\varphi_k(x)\,\mathrm{d}x \approx \tfrac{h}{2}\Big(\bar{f}(x_{k-1})\varphi_k(x_{k-1}) + \bar{f}(x_k)\varphi_k(x_k)\Big) = \tfrac{h}{2}\bar{f}(x_k) = f_1^{(k)},\\
&\int_{T_n} \bar{f}(x)\varphi_{n-1}(x)\,\mathrm{d}x \approx \tfrac{h}{2}\Big(\bar{f}(x_{n-1})\varphi_{n-1}(x_{n-1}) + \bar{f}(x_n)\varphi_{n-1}(x_n)\Big) = \tfrac{h}{2}\bar{f}(x_{n-1}) = f_0^{(n)}.
\end{aligned}
$$

From this we can assembly the loadvector as:

$$
\underline{f}_h = \begin{pmatrix} f_1^{(1)} + f_0^{(2)} \\ f_1^{(2)} + f_0^{(3)} \\ \vdots \\ f_1^{(n-1)} + f_0^{(n)} \end{pmatrix} = \begin{pmatrix} \frac{h}{2}\bar{f}(x_1) + \frac{h}{2}\bar{f}(x_1) \\ \frac{h}{2}\bar{f}(x_2) + \frac{h}{2}\bar{f}(x_2) \\ \vdots \\ \frac{h}{2}\bar{f}(x_{n-1}) + \frac{h}{2}\bar{f}(x_{n-1}) \end{pmatrix} = h \begin{pmatrix} \bar{f}(x_1) \\ \bar{f}(x_2) \\ \vdots \\ \bar{f}(x_{n-1}) \end{pmatrix}.
$$

So we have to solve the following equations:

$$
\begin{aligned}
\tfrac{Et^3}{12} K_h \underline{\theta}_h - \kappa G t N_h \underline{w}_h + \kappa G t M_h \underline{\theta}_h &= 0,\\
\kappa G t K_h \underline{w}_h - \kappa G t N_h^{\mathsf{T}} \underline{\theta}_h &= \underline{f}_h.
\end{aligned}
$$

We can write this down also as one equation with a block matrix as:

$$
\begin{pmatrix} \frac{Et^3}{12} K_h + \kappa G t M_h & -\kappa G t N_h \\ -\kappa G t N_h^{\mathsf{T}} & \kappa G t K_h \end{pmatrix} \begin{pmatrix} \underline{\theta}_h \\ \underline{w}_h \end{pmatrix} = \begin{pmatrix} 0 \\ \underline{f}_h \end{pmatrix}.
$$

Chapter 5

Results

For the concrete values

$$L = 1000,\ E = 210,\ G = 80,\ \kappa = 1,\ \bar{f} = \frac{t^3}{1000},$$

the equations system was then solved with the preconditioned CG-method (cf. [2]), with the tridiagonal preconditioner matrix:

$$P = \begin{pmatrix} \frac{Et^3}{12} K_h + \kappa G t M_h & | & 0 \\ - - - - - - - - - - & - & - - - - \\ 0 & | & \kappa G t K_h \end{pmatrix}.$$

5.1 Results for $h \to 0$

At first we assume, that we have a fixed value for the thickness t of the beam, and we look at the result for different stepsizes h.

The analytical solution to this problem in terms of x, t and $\bar{f}$ can be written as:

$$\begin{aligned} \theta(x,t,\bar{f}) &= \frac{\bar{f}}{105\,t^3}\,x^3 - \frac{100\,\bar{f}}{7\,t^3}\,x^2 + \frac{100000\,\bar{f}}{21\,t^3}\,x, \\ w(x,t,\bar{f}) &= \frac{\bar{f}}{420\,t^3}\,x^4 - \frac{100\,\bar{f}}{21\,t^3}\,x^3 - \frac{\bar{f}\,(-8000000 + 21\,t^2)}{3360\,t^3}\,x^2 + \frac{25\,\bar{f}}{4\,t}\,x, \end{aligned}$$

so for our concrete value for $\bar{f}$ we get:

$$\begin{aligned} \theta(x) &= \frac{1}{105000}\,x^3 - \frac{1}{70}\,x^2 + \frac{100}{21}\,x, \\ w(x,t) &= \frac{1}{420000}\,x^4 - \frac{1}{210}\,x^3 + \frac{8000000 - 21\,t^2}{3360000}\,x^2 + \frac{t^2}{160}\,x. \end{aligned}$$

In figure 5.1 we can see the analytical solution for θ, which is independent from t.

In figure 5.2 we can see, that for w all the curves are really close for different values of t.

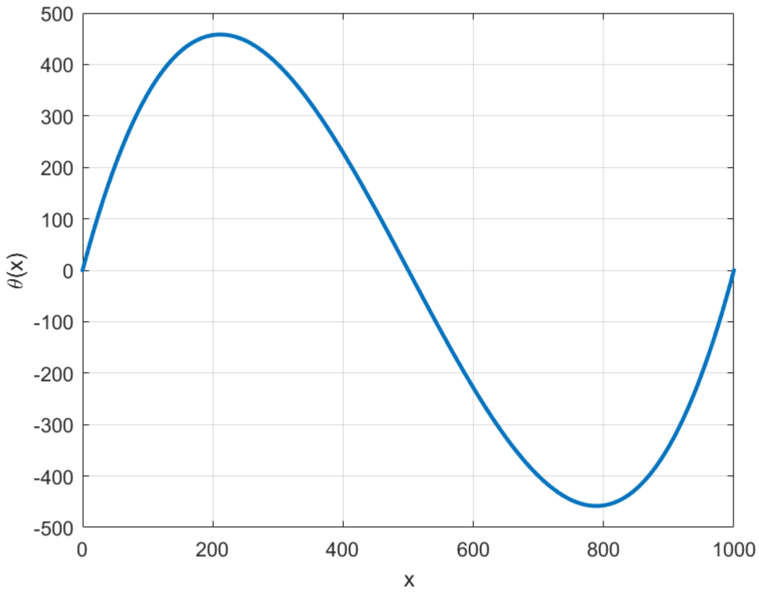

Figure 5.1: $\theta(x)$ (independent from t)

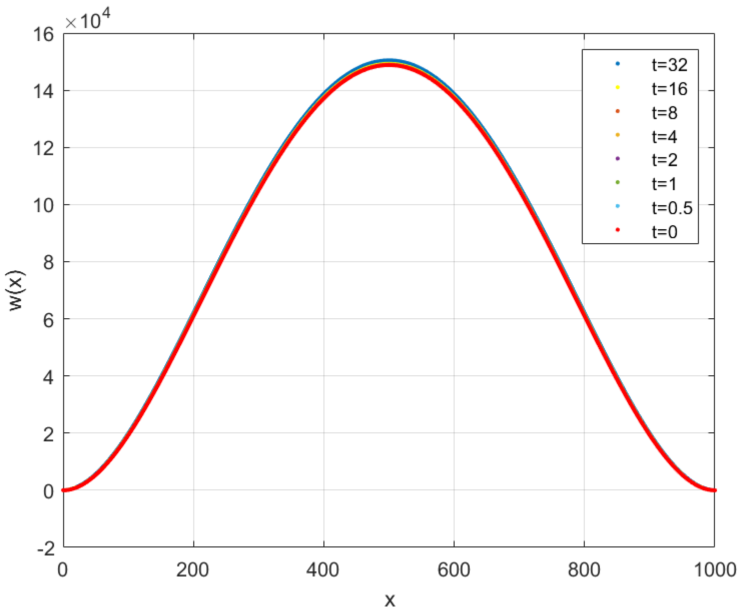

Figure 5.2: $w(x)$ for different t

Now we look at the numerical solutions.

First we set $t = 10$, then we get the following results for θ_h:
In figure 5.3 we can see, that for smaller h, the dotted numerical solutions always get closer to the blue solid line, which represents the analytical solution.

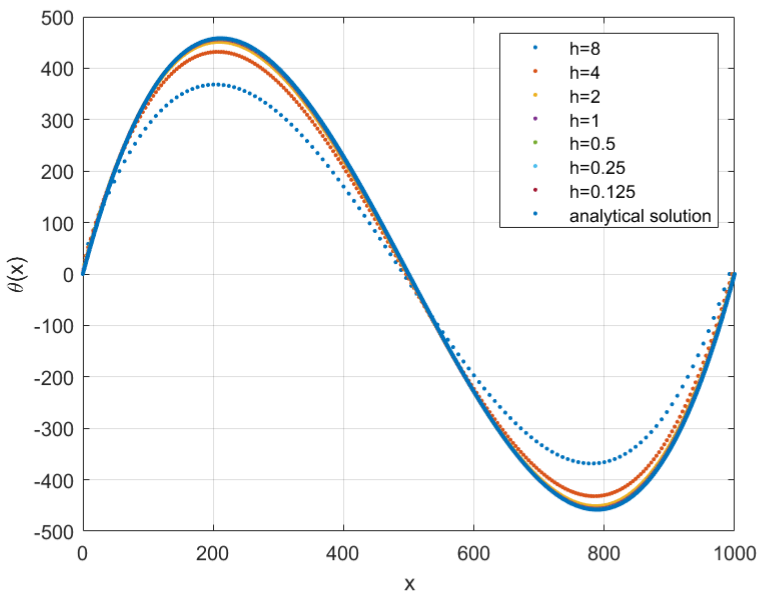

Figure 5.3: θ_h for different h and $t = 10$ fixed

In figure 5.4, we can see the same result for w_h: The dotted numerical results always get closer to the analytical result, which is again represented by the blue solid line.

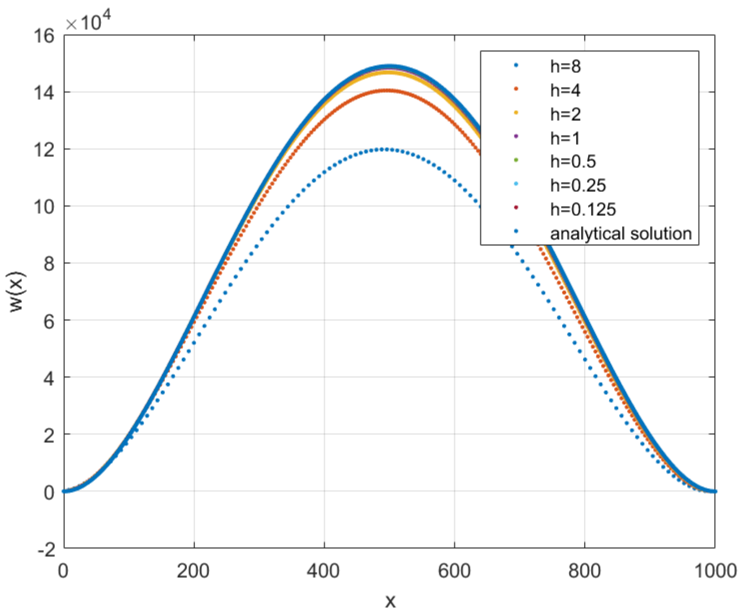

Figure 5.4: w_h for different h and $t = 10$ fixed

For $t = 1$ we can see in figure 5.5 and figure 5.6, that for small h the numerical solutions for θ_h and w_h are really bad, but as h decreases, the numerical solutions get better and they converge to the analytical solutions for θ and w:

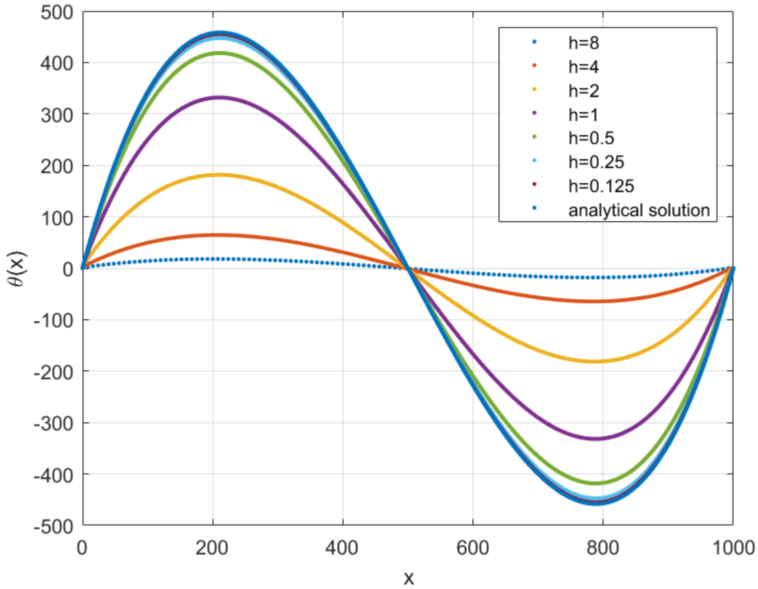

Figure 5.5: θ_h for different h and $t = 1$ fixed

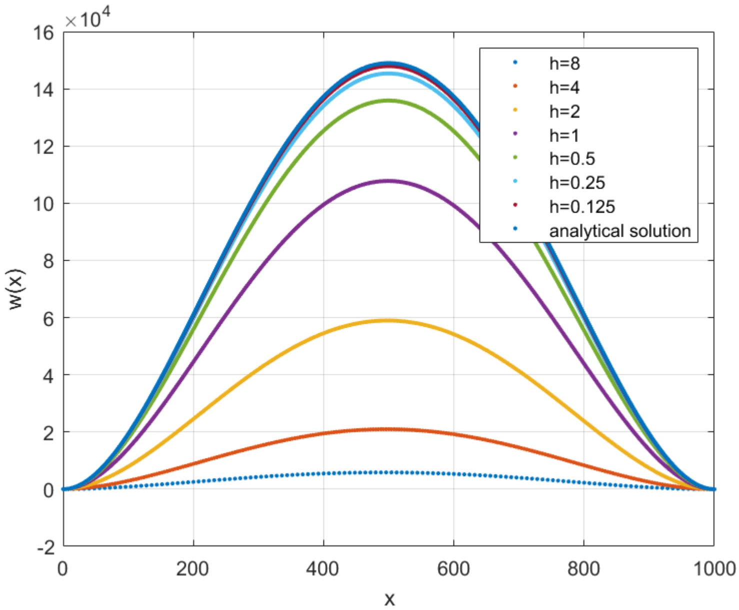

Figure 5.6: w_h for different h and $t = 1$ fixed

If we now look at the error $\|\theta(x) - \theta_h(x)\|_{H^1(0,L)}$, shown in figure 5.7, we can also see, that the numerical solutions get better, if h gets smaller:

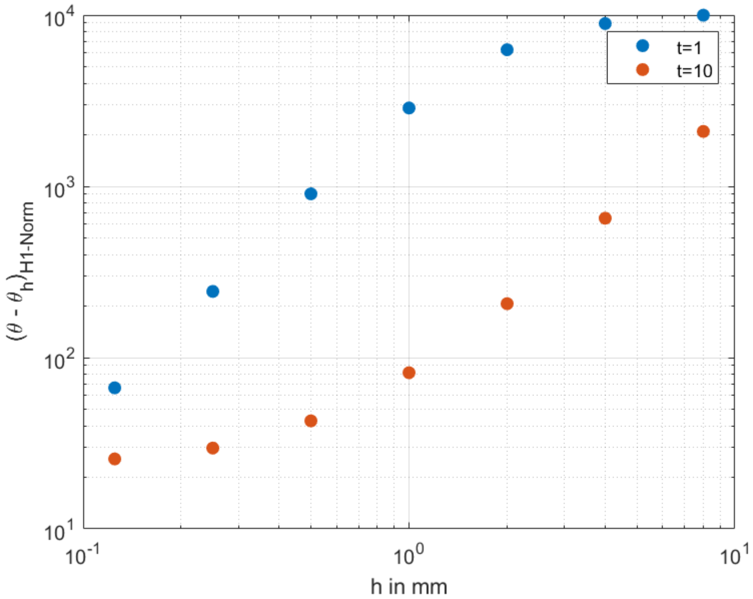

Figure 5.7: $\|\theta(x) - \theta_h(x)\|_{H^1(0,L)}$

The same is also true if we look at the error $\|w(x) - w_h(x)\|_{H^1(0,L)}$, shown in figure 5.8: We can again see, that the error gets smaller, if h gets smaller.

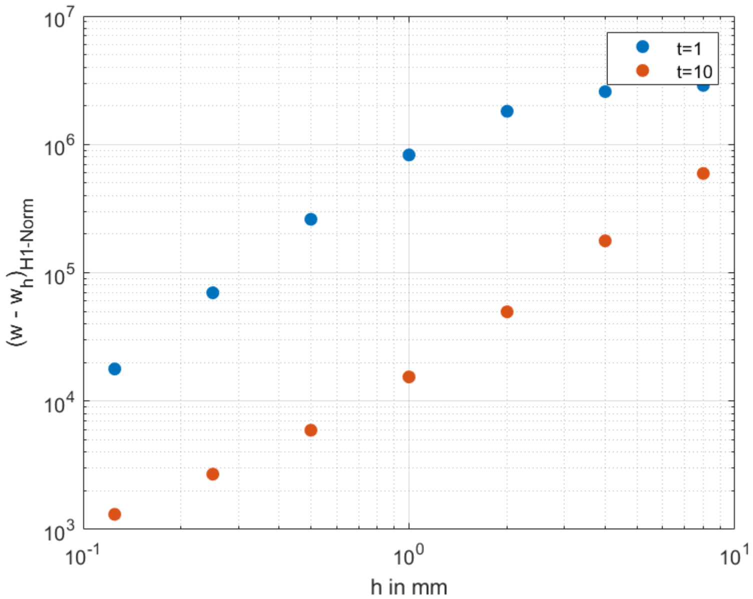

Figure 5.8: $\|w(x) - w_h(x)\|_{H^1(0,L)}$

Then we also want to look at the iteration numbers. The iteration numbers for $t = 10$ are shown in figure 5.9. We can see that the iteration number gets slightly higher, if the stepsize h gets smaller.

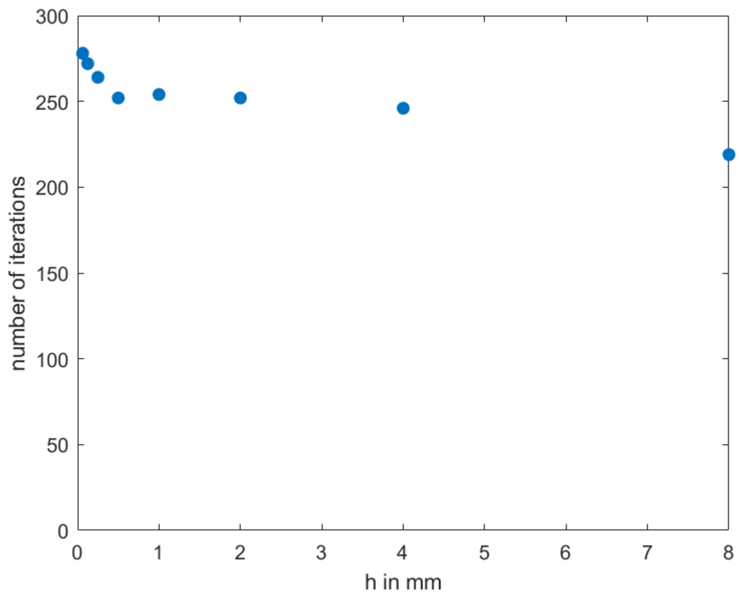

Figure 5.9: Number of iterations for $t = 10$

For $t = 1$ the situation is a little different. In figure 5.10 we can see, that the iteration number gets visibly bigger if h gets smaller.

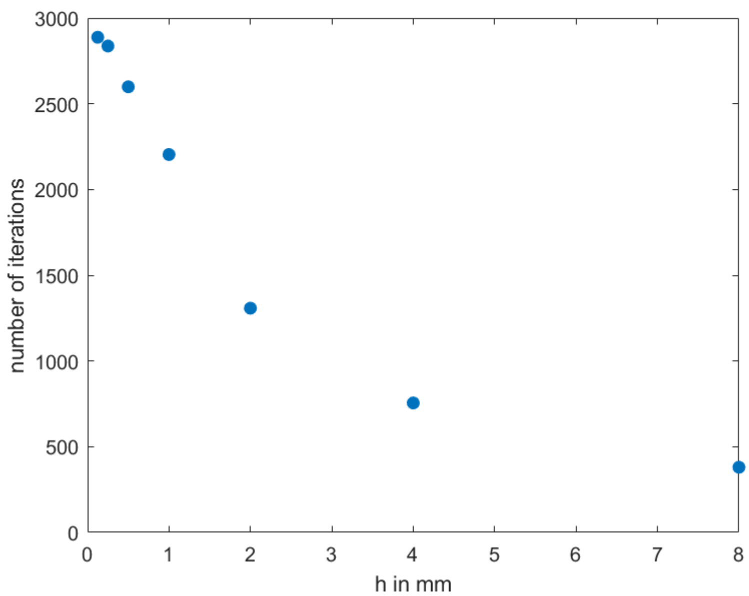

Figure 5.10: Number of iterations for $t = 1$

So as a result we can say, that for a smaller stepsize h the numerical solution approximates the analytical solution always better, so the H^1-error between the numerical and the analytical solutions gets smaller with smaller h. The condition number of the matrix gets slightly bigger, if h gets smaller.

5.2 Results for $t \to 0$

Now we also want to look at the results for a fixed stepsize h for the discretization and a variation the thickness t of the beam.

First we look at the numerical solutions for θ, if we set $h = 10$ and choose different values for t. If we look at the dotted lines in figure 5.11, we recognize, that the numerical solutions get always smaller for smaller t.

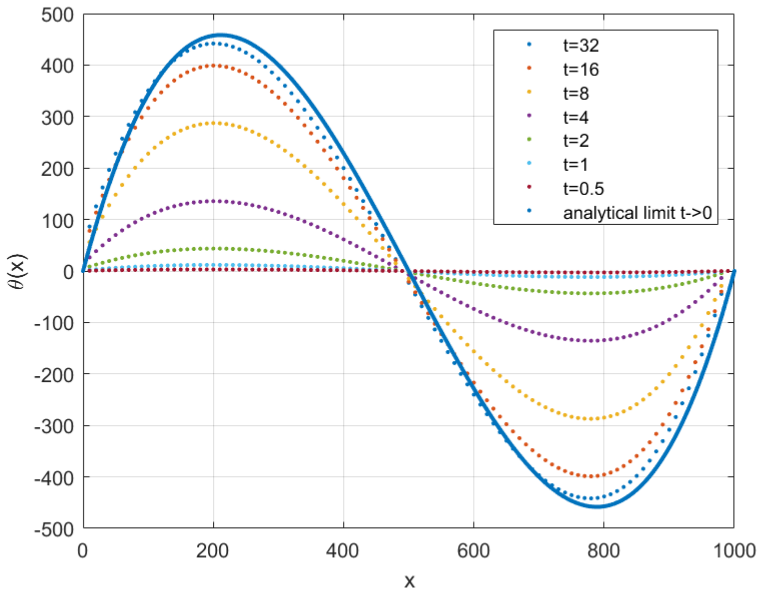

Figure 5.11: $\theta(x)$ for different t and h=10 fixed

If we now look at the results for w for different values for t and $h = 10$ fixed, pictured in figure 5.12, we can basically see the same result. Again, the numerical solutions get smaller, for smaller t, so for $t \to 0$ the numerical approximations get worse.

We can see in figure 5.13, that for a finer discretization with $h = 1$, the numerical solutions for θ also get smaller for $t \to 0$, and for really small t they are a bad approximation. But we can also see that this time the numerical solutions for big values of t seem to be quite good approximations for the analytical solution θ.

Again, we can observe the same result if we look at the numerical solutions for w for $h = 1$, which are represented in figure 5.14. For $t \to 0$ we always get smaller numerical results, and for really small t, the numerical solution is a bad approximation. But again, for big t, the numerical solutions are quite good approximations for the analytical solutions.

Now we also have a look at the iteration number. For $h = 10$ we can see in figure 5.15, that for smaller t there is a visible increase of the iteration number.

In figure 5.16 we can see, that for $h = 1$ the iteration number gets much bigger, if t gets smaller.

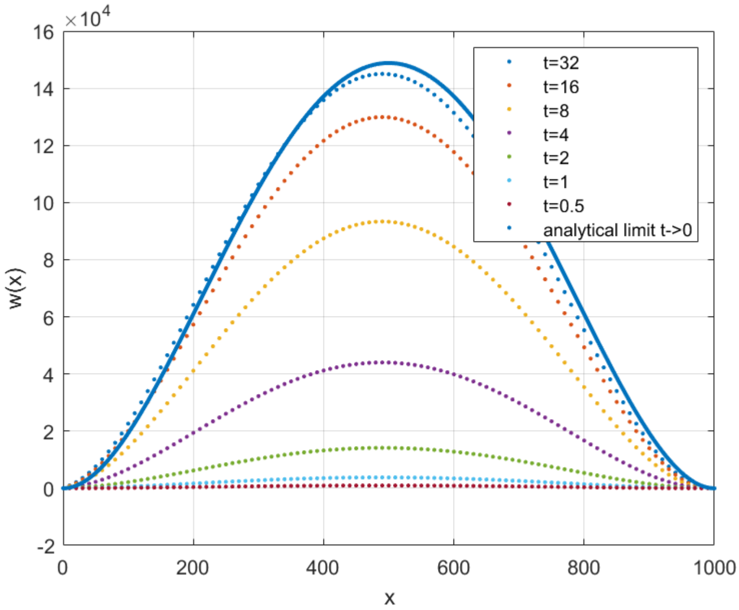

Figure 5.12: $w(x)$ for different t and h=10 fixed

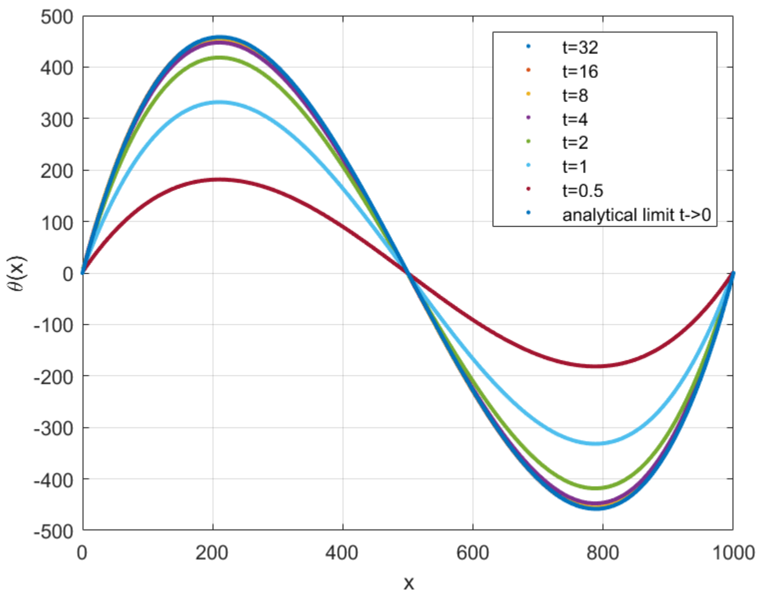

Figure 5.13: $\theta(x)$ for different t and h=1 fixed

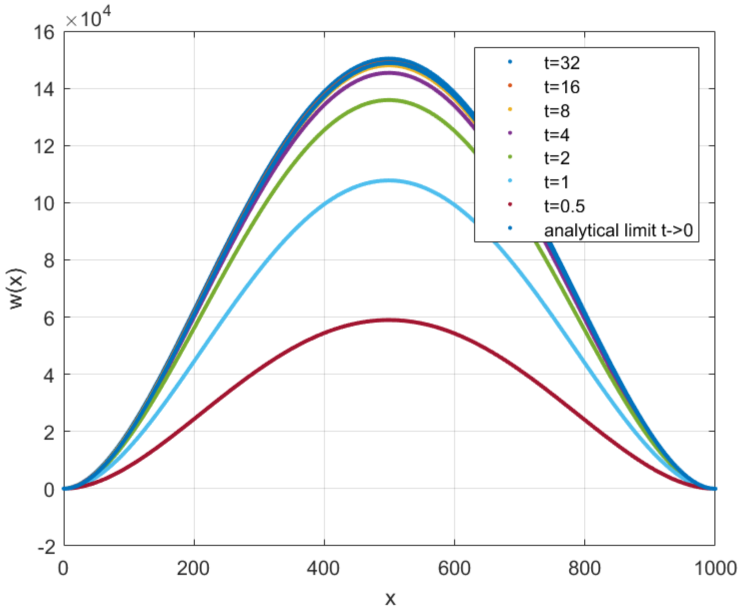

Figure 5.14: $w(x)$ for different t and h=1 fixed

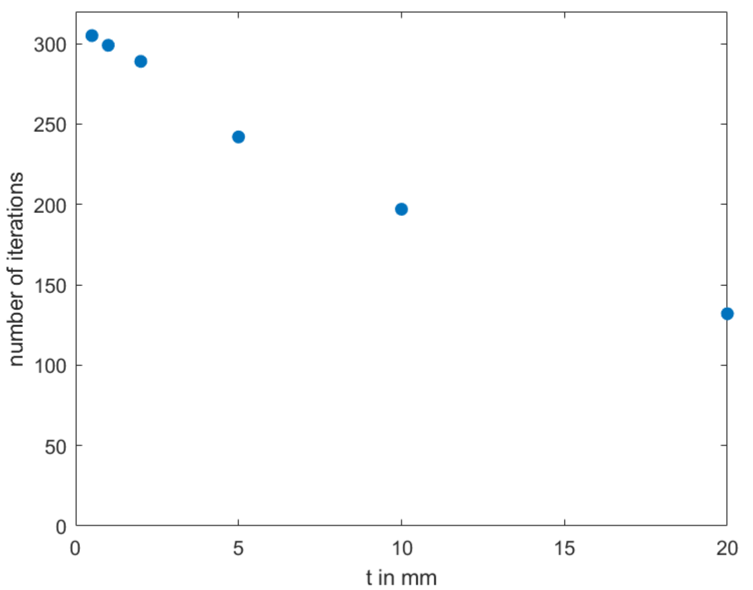

Figure 5.15: Number of iterations for $h = 10$

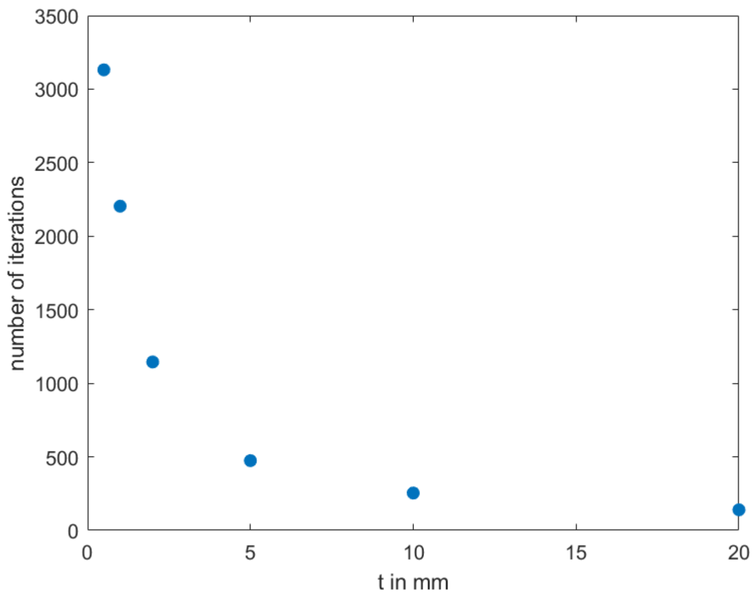

Figure 5.16: Number of iterations for $h = 1$

So the condition number gets bigger, if t gets smaller.

If we choose t always smaller, also the numerical solutions get smaller, so for $t \to 0$ the numerical solution is way too small. This has to do with the Locking effect, which lets the beam appear to be stiffer than it actually is. If we want to make t smaller, we also have to make h smaller, to get a good approximation. For example if we look at the figure 5.17, we can see, that if we have a certain error for $t = 2$ for a certain h and we then want to have about the same error for $t = 1$, we have to choose $\frac{h}{2}$ to achieve this.

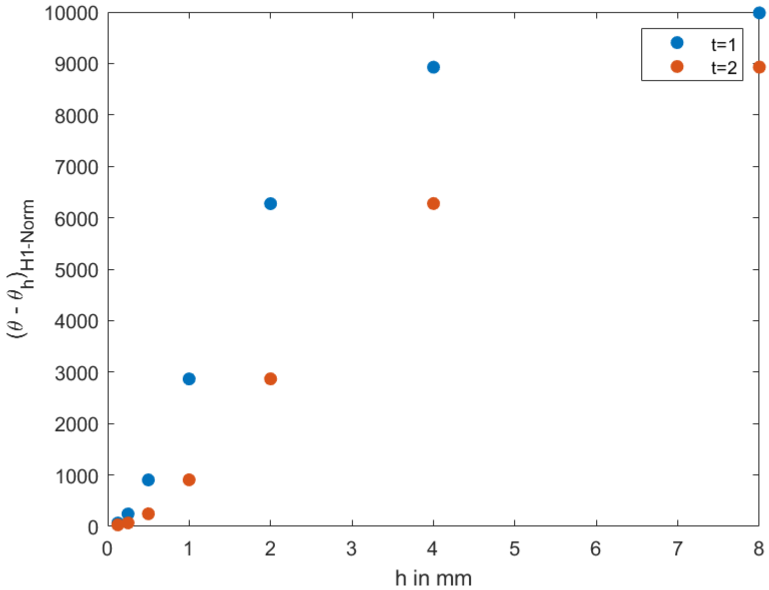

Figure 5.17: $\|\theta - \theta_h\|_{H^1(0,L)}$ for $t = 1$ and $t = 2$

Bibliography

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