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Institute of Computational Mathematics

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Clemens Hofreither
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

Stefan Takacs
Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences
Altenberger Str. 69, 4040 Linz, Austria

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Robust Multigrid for Isogeometric Analysis Based on Stable Splittings of Spline Spaces

Clemens Hofreither Stefan Takacs

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Abstract

We present a robust and efficient multigrid method for isogeometric discretizations using tensor product B-splines of maximum smoothness. Our method is based on a stable splitting of the spline space into a large subspace of "interior" splines which satisfy a robust inverse inequality, as well as one or several smaller subspaces which capture the boundary effects responsible for the spectral outliers known to occur in Isogeometric Analysis. We then construct a multigrid smoother based on a subspace correction approach, applying a different smoother to each of the subspaces. For the interior splines, we use a mass smoother, whereas the remaining components are treated with suitably chosen Kronecker product smoothers or direct solvers.

We prove that the resulting multigrid method exhibits iteration numbers which are robust with respect to the spline degree and the mesh size. Furthermore, it can be efficiently realized for discretizations of problems in arbitrarily high geometric dimension. Some numerical examples illustrate the theoretical results and show that the iteration numbers also scale relatively mildly with the problem dimension.

1 Introduction

Isogeometric Analysis (IgA) is method for the numerical solution of partial differential equations (PDEs) introduced in the seminal paper [17] which has since attracted a sizable research community. Spline spaces, such as spaces spanned by tensor product B-splines or NURBS, are commonly used for geometry representation in industrial CAD systems. The foundational idea in IgA is to use such spline spaces both for the representation of the computational domain and for the discretization of the quantities of interest when solving a PDE. The overall goal is to create a tighter integration between geometric design and analysis.

Our interest lies in building efficient solvers for the large, sparse linear systems which arise when applying isogeometric discretizations to boundary value problems. By now, most established solution strategies known from the Finite Element literature have been applied in one way or another to IgA: among these, direct solvers [2], non-overlapping and overlapping domain decomposition methods [18, 4, 5, 6], and multilevel and multigrid methods [1, 11, 16, 10, 14]. A recent contribution [19] constructs preconditioners based on fast solvers for Sylvester equations. The above list is certainly not comprehensive.

In IgA, we typically encounter as discretization parameters the mesh size and the spline degree. In the early IgA solver literature, the focus was on translating solvers from the FEM world to IgA with minimal adaptations. As a rule, it was found that such an approach results in methods that work well for low spline degrees, but deteriorate in performance as the degree is increased; often dramatically so. This motivated the search for IgA solvers that are robust not only with respect to the mesh size (which is often easy to achieve), but also with respect to the spline degree.

Within the class of multigrid methods for IgA, advances towards a robust method were made using two approaches. In [9], a careful analysis of the symbol of isogeometric stiffness matrices served as the basis for the construction of multigrid methods. This theoretical approach is somewhat related to the technique known as Local Fourier Analysis (LFA) in the multigrid literature (see, e.g., [21]). It appears that the method presented in [9] is roughly comparable to the one studied in [15], which uses mass matrices as multigrid smoothers, an approach that was itself motivated by LFA. For both methods, an increase in the number of smoothing steps, roughly linearly with the spline degree, is required in order to maintain robust convergence. They can thus not be considered totally robust and efficient in the strict meaning that we will use in the present work.

A second approach towards robust and efficient multigrid was presented in [14]. Based on a robust inverse inequality and approximation error estimate in a large subspace of maximally smooth spline spaces derived in [20], it was shown that mass matrices can be used as robust smoothers in this large subspace. For the remaining, relatively few degrees of freedom, a low-rank correction was constructed. (These degrees of freedom are associated with the boundary of the domain and cannot be captured by LFA, which assumes periodic boundary conditions.) This approach resulted in a provably robust and efficient multigrid method for two-dimensional problems with splines of maximum smoothness. It was however not clear how to extend this approach efficiently to three and higher dimensions.

The present work can be viewed as a continuation of [14]. Based on the theoretical results from [20], we construct a splitting of the tensor product spline space into a large, regular interior part and several smaller spaces which capture boundary effects. The splitting is L_2-orthogonal and H^1-stable with respect to both the mesh size and the spline degree. This stability enables us to construct a multigrid smoother based on an additive subspace correction approach, applying a different smoother in each of the subspaces. In the regular interior subspace, we use a mass smoother. In the other subspaces, we construct smoothers which exploit the particular structure of the subspaces while still permitting an efficient application through a Kronecker product representation. In one small subspace associated with the corners of the domain, we apply an exact solver.

Unlike the low-rank correction approach from [14], the subspace correction approach generalizes easily to three-dimensional problems, and indeed to problems of arbitrary space dimension. We show that the method converges robustly with respect to mesh size and spline degree, and that one iteration is asymptotically not more expensive than an application of the stiffness matrix. The result is a quasi-optimal solution method for problems of arbitrary space dimensions.

We emphasize that the stable splitting presented in Section 3 is an interesting theoretical result in its own right, and we anticipate future applications of this idea to different solution strategies, or to the solution of eigenvalue problems while avoiding the spectral outliers commonly observed in IgA (cf. [3]).

The remainder of the paper is organized as follows. In Section 2, we introduce the needed spline spaces and present an isogeometric model problem. We also present an algorithmic multigrid framework and an abstract convergence result which forms the basis of our later analysis. In Section 3, we derive the main new theoretical result used in our construction: the L_2-orthogonal and H^1-stable splitting of the spline space into a large, regular interior part and smaller spaces which capture boundary effects. In Section 4, we use this space splitting to construct a multigrid smoother based on the idea of additive subspace correction and show that it results in a robust solver. In Section 5, we present details on the numerical realization of the proposed smoother and show that it permits an efficient implementation in arbitrary space dimensions. In Section 6, we present numerical experiments which demonstrate the performance of the proposed method in practice.

2 Preliminaries

2.1 Spline spaces and B-splines

Consider a subdivision of the interval $(0, 1)$ into $m \in \mathbb{N}$ intervals of length $h = 1/m$. We introduce the spline space of degree $p \in \mathbb{N}$ with maximum smoothness,

$$S := \{u \in C^{p-1}(0, 1) : u|_{((j-1)h,jh)} \in \mathcal{P}^p \quad \forall j = 1, \dots, m\},$$

where $C^{p-1}(0, 1)$ are the $p - 1$ times continuously differentiable functions on $(0, 1)$ and $\mathcal{P}^p$ are the polynomials of degree at most p. We have

$$n := \dim S = m + p.$$

Whenever we require a basis for S, we will use the normalized (i.e., satisfying a partition of unity) B-splines (see, e.g., [8]) and denote them by

$$\mathcal{B} := \{\varphi_1, \dots, \varphi_n\}.$$

In higher dimensions $d > 1$, we introduce the space of tensor product splines and the tensor product B-spline basis

$$S^d := S \otimes \dots \otimes S, \quad \varphi_{j_1,\dots,j_d}(x) := \varphi_{j_1}(x_1) \cdots \varphi_{j_d}(x_d), \quad j_k \in \{1, \dots, n\},$$

defined over $(0, 1)^d$. For notational convenience, we assume that the same spline space is used in each of the d coordinate directions. Both our construction and our analysis are however straightforward to generalize to the case where different spline spaces are used in different coordinate directions.

2.2 Isogeometric model problem

Let $\Omega = (0, 1)^d$ with $d \in \mathbb{N}$. As a model problem for d-dimensional boundary value problems for elliptic second-order partial differential equations, we consider the following: find $u \in H^1(\Omega)$ such that

$$a(u, v) = \langle f, v\rangle \qquad \forall v \in H^1(\Omega), \tag{1}$$

where

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx \qquad \forall u, v \in H^1(\Omega) \tag{2}$$

and f is a linear functional on $H^1(\Omega)$. This is the variational formulation of a pure Neumann boundary value problem for the partial differential operator $-\Delta u + u$. In the following, we will sometimes refer to the operator $A : H^1(\Omega) \to H^1(\Omega)'$ given by $Av = a(v, \cdot)$. Also note that $\|v\|_A^2 = a(v,v) = \|v\|_{H^1(\Omega)}^2$.

Discretizing (1) using tensor product B-splines, we obtain the discrete problem: find $u_h \in S^d$ such that

$$a(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in S^d. \tag{3}$$

We are interested in robust and efficient iterative solvers for the discrete problem (3). Here, by "robust" we mean that the number of iterations to solve the problem should stay uniformly bounded with respect to both the mesh size h and the spline degree p, and by "efficient" we mean that one iteration of the method should not be asymptotically more expensive than one application of the stiffness matrix. Combined, these properties allow us to solve (3) in quasi-optimal time.

In IgA, it is common to introduce a bijective geometry map from Ω to the actual domain of interest in order to be able to treat more complicated computational domains. Basis functions on the transformed domain are defined by composing the basis functions on the reference domain with the inverse of the geometry map. Furthermore, one often considers more general PDEs with varying and possibly matrix-valued coefficients. Discretizations for such more general problems can be preconditioned with a solver for the model problem (3), and the resulting condition number depends only on the geometry map and the coefficient functions, but not on discretization parameters like the mesh size h or the spline degree p. This principle has been widely used in the literature on IgA solvers (see, e.g., [9, 14]) and formalized in [19]. Therefore, a robust and efficient solver for the model problem (3) immediately yields robust and efficient solvers for a more general class of problems with "benign" geometry maps and mildly varying coefficients. This justifies the study of solvers for the model problem.

Three different refinement strategies for IgA discretizations were proposed in [17]: h-refinement (reducing the mesh size), p-refinement (increasing the spline degree), and the so-called k-refinement. In the latter, the idea is to refine a spline space by inserting only single new knots at the midpoints of the knot spans to be refined. This strategy maintains the maximum possible smoothness C^{p-1} for the spline space of degree p. Already in [17], it was observed that this strategy exhibits favorable practical performance since it allows the use of a high spline degree (and the associated high approximation power) while keeping the number of degrees of freedom relatively low. In the wider IgA literature, k-refinement appears to be the most popular refinement strategy, which motivates the study of solvers for spline spaces with maximum smoothness.

2.3 A multigrid method framework

Given a discretization space V and a coarse space $V_c \subset V$, we denote by $P : V_c \to V$ the canonical embedding. Let $A : V \to V'$ denote the operator in a (discretized) equation

$$Au = f$$

to be solved for $u \in V$. The corresponding coarse-space operator is given by $A_c := P'AP$. Furthermore, we assume that we are given a self-adjoint and positive definite smoothing operator $L : V \to V'$.

Given a previous iterate $u^{(k)}$, we let $u^{(k,0)} := u^{(k)}$ and perform $\nu \in \mathbb{N}$ *smoothing steps* given by

$$u^{(k,j)} := u^{(k,j-1)} + \tau L^{-1}(f - Au^{(k,j-1)}), \qquad j = 1, \dots, \nu,$$

where $\tau > 0$ is a damping parameter. Then, we perform one *coarse-grid correction step* given by

$$u^{(k+1)} := u^{(k,\nu)} + PA_c^{-1}P'(f - Au^{(k,\nu)}).$$

Together, these updates describe one iteration $u^{(k)} \mapsto u^{(k+1)}$ of a *two-grid method.* Given an entire sequence of nested spaces $V_0 \subset \ldots \subset V_L = V$, we can replace the exact inversion of A_c in the coarse-grid correction step by one or two recursive applications of the two-grid method on the next coarser level V_{L-1}, and so on until we reach the coarsest level V_0, where an exact solver is used. Using one respectively two recursive iteration steps results in the *V-cycle* respectively *W-cycle multigrid method.*

An abstract convergence result for the two-grid method is given below. It was derived in [14] using relatively standard arguments based on the multigrid theory as developed by Hackbusch [13]. Under the same assumptions, it was shown in [14] that convergence of the W-cycle multigrid method follows as well. Therefore, we restrict ourselves to the analysis of the two-grid method in the present work.

Theorem 1 ([14])**.** *Assume that there are constants C_A and C_I such that the inverse inequality*

$$\|u\|_A^2 \le C_I \|u\|_L^2 \qquad \forall u \in V \tag{4}$$

and the approximation property for the A-orthogonal projector $\Pi_c : V \to V_c$

$$\|(I - \Pi_c)u\|_L^2 \le C_A \|u\|_A^2 \qquad \forall u \in V \tag{5}$$

hold. Then the two-grid method converges for any damping parameter $\tau \in (0, C_I^{-1}]$ and any number of smoothing steps $\nu > \nu_0 := \tau^{-1} C_A$ with rate $q = \nu_0/\nu < 1$.

In particular, the above result states that if C_A and C_I do not depend on the mesh size h and the spline degree p, then the two-grid method converges with a rate $q < 1$ which does not depend on h and p. In other words, the two-grid method is then robust.

In addition to properties (4) and (5), care must be taken that the smoother can be realized efficiently. In other words, it should be possible to apply the inverse L^{-1} with a computational complexity which is roughly comparable to that for applying A.

3 Stable splittings of spline spaces

Consider first the univariate case, $d = 1$, with $\Omega = (0, 1)$. In [20], the subspace

$$S_0 := \left\{ u \in S : u^{(2l+1)}(0) = u^{(2l+1)}(1) = 0 \quad \forall l \in \mathbb{N}_0 \text{ with } 2l + 1 < p \right\}$$

of splines with vanishing odd derivatives of order less than p at the boundaries was introduced (denoted there by $\widetilde{S}_{p,h}(\Omega)$). It is a large subspace of S in the sense that

$$\dim S_0 \geq \dim S - p.$$

The subspace S_0 has the very desirable property of satisfying both a (first-order) approximation property and an inverse inequality, both with constants which are independent of the spline degree p. To formulate these results, we let $\Pi_0 : H^1(\Omega) \to S_0$ and $Q_0 : L_2(\Omega) \to S_0$ denote the H^1-orthogonal and the L_2-orthogonal projector into S_0, respectively.

Theorem 2 ([20])**.** *For any spline degree $p \in \mathbb{N}$, we have the inverse inequality*

$$|u|_{H^1(\Omega)} \leq 2\sqrt{3}h^{-1}\|u\|_{L_2(\Omega)} \qquad \forall u \in S_0.$$

Theorem 3 ([20, 14])**.** *For any spline degree $p \in \mathbb{N}$ and any $u \in H^1(\Omega)$, we have the approximation error estimates*

$$\|(I - Q_0)u\|_{L_2(\Omega)} \leq \sqrt{2}h|u|_{H^1(\Omega)},$$
$$\|(I - \Pi_0)u\|_{L_2(\Omega)} \leq 2\sqrt{2}h\|u\|_{H^1(\Omega)}.$$

Contrast these properties with the entire spline space S, which does satisfy a robust approximation property, but whose inverse inequality deteriorates with increasing degree p ([20]). On the other hand, a smaller space of only "interior" splines, built by discarding the p leftmost and p rightmost B-splines, does satisfy a robust inverse inequality but loses the approximation property.

We remark that the non-robustness of the inverse inequality in S is the root cause of the spectral "outliers" commonly observed when solving eigenvalue problems using IgA (cf. [3]). No such outliers appear in the space S_0.

3.1 A stable splitting in 1D

Let $S_1 := S_0^{\perp_{L_2}}$ denote the L_2-orthogonal complement of S_0 in S. Consider the splitting of S into the direct sum

$$S = S_0 \oplus S_1 \quad \longleftrightarrow \quad u = Q_0 u + (I - Q_0)u$$

of S_0 and its complement, illustrated in Figure 1. Due to orthogonality, we have

$$\|u\|_0^2 = \|Q_0 u\|_0^2 + \|(I - Q_0)u\|_0^2.$$

Crucially, we can prove that this splitting is stable also in the H^1-norm. This is a direct result of the space S_0 satisfying both an approximation property and an inverse inequality.

Here and in the sequel, we abbreviate the $L_2(\Omega)$-norm by $\|\cdot\|_0$, and the full $H^1(\Omega)$-norm and the seminorm, respectively, by $\|\cdot\|_1$ and $|\cdot|_1$. Furthermore, we write c for a generic positive constant which does not depend on the mesh size h or the spline degree p.

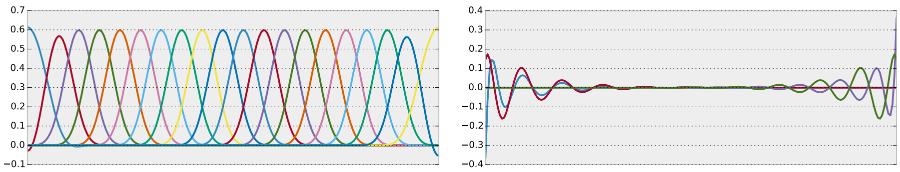

Figure 1: Bases for the space S_0 (left) and its orthogonal complement S_1 (right) for $p = 4$, $h = 1/20$. Here, $\dim S_0 = 20$ and $\dim S_1 = 4$.

Theorem 4. *For any spline $u \in S$, we have*

$$c^{-1}\|u\|_1^2 \leq \|Q_0 u\|_1^2 + \|(I - Q_0)u\|_1^2 \leq c\|u\|_1^2.$$

Proof. The left inequality follows immediately from the Cauchy-Schwarz inequality with $c = 2$. For the right inequality, we observe that

$$\begin{aligned}\|Q_0 u\|_1 &\leq \|\Pi_0 u\|_1 + \|(\Pi_0 - Q_0)u\|_1 \\ &\leq \|u\|_1 + ch^{-1}\left(\|(I - \Pi_0)u\|_0 + \|(I - Q_0)u\|_0\right),\end{aligned}$$

where we used the triangle inequality, stability of the H^1-projector Π_0 in the H^1-norm and the robust inverse inequality in S_0 (Theorem 2). Due to Q_0 producing the best L_2-approximation in S_0, we have $\|(I - Q_0)u\|_0 \leq \|(I - \Pi_0)u\|_0$. Thus, with the approximation result Theorem 3 we obtain H^1-stability of the L_2-projector,

$$\|Q_0 u\|_1 \leq c\|u\|_1.$$

Using the triangle inequality, the desired result follows from

$$\|(I - Q_0)u\|_1 \leq \|u\|_1 + \|Q_0 u\|_1 \leq (1 + c)\|u\|_1. \qquad \square$$

Theorem 5. *We have stability of the above splitting in the H^1-seminorm, that is, for any spline $u \in S$,*

$$c^{-1}|u|_1^2 \leq |Q_0 u|_1^2 + |(I - Q_0)u|_1^2 \leq c|u|_1^2.$$

Proof. By definition, the constant functions are contained in S_0, and therefore $Q_0\overline{u} = \overline{u}$ and $(I - Q_0)\overline{u} = 0$ for any $\overline{u} \in \mathbb{R}$. In the following, we choose the mean value $\overline{u} := \int_\Omega u = \int_\Omega Q_0 u$. By Theorem 4, we have

$$\begin{aligned}|u|_1^2 = |u - \overline{u}|_1^2 &\leq c\left(\|Q_0(u - \overline{u})\|_1^2 + \|(I - Q_0)(u - \overline{u})\|_1^2\right) \\ &= c\left(\|Q_0 u - \overline{u}\|_1^2 + \|(I - Q_0)u\|_1^2\right).\end{aligned}$$

Due to the Poincaré inequality and $\overline{u} = \int_\Omega Q_0 u$, we have

$$\|Q_0 u - \overline{u}\|_1 \leq c|Q_0 u - \overline{u}|_1 = c|Q_0 u|_1.$$

For the second term, we have from Theorem 3

$$\|(I - Q_0)u\|_1^2 = \|(I - Q_0)(I - Q_0)u\|_0^2 + |(I - Q_0)u|_1^2 \leq (1 + ch)|(I - Q_0)u|_1^2.$$

Since $h \leq 1$, we obtain overall

$$|u|_1^2 \leq c\left(|Q_0 u|_1^2 + |(I - Q_0)u|_1^2\right).$$

For the second inequality, we obtain from Theorem 4

$$|Q_0 u|_1^2 + |(I - Q_0)u|_1^2 = |Q_0(u - \overline{u})|_1^2 + |(I - Q_0)(u - \overline{u})|_1^2 \le c\|u - \overline{u}\|_1^2.$$

Again with the Poincaré inequality, we obtain

$$|Q_0 u|_1^2 + |(I - Q_0)u|_1^2 \le c|u|_1^2. \qquad \square$$

3.2 A stable splitting in 2D

The two-dimensional tensor product spline space is given by $S^2 = S \otimes S$, and since the tensor product distributes over direct sums, we obtain the splitting

$$S^2 = (S_0 \otimes S_0) \oplus (S_0 \otimes S_1) \oplus (S_1 \otimes S_0) \oplus (S_1 \otimes S_1) = S_{00} \oplus S_{01} \oplus S_{10} \oplus S_{11}$$

with the abbreviations

$$S_{\alpha_1,\alpha_2} := S_{\alpha_1} \otimes S_{\alpha_2} \quad (\alpha_j \in \{0, 1\}).$$

A visualization of this splitting is shown in Figure 2. Note that the shaded regions do not correspond to the supports of the function spaces; in fact, each of the subspaces has global support. However, the shaded regions roughly correspond to regions where the corresponding functions are "largest", and their areas roughly correspond to the space dimensions. In view of this, it makes sense to think of S_{00} as an "interior" space, of S_{01} and S_{10} as "edge" spaces, and of S_{11} as a "corner" space.

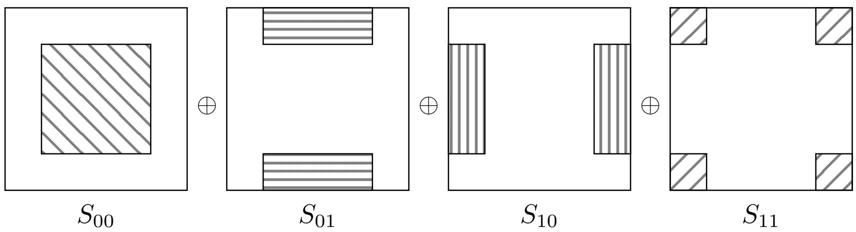

Figure 2: Visualization of the splitting in 2D.

Again, we can prove that the splitting is H^1-stable. In the following, we let $M : S \to S'$, $K : S \to S'$ denote the operators in the univariate spline space associated with the bilinear forms

$$\langle Mu, v\rangle := \int_0^1 u(x)v(x)\,dx, \quad \langle Ku, v\rangle := \int_0^1 u'(x)v'(x)\,dx \quad \forall u, v \in S,$$

that is, the one-dimensional mass and stiffness operators, respectively. For any $(\alpha_1, \alpha_2) \in \{0, 1\}^2$, we furthermore introduce the abbreviations

$$Q_1 := I - Q_0 : S \to S_1, \qquad Q_{\alpha_1,\alpha_2} := Q_{\alpha_1} \otimes Q_{\alpha_2} : S^2 \to S_{\alpha_1,\alpha_2}$$
$$K_{\alpha_j} := Q'_{\alpha_j} K Q_{\alpha_j} : S_{\alpha_j} \to S'_{\alpha_j}, \qquad M_{\alpha_j} := Q'_{\alpha_j} M Q_{\alpha_j} : S_{\alpha_j} \to S'_{\alpha_j}.$$

As tensor products of $L_2(0, 1)$-orthogonal projectors, the projectors Q_{α_1,α_2} are $L_2(\Omega)$-orthogonal, as one easily verifies. Thus the splitting of S^2 given above is a direct sum of L_2-orthogonal subspaces, and we have

$$\|u\|_0^2 = \sum_{(\alpha_1,\alpha_2)} \|Q_{\alpha_1,\alpha_2} u\|_0^2, \tag{6}$$

where here and below sums over (α_1, α_2) are taken to run over the set $\{0,1\}^2$.

Theorem 6. *For any tensor product spline $u \in S^2$, we have*

$$c^{-1}|u|_1^2 \le \sum_{(\alpha_1,\alpha_2)} |Q_{\alpha_1,\alpha_2} u|_1^2 \le c|u|_1^2,$$
$$c^{-1}\|u\|_1^2 \le \sum_{(\alpha_1,\alpha_2)} \|Q_{\alpha_1,\alpha_2} u\|_1^2 \le c\|u\|_1^2.$$

Proof. We begin with the first statement. The left inequality again follows by the Cauchy-Schwarz inequality. For the right one, fix $(\alpha_1, \alpha_2) \in \{0,1\}^2$ and observe that the H^1-seminorm can be written using tensor products of one-dimensional operators as

$$|Q_{\alpha_1,\alpha_2} u|_1^2 = |Q_{\alpha_1,\alpha_2} u|_{K\otimes M}^2 + |Q_{\alpha_1,\alpha_2} u|_{M\otimes K}^2. \tag{7}$$

The first term can be rewritten, using the definitions and basic identities for tensor products of operators, as

$$|Q_{\alpha_1,\alpha_2} u|_{K\otimes M}^2 = \langle Q'_{\alpha_1,\alpha_2}(K \otimes M) Q_{\alpha_1,\alpha_2} u, u\rangle = \langle (K_{\alpha_1} \otimes M_{\alpha_2}) u, u\rangle.$$

Due to orthogonality and Theorem 5, we have

$$M_0 + M_1 = M, \qquad K_0 + K_1 \le cK,$$

where all summands are positive semidefinite operators. This implies that we can estimate, in the spectral sense, $K_{\alpha_1} \le cK$ and $M_{\alpha_2} \le M$, and we obtain

$$|Q_{\alpha_1,\alpha_2} u|_{K\otimes M}^2 \le c|u|_{K\otimes M}^2.$$

Treating the second term in (7) analogously, we obtain

$$|Q_{\alpha_1,\alpha_2} u|_1^2 \le c(|u|_{K\otimes M}^2 + |u|_{M\otimes K}^2) = c|u|_1^2.$$

The first statement now follows by summing up over all (α_1, α_2). The second statement follows by adding the identity (6). □

3.3 Stable splitting in arbitrary dimensions

For any $d \in \mathbb{N}$, we define multiindices $\alpha \in \{0,1\}^d$ and generalize the notations from Section 3.2 in the straightforward way to higher dimensions. We obtain the splitting into the direct sum of 2^d subspaces

$$S^d = \bigoplus_\alpha S_\alpha, \qquad \text{where} \qquad S_\alpha = S_{\alpha_1} \otimes \ldots \otimes S_{\alpha_d}.$$

The L_2-orthogonal projectors into the subspaces are given by

$$Q_\alpha = Q_{\alpha_1} \otimes \ldots \otimes Q_{\alpha_d} : S^d \to S_\alpha.$$

We obtain that the splitting is H^1-stable completely analogously to the two-dimensional case.

Theorem 7. *For any d-dimensional tensor product spline $u \in S^d$, we have*

$$c^{-1}\|u\|_1^2 \le \sum_{\alpha=(0,\ldots,0)}^{(1,\ldots,1)} \|Q_\alpha u\|_1^2 \le c\|u\|_1^2.$$

The same statement holds in the H^1-seminorm.

Proof. Completely analogous to Theorem 6. □

4 Construction of a robust multigrid method

Recall that S was a spline space of degree p and mesh size h. Let $S_c \subset S$ be the analogous spline space with uniform mesh size $2h$. For the construction of our two-grid method in d dimensions in accordance with the framework introduced in Section 2.3, we let

$$V := S^d, \qquad V_c := (S_c)^d \subset V.$$

The prolongation $P : V_c \to V$ is the canonical embedding of the coarse tensor product spline space in the fine one. It can be represented as the d-fold tensor product of prolongations for the univariate spline spaces, $I : S_c \to S$.

The following result states that a robust approximation error estimate holds for the Galerkin projector to the coarse spline space. It was proved for $d = 1$ and $d = 2$ in [14]. We extend the proof to arbitrary dimensions in the Appendix.

Lemma 8. *The A-orthogonal projector $\Pi_c : S^d \to (S_c)^d$ satisfies the approximation error estimate*

$$\|(I - \Pi_c)u\|_{L_2(\Omega)} \leq ch\|u\|_A \qquad \forall u \in S^d$$

with a constant c which is independent of h and p (but may depend on d).

In the following subsections, we construct a smoother for the two-grid method on these nested spline spaces which leads to a robust and efficient iterative method.

4.1 A multigrid smoother based on subspace correction

In each of the 2^d subspaces $S_\alpha \subset S^d$, $\alpha \in \{0,1\}^d$, we prescribe a local, symmetric and positive definite smoothing operator $L_\alpha : S_\alpha \to S'_\alpha$. The overall smoothing operator is then given by the additive subspace operator

$$L := \sum_\alpha Q'_\alpha L_\alpha Q_\alpha : S^d \to S^{d'}, \tag{8}$$

and its inverse has the form

$$L^{-1} = \sum_\alpha L_\alpha^{-1} Q'_\alpha : S^{d'} \to S^d.$$

The assumptions of Theorem 1 for L, and thus the convergence of the two-grid method with such a smoother, can be guaranteed under simple assumptions on the subspace operators L_α, as the following two lemmas show. The stability of the space splitting is crucial to both proofs.

Lemma 9. *Assume that for every $\alpha \in \{0,1\}^d$, we have*

$$\langle A v_\alpha, v_\alpha \rangle \leq c \langle L_\alpha v_\alpha, v_\alpha \rangle \qquad \forall v_\alpha \in S_\alpha.$$

Then the subspace correction smoother satisfies

$$\langle A v, v \rangle \leq c \langle L v, v \rangle \qquad \forall v \in S^d.$$

Proof. Due to Theorem 7 and the assumption, we have

$$\langle Av, v\rangle \leq c\sum_{\alpha}\langle AQ_\alpha v, Q_\alpha v\rangle \leq c\sum_{\alpha}\langle L_\alpha Q_\alpha v, Q_\alpha v\rangle = c\langle Lv, v\rangle. \qquad \square$$

Lemma 10. *Assume that for every $\alpha \in \{0,1\}^d$, we have*

$$\langle L_\alpha v_\alpha, v_\alpha\rangle \leq c\langle (A + h^{-2}M^d)v_\alpha, v_\alpha\rangle \qquad \forall v_\alpha \in S_\alpha,$$

where $M^d : S^d \to S^{d'}$ is the mass operator in the tensor product spline space. Then the subspace correction smoother satisfies

$$\|(I - \Pi_c)v\|_L \leq c\|v\|_A \qquad \forall v \in S^d.$$

Proof. From the assumption and Theorem 7, we obtain

$$\langle Lv, v\rangle \leq c\sum_{\alpha}\langle (A + h^{-2}M^d)Q_\alpha v, Q_\alpha v\rangle \leq c\langle (A + h^{-2}M^d)v, v\rangle.$$

Thus, it follows

$$\|(I - \Pi_c)v\|_L^2 \leq c\|(I - \Pi_c)v\|_A^2 + ch^{-2}\|(I - \Pi_c)v\|_{M^d}^2 \leq c\|v\|_A^2,$$

where we used the stability of the coarse grid projector and the coarse grid approximation property Lemma 8. $\square$

4.2 Construction in 2D

In the two-dimensional case, the operator associated with the bilinear form (2) admits the representation

$$A = K \otimes M + M \otimes K + M \otimes M$$

in terms of the stiffness and mass operators for the univariate case. Restricting A to a subspace $S_\alpha = S_{\alpha_1,\alpha_2}$, we obtain

$$A_\alpha := Q'_\alpha A Q_\alpha = K_{\alpha_1} \otimes M_{\alpha_2} + M_{\alpha_1} \otimes K_{\alpha_2} + M_{\alpha_1} \otimes M_{\alpha_2}.$$

The inverse inequality in S_0 (Theorem 2) allows us to estimate

$$K_0 \leq ch^{-2}M_0.$$

Using this inequality, we obtain subspace smoothers by estimating

$$\begin{aligned}
A_{00} &\lesssim h^{-2}M_0 \otimes M_0 &&=: L_{00},\\
A_{01} &\lesssim M_0 \otimes (h^{-2}M_1 + K_1) &&=: L_{01},\\
A_{10} &\lesssim (h^{-2}M_1 + K_1) \otimes M_0 &&=: L_{10},\\
A_{11} &= Q'_{11}AQ_{11} &&=: L_{11}.
\end{aligned}$$

By construction, the operators L_α satisfy the assumption of Lemma 9,

$$\langle Av_\alpha, v_\alpha\rangle \leq c\langle L_\alpha v_\alpha, v_\alpha\rangle \qquad \forall v_\alpha \in S_\alpha.$$

Furthermore, it is easy to see that each L_α can be spectrally bounded from above by a constant times $Q'_\alpha(A + h^{-2}M \otimes M)Q_\alpha$, which proves the assumption of Lemma 10. This proves the sufficient conditions for two-grid convergence from Theorem 1 and thus the following result.

Theorem 11. *There exist choices for τ and ν, independent of h and p, such that the two-grid method in S^2 with the smoother induced by the subspace operators L_α as given above converges with a rate $q < 1$ which does not depend on the grid size h or the spline degree p.*

4.3 Construction in 3D

In the three-dimensional case, we have

$$A = K \otimes M \otimes M + M \otimes K \otimes M + M \otimes M \otimes K + M \otimes M \otimes M.$$

Analogously to Section 4.2, we use the inverse inequality in S_0 to estimate

$$\begin{aligned}
A_{000} &\lesssim h^{-2} M_0 \otimes M_0 \otimes M_0 && =: L_{000},\\
A_{001} &\lesssim M_0 \otimes M_0 \otimes (h^{-2} M_1 + K_1) && =: L_{001},\\
A_{010} &\lesssim M_0 \otimes (h^{-2} M_1 + K_1) \otimes M_0 && =: L_{010},\\
A_{100} &\lesssim (h^{-2} M_1 + K_1) \otimes M_0 \otimes M_0 && =: L_{100},\\
A_{011} &\lesssim M_0 \otimes (h^{-2} M_1 \otimes M_1 + K_1 \otimes M_1 + M_1 \otimes K_1) && =: L_{011},\\
A_{110} &\lesssim (h^{-2} M_1 \otimes M_1 + K_1 \otimes M_1 + M_1 \otimes K_1) \otimes M_0 && =: L_{110},\\
A_{101} &\lesssim K_1 \otimes M_0 \otimes M_1 + h^{-2} M_1 \otimes M_0 \otimes M_1 + M_1 \otimes M_0 \otimes K_1 && =: L_{101},\\
A_{111} &= Q_{111}' A Q_{111} && =: L_{111}.
\end{aligned}$$

We point out that, whereas L_{011} and L_{110} permit a tensor product factorization, the operator L_{101} cannot directly be factorized due to the ordering of the involved spaces. However, the tensor product space S_{101} is isomorphic to S_{011} by a simple swapping of the order of the involved tensor products. We exploit this in Section 5.3 below by a simple renumbering of the degrees of freedom in order to obtain an efficient method for inverting L_{101}.

By the same arguments as in Section 4.2, we see that the resulting subspace correction smoother satisfies the assumptions of Theorem 1 and thus that the resulting two-grid method converges robustly with respect to h and p.

Using the technique of reordering the tensor products, this construction extends in a straightforward manner to higher space dimensions d. Furthermore, the proof of robustness for arbitrary d proceeds using the same arguments.

5 Numerical realization

In Section 4, we have constructed a multigrid smoother and shown that it leads to a robust two-grid method. In this section, we provide details on the numerical realization of the method and show that it permits an efficient implementation.

5.1 Computation of a basis for S_0 and S_1

In order to be able to work with the space S_0 and its orthogonal complement, we require bases for them. The aim of this subsection is to provide an algorithm for computing such bases as linear combinations of B-splines.

Recall that the univariate spline space S with m knot spans of width $h = 1/m$, degree p and maximum smoothness C^{p-1} has dimension $n = m + p$. Let

$$\mathcal{B} := \{\varphi_1, \dots, \varphi_n\}$$

denote the normalized (i.e., satisfying a partition of unity) B-spline basis of S. We have $\operatorname{supp} \varphi_j = [(j - p - 1)h, jh] \cap [0, 1]$. All interior B-splines

$$\mathcal{B}^I := \{\varphi_{p+1}, \dots, \varphi_{n-p}\}$$

vanish with all their derivatives up to the $p - 1$st at the boundaries of the interval $[0, 1]$ and therefore lie in S_0. (Here and in the following we assume that $p + 1 \leq m$ such that $\mathcal{B}^I$ is nonempty.)

It remains to find linear combinations of the first and last p B-splines which complete $\mathcal{B}^I$ to a basis of S_0. Recall that $u \in S$ lies in S_0 iff

$$u^{(2l+1)}(0) = u^{(2l+1)}(1) = 0 \quad \forall l \in \mathbb{N}_0 \text{ with } 2l + 1 < p.$$

Consider first the left boundary. We need to satisfy $k := \lfloor p/2 \rfloor$ conditions on the derivatives of the splines. Let

$$\tilde{D} = \left(h^{2i-1} \varphi_j^{(2i-1)}(0) \right)_{i=1,\dots,k,\ j=1,\dots,p} \in \mathbb{R}^{k \times p}$$

denote the matrix of the relevant B-spline derivatives at 0, scaled with a suitable power of h in order to avoid numerical instabilities. We pad $\tilde{D}$ with zero rows to obtain a square matrix $D \in \mathbb{R}^{p \times p}$. Computing the singular value decomposition (SVD), we obtain

$$D = U \Sigma V^\top$$

with $U, V \in \mathbb{R}^{p \times p}$ orthogonal and $\Sigma \in \mathbb{R}^{p \times p}$ diagonal. By construction, Σ contains k nonzero and $p - k$ zero singular values. Therefore, the rightmost $p - k$ columns of V span the kernel of D, and the linear combinations

$$\mathcal{B}_0^L := \left\{ \sum_{i=1}^{p} V_{i,j} \varphi_i : j = p - k + 1, \dots, p \right\}$$

lie in S_0. By the analogous procedure at the right boundary, we compute a set $\mathcal{B}_0^R$ of $p - k$ linear combinations of the last p B-splines. Then, the functions in the set

$$\mathcal{B}_0 := \mathcal{B}_0^L \cup B^I \cup \mathcal{B}_0^R$$

are by construction linearly independent and lie in S_0. Since $n_0 := |\mathcal{B}_0| = n - 2k = \dim S_0$, we have

$$\operatorname{span} \mathcal{B}_0 = S_0.$$

In practice, we collect the coefficients in a sparse block diagonal matrix

$$P_0 = \begin{bmatrix} V^L[:, p - k + 1 : p] & & \\ & I_{n-2p} & \\ & & V^R[:, p - k + 1 : p] \end{bmatrix} \in \mathbb{R}^{n \times n_0},$$

where $V^L[:, p - k + 1 : p] \in \mathbb{R}^{p \times (p-k)}$ denotes the last $p - k$ columns of the matrix V computed for the left boundary, analogously V^R that for the right boundary, and I_d is the $d \times d$ identity matrix. Then clearly, splines in S_0 can be uniquely represented in terms of the B-spline basis as

$$u \in S_0 \quad \Longleftrightarrow \quad \exists \underline{u} \in \mathbb{R}^{n_0} : u = \sum_{j=1}^{n} (P_0 \underline{u})_j \varphi_j.$$

Due to the SVD producing an orthonormal basis, collecting the remaining columns of V^L and V^R in a second sparse block matrix

$$P_\perp = \begin{bmatrix} V^L[:, 1:k] & 0 \\ 0 & 0 \\ 0 & V^R[:, 1:k] \end{bmatrix} \in \mathbb{R}^{n \times 2k}$$

satisfies $P_0^\top P_\perp = 0$. In fact, the columns of the concatenation $\begin{bmatrix} P_0 & P_\perp \end{bmatrix}$ form an orthonormal basis of $\mathbb{R}^n$. Let

$$P_1 := \underline{M}^{-1} P_\perp \in \mathbb{R}^{n \times 2k},$$

where $\underline{M}$ denotes the $\mathcal{B}$-mass matrix. Note that P_1 is no longer sparse. Furthermore, let $\underline{u} \in \mathbb{R}^{n_0}$ and $\underline{v} \in \mathbb{R}^{2k}$ with associated splines

$$u = \sum_{j=1}^{n} (P_0 \underline{u})_j \varphi_j, \qquad v = \sum_{j=1}^{n} (P_1 \underline{v})_j \varphi_j.$$

By construction, $u \in S_0$. We have

$$\langle u, v \rangle_{L_2(\Omega)} = \langle \underline{M} P_0 \underline{u}, \underline{M}^{-1} P_\perp \underline{v} \rangle = \underline{u}^\top P_0^\top P_\perp \underline{v} = 0,$$

which means that v lies in the L_2-orthogonal complement of S_0. All in all, we have constructed basis representations or "prolongation matrices"

$$P_0 \in \mathbb{R}^{n \times (n-2k)}, \qquad P_1 = \underline{M}^{-1} P_\perp \in \mathbb{R}^{n \times 2k}$$

for S_0 and its L_2-orthogonal complement S_1, respectively.

For $d > 1$, we let $\alpha \in \{0, 1\}^d$ and introduce the Kronecker products

$$P_\alpha := P_{\alpha_1} \otimes \ldots \otimes P_{\alpha_d} \in \mathbb{R}^{n^d \times n_\alpha},$$

where $n_\alpha = \dim S_\alpha$, which represent bases for the spaces S_α in terms of the coefficients of linear combinations of the tensor product B-spline basis $\mathcal{B}^{\otimes d}$.

5.2 Implementation of the subspace correction smoother

For $\alpha \in \{0, 1\}^d$, let the symmetric and positive definite matrix $\underline{L}_\alpha \in \mathbb{R}^{n_\alpha \times n_\alpha}$ be the matrix representation of $L_\alpha : S_\alpha \to S'_\alpha$ with respect to the basis for S_α given by P_α as defined in Section 5.1. Then the matrix representation of

$$L^{-1} = \sum_\alpha L_\alpha^{-1} Q'_\alpha = \sum_\alpha I_{S_\alpha \to S^d} L_\alpha^{-1} I_{S^{d'} \to S'_\alpha} Q'_\alpha I_{S^{d'} \to S'_\alpha}$$

is given by

$$\underline{L}^{-1} = \sum_\alpha P_\alpha \underline{L}_\alpha^{-1} P_\alpha^\top \underline{M} P_\alpha \underline{M}_\alpha^{-1} P_\alpha^\top = \sum_\alpha P_\alpha \underline{L}_\alpha^{-1} P_\alpha^\top, \tag{9}$$

where we used that the matrix representation of the embedding $I_{S_\alpha \to S^d}$ is P_α and the matrix representation of the L_2-projector Q_α is

$$\underline{M}_\alpha^{-1} P_\alpha^\top \underline{M}, \quad \text{where} \quad \underline{M}_\alpha = P_\alpha^\top \underline{M} P_\alpha.$$

Hence (9) can be used to implement the subspace correction smoother using only the prolongation matrices P_α and a fast method for applying $\underline{L}_\alpha^{-1}$. It is never necessary to explicitly apply the L_2-projectors Q_α. Furthermore, due to the use of additive subspace correction, the residual needs to be computed only once, and the individual subspace corrections may be done in parallel.

5.3 Inversion of the subspace operators

The final required algorithmic component is a fast method for applying the inverse of the local smoothing matrices $\underline{L}_\alpha \in \mathbb{R}^{n_\alpha \times n_\alpha}$. We illustrate this in the three-dimensional setting as described in Section 4.3, but the principles are the same regardless of dimension. A detailed discussion of the computational costs for arbitrary dimension is given in Section 5.4.

Interior space and face spaces. The interior space S_{000} and the face spaces $S_{001}, S_{010}, S_{100}$ contain the complement space S_1 as a factor space at most once, and thus the matrices associated with their smoothing operators can be represented as Kronecker products of three one-dimensional discretization matrices, e.g.,

$$\underline{L}_{000} = h^{-2} \underline{M}_0 \otimes \underline{M}_0 \otimes \underline{M}_0, \quad \underline{L}_{001} = \underline{M}_0 \otimes \underline{M}_0 \otimes (h^{-2} \underline{M}_1 + \underline{K}_1).$$

Here the symmetric matrices $\underline{M}_\beta, \underline{K}_\beta \in \mathbb{R}^{\dim S_\beta \times \dim S_\beta}$, $\beta \in \{0, 1\}$, are the matrix representations of M_β and K_β, respectively, with respect to the bases described by P_β as computed in Section 5.1 above. For $\beta = 0$, they have dimension $\mathcal{O}(n)$ and are banded, whereas for $\beta = 1$ they have dimension $\mathcal{O}(p)$ and are dense.

Since the Kronecker product can be inverted componentwise, we obtain, e.g.,

$$\underline{L}_{001}^{-1} = \underline{M}_0^{-1} \otimes \underline{M}_0^{-1} \otimes (h^{-2} \underline{M}_1 + \underline{K}_1)^{-1}.$$

Instead of computing this (dense) inverse explicitly, we employ the algorithm described by de Boor [7] for computing the application of a Kronecker product of matrices to a vector, given only routines for applying the individual Kronecker factors. For the latter, we use Cholesky factorization.

Edge spaces. The spaces $S_{011}, S_{110}, S_{101}$ contain the complement space S_1 as a factor twice. In S_{011}, the matrix to be inverted has the form

$$\underline{L}_{011} = \underline{M}_0 \otimes (h^{-2} \underline{M}_1 \otimes \underline{M}_1 + \underline{K}_1 \otimes \underline{M}_1 + \underline{M}_1 \otimes \underline{K}_1).$$

It again has Kronecker product structure and can be inverted using the algorithm described in the previous case. The same holds for S_{110}.

In the case of the space S_{101}, the associated matrix

$$\underline{L}_{101} = \underline{K}_1 \otimes \underline{M}_0 \otimes \underline{M}_1 + h^{-2} \underline{M}_1 \otimes \underline{M}_0 \otimes \underline{M}_1 + \underline{M}_1 \otimes \underline{M}_0 \otimes \underline{K}_1$$

does not permit a Kronecker product factorization due to the order of the involved spaces. However, by a simple renumbering of the degrees of freedom,

S_{101} can be identified with S_{011}, and then $\underline{L}_{011}$ can be applied as above. Alternatively, the matrix $\underline{L}_{101}$ with dimension $\mathcal{O}(np^2)$ can be directly computed and inverted in its entirety using Cholesky factorization. This has the higher computational complexity $\mathcal{O}(n^3p^6)$ but remains quasi-optimal in n since n^3 is the total number of degrees of freedom.

Corner space. The space S_{111} is the tensor product of the three complement spaces and has dimension $\dim(S_1)^3 \leq p^3$. The associated matrix

$$\underline{L}_{111} = P_{111}^\top \underline{A} P_{111}$$

is dense and is inverted by means of Cholesky factorization.

5.4 Computational costs

We now study the computational complexity for applying the subspace correction smoother in the general d-dimensional setting. In our analysis, we ignore multiplicative constants which depend only on d. Repeatedly, we make use of the fact that the Cholesky factorization of a symmetric matrix of dimension n and bandwidth $p \ll n$ can be computed in $\mathcal{O}(np^2)$ operations, and its inverse can then be applied in $\mathcal{O}(np)$ operations. If the matrix is not banded but dense, the factorization and inversion require $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2)$ operations, respectively (cf. [12]).

By the renumbering of degrees of freedom described in Section 5.3, we can always rearrange the factor spaces in such a way that we only need to consider spaces of the form

$$\underbrace{S_0 \otimes \ldots \otimes S_0}_{k \text{ times}} \otimes \underbrace{S_1 \otimes \ldots \otimes S_1}_{d-k \text{ times}}.$$

The matrices to be inverted in these spaces, constructed as in Section 5.3, have the form

$$\underline{L}_{\{k,d-k\}} := \underbrace{\underline{M}_0 \otimes \ldots \otimes \underline{M}_0}_{k \text{ times}} \otimes \underline{X}_{d-k},$$

where $\underline{X}_j \in \mathbb{R}^{(\dim S_1)^j \times (\dim S_1)^j}$ is a dense and symmetric matrix. Recall that $\dim S_1 = \dim S_0^{\perp_{L_2}} \leq p$ is the size of the complement space.

Setup costs. The computation of the basis for S_0 and its L_2-orthogonal complement as described in Section 5.1 requires computing the SVD of two matrices of dimension $\mathcal{O}(p)$ as well as $\mathcal{O}(p)$ applications of the inverse of $\underline{M}$, which has dimension n and bandwidth $\mathcal{O}(p)$. The costs for this step are thus $\mathcal{O}(p^3 + np^2)$.

The one-dimensional mass matrix in S_0, $\underline{M}_0$, has dimension $\mathcal{O}(n)$ and bandwidth $\mathcal{O}(p)$ and thus requires $\mathcal{O}(np^2)$ operations to factorize.

The matrices $\underline{X}_j$, $j = 1, \ldots, d$, are dense and therefore require $\mathcal{O}(p^{3j})$ operations to factorize.

The overall setup costs are therefore $\mathcal{O}(np^2 + p^{3d})$. For $d \geq 2$, this is no more than the cost $\mathcal{O}(n^dp^d)$ for applying the stiffness matrix if $p^2 \lesssim n$.

Application costs. After factorization, the cost for applying the inverse $\underline{M}_0^{-1}$ is $\mathcal{O}(np)$, and for $\underline{X}_j^{-1}$, it is $\mathcal{O}(p^{2j})$. To apply $\underline{L}_{\{k,d-k\}}^{-1}$ using the Kronecker product algorithm from [7], we need to perform n^{d-1} applications of each of the k factors $\underline{M}_0^{-1}$ and n^k applications of $\underline{X}_j^{-1}$. Thus, the cost is $\mathcal{O}(kn^dp + n^kp^{2(d-k)})$.

The inverse of $\underline{L}_{\{k,d-k\}}$ needs to be applied $\binom{d}{k}$ times since that is the number of multiindices $\alpha \in \{0,1\}^d$ which permute to $(0,\dots,0,1,\dots,1)$ with exactly k leading zeros. The binomial coefficient satisfies $\binom{d}{k} = \mathcal{O}(2^d/\sqrt{d})$ and in particular can be bounded from above by a constant which depends only on d. The overall cost for one application of the subspace correction smoother is then

$$\sum_{k=0}^{d} \binom{d}{k} \mathcal{O}(kn^d p + n^k p^{2(d-k)}) = \mathcal{O}\left(n^d p + \max_{k=0,\dots,d} n^k p^{2(d-k)}\right).$$

The first term is less than the cost $\mathcal{O}(n^d p^d)$ for applying the stiffness matrix, and so is the second under the assumption $p \lesssim n$.

Overall costs. In summary, if $d \geq 2$ and $p^2 \lesssim n$, both setup and application of the smoother are asymptotically not more expensive than one application of the stiffness matrix, which has complexity $\mathcal{O}(n^d p^d)$.

6 Numerical experiments

We solve the model problem

$$\begin{aligned} -\Delta u + u &= f \qquad \text{in } \Omega = (0,1)^d, \\ \partial_n u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

for $d = 1, 2, 3$ with the right-hand side

$$f(\mathbf{x}) = d\pi^2 \prod_{j=1}^{d} \sin(\pi(x_j + \tfrac{1}{2})).$$

We perform a (tensor product) B-spline discretization using equidistant knot spans and maximum-continuity splines for varying spline degrees p. We refer to the coarse discretization with only a single interval as level $\ell = 0$ and perform uniform, dyadic refinement to obtain the finer discretization levels ℓ with $2^{\ell d}$ elements and knot spacing $h = 2^{-\ell}$.

We set up a V-cycle multigrid method as described in Section 2.3 and using on each level the proposed smoother (8) as constructed in Section 4. We always use one pre- and one post-smoothing step. The iteration is stopped once the ℓ^2-norm of the initial residual has been reduced by a factor of 10^{-8}.

The method was implemented in C++ based on the G+SMO library[1] which is developed in the framework of the National Research Network "Geometry + Simulation" at Johannes Kepler University, Linz. The results have been obtained on a standard Linux workstation with an Intel® Core™ i7-2600 CPU with 3.40GHz and 8GB RAM. A single CPU core was used per experiment.

The V-cycle multigrid iteration numbers for the 1D, 2D and 3D problem are given in Tables 1–3. For the construction of the smoother, we required that the space S_0 be non-empty. This is the reason that for higher degrees p, the coarse space requires additional refinement steps in some cases. In each column, the bottom-most iteration number corresponds to a two-grid method, the next higher one to a three-grid method, and so on. In the 3D case, memory was insufficient to complete some experiments on level $\ell = 6$.

[1] `http://www.gs.jku.at/gismo`

$\ell \diagdown p$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
9	27	33	34	34	33	33	33	32	31	31	31	28	28	29	29
8	27	33	34	34	32	33	33	31	30	30	31	28	28	27	28
7	27	33	34	34	32	33	33	31	28	30	29	28	25	26	26

Table 1: V-cycle iteration numbers: 1D.

$\ell \diagdown p$	1	2	3	4	5	6	7	8	9	10
8	34	38	39	39	39	38	38	37	37	36
7	34	38	39	39	38	38	37	36	36	34
6	34	38	38	38	37	37	35	34	34	32
5	34	36	37	34	34	32	30	28	26	24
4	34	33	32	28	25	21	19	16	13	11
3	38	25	21	15	11	9	7	-	-	-
2	40	14	10	-	-	-	-	-	-	-
1	43	-	-	-	-	-	-	-	-	-

Table 2: V-cycle iteration numbers: 2D.

$\ell \diagdown p$	2	3	4	5	6	7
6	46	44	43			
5	44	43	42	39	38	35
4	39	36	32	29	25	23
3	30	42	18	23	12	17
2	16	23	-	-	-	-

Table 3: V-cycle iteration numbers: 3D.

We observe that the iteration numbers are robust with respect to both the discretization level ℓ (and thus h) and the spline degree p. They do increase with the space dimension d, but this dependence, which we have not fully analyzed, appears to be relatively mild. In particular, the 2D iteration numbers are significantly lower than those obtained using the boundary-corrected mass smoother in [14].

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Appendix

The aim of this section is to prove Lemma 8, an approximation result for the coarse spline space Galerkin projector in d dimensions. It was shown in [14] for $d = 1$ and $d = 2$, and here we extend it to arbitrary dimensions by induction.

Before we give the proof, we need an auxiliary lemma which is a variant of the Aubin-Nitsche duality argument in a finite-dimensional Hilbert space V. By the choice of a suitable basis, we can identify V with $\mathbb{R}^n$, and operators A on V with matrices. We use this matrix representation implicitly in the following, and operations like $A^{1/2}$ and $A^\top$ are to be understood in the matrix sense.

Lemma 12. *Let A and M be self-adjoint and positive definite linear operators on V, $\Pi : V \to V$ an A-orthogonal projector, and $\theta > 0$. Then, the statements*

$$\|\Pi u\|_M \leq \theta \|u\|_A \qquad \forall u \in V \tag{10}$$

and

$$\|\Pi u\|_A \leq \theta \|u\|_{AM^{-1}A} \qquad \forall u \in V. \tag{11}$$

are equivalent.

Proof. We first observe that (10) and (11) are equivalent to

$$\|M^{1/2}\Pi A^{-1/2}\| \leq \theta, \qquad \|A^{1/2}\Pi A^{-1}M^{1/2}\| \leq \theta, \tag{12}$$

respectively. Since Π is self-adjoint in the scalar product $(\cdot,\cdot)_A$, we have that $A\Pi = \Pi^\top A$ and therefore

$$\Pi A^{-1} = A^{-1}\Pi^\top. \tag{13}$$

Using the self-adjointness of M and A as well as (13), we obtain

$$\begin{aligned}\|A^{1/2}\Pi A^{-1}M^{1/2}\| &= \|(A^{1/2}\Pi A^{-1}M^{1/2})^\top\| \\ &= \|M^{1/2}A^{-1}\Pi^\top A^{1/2}\| = \|M^{1/2}\Pi A^{-1/2}\|.\end{aligned}$$

This proves that the two statements in (12) are equivalent. □

Proof of Lemma 8. Within this proof, we denote the dimensions explicitly and use a recursive representation of the stiffness operator,

$$\begin{aligned} M_d &:= M \otimes \ldots \otimes M \qquad (d \text{ times}), \\ A_1 &:= K + M, \\ A_d &:= A_{d-1} \otimes M + M_{d-1} \otimes K. \end{aligned}$$

Furthermore we let Π_d denote the A_d-orthogonal projector into $(S_c)^d$.

In [14], the desired result was proved for $d = 1$, namely,

$$\|(I - \Pi_1)u\|_M \leq ch\|u\|_{A_1} \qquad \forall u \in S. \tag{14}$$

By Lemma 12, this is equivalent to

$$\|(I - \Pi_1)u\|_{A_1} \leq ch\|u\|_{A_1 M^{-1} A_1} \qquad \forall u \in S. \tag{15}$$

Stability of the A_1-orthogonal projector means that

$$\|(I - \Pi_1)u\|_{A_1} \leq \|u\|_{A_1} \qquad \forall u \in S. \tag{16}$$

We now show the desired result using induction. Assume that we have already shown, for some $d > 1$,

$$\|(I - \Pi_{d-1})u\|_{M_{d-1}} \leq ch\|u\|_{A_{d-1}} \qquad \forall u \in S^{d-1}. \tag{17}$$

Using Lemma 12, this implies

$$\|(I - \Pi_{d-1})u\|_{A_{d-1}} \leq ch\|u\|_{A_{d-1} M_{d-1}^{-1} A_{d-1}} \qquad \forall u \in S^{d-1}. \tag{18}$$

Stability of the A_{d-1}-orthogonal projector means that

$$\|(I - \Pi_{d-1})u\|_{A_{d-1}} \leq \|u\|_{A_{d-1}} \qquad \forall u \in S^{d-1}. \tag{19}$$

Using equations (14)–(19) and the fact that the operator norm of a tensor product is the product of the individual operator norms, we obtain for all $u \in S^d$

$$\begin{aligned} &\|(I - \Pi_{d-1}) \otimes (I - \Pi_1)u\|_{A_{d-1} \otimes M + M_{d-1} \otimes A_1} \leq ch\|u\|_{A_{d-1} \otimes A_1}, \\ &\|(I - \Pi_{d-1}) \otimes Iu\|_{A_{d-1} \otimes M + M_{d-1} \otimes A_1} \leq ch\|u\|_{A_{d-1} M_{d-1}^{-1} A_{d-1} \otimes M + A_{d-1} \otimes A_1}, \\ &\|I \otimes (I - \Pi_1)u\|_{A_{d-1} \otimes M + M_{d-1} \otimes A_1} \leq ch\|u\|_{A_{d-1} \otimes A + M_{d-1} \otimes A_1 M^{-1} A_1}. \end{aligned}$$

As

$$I - \Pi_{d-1} \otimes \Pi_1 = (I - \Pi_{d-1}) \otimes I + I \otimes (I - \Pi_1) - (I - \Pi_{d-1}) \otimes (I - \Pi_1),$$

this implies using the triangle inequality

$$\begin{aligned} &\|(I - \Pi_{d-1} \otimes \Pi_1)u\|_{A_{d-1} \otimes M + M_{d-1} \otimes A_1} \\ &\qquad \leq ch\|u\|_{A_{d-1} M_{d-1}^{-1} A_{d-1} \otimes M + A_{d-1} \otimes A_1 + M_{d-1} \otimes A_1 M^{-1} A_1}. \end{aligned}$$

As the norm on the left-hand side is bounded from below by $\| \cdot \|_{A_d}$ and the norm on the right-hand side is bounded from above by $c\| \cdot \|_{A_d M_d^{-1} A_d}$, we further obtain

$$\|(I - \Pi_{d-1} \otimes \Pi_1)u\|_{A_d} \leq ch\|u\|_{A_d M_d^{-1} A_d} \qquad \forall u \in S^d.$$

Both $\Pi_{d-1} \otimes \Pi_1$ and Π_d are projectors into $(S_c)^d$. Since the latter projector produces the best approximation in the A_d-norm, we have

$$\|(I - \Pi_d)u\|_{A_d} \leq ch\|u\|_{A_d M_d^{-1} A_d} \qquad \forall u \in S^d,$$

which by Lemma 12 is equivalent to the desired result

$$\|(I - \Pi_d)u\|_{M_d} \leq ch\|u\|_{A_d} \qquad \forall u \in S^d. \qquad \square$$

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