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Interior point method for the numerical simulation of the obstacle problem

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Abstract

In this work the interior point method is considered for the numerical solution of the obstacle problem. The scheme follows the predictor-corrector approach. For the numerical realization the unknowns are approximated by using first order finite elements. Results for several 2D examples are presented.

Contents

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Introduction

The unknown of the obstacle problem is the deflection of the membrane which is stretched downward under an acting force and the deformation of which is restricted from below by an obstacle. Mathematically this problem can be posed as a constrained convex minimization problem, as a variational inequality, free boundary problem or as a linear complementarity problem. The mathematical formulations of this problem appears in many other applications: fluid filtration in porous media, elasto-plasticity, optimal control and financial mathematics.

The formulations of the obstacle problem and existence of the solution have been discussed in many works e.g. by G.Stampacchia, L.A.Caffarelli, A.Friedman. Although there are results on the existence of the solution, it's difficult to find the analytical solution in general case. That's why the effective methods of the finding of numerical solution of this problem take important place in applications.

In this work we consider an interior point method for the approximate solution of the obstacle problem. In chapter 1 we derive the mathematical model and discuss the existence and uniqueness of the solution. This chapter is based mainly on the works [3] and [4]. In the chapter 2 we construct the interior point method for the obstacle problem when it's considered as a minimization problem. In the third chapter the algorithm for the implementation is formulated. We use piecewise linear finite elements for the approximation of the solution. The part of this chapter about the finite element error estimate is based on the work [8]. And, finally, we discuss numerical results of several examples.

Chapter 1

Mathematical modeling of the obstacle problem

1.1 Statement of the problem

We consider the membrane problem, when an elastic membrane is attached to a flat wireframe and force is acting on it only in vertical direction. By $\Omega \subset \mathbb{R}^2$ we denote the domain which is enclosed by the wireframe. We denote by $v(x,y)$ the deflection of the membrane. We choose the Cartesian coordinate system such that the Oxy plane coincides with the plane of the wireframe, so $v(x,y) = 0$ on $\partial\Omega$. We assume that the rigid body which we call "the obstacle" in the following is placed under the membrane. Denoting by $\psi(x,y)$ the surface of the obstacle we make assumption that $\psi(x,y) \leq 0 \quad \forall (x,y) \in \partial\Omega$.

Total energy of the deformed membrane is :

$$J(v) = P(v) - E(v),$$

where $P(v)$ - is the potential energy and $E(v)$ is the energy due to the external forces. Assuming that the potential energy is proportional to the change of the membrane's surface area we can approximate it by using of Taylor expansion:

$$P(v) = \int_\Omega \sqrt{1 + \frac{dv}{dx}^2 + \frac{dv}{dy}^2}\, d\omega - \mu(\Omega) \approx \frac{1}{2}\int_\Omega |\nabla v|^2\, d\omega,$$

here and in the following μ stays for the Lebesgue measure. Then the total energy is:

$$J(v) = \frac{1}{2}\int_{\Omega} \mid \nabla v \mid^2 d\omega - \int_{\Omega} f v d\omega.$$

The "obstacle problem" consists in finding the equilibrium state of the membrane, i.e. in minimizing the energy functional $J(v)$, when the deflection of the membrane is restricted from below by the obstacle. Then the set of admissible deflections is given as :

$$K = \{v \in H_0^1(\Omega) | v \geq \psi \text{ a.e. in } \Omega\}.$$

We see that the K is not a linear set. Throughout this work we assume that $\psi \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)$.

Thus, we come up with the following:

Problem 1.1.1 *Given a bounded domain $\Omega \subset \mathbb{R}^2$ and functions $f \in H^{-1}$and$\psi \in L^2(\Omega)$, find a solution $u \in H_0^1$ such that*

$$J(u) = \min_{v \in K} J(v) \quad \forall v \in K,$$

where the functional $J(v) : H_0^1 \to \mathbb{R}$ is represented by :

$$J(v) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 d\omega - \langle f, v \rangle.$$

1.2 Existence and uniqueness of the solution

For the existence and uniqueness of the solution of the problem (1.1.1) we bring here the following statement for more general problem:

Theorem 1.2.1 *Let H be a Hilbert space, $K \subset H$ be closed and convex, the continuous bilinear form $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ be symmetric and coercive, i.e.*

$$\exists \alpha > 0 : \quad a(v,v) \geq \alpha \|v\|^2 \quad v \in H,$$

and $f \in H^*$.

Then there exists unique solution to the minimization problem:

$$\text{to find } u \in K : J(u) = \min_{v \in K} J(v) \quad \forall v \in K,$$

where the functional $J : H \to \mathbb{R}$ *is defined by*

$$J(v) = \frac{1}{2} a(v, v) - \langle f, v \rangle.$$

Remark Here $\langle f, v \rangle$ is the pairing between f and v, i.e. $\langle f, v \rangle = f(v)$.

Definition 1.2.2 *The point* $y \in K$ *such that*

$$\|x - y\| \le \|x - z\| \; \forall z \in K$$

is called the projection of x *onto* K.

For the proof of the theorem we need the following lemma:

Lemma 1.2.3 *If* K *is closed and convex subset of a Hilbert space then each* $x \in H$ *admits unique projection on* K.

Proof of the lemma (1.2.3) Let $d = \inf_{z \in K} \|x - z\|$. Then we can find a sequence $\{\eta_k\} \in K : \lim_{k \to \infty} \|\eta_k - x\| = d$. Since K is convex, for any $\eta_m, \eta_n \in \{\eta_k\}$ it holds $\frac{1}{2}(\eta_m + \eta_n) \in K$ and

$$d^2 \le \left\| x - \frac{1}{2}(\eta_m + \eta_n) \right\|^2.$$

Applying the parallelogram law for Hilbert space:

$$2\|x - \eta_m\|^2 + 2\|x - \eta_n\|^2 = \|\eta_n - \eta_m\|^2 + 2\left\| x - \frac{1}{2}(\eta_n + \eta_m) \right\|^2.$$

Hence:

$$\|\eta_n - \eta_m\|^2 \le 2\|x - \eta_m\|^2 + 2\|x - \eta_n\|^2 - 4d^2.$$

From this it follows that $\{\eta_k\}$ is Cauchy sequence, and since H is complete, it converges to an element $y \in H$. And, since K is closed, $y \in K$ and $\|x - y\| = d$.

Now, let's suppose that there are two projections y_1 and y_2 of the element x. Then from the previous discussions it follows that :

$$\|y_1 - y_2\|^2 \leq 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - 4d^2 = 0.$$

Therefore, $y_1 = y_2$.■

Proof of the theorem (1.2.1) We can consider the bilinear form $a(\cdot,\cdot)$ as the inner product in Hilbert space H, then the norm $\|v\|_a = \sqrt{a(v,v)}$ is equivalent to the given norm in H. By the Riesz representation lemma we can find an element $v^* \in H$ such that

$$\langle f, v\rangle = a(v^*, v) \; \forall v \in H.$$

Then

$$J(v) = \frac{1}{2}a(v,v) - a(v^*,v) = \frac{1}{2}\|v - v^*\|_a^2 - \frac{1}{2}\|v^*\|_a^2.$$

Thus, the problem of minimization of $J(v)$ reduces to finding of the projection of v^* on the closed convex set K. By lemma (1.2.3) there exist unique projection, therefore unique solution of the minimization problem.■

To apply this theorem for the problem (1.1.1) we need to show respective properties of the set K and $a(u,v) = \int_\Omega \nabla u \nabla v d\omega$.

Lemma 1.2.4 *K is convex and closed.*

Proof Let $u, v \in K$. Then for $0 < t < 1$: $tu + (1-t)v \in V$ and $tu + (1-t)v \geq t\psi + (1-t)\psi = \psi$. This shows convexity of K.

Now let $\{v_k\} \in K$ be a convergent sequence. Since from the convergence in V follows convergence in $L^2(\Omega)$, it contains an a.e. pointwise convergent subsequence $\{v_{k_n}\}$. Let $\{v_{k_n}\} \longrightarrow v$ pointwise. Suppose that v is not from K.

Then there exist subset $A \subset K : \mu(A) > 0$ such that $v < \psi$ on A, more precisely, $\exists \varepsilon > 0 : v \le \psi - \varepsilon$ on A. But then

$$\int_\Omega |v_{k_n} - v|^2 dxdy \ge \int_A |v_{k_n} - v|^2 dxdy \ge \varepsilon^2 \mu(A) > 0.$$

This contradicts to the above mentioned a.e. pointwise convergence. ■

$a(u,v) = \int_\Omega \nabla u \nabla v d\omega$ is bilinear and symmetric. It's continuous:

$$\int_\Omega \nabla u \nabla v d\omega \le |u|_{1,\Omega} |v|_{1,\Omega} \le \|u\|_{1,\Omega} \|v\|_{1,\Omega},$$

and it's coercitivity follows from Poincaré-Friedrichs inequality:

$$\exists C(\Omega) > 0 : \quad a(v,v) = |v|^2_{1,\Omega} \ge C(\Omega) \|v\|^2_{1,\Omega} \quad \forall v \in H^1_0.$$

Thus, we have all prerequisites for the following statement:

Theorem 1.2.5 *Assume that $\exists\, \tilde{u} \in H^1_0 : \tilde{u} \ge \psi$ a.e. in Ω. Then there exists unique solution $\bar{u}$ to the problem (1.1.1).*

1.3 Alternative equivalent formulations

1.3.1 Variational inequality

One of the most useful approaches to obtain the properties of the solution to the problem (1.1.1) is using its equivalent formulations by variational inequality. The problem (1.1.1) can be reformulated as follows:

Problem 1.3.1 *Let $K \subset H^1_0$ be closed and convex and $f \in H^{-1}$. To find:*

$$u \in K : \int_\Omega \nabla u \nabla (v - u) d\omega \ge \langle f, v - u \rangle \ \forall v \in K.$$

Theorem 1.3.2 *$u \in K$ solves (1.1.1) if and only if it solves (1.3.1).*

Proof Let $u \in K$ be the solution to the minimization problem (1.1.1). Then for $t \in [0,1] : u + t(v-u) \in K \; \forall v \in K$. Function defined by $\phi(t) = J(u+t(v-u))$, $t \in [0,1]$ attains its minimum at the point $t = 0$, i.e.

$$\phi(0) \leq \phi(t) \; \forall t \in [0,1].$$

Then

$$0 \leq \lim_{t \to 0+} \frac{\phi(t) - \phi(0)}{t} = \int_\Omega \nabla u \nabla (v-u) d\omega - \langle f, v-u \rangle.$$

Let $u \in K$ be such that $\int_\Omega \nabla u \nabla (v-u) d\omega \geq \langle f, v-u \rangle \; \forall v \in K$. Then for any $v \in K$ it holds :

$$J(v) - J(u) = \phi(1) - \phi(0) = \int_\Omega \nabla u \nabla (v-u) d\omega - \langle f, v-u \rangle + \frac{1}{2} \int_\Omega |\nabla (v-u)|^2 d\omega \geq 0.$$

■

The next theorem states about the well-posedness of the problem:

Theorem 1.3.3 *There exist unique solution to the problem (1.3.1). In addition, the mapping $f \to u$ is Lipschitz, that is, if u_1, u_2 are solutions to the problem (1.3.1) corresponding to f_1, $f_2 \in H^{-1}$, then*

$$\|u_1 - u_2\| \leq L \|f_1 - f_2\|_{H^{-1}}, \tag{1.1}$$

where $L > 0$ constant.

Proof Existence of the unique solution results from the previous discussions, namely, from the equivalence of the variational inequality (1.3.1) to the minimization problem (1.1.1).

We demonstrate validity of (1.1). We set $v = u_2$ in the variational inequality for the solution u_1 and $v = u_1$ in the inequality for u_2. Upon adding we obtain:

$$\int_\Omega |\nabla (u_1 - u_2)|^2 d\omega \leq \langle f_1 - f_2, u_1 - u_2 \rangle.$$

From the coercitivity of the form $a(u,v) = \int_\Omega \nabla u \nabla v d\omega$, it follows that

$$C(\Omega) \|u_1 - u_2\|^2 \leq \langle f_1 - f_2, u_1 - u_2 \rangle \leq \|f_1 - f_2\|_{H^{-1}} \|u_1 - u_2\|. \quad ■$$

1.3.2 Free boundary problem

Now we assume that $u \in H^2(\Omega) \cap K$ is the solution of the obstacle problem. We divide domain Ω into the set $\Omega_+ = \{x \in \Omega : u(x) > \psi(x)\}$, which we call *nonco-incidence set*, and *coincidence set* $\Omega_0 = \{x \in \Omega : u(x) = \psi(x)\}$.

Since u solves the variational inequality (1.3.1), applying Green's formula to the left hand side of the inequality we obtain

$$-\int_\Omega \Delta u(v-u)d\omega + \int_{\partial\Omega} \frac{\partial u}{\partial n}(v-u)ds \geq \int_\Omega f(v-u)d\omega \quad \forall v \in K. \tag{1.2}$$

Both $u, v \in K \subset H_0^1$, so the boundary term vanishes. Let's take any nonnegative function $\zeta \in C_0^\infty(\Omega)$. Then $v = u + \varepsilon\zeta \in K$ for $\varepsilon \geq 0$. Substituting this in the latter inequality, we obtain:

$$-\int_\Omega \Delta u \zeta d\omega \geq \int_\Omega f \zeta d\omega, \quad \forall \zeta \in C_0^\infty,\ \zeta \geq 0.$$

From this it follows that :

$$-\Delta u \geq f \text{ a.e. in } \Omega.$$

Let's assume that $\psi \in C(\overline{\Omega})$. Then the set Ω_+ is open, since $u \in H^2 \hookrightarrow C(\overline{\Omega})$. We consider a point $x \in \Omega_+$, for which we can choose a neighborhood $U_\delta(x)$ such that $U_\delta(x) \subset \Omega_+$. For any $\zeta \in C_0^\infty(U_\delta(x))$ we may find an $\varepsilon > 0$ such that $v = u + \varepsilon\zeta \in K$. Substituting this v in (1.2) and dividing by ε we find that

$$-\int_{U_\delta(x)} \Delta u \zeta d\omega \geq \int_{U_\delta(x)} f \zeta d\omega \quad \forall \zeta \in C_0^\infty(U_\delta(x).)$$

In particular this holds for $-\zeta$, therefore

$$-\Delta u = f \text{ in } U_\delta(x),$$

and also a.e. in Ω_+.

The function $u - \psi \in H^2(\Omega)$, where $u \in K$ is the solution of the obstacle problem, attains its minimum in Ω in the coincidence set Ω_0. Thus, using the

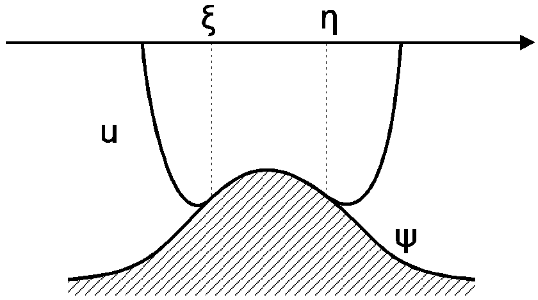

Figure 1.1: Free boundary for 1D problem.

necessary condition for extremal point, we have then:

$$u = \phi \quad \text{on } \Gamma^*,$$

$$\nabla u = \nabla \phi \quad \text{on } \Gamma^*,$$

where $\Gamma^* = \partial\Omega_+ \cap \Omega$ is called the *free boundary* of the problem.

We have shown that the solution of the obstacle problem with the appropriate data corresponds to the formal solution of the following boundary problem:

$$\begin{cases} -\Delta u \geq f & \text{a.e. in } \Omega, \\ u \geq \psi & \text{a.e. in } \Omega, \\ \text{if } u(x) > \psi(x) & \text{then } -\Delta u(x) = f(x), \\ u = \phi & \text{on } \Gamma^*, \\ \nabla u = \nabla \phi & \text{on } \Gamma^*, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that the free boundary of the problem is not known in advance. This kind of formulations may be useful only in the one-dimensional case of the problem, since we don't know about the smoothness of the Γ^*.

1.3.3 Linear complementarity problem

We can reformulate the free boundary problem so that the free boundary conditions need not be handled explicitly as follows:

$$\begin{cases} -\Delta u \geq f & \text{a.e. in } \Omega, \\ u \geq \psi & \text{a.e. in } \Omega, \\ (u-\psi)(-\Delta u - f) = 0 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Some results for this form of the obstacle problem interested reader can find in the work of Brézis and Stampacchia: if $f \in L^2(\Omega)$, and $\psi \in H^2(\Omega)$ then the problem (1.3.1) has a unique solution $u \in H^2(\Omega)$ and it satisfies (1.3).

Chapter 2

Interior point method for the numerical solution of the obstacle problem

2.1 The central path

The idea of the interior point methods is to replace the constrained minimization problem by a sequence of unconstrained minimization problems, for solving of which we can use Newton's method. An objective functional of the unconstrained problem we generate by adding barrier functional to the objective functional of the original constrained problem. Barrier function serves as barrier against leaving of the elements the feasible region K. Each of the problems in the sequence corresponds to the objective functional $J_\kappa(v)$ depending on the nonnegative penalty parameter κ. For our problem we construct $J_\kappa(v)$ in the following way:

$$J_\kappa(v) = J(v) - \kappa \int_\Omega \ln(v - \psi) d\omega.$$

We extend the definition of the ln to the whole real domain axis by setting $\ln z = -\infty, \quad z \leq 0$. Then barrier function approaches infinity as the elements from the interior approach the boundary. Thus we obtained the family of the following unconstrained minimization problem:

Problem 2.1.1 *Given a bounded domain $\Omega \subset \mathbb{R}^2$ and functions $f \in H^{-1}$and$\psi \in L^2(\Omega)$, find a solution $u_\kappa \in H_0^1(\Omega)$ such that*

$$J_\kappa(u_\kappa) = \min_{v \in H_0^1(\Omega)} J_\kappa(v) \quad \forall v \in H_0^1(\Omega),$$

where the functional $J_\kappa(v) : H_0^1 \to \mathbb{R}$ is represented by :

$$J_\kappa(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 d\omega - \langle f, v \rangle - \kappa \int_\Omega \ln(v - \psi).$$

Now we will determine the existence and uniqueness of the solution for the auxiliary problems (2.1.1). We use the following theorem:

Theorem 2.1.2 *Let V be a Hilbert space and let $F : V \to \mathbb{R} \cup \{\pm\infty\}$ be a proper lower semicontinuous function. If F is coercive, i.e. that*

$$\lim_{\|u\| \to \infty} F(u) = \infty,$$

then the problem

$$\min_{u \in V} F(u)$$

has at least one solution. If F is strictly convex, then the solution is unique.

Proof Let $\{u_n\} \in V$ be the minimizing sequence :

$$c = \lim_{n \to \infty} F(u_n) = \inf_{u \in V} F(u).$$

Since $c < +\infty$ and $F(u)$ is coercive:

$$\|u_n\| < \text{const}.$$

Then we can find subsequence $\{u_{n_k}\} \subset \{u_n\}$ weakly converging in V :

$$u_{n_k} \rightharpoonup \bar{u} \in V.$$

By virtue of lower semicontinuity of $F(u)$:

$$\lim_{n\to\infty} F(u_n) \geq F(\bar{u}).$$

Then it holds that

$$F(\bar{u}) \leq \inf_{u\in V} F(u),$$

so $u = \bar{u}$ is the solution to the problem.

Let now $u_1, u_2 \in V$ be two solutions to the problem. Then $\frac{1}{2}(u_1 + u_2) \in V$. If $F(u)$ is strictly convex function, then

$$F\left(\frac{1}{2}(u_1+u_2)\right) < \min_{u\in V} F(u),$$

so only one solution exists for the strictly convex $F(u)$. ■

We make an assumption (A1):

$$\exists \tilde{u} \in H_0^1(\Omega): \quad -\int_\Omega \ln(\tilde{u}-\psi)d\omega < \infty.$$

Theorem 2.1.3 *With the assumption (A1) for each $\kappa > 0$ the problem (2.1.1) has a unique solution u_κ.*

Proof Let $\kappa > 0$ be an arbitrary fixed number. We apply theorem 2.1.2 with $V = H_0^1(\Omega)$ and $F(u) = J_\kappa(u)$. $J_\kappa(u)$ is the sum of the strictly convex continuous function $J(u)$ and the function $\phi(u) = -\int_\Omega \ln(u-\psi)d\omega$. It's clear that $\phi(u)$ is a convex function and consequently $J_\kappa(u)$ is strictly convex. Using the inequality $\ln z \leq z$ for $z > 0$ we obtain

$$\phi(u) \geq -\int_\Omega \max\{u-\psi, 0\} \geq -\|u-\psi\|_{L^2(\Omega)}\mu(\Omega)^{\frac{1}{2}}.$$

Since $\|u-\psi\| \geq \|u-\psi\|_{L^2(\Omega)}$, together with the assumption (A1) it follows that ϕ and hence also J_κ is proper. Moreover, we obtain

$$J_\kappa(u) \geq c\|u\|^2 - \|f\|\|u\| - \kappa\|u-\psi\|\mu(\Omega)^{\frac{1}{2}}$$

for some $c > 0$. Hence

$$\lim_{\|u\|\to\infty} J_\kappa(u) = \infty$$

follows. It remains to show lower semicontinuity of J_κ. It suffices to show that ϕ is lower semicontinuous.

From the definition of the Lebesgue integral,

$$\phi(u) = \sup_{\varepsilon>0} \phi_\varepsilon(u)$$

follows, where $\phi_\varepsilon(u) = -\int_\Omega \ln(\max\{u-\psi,\varepsilon\})d\omega$. Since the pointwise supremum of a family of continuous functions is lower semicontinuous, lower semicontinuity of ϕ follows. ■

Definition 2.1.4 *The mapping $\kappa \to u_\kappa$ is called the* ***central path****.*

The idea behind the interior point method is to follow the central path. This means we begin with some value of κ and find the solution to the corresponding unconstrained problem. Then we decrease κ and solve again the auxiliary problem, and so on. Next theorem states about the convergence of u_κ to $\bar{u}$ as $\kappa \to 0$.

Theorem 2.1.5 *For all $\kappa > 0$ one has $J(u_\kappa) \le J(\bar{u}) + \kappa\mu(\Omega)$.*

Proof We denote again by $\phi(u) = -\int_\Omega \ln(u-\psi)d\omega$.

For fixed $\kappa > 0$, let $\eta(t) = J_\kappa(v_t)$ for $t \in [0,1)$, where $v_t = t\bar{u} + (1-t)u_\kappa$. Since

$$v_t - \psi = t(\bar{u}-\psi) + (1-t)(u_\kappa - \psi) \ge (1-t)(u_\kappa - \psi) \ge 0 \text{ a.e. in } \Omega,$$

together with the monotonicity of ln, we have $\eta(0) \le \eta(t) \le J(v_t) + \kappa\phi(u_\kappa) - \kappa\ln(1-t)\mu(\Omega) < +\infty$, $t \in [0,1)$. It's easy to see that $\eta(t)$ is convex on $[0,1)$.

Then the function $\theta(t) = \frac{\eta(t)-\eta(0)}{t}$ is monotone nondecreasing and it's bounded from below by 0. Hence there exists

$$\eta'(0) = \lim_{t\to 0+} \frac{\eta(t)-\eta(0)}{t} \geq 0, \quad t \in (0,1).$$

Further we have

$$0 \leq \eta'(0) = \int_\Omega \nabla u_\kappa \nabla(\bar{u}-u_\kappa)d\omega + \langle f, \bar{u}-u_\kappa\rangle + \kappa \lim_{t\to 0} \frac{\phi(v_t)-\phi(0)}{t}.$$

It also follows

$$\frac{\phi(v_t)-\phi(v_0)}{t} = \kappa \int_\Omega t^{-1}(\ln(v_0-\psi)-\ln(v_t-\psi))d\omega.$$

As $t \to 0$, the integrand converges to $\frac{u_\kappa-\bar{u}}{u_\kappa-\psi}$ a.e. in Ω. Further, on the set $\{u_\kappa \geq \bar{u}\}$ we have

$$0 \leq t^{-1}(\ln(v_0-\psi)-\ln(v_t-\psi)) \leq \frac{u_\kappa-\bar{u}}{v_t-\psi} \leq \frac{u_\kappa-\psi}{v_t-\psi} \leq \frac{1}{1-t},$$

and hence, by theorem of dominated convergence, we obtain

$$\lim_{t\to 0+} \int_{\{u_\kappa \geq \bar{u}\}} t^{-1}(\ln(v_0-\psi)-\ln(v_t-\psi))d\omega = \int_{\{u_\kappa\geq\bar{u}\}} \frac{u_\kappa-\bar{u}}{u_\kappa-\psi}d\omega.$$

On the other hand, on the set $u_\kappa < \bar{u}$ we have

$$t^{-1}(\ln(v_0-\psi)-\ln(v_t-\psi)) \searrow \frac{u_\kappa-\bar{u}}{u_\kappa-\psi} < 0, \text{ for } t \to 0+.$$

Since

$$\lim_{t\to 0+} \int_{\{u_\kappa<\bar{u}\}} \frac{\ln(v_0-\psi)-\ln(v_t-\psi)}{t}d\omega =$$

$$= \frac{\eta'(0) - \int_\Omega \nabla u_\kappa \nabla(\bar{u}-u_\kappa)d\omega - \langle f, \bar{u}-u_\kappa\rangle}{\kappa} - \int_{\{u_\kappa\geq\bar{u}\}} \frac{u_\kappa-\bar{u}}{u_\kappa-\psi}d\omega$$

we have

$$0 \geq \lim_{t\to 0+} \int_{\{u_\kappa<\bar{u}\}} \frac{\ln(v_0-\psi)-\ln(v_t-\psi)}{t}d\omega = \int_{\{u_\kappa<\bar{u}\}} \frac{u_\kappa-\bar{u}}{u_\kappa-\psi}d\omega > -\infty$$

by the theorem of Beppo-Levi. Hence, $\int_\Omega \frac{u_\kappa - \bar{u}}{u_\kappa - \psi} d\omega$ exists and

$$0 \leq \eta'(0) = \int_\Omega \nabla u_\kappa \nabla(\bar{u} - u_\kappa) d\omega + \langle f, \bar{u} - u_\kappa \rangle + \kappa \int_\Omega \frac{u_\kappa - \bar{u}}{u_\kappa - \psi} d\omega.$$

It follows that $t = 0$ is the solution of the optimization problem

$$\min_{t \geq 0} J(v_t) + \kappa t \int_\Omega \frac{u_\kappa - \bar{u}}{u_\kappa - \psi} d\omega.$$

Therefore,

$$J(v_0) \leq J(v_1) + \kappa \int_\Omega 1 - \frac{\bar{u} - \psi}{u_\kappa - \psi} d\omega \leq J(v_1) + \kappa \mu(\Omega)$$

and since $v_0 = u_\kappa$, $v_1 = \bar{u}$, the theorem is proved. ■

Using in this inequality Taylor expansion of $J(u_\kappa)$ at the point $\bar{u}$, we obtain:

$$\frac{1}{2} \int_\Omega |\nabla(u_\kappa - \bar{u})|^2 d\omega \leq \kappa \mu(\Omega),$$

or, equivalently,

$$\|u_\kappa - \bar{u}\|_{H_0^1(\Omega)} = O(\sqrt{\kappa}).$$

2.2 A predictor-corrector approach for the following the central path

All the results from the preceding section remain valid if we replace $H_0^1(\Omega)$ by a closed subspace $V \subset H_0^1(\Omega)$, e.g. if $\exists\, \tilde{u} \in V \ : \ \int_\Omega \ln(\tilde{u} - \psi) d\omega < \infty$ then for each $\kappa > 0$ the problem

$$\min_{v \in V} J_\kappa(v) \quad \forall v \in V,$$

with

$$J_\kappa(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 d\omega - \langle f, v \rangle - \kappa \int_\Omega \ln(v - \psi),$$

has a unique solution which is again denoted by u_κ.

2.2.1 The corrector step

For given $\kappa > 0$ we are given an approximation $u \in V$ of u_κ and we want to find a better approximation u^+ by means of Newton's method, i.e. $u^+ = u + \delta u$ where the so called Newton corrector δu is a solution of the problem

$$\min_{v \in V} \phi_{\kappa,u}(v) \tag{2.1}$$

with

$$\phi_{\kappa,u}(v) = J_\kappa(u) + J'_\kappa(u)v + \frac{1}{2} J''_\kappa(v,v).$$

Here $J'_\kappa(u) : V \to \mathbb{R}$, and $J''_\kappa(u) : V \times V \to \mathbb{R}$ are

$$J'_\kappa(u)v = \int_\Omega \nabla u \nabla v d\omega - \langle f, v \rangle - \kappa \int_\Omega \frac{v}{u - \psi} d\omega$$

and

$$J''_\kappa(u)(v,w) = \int_\Omega \nabla v \nabla w d\omega + \kappa \int_\Omega \frac{vw}{(u-\psi)^2} d\omega.$$

For ease of presentation we assume that $u - \psi \geq \varepsilon > 0$ a.e. in Ω. Then $J'(u)$ and $J''(u)$ are well defined. Further, it easily follows that $\phi_{\kappa,u}$ is a continuous strictly convex functional on V and a unique minimizer δu of $\phi_{\kappa,u}$ exists and satisfies

$$J''_\kappa(u)(\delta u, v) = -J'_\kappa(u)v, \quad \forall v \in V.$$

So at this step we replaced the objective functional J_κ by its quadratic approximation $\phi_{\kappa,u}$. Since we used the logarithmic barrier functions it's sufficient here to use Taylor expansion up to the second order. Thus we can rely on Newton's method for solving this problem.

Now let us analyze the next corrector step given by the unique solution δu^+ of the optimization problem

$$\min_{v \in V} \phi_{\kappa,u^+}(v).$$

Since

$$\begin{aligned} J'_\kappa(u^+)v &= \int_\Omega \nabla u^+ \nabla v d\omega - \langle f, v\rangle - \kappa \int_\Omega \frac{v}{u^+ - \psi} d\omega \\ &= J'_\kappa(u)v + J''_\kappa(u)(\delta u, v) - \kappa \int_\Omega \left(\frac{v}{u^+ - \psi} - \frac{v}{u - \psi} + \frac{\delta u\, v}{(u-\psi)^2} \right) d\omega \\ &= -\kappa \int_\Omega \frac{(\delta u)^2 v}{(u^+ - \psi)(u-\psi)^2} d\omega \end{aligned}$$

we obtain

$$J''_\kappa(u^+)(\delta u^+, v) = -J'_\kappa(u^+)v = \kappa \int_\Omega \frac{(\delta u)^2 v}{(u^+ - \psi)(u-\psi)} d\omega, \ \forall v \in V.$$

By taking $v = \delta u^+$ it follows that

$$\int_\Omega |\nabla \delta u^+|^2 d\omega + \kappa \int_\Omega \frac{(\delta u^+)^2}{(u^+ - \psi)^2} d\omega = \kappa \int_\Omega \frac{\delta u^+ (\delta u)^2}{(u^+ - \psi)(u-\psi)^2} d\omega. \tag{2.2}$$

Using the Cauchy-Schwartz inequality we conclude that

$$\kappa \int_\Omega \frac{(\delta u^+)^2}{(u^+ - \psi)^2} d\omega \le \kappa \left(\int_\Omega \frac{(\delta u^+)^2}{(u^+ - \psi)^2} d\omega \right)^{\frac{1}{2}} \left(\int_\Omega \frac{(\delta u)^4}{(u-\psi)^4} d\omega \right)^{\frac{1}{2}}$$

and

$$\int_\Omega \frac{(\delta u^+)^2}{(u^+ - \psi)^2} d\omega \le \int_\Omega \frac{(\delta u)^4}{(u-\psi)^4} d\omega. \tag{2.3}$$

Now consider Newton's step defined by $u^{n+1} = u^n + \delta u^n$, $n = 0, 1, 2, ...$, where δu^n is the unique minimizer of $\phi_{\kappa, u^n}(\cdot)$.

Definition 2.2.1 *We say the sequence $\{\gamma^n\}$ converges R-superlinearly to γ if*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|\gamma^n - \gamma\|} = 0.$$

Theorem 2.2.2 *Let $\kappa > 0$ be fixed and assume that there exist some reals $C > 0$ such that*

$$\left\| \frac{\delta u}{u-\psi} \right\|_{L^\infty(\Omega)} \le C \left\| \frac{\delta u}{u-\psi} \right\|_{L^2(\Omega)} \forall u > \psi.$$

Then for all starting points $u^0 > \psi$ with $\|\frac{\delta u^0}{u^0 - \psi}\|_{L^2(\Omega)} < \frac{1}{C}$ Newton iterations converge to u_κ. Moreover, the convergence is R-superlinear.

Proof From (2.3) we obtain

$$\left\|\frac{\delta u^{n+1}}{u^{n+1}-\psi}\right\|_{L^2(\Omega)} \leq \left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^\infty(\Omega)}\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^2(\Omega)} \leq C\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|^2_{L^2(\Omega)}.$$

With the starting point $\left\|\frac{\delta u^0}{u^0-\psi}\right\|_{L^2(\Omega)} < \frac{\tau}{C}$, $0<\tau<1$ it follows then

$$\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^2(\Omega)} < \frac{\tau^{2^n}}{C}.$$

From (2.2) we obtain for $\nabla\delta u^{n+1}$

$$\int_\Omega |\nabla\delta u^{n+1}|^2 \leq \kappa\int_\Omega \frac{\delta u^{n+1}(\delta u^n)^2}{(u^{n+1}-\psi)(u^n-\psi)^2}d\omega$$

$$\begin{aligned}
\Rightarrow\quad |\delta u^{n+1}|^2_{1,\Omega} &\leq \kappa\left\|\frac{\delta u^{n+1}}{u^{n+1}-\psi}\right\|_{L^2(\Omega)}\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^2(\Omega)}\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^\infty(\Omega)}\\
\Rightarrow\quad |\delta u^{n+1}|_{1,\Omega} &\leq \frac{\sqrt{\kappa}}{C}\tau^{2^{n+1}}
\end{aligned}$$

This implies that $\|\delta u^{n+1}\| \to 0$ as $n\to\infty$, so the sequence $\{u^n\}$ is Cauchy sequence and it converges to some element $\bar{u}\in V$:

$$\bar{u} = \lim_{n\to\infty} u^n.$$

Due to the strict convexity of J_κ it holds:

$$J_\kappa(v) \geq J_\kappa(u^n) + J'_\kappa(u^n)(v-u^n) \quad \forall v\in V. \tag{2.4}$$

Consider $J'_\kappa(u^n)(v-u^n)$:

$$\begin{aligned}
J'_\kappa(u^n)(v-u^n) &= -J''_\kappa(\delta u^n, v-u^n)\\
&= \int_\Omega \nabla\delta u^n\nabla(v-u^n) + \kappa\int_\Omega \frac{\delta u^n(v-u^n)}{(u^n-\psi)^2}\\
&\leq |\delta u^n|_{1,\Omega}\|v-u^n\|_{1,\Omega} + \kappa\left\|\frac{\delta u^n}{u^n-\psi}\right\|_{L^2(\Omega)}\left\|\frac{v-u^n}{u^n-\psi}\right\|_{L^2(\Omega)}.
\end{aligned}$$

From this we can conclude that $J'_\kappa(u^n)(v-u^n)\to 0$ as $n\to\infty$. Taking the limit in (2.4) we obtain

$$J(v) \geq J(\bar{u}) \quad \forall v\in V,$$

thus $\bar{u} = u_\kappa$.

It remains to justify R-superlinear convergence:

$$|u^n - \bar{u}|_{1,\Omega} \leq \sum_{k=n}^{\infty} |\delta u^k|_{1,\Omega} \leq \frac{\sqrt{\kappa}}{C} \sum_{k=n}^{\infty} \tau^{2^k} \leq \frac{\sqrt{\kappa}}{C} \frac{\tau^{2^n}}{1-\tau^2}.$$

Consequently,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\|u^n - \bar{u}\|} = 0.$$

■

2.2.2 The predictor step

For $\kappa > 0$ let $u^+ = u + \delta u$ be the result of one corrector step. Under the assumptions of theorem 2.2.2 u^+ is sufficiently near u_κ, when $\left\| \frac{\delta u}{u-\psi} \right\|_{L^2(\Omega)} \leq \frac{\tau}{C}$ for some $\tau < 1$.

Now we want to approximate u_{κ^+} for the smaller penalty parameter $\kappa^+ = (1-\rho)\kappa$. To do this we compute a predictor p as the unique solution of the minimization problem

$$\min_{v \in V} J(u^+) + J'(u^+)v + \frac{1}{2} J''_\kappa(u^+)(v,v) \tag{2.5}$$

and take $\tilde{u} = u^+ + \rho p$ as starting point for the next Newton's method to approximate u_{κ^+}. We want to stay near the central path by performing single Newton's step. For a given accuracy of the Newton method τ, in view of theorem 2.2.2 we want to choose ρ such that the next corrector $\delta\tilde{u}$ satisfies

$$\left\| \frac{\delta \tilde{u}}{\tilde{u} - \psi} \right\|_{L^2(\Omega)} \leq \frac{\tau}{C}.$$

We present now an analysis for a proper choice of ρ. As the solution to the minimization problem (2.5) the predictor p satisfies

$$J''_\kappa(u^+)(p,v) = -J'(u^+)v \quad \forall v \in V. \tag{2.6}$$

And for the next corrector step it holds

$$J''_\kappa(\tilde{u})(\delta\tilde{u},v) = -J'_\kappa(\tilde{u})v \quad \forall v \in V. \tag{2.7}$$

We have

$$J'(\tilde{u})v = J'(u^+)v + \rho\int_\Omega \nabla p \nabla v d\omega \quad \forall v \in V,$$

then by use of (2.6) we obtain

$$\begin{aligned}
\forall v \in V \quad J'(\tilde{u})v &= J'(u^+)v + \rho\left(-J'(u^+)v - \kappa\int_\Omega \frac{pv}{(u^+-\psi)^2}d\omega\right)\\
&= (1-\rho)J'(u^+)v - \rho\kappa\int_\Omega \frac{pv}{(u^+-\psi)^2}\\
&= (1-\rho)\left(J'_\kappa(u^+)v + \kappa\int_\Omega \frac{v}{u^+-\psi}d\omega\right) - \rho\kappa\int_\Omega \frac{pv}{(u^+-\psi)^2}\\
&= \int_\Omega -\kappa^+\frac{(\delta u)^2 v}{(u^+-\psi)(u-\psi)^2} + \kappa^+\frac{v}{u^+-\psi} - \rho\kappa\frac{pv}{(u^+-\psi)^2}d\omega
\end{aligned}$$

and consequently, denoting by $\xi = \frac{\delta u}{u-\psi}$,:

$$\begin{aligned}
\forall v \in V \quad -J'_{\kappa^+}(\tilde{u})v &= -J'(\tilde{u})v + \kappa^+\int_\Omega \frac{v}{\tilde{u}-\psi}d\omega\\
&= \int_\Omega \kappa^+\left(\frac{\xi^2 v}{u^+-\psi} - \frac{v}{u^+-\psi} + \frac{v}{\tilde{u}-\psi}\right) + \rho\kappa\frac{pv}{(u^+-\psi)^2}d\omega\\
&= \int_\Omega \kappa^+\frac{\xi^2 v}{u^+-\psi} - \rho\kappa^+\frac{pv}{(u^+-\psi)(\tilde{u}-\psi)} + \rho\kappa\frac{pv}{(u^+-\psi)^2}d\omega\\
&= \int_\Omega \kappa^+\frac{\xi^2 v}{u^+-\psi} + \rho^2\kappa\frac{p(u^+ + p - \psi)v}{(u^+-\psi)^2(\tilde{u}-\psi)}d\omega.
\end{aligned}$$

Then using (2.7) with $v = \delta\tilde{u}$ and application of the Cauchy-Schwartz inequality yield

$$\int_\Omega |\nabla\delta\tilde{u}|^2 d\omega + \kappa^+\int_\Omega \frac{(\delta\tilde{u})^2}{(\tilde{u}-\psi)^2}d\omega \le \kappa^+\left(\Xi\int_\Omega \frac{(\delta\tilde{u})^2}{(\tilde{u}-\psi)^2}d\omega\right)^{\frac{1}{2}},$$

where

$$\Xi = \int_\Omega \left(\xi^2(1+\rho\frac{p}{u^+-\psi}) + \frac{\rho^2}{1-\rho}\frac{p}{u^+-\psi}\left(1+\frac{p}{u^+-\psi}\right)\right)^2 d\omega.$$

It follows then
$$\int_{\Omega} \frac{(\delta \tilde{u})^2}{(\tilde{u}-\psi)^2} d\omega \leq \Xi.$$
Thus we constructed a policy for a choice of a suitable decreasing parameter ρ for κ: ρ should satisfy the following conditions

1. $\rho < 1$
2. $u^{+} - \psi + \rho p > 0$ in Ω
3. $\Xi \leq \left(\frac{\tau}{C}\right)^2$.

Chapter 3

Numerical implementation

3.1 Algorithm

Interior point method consists of outer cycle : corrector-predictor steps and inner cycle for solely corrector step. As starting point for the approximate solution we can choose $u_h = 0$.

As we already noted the principle of the interior point method is to follow the central path. Ideally, we would like the iterations u_i corresponding to κ_i would lie on the central path, i.e. $u_i = u_{\kappa_i}$, where u_{κ_i} is the exact solution of the corresponding UP. But to achieve high accuracy on each corrector step would cost us a number of iterations. So, instead, we try our approximate solutions in each corrector step to remain in the predefined neighborhood of the exact solution (picture 3.1).

Under the assumption that $\left\|\frac{\delta u^0}{u^0-\psi}\right\|_{L^2(\Omega)} < \frac{\tau}{C}$, from the proof of theorem 2.2.2 we have:

$$\begin{aligned}
\left\|\frac{\delta u^{n+1}}{u^{n+1}-\psi}\right\|_{L^\infty(\Omega)} &\leq C\left\|\frac{\delta u^{n+1}}{u^{n+1}-\psi}\right\|_{L^2(\Omega)} \\
&\leq C\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^\infty(\Omega)}\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^2(\Omega)} \\
&\leq \tau^{2^n}\left\|\frac{\delta u^{n}}{u^{n}-\psi}\right\|_{L^\infty(\Omega)}
\end{aligned}$$

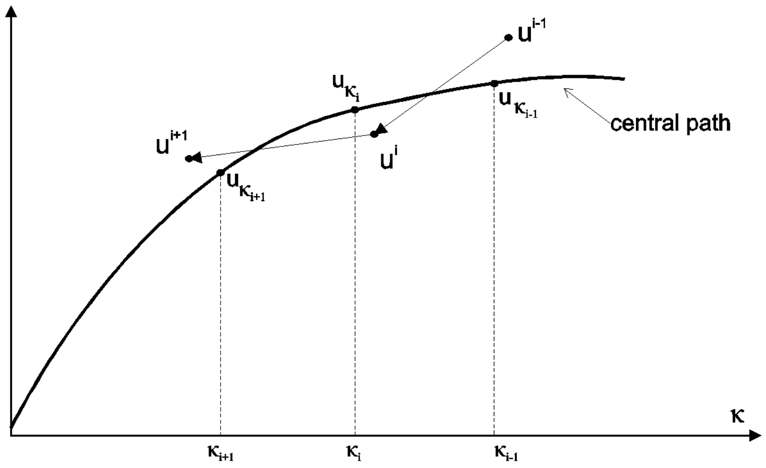

Figure 3.1: Staying close to the central path by single Newton iterations

Thus, if we choose at each predictor step decreasing factor ρ of the κ such that the starting point $\tilde{u}$ for the next Newton iterations satisfies $\left\|\frac{\delta\tilde{u}}{\tilde{u}-\psi}\right\|_{L^2(\Omega)} < \frac{\tau}{C}$ and define to stop Newton iterates as the condition $\left\|\frac{\delta u}{u-\psi}\right\|_{L^2(\Omega)} < \tau$ is satisfied, then we reach this kind of accuracy τ for the corrector step just in single step:

$$\left\|\frac{\delta u^1}{u^1-\psi}\right\|_{L^\infty(\Omega)} \leq C\left\|\frac{\delta u^0}{u^0-\psi}\right\|_{L^\infty(\Omega)}\left\|\frac{\delta u^0}{u^0-\psi}\right\|_{L^2(\Omega)} \leq C^2\frac{\tau^2}{C^2} < \tau.$$

We also need to determine the constant C for the assumptions of the theorem 2.2.2 to be fulfilled. This constant can be found by updating it at each Newton iterate as the maximal value of the old one and the ratio $\left\|\frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} / \left\|\frac{\delta u}{u-\psi}\right\|_{L^2(\Omega)}$.

After solving of the elliptic equations for the corrector δu we need to choose the step-size α such that updated value of the approximate solution would be feasible, i.e. it should hold $u+\alpha\delta > \psi$. This is equivalent to $\alpha\frac{\delta u}{u-\psi} > -1$, and since under the appropriate assumptions $\left\|\frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} \leq \tau$ holds, for the implementation we write it as $\left\|\alpha\frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} < 1$.

Algorithm 1 Interior points method

precision ε, precision τ, κ, $u^{(0)}$
k=0
repeat
  $u \leftarrow u^{(k)}$
  repeat
    solve for δu : $J''_\kappa(u)(\delta u, v) = -J'_\kappa(u)v \quad \forall v \in V$
    $C = \max\left\{C, \left\|\frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} / \left\|\frac{\delta u}{u-\psi}\right\|_{L^2(\Omega)}\right\}$
    choose step-size $0 < \alpha \leq 1$ such that : $\left\|\alpha \frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} \leq 0.99$
    $u = u + \alpha \delta u$
  until $\|\frac{\delta u}{u-\psi}\|_{L^\infty(\Omega)} \leq \tau$
  solve for a predictor p : $J''_\kappa(u)(p, v) = -J'(u)v, \quad \forall v \in V.$
  choose $0 < \rho < 0.99$ such that :

- $\left\|\rho \frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)} \leq 0.99$
- $\Xi \leq \left(\frac{\tau}{C}\right)^2$

  $\kappa = (1-\rho)\kappa$
  k=k+1
  $u^{(k+1)} = u + \rho p$
until $\kappa \leq \varepsilon$

As we established in the preceding section we also need choose the step-size ρ for the predictor such that the condition

$$\Xi \leq \left(\frac{\tau}{C}\right)^2$$

with

$$\Xi = \int_\Omega \left(\left(\frac{\delta u}{u-\psi}\right)^2 \left(1 + \rho \frac{p}{u^+ - \psi}\right) + \frac{\rho^2}{1-\rho} \frac{p}{u^+ - \psi} \left(1 + \frac{p}{u^+ - \psi}\right)\right)^2 d\omega \tag{3.1}$$

would be satisfied. For realization of this condition we set the $\rho = 0.99$, and then decrease it consecutively until it's appropriate. For the saving of the computation time it makes sense to divide the integral (3.1) into the several parts as follows:

$$\Xi = I_1 + 2\rho I_2 + \rho^2 I_3 + 2\frac{\rho^2}{1-\rho} I_4 + 2\rho\frac{\rho^2}{1-\rho} I_5 + \frac{\rho^4}{(1-\rho)^2} I_6,$$

where

$$\begin{aligned}
I_1 &= \int_\Omega \left(\frac{\delta u}{u-\psi}\right)^4 d\omega \\
I_2 &= \int_\Omega \left(\frac{\delta u}{u-\psi}\right)^4 \frac{p}{u^+-\psi} d\omega \\
I_3 &= \int_\Omega \left(\frac{\delta u}{u-\psi}\right)^4 \left(\frac{p}{u^+-\psi}\right)^2 d\omega \\
I_4 &= \int_\Omega \left(\frac{\delta u}{u-\psi}\right)^2 \frac{p}{u^+-\psi} \left(1+\frac{p}{u^+-\psi}\right) d\omega \\
I_5 &= \int_\Omega \left(\frac{\delta u}{u-\psi}\right)^2 \left(\frac{p}{u^+-\psi}\right)^2 \left(1+\frac{p}{u^+-\psi}\right) d\omega \\
I_6 &= \int_\Omega \left(\frac{p}{u^+-\psi}\right)^2 \left(1+\frac{p}{u^+-\psi}\right)^2 d\omega
\end{aligned}$$

3.2 Finite element discretization

3.2.1 Corrector step

At the corrector step we solve the variational elliptic problem for the unknown corrector:

$$\text{find } \delta u \in H_0^1 : \; J''_\kappa(u)(\delta u, v) = -J'_\kappa(u)v \quad \forall v \in H_0^1.$$

If we write it explicitly :

$$\text{find } \delta u \in H_0^1 : \int_\Omega \nabla \delta u \nabla v d\omega + \kappa \int_\Omega \frac{\delta u v}{(u-\psi)^2} d\omega = -\int_\Omega \nabla u \nabla v d\omega + \int_\Omega f v d\omega + \kappa \int_\Omega \frac{v}{u-\psi} d\omega. \tag{3.2}$$

According to Lax-Milgram Lemma there exist unique solution for this variational problem.

We use standard Galerkin method. Let $\mathscr{T}_h$ be regular triangulation of $\bar{\Omega}$:
$\bar{\Omega} = \cup_{T\in\mathscr{T}_h}\overline{T}$, T - triangle of the mesh;
for any two distinct triangles T_1 and T_2 : $\mathrm{int}T_1 \cap \mathrm{int}T_2 = \varnothing$;
any non-empty intersections of two distinct triangles equals one common edge $E \in \mathscr{E}(\mathscr{T}_h)$ or a node $x \in \mathscr{N}(\mathscr{T}_h)$.

Let $V_h = \left\{ v_h \in C(\bar{\Omega}) |\, v_h\Big|_T \in P^1(T)\ \forall T \in \mathscr{T}_h,\ v_h\Big|_{\partial\Omega} = 0 \right\} \subset V = H_0^1(\Omega)$.
Let $\{\phi_j\},\ j = \overline{1,N}$ be piecewise linear basis functions in V_h with compact support: $\forall x_i \in \mathscr{N}(\mathscr{T}_h) \quad \phi_j(x_i) = \delta_{ij}$; $N < \infty$ number of nodes in the mesh, dimension of V_h. Then we can use the representation

$$u_h = \sum_j u_j\phi_j, \quad \delta u_h = \sum_j \delta u_j \phi_j.$$

Substituting this in (3.2) we obtain the linear system of equations :

$$\mathbf{A}\underline{\delta u_h} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{N\times N} : V_h \to \mathbb{R}^N$ has components:

$$A_{ij} = \int_\Omega \nabla\phi_i\nabla\phi_j d\omega + \kappa\int_\Omega \frac{\phi_i\phi_j}{(u_h-\psi)^2}d\omega,$$

$\mathbf{b} \in \mathbb{R}^N$ with components

$$b_i = -\sum_j u_j \int_\Omega \nabla\phi_i\nabla\phi_j d\omega + \int_\Omega f\phi_i d\omega + \kappa\int_\Omega \frac{\phi_i}{u_h-\psi}d\omega,$$

and $\underline{\delta u_h} \in \mathbb{R}^N$ with components δu_j.

Assembling the stiffness matrix

For a triangular element $T \in \mathscr{T}_h$ let $(x_1,y_1),(x_2,y_2),(x_3,y_x)$ be the vertices and ϕ_1, ϕ_2, ϕ_3 be the corresponding basis functions in V_h. We denote by $|T|$ the area of the triangle. Then

$$|T| = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

Since

$$\begin{pmatrix}\phi_1(x,y)\\ \phi_2(x,y)\\ \phi_3(x,y)\end{pmatrix} = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix}^{-1} \begin{pmatrix}1\\ x\\ y\end{pmatrix},$$

where λ_1, λ_2, λ_3 are the corresponding barycentric coordinates at $(x,y) \in T$, it can be easily computed that

$$\nabla\phi_i(x,y) = \frac{1}{2}\begin{pmatrix}y_{i+1} - y_{i+2}\\ x_{i+2} - x_{i+1}\end{pmatrix}.$$

Here, the indices are to be understood modulo 3.

Then

$$\int_T \nabla\phi_i \nabla\phi_j d\omega = \frac{1}{4|T|}(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1}) \begin{pmatrix}y_{j+1} - y_{j+2}\\ x_{j+2} - x_{j+1}\end{pmatrix}.$$

Thus, using this, for the first term $\int_T \nabla\phi_i \nabla\phi_j d\omega$ of the local stiffness matrix we can write as

$$M_1 = \frac{|T|}{2} G G^T \quad \text{with } G = \begin{pmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{pmatrix}^{-1} \begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}$$

Using the trapezoidal rule for integration, we obtain for the second term of the local stiffness matrix:

$$M_2 = \frac{\kappa|T|}{3} \begin{pmatrix}\frac{1}{(u_1-\psi_1)^2} & 0 & 0\\ 0 & \frac{1}{(u_2-\psi_2)^2} & 0\\ 0 & 0 & \frac{1}{(u_3-\psi_3)^2}\end{pmatrix},$$

where u_1, u_2, u_3 and ψ_1, ψ_2 , ψ_3 values of u_h and ψ respectively at the corresponding nodes of the triangle.

Assembling the right hand side

For the terms of the right-hand side we easily obtain using the trapezoidal rule

$$\int_T f\phi_i d\omega = \frac{|T|}{3} f(x_i, y_i),$$

$$\kappa \int_T \frac{\phi_i}{u_h - \psi} d\omega = \frac{\kappa|T|}{3} \frac{1}{u_i - \psi(x_i, y_i)}.$$

3.2.2 Predictor step

At the predictor step we need to solve for p the following variational problem:

$$J''_\kappa(u^+)(p,v) = -J'(u^+)v \quad \forall v \in V.$$

Writing explicitly,

$$\text{find } p \in V: \int_\Omega \nabla p \nabla v d\omega + \kappa \int_\Omega \frac{pv}{(u^+ - \psi)^2} d\omega = -\int_\Omega \nabla u^+ \nabla v d\omega + \int_\Omega f v d\omega.$$

Discretization by using piecewise linear functions leads to the system:

$$\mathbf{A}^+ \underline{p_h} = \mathbf{b}^+,$$

where $\mathbf{A}^+ \in \mathbb{R}^{N\times N} : V_h \to \mathbb{R}^N$ has components:

$$A^+_{ij} = \int_\Omega \nabla\phi_i \nabla\phi_j d\omega + \kappa \int_\Omega \frac{\phi_i\phi_j}{(u^+_h - \psi)^2},$$

$\mathbf{b}^+ \in \mathbb{R}^N$ with components

$$b^+_i = -\sum_j u^+_j \int_\Omega \nabla\phi_i \nabla\phi_j d\omega + \int_\Omega f\phi_i d\omega,$$

and $\underline{p_h} \in \mathbb{R}^N$ with components p_j.

3.3 Error estimate for Finite Element solution

First we bring the theorem about the error estimate for the approximation which is valid for a general class of approximations schemes for variational inequalities.

We consider in the Hilbert space V the problem :

$$\text{find } u \in K : a(u, v-u) \geq \langle f, v-u\rangle \quad \forall v \in K, \tag{3.3}$$

where $K \subset V$ is a convex set, $a(\cdot,\cdot)$ is continuous bilinear form, $f \in V^*$.

Let V_h be a finite-dimensional subspace of the space V. We replace K by a convex set $K_h \subset V_h$. The corresponding discrete problem is

$$\text{find } u_h \in K_h : a(u_h, v_h - u_h) \geq \langle f, v_h - u_h \rangle \quad \forall v_h \in K_h, \tag{3.4}$$

We define the linear mapping : $A : V \to V^*$ such that $\langle Au, v \rangle = a(u, v) \quad \forall v \in V$.

Theorem 3.3.1 *If u and u_h satisfy (3.3) and (3.4) respectively, then*

$$a(u-u_h, u-u_h) \leq a(u-u_h, u-v_h) + \langle Au - f, v_h - u_h \rangle \quad \forall v_h \in K_h \tag{3.5}$$

$$a(u-u_h, u-u_h) \leq a(u-u_h, u-v_h) + \langle Au - f, v_h - u \rangle + \langle Au - f, v - u_h \rangle \tag{3.6}$$

$$v_h \in K_h, \ \forall v \in K$$

Proof We have

$$\begin{aligned} a(u-u_h, u-u_h) &= a(u-u_h, u-v_h) + a(u-u_h, v_h-u_h) \\ &= a(u-u_h, u-v_h) + \langle Au - f, v_h - u_h \rangle + \langle f, v_h - u_h \rangle - a(u_h, v_h - u_h) \\ \text{(from (3.4))} &\leq a(u-u_h, u-v_h) + \langle Au - f, v_h - u_h \rangle. \end{aligned} \tag{3.7}$$

(3.3) implies $\langle Au - f, v - u \rangle \geq 0 \quad \forall v \in K$, then

$$\begin{aligned} \langle Au - f, v_h - u_h \rangle &= \langle Au - f, v_h - u \rangle + \langle Au - f, u - v \rangle + \langle Au - f, v - u_h \rangle \\ &\leq \langle Au - f, v_h - u \rangle + \langle Au - f, v - u_h \rangle. \end{aligned}$$

From this and (3.7) follows (3.6). ■

Now, we replace Ω with its polygonal approximation Ω_h. We choose shape regular triangulation of Ω_h such that all the vertices of $\mathscr{T}_h$ which are on the boundary of the set Ω_h are also on $\partial\Omega$. Let h be triangulation parameter, i.e. the diameter of the biggest triangle is less than h. With such a triangulation $\mathscr{T}_h$ we associate subspace $V_h \subset H_0^1(\Omega)$ with piecewise linear elements defined by function values at the triangle vertices. We denote by $\Pi_h u$ piecewise linear interpolate of u.

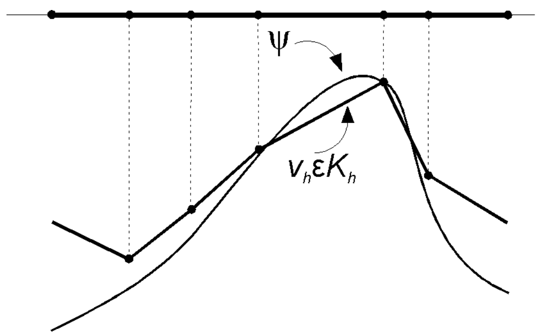

Figure 3.2: The set K_h is not in general contained in the set K.

Let K_h be approximation of the convex set K :

$$K_h = \{v_h \in V_h | \forall b \in \mathcal{N}_h : \; v_h(b) \geq \psi(b), \; v_h = \Pi_h u \text{ on } \Omega - \Omega_h\}.$$

Note that the set K_h is not in general contained in the set K.

Theorem 3.3.2 *Assume that the solution u is in the space $H^2(\Omega)$. Then the continuous piecewise linear approximation u_h satisfies $\|u_h - u\|_{1,\Omega} = O(h)$.*

Proof Using integration by parts and denoting by $\mu = -(\Delta u + f)$, we find

$$\begin{aligned} \langle Au - f, v \rangle &= \int_\Omega \nabla u \nabla v d\omega - \langle f, v \rangle \\ &= -\int_\Omega \Delta u \, v d\omega - \langle f, v \rangle \\ &= \langle \mu, v \rangle. \end{aligned}$$

Taking in (3.5) instead of $v_h = \Pi_h u$ gives:

$$|u - u_h|_{1,\Omega} \leq a(u - u_h, u - \Pi_h u) + \langle \mu, \Pi_h u - u_h \rangle_{\Omega_h}, \tag{3.8}$$

since $u_h = \Pi_h u$ on $\Omega - \Omega_h$.

From [7] the variational inequality (3.3) implies the following pointwise relations: $\mu \geq 0$ and $\mu(\psi - u) = 0$ a.e. on Ω.

Hence

$$\begin{aligned}\langle\mu,\Pi_h u-u_h\rangle_{\Omega_h} &= \langle\mu,\Pi_h(u-\psi)-(u-\psi)\rangle_{\Omega_h}+\langle\mu,u-\psi\rangle+\langle\mu,\Pi_h\psi-u_h\rangle_{\Omega_h}\\ &\leq \langle\mu,\Pi_h(u-\psi)-(u-\psi)\rangle_{\Omega_h} \qquad (3.9)\\ &\leq \|\mu\|_{L^2(\Omega_h)}\|\Pi_h(u-\psi)-(u-\psi)\|_{L^2(\Omega_h)}=O(h^2). \qquad (3.10)\end{aligned}$$

Here (3.9) was derived from the observation: from $u_h(b)\geq\Pi_h\psi(b)\ \forall b\in\mathscr{N}_h$ follows that $u_h\geq\Pi_h\psi$ on Ω_h; hence $\langle\mu,\Pi_h\psi-u_h\rangle_{\Omega_h}\leq 0$.

Thus, from (3.8) we can conclude:

$$\|u-u_h\|_{1,\Omega}^2\leq C\|u-u_h\|_{1,\Omega}\|u-\Pi_h u\|_{1,\Omega}+O(h^2).$$

Taking into account that linear interpolate is of the first order approximation in H^1, we obtain the error estimate

$$\|u-u_h\|_{1,\Omega}=O(h). \qquad \blacksquare$$

3.4 Numerical experiments

In this section we present results for several examples. For all these examples the initial iterate was taken $u^{(0)}=0$.

We know that the convergence rate of the solutions of the sequence of unconstrained minimization problems (UP) to the solution of the original problem is $\|u_\kappa-\bar{u}\|=O(\sqrt{\kappa})$. And accuracy of approximation by using piecewise finite elements will not be better than $O(h)$. Consequently, we should relate the tolerance ε for κ with h in a proper way: e.g. $\varepsilon=0.1/N$, where N - number of unknowns.

Initial barrier parameter was chosen $\kappa_0=1$.

For Newton iterations stopping criterion was $\left\|\frac{\delta u}{u-\psi}\right\|_{L^\infty(\Omega)}\leq 0.5$.

For the integrations in computing of $L^2(\Omega)$-norms and checking of the conditions $\Xi\leq\left(\frac{\tau}{C}\right)^2$ trapezoidal rule was used.

Implementations were performed on WindowsXP platform with the 1595 MHz speed Intel x86 processor by using MatLab@ programming language.

3.4.1 Example 1

In this example : domain Ω is a unit circle; force acting on the membrane is constant: $f = -32$; and obstacle is a plane $z = -1$.

The exact solution to this problem is :

$$u = \begin{cases} -1, & \text{if } r < 0.7374 \\ 8r^2 - 8.699 \ln r - 8, & \text{if } r \geq 0.7374 \end{cases}$$

In table 3.4.1 N means number of nodes in the triangulation mesh, the column 'iterations' shows the number of predictor-corrector steps; the next column - the number of Newton steps at the first iteration (at the other iterations we have only one Newton corrector step). The error is considered in H^1-norm; $e_h = |u - u_h|$, where u is the exact solution. The column 'κ' shows the final value of κ when iterations stopped.

Table 3.1: Results for the flat obstacle

N	iter.	Newton steps (first it.)	CPU, s	$\|u-u_h\|_{1,\Omega}$	$\max_j e_h$	κ	C
144	5	6	0.437	0.48196	0.06951	0.0079	4.316
544	6	6	1.31	0.15497	0.01686	0.00235	9.2418
1130	6	6	2.75	0.10952	0.00974	0.00236	9.9739
1506	6	6	3.718	0.08702	0.00666	0.00219	16.9396
2173	7	6	6.062	0.08183	0.00625	0.00065	14.4017
4421	7	6	13.422	0.03599	0.00312	0.00066	20.6203
8257	8	6	40.437	0.01516	0.00167	0.00054	26.3749
17489	8	6	121.68	0.01338	0.00104	0.0005	19.3735
33985	9	6	335.4	0.01137	0.0007	0.00031	34.9717

From this table we see that the number of iterations do not increase unpredictably as the number of unknowns is increased, and the number of Newton iterations for the correction of solution at the first iterate is the same for all considered here cases.

In the picture 3.3 the final iterate of the solution u is depicted. In the picture 3.4 on the left : the red zone is the coincidence set and the blue zone - non-coincidence set. On the right distribution of error $e_h = |u - u_h|$ is depicted. We observe that the approximate solution is well-behaved in the contact zone and the larger values of errors come to the non-coincidence set.

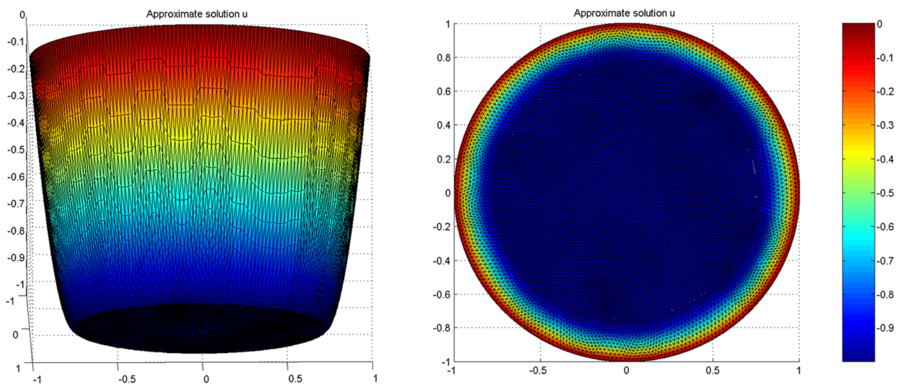

Figure 3.3: Approximate solution for the plane-obstacle

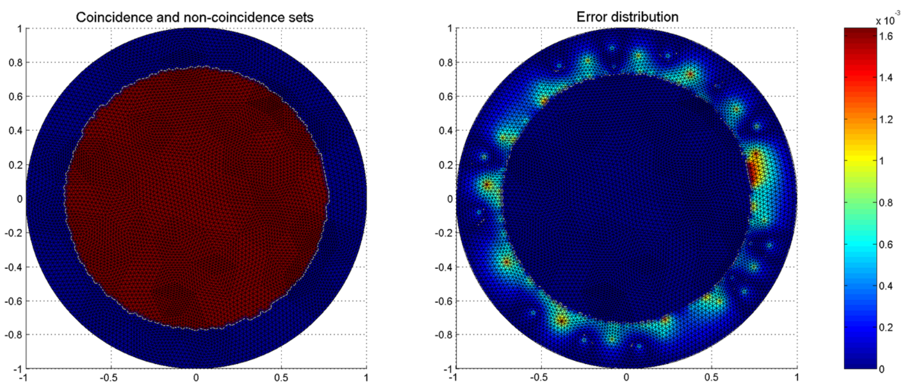

Figure 3.4: On the left: coincidence and non-coincidence sets, on the right: error distribution.

3.4.2 Example 2

We consider another radially symmetric problem when: Ω is a unit circle, constant force is acting on the membrane $f = -10$, and the obstacle is described as

$$\psi(x,y) = \begin{cases} \sqrt{R^2 - x^2 - y^2} - R - 1, & \text{if } x^2 + y^2 \le R^2, \\ -5, & \text{if } x^2 + y^2 > R^2 \end{cases}$$

where $R = 0.7$

The exact solution to this problem is :

$$u = \begin{cases} \sqrt{R^2 - x^2 - y^2} - R - 1, & \text{if } x^2 + y^2 < r \\ 5(x^2 + y^2)/2 - a \ln r - 5/2, & \text{if } x^2 + y^2 \ge r \end{cases}$$

where $r = 0.3976$, $a = r^2(5 + 1/\sqrt{R^2 - r^2})$.

Figure 3.5 depicts the final iterate of the approximate solution.

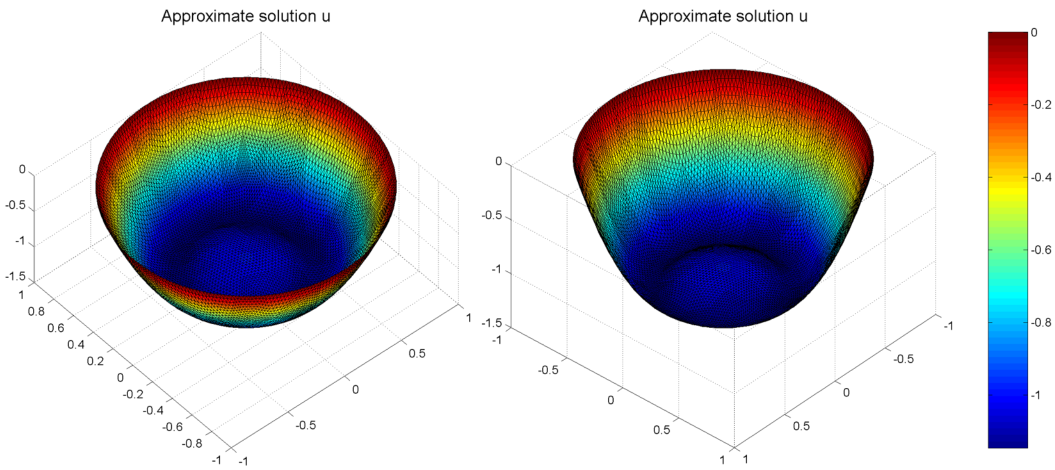

Figure 3.5: Approximate solution for the obstacle with the spherical surface

The table 3.4.2 reports the results when the stopping criterion for the Newton iterations was changed to $\frac{\delta u}{u - \psi} \le 0.1$. It illustrates that stricter stopping criterion doesn't improve the accuracy of the solution for the original constrained problem, but only increases the number of iterations. This confirms the fact that we don't need to solve each corrector problem with high accuracy but it's sufficient

Table 3.2: Results for the example 2

N	iter.	Newton steps (first it.)	CPU, s	$\|u-u_h\|_{1,\Omega}$	$\max_j e_h$	κ	C
144	10	6	0.841	0.11367	0.01785	0.0055	5.4551
544	12	6	1.8	0.05661	0.00797	0.00203	7.5279
1130	13	6	4.65	0.04224	0.00439	0.00104	11.7238
1506	13	6	6.125	0.03597	0.00358	0.0011	17.0023
2173	14	6	9.656	0.03439	0.003395	0.00145	15.4000
4421	15	6	17.828	0.02136	0.00240	0.00067	17.9878
8257	15	6	47	0.01776	0.00170	0.00053	25.6321
17489	16	6	114	0.01966	0.00114	0.0007	34.7167
33985	16	6	379.8	0.02171	0.00136	0.00028	40.4416

Table 3.3: Results for the example 2 with $\tau = 0.1$

N	iter.	Newton st. (first iter.)	$\|u-u_h\|_{1,\Omega}$
144	23	7	0.11973
544	28	7	0.05667
1130	29	7	0.04222
1506	29	7	0.03589
2173	31	7	0.03426
4421	32	7	0.02135
8257	33	7	0.01772
17489	35	7	0.01964
33985	37	7	0.02169

the approximate solutions to remain in the predefined neighborhood of the corresponding point on the central path. Figure 3.6 demonstrates how $\|u-u_h\|_{1,\Omega}$ changes with iterations for different number of nodes in the mesh. We can observe that below the some value of κ it doesn't decrease further with the next iterations. Thus, sufficient number of iterations was done in order to approximate the solution for the given mesh size.

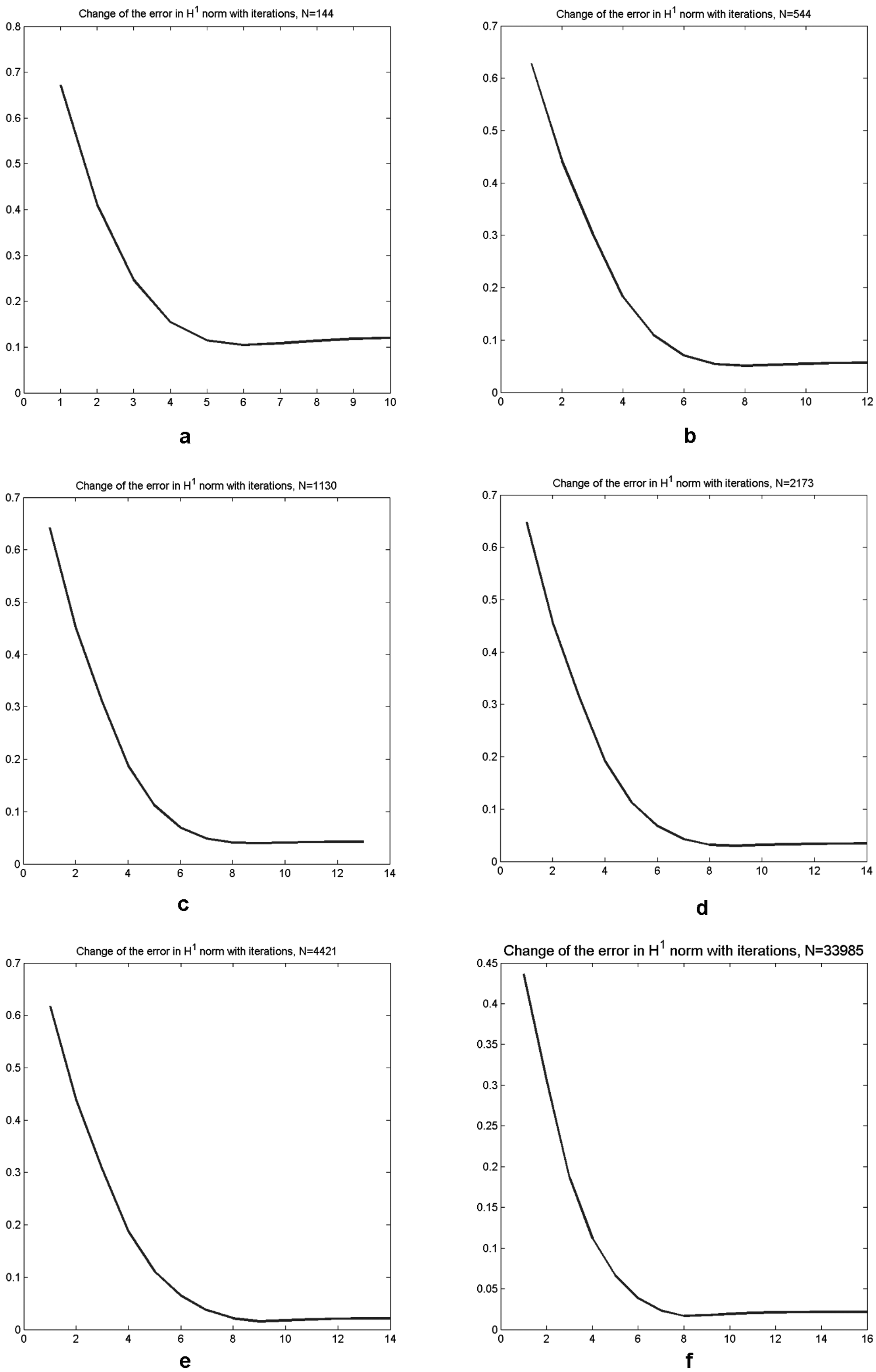

Figure 3.6: Convergence of the solution for a: $N = 144$, b: $N = 544$, c: $N = 1130$, d: $N = 2173$, e: $N = 4421$, f: $N = 33985$

In the picture 3.7 on the left coincidence (in red) and non-coincidence (in blue) sets are shown. On the right distribution of the error considered as the absolute of the difference between exact and approximate solutions value is shown. We see that again the highest values of such error comes to the region in non-coincidence set closer to the free boundary.

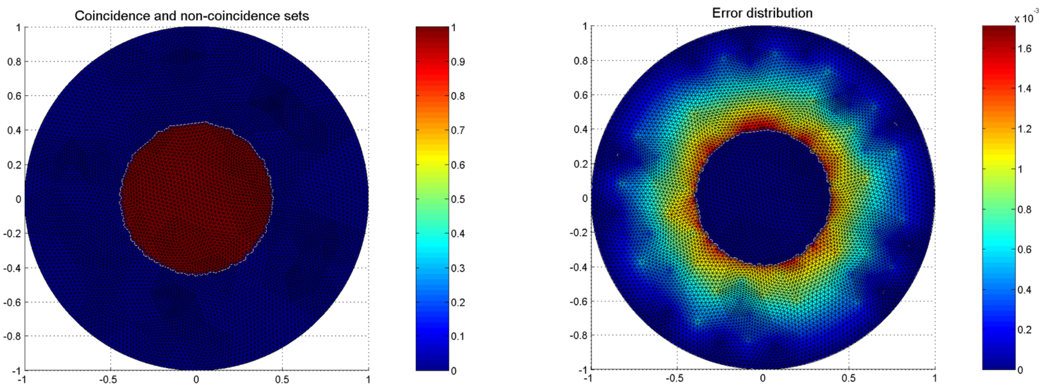

Figure 3.7: On the left: coincidence and non-coincidence sets, on the right: error distribution.

3.4.3 Example 3

For this example we take Ω as unit circle, obstacle is described as

$$\psi(x,y) = \begin{cases} \sqrt{0.64-(x+0.3)^2}-1, & \text{if } |x+0.3| \leq 0.8, \\ -1, & \text{if } |x+0.3| > 0.8 \end{cases}$$

We considered this problem with different constant forces: $f=-5$, $f=-10$, $f=-30$ and $f=-100$ (pictures 3.8, 3.9, 3.10, 3.11) and observed that the number of iterations remains stable also with respect to the number of contact points.

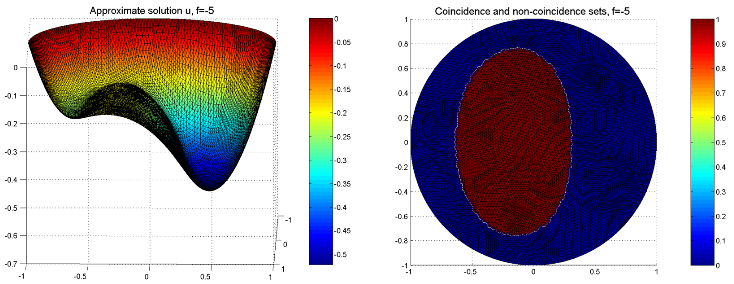

Figure 3.8: $f = -5$, 13 corrector-predictor iterations, 2 Newton iterations at the first corrector step

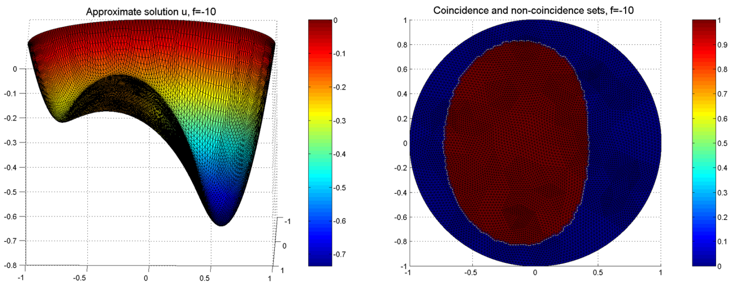

Figure 3.9: $f = -10$, 14 corrector-predictor iterations, 4 Newton iterations at the first corrector step

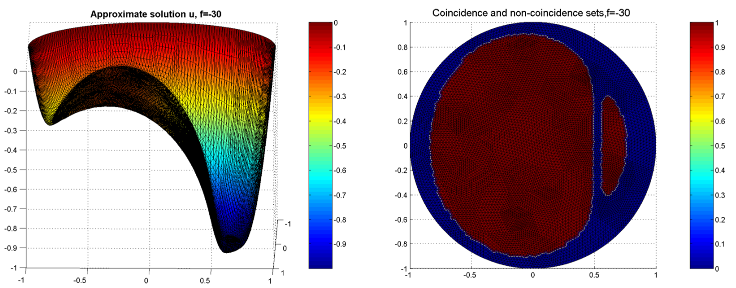

Figure 3.10: $f = -30$, 18 corrector-predictor iterations, 7 Newton iterations at the first corrector step

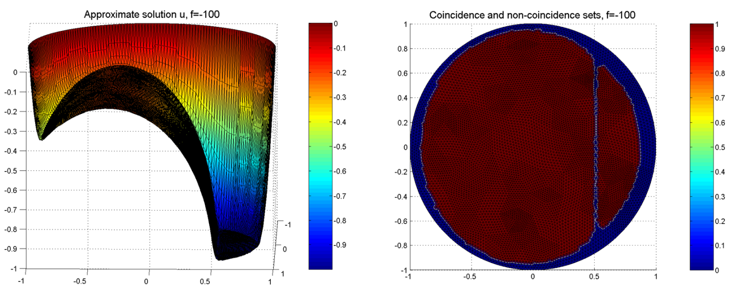

Figure 3.11: $f = -100$, 13 corrector-predictor iterations, 13 Newton iterations at the first corrector step

3.4.4 Example 4

In this example Ω is a unit circle, force has intensity $f = -30$ and discontinuous obstacle:

$$\psi(x,y) = \begin{cases} -1, & \text{if } |x-0.3| \leq 0.3 \text{ and } |y| \leq 0.4, \\ -100, & \text{otherwise} \end{cases}$$

Here we can motivate our choice of trapezoidal rule for the integration: using this rule the method copes with such discontinuous obstacles as well.

The picture 3.12 depicts the result of the final iteration, the picture 3.13 illustrates Lagrange multipliers and picture 3.14 – subdivision of Ω into coincidence and non-coincidence sets. Lagrange multipliers computed as $\mu = \frac{\kappa}{u-\psi}$ at the final iteration can be interpreted as the reaction forces at the corresponding points of the obstacle.

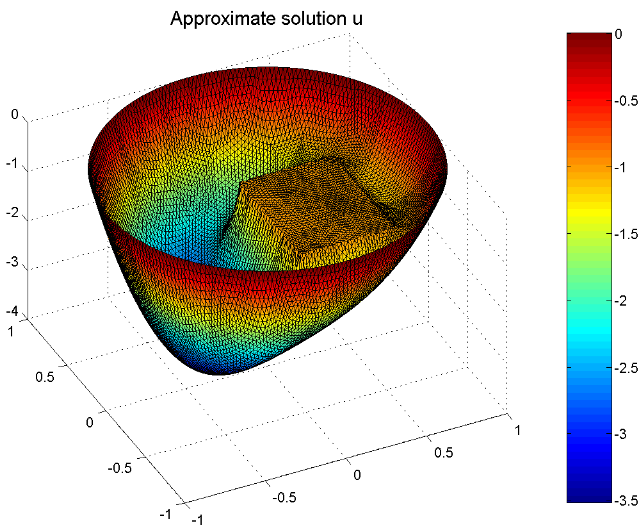

Figure 3.12: Numerical solutions for the discontinuous obstacle

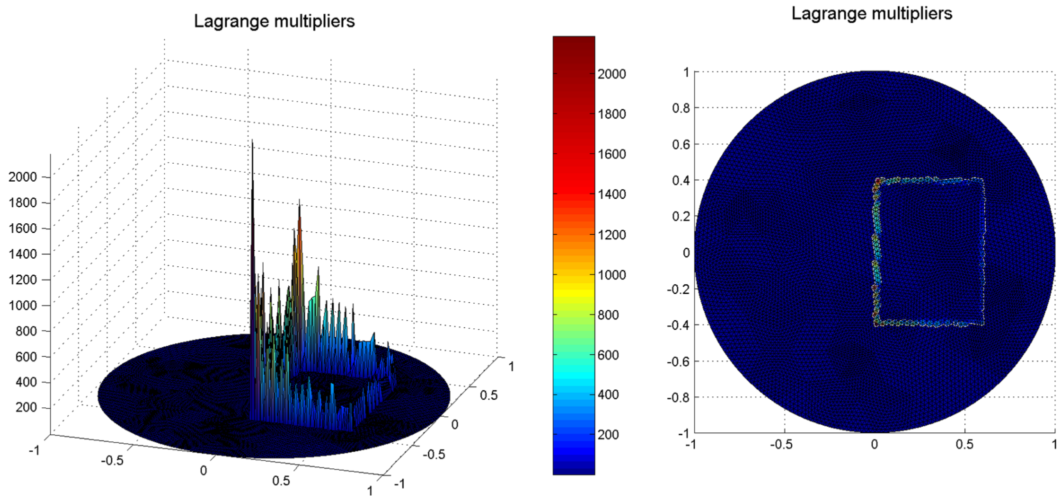

Figure 3.13: Lagrange multipliers

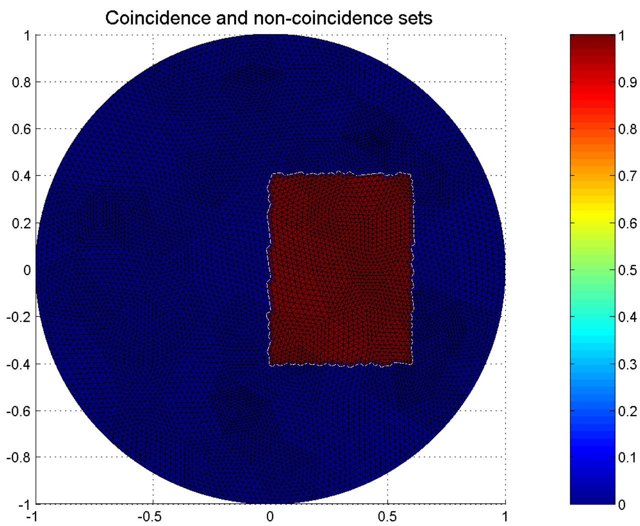

Figure 3.14: In red is coincidence set and and in blue is non-coincidence set

Conclusion

In this work we considered the mathematical model of the obstacle problem for the numerical simulation of it by using the interior point method. We discussed the existence and uniqueness of the solution of the primal formulation of the obstacle problem. Constrained minimization problem was replaced by a sequence of unconstrained minimization problems (UP) by adding the barrier function multiplied with the barrier parameter and showed that the solution of the sequence of UP converges to the solution of the original problem as the barrier parameter converges to zero.

The algorithm for the implementation follows the predictor-corrector method: we begin with a feasible guess of the solution of UP for the current value of the barrier parameter and correct that 'guess' by using of Newton method. For the numerical implementation of the method we used finite element discretization with piecewise linear elements.

We performed several numerical tests which gave satisfactory results. We observed that the number of iteration doesn't increase unpredictably with the increase of the mesh nodes or contact points. We observed also that numerical solution behaves well in the coincidence set and higher values of error occur in the non-coincidence set near the free boundary. Thus, for the improvement of the accuracy it seems to be of interest using of adaptive finite elements for refining the mesh in the region where the solution is of least accuracy.

Appendix A

List of notations

$v(\cdot)$, $v(\cdot,\cdot)$	function v of one variable, two variables
int$A$	interior of A
$J'(u)v$	first Fréchet derivative of a functional J at a point u along v
$J''(u)(v,w)$	second Fréchet derivative of a functional J at a point u
$H^m(\Omega)$	$= \{v \in L^2(\Omega) \mid \forall\alpha : \ |\alpha| \leq m, \ \partial^\alpha v \in L^2(\Omega)\}$
$H^1_0(\Omega)$	$= \{v \in H^1 \mid v = 0 \text{ on } \partial\Omega\}$
H^*	dual of a space H
H^{-1}	dual of the space H^1
$\hookrightarrow$	continuous embedding
$\|v\|_{m,\Omega}$	$= \left(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha v|^2\right)^{1/2}$
$|v|_{m,\Omega}$	$= \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2\right)^{1/2}$

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Eidesstattliche Erklarung

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