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# The Radius of Metric Subregularity

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*Dedicated to Professor Alexander Ioffe on the occasion of his 80$^{th}$ birthday*

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**Abstract** There is a basic paradigm, called here the *radius of well-posedness*, which quantifies the "distance" from a given well-posed problem to the set of ill-posed problems of the same kind. In variational analysis, well-posedness is often understood as a regularity property, which is usually employed to measure the effect of perturbations and approximations of a problem on its solutions. In this paper we focus on evaluating the radius of the property of metric subregularity which, in contrast to its siblings, metric regularity, strong regularity and strong subregularity, exhibits a more complicated behavior under various perturbations. We consider three kinds of perturbations: by Lipschitz continuous functions, by semismooth functions, and by smooth functions, obtaining different expressions/bounds for the radius of subregularity, which involve generalized derivatives of set-valued mappings. We also obtain different expressions when using either Frobenius or Euclidean norm to measure the radius. As an application, we evaluate the radius of subregularity of a general constraint system. Examples illustrate the theoretical findings.

**Keywords** well-posedness · metric subregularity · generalized differentiation · radius theorems · constraint system

**Mathematics Subject Classification (2000)** 49J52 · 49J53 · 49K40 · 90C31

## 1 Introduction

According to the classical definition of Hadamard, a mathematical problem is well-posed when it has a unique solution which is a continuous function of the data of the problem. Establishing the well-posedness is a basic task, but there are other questions around it such as how "robust" the well-posedness property is under perturbations, or how "far" from a

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given well-posed problem the ill-posed problems are. The formalization of the latter question leads to the concept of the *radius of well-posedness*, which quantifies the distance from a given well-posed problem to the set of ill-posed problems of the same kind.

To be specific, consider the problem of solving the linear equation $Ax = b$, where $A$ is an $n \times n$ matrix and $b \in \mathbb{R}^n$. This problem is well-posed in the sense of Hadamard exactly when the matrix $A$ is nonsingular. The radius of well-posedness of this problem is well known, thanks to the *Eckart–Young theorem* [7], which says the following: for any nonsingular $n \times n$ matrix $A$,

$$\inf_{B \in L(\mathbb{R}^n,\mathbb{R}^n)} \{ \|B\| \mid A + B \text{ singular} \} = \frac{1}{\|A^{-1}\|}, \tag{1.1}$$

where $L(\mathbb{R}^n,\mathbb{R}^m)$ denotes the set of $n \times m$ matrices, and $\|\cdot\|$ is the usual operator norm. In numerical linear algebra this theorem is intimately connected with the *conditioning* of the matrix $A$. Namely, the expression on the right-hand side of (1.1) is the reciprocal of the *absolute condition number* of $A$; dividing by $\|A\|$ would give us a similar expression for the *relative condition number*. Thus, the radius equality (1.1) is in line with the idea of conditioning; the farther a matrix is from the set of singular matrices, the better its conditioning is. The reader can find a broad coverage of the mathematics around condition numbers and conditioning in the monograph [1].

A far reaching generalization of the Eckart–Young theorem was proved in [3] for the property of *metric regularity* of a set-valued mapping $F$ acting generally between metric spaces, which is the same as nonsingularity when $F$ is a square matrix. This generalization was later extended in [4] to the properties of *strong metric regularity* and *strong metric subregularity*, see also [5, Section 6A]. In this paper we deal with the radius of *metric subregularity*, a property which turns out to be quite different from its siblings.

We proceed now with the definitions of these properties; more details regarding the notation and the definitions used in the paper are given in Section 2.

A set-valued mapping $F$ acting from $\mathbb{R}^n$ to $\mathbb{R}^m$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ if $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and there exists a number $\kappa \in [0,+\infty)$ together with neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that

$$d(x,F^{-1}(y)) \le \kappa d(y,F(x)) \text{ for all } x \in U,\ y \in V. \tag{1.2}$$

Here $d(x,C)$ is the distance from a point $x$ to a set $C$: $d(x,C) = \inf_{y \in C} \|x - y\|$. The infimum of the set of values $\kappa$ for which (1.2) holds is called the *modulus of metric regularity*, denoted by $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$. A mapping $F$ is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if its inverse $F^{-1}$ has the *Aubin property* at $\bar{y}$ for $\bar{x}$, a property which in the single-valued case reduces to the Lipschits continuity.

A mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is said to have a *single-valued localization* around $\bar{x}$ for $\bar{y}$ if there exist neighborhoods $U$ of $\bar{x}$ and $V$ of $\bar{y}$ such that the truncated mapping $U \ni x \mapsto F(x) \cap V$ is single-valued, a function on $U$.

If the inverse $F^{-1}$ of a mapping $F$ has a localization at $\bar{y}$ for $\bar{x}$ which is Lipschitz continuous, then $F$ is said to be *strongly metrically regular*, or simply *strongly regular*; in this case $F$ is automatically metrically regular at $\bar{x}$ for $\bar{y}$ and the Lipschitz modulus of the localization at $\bar{y}$ equals $\operatorname{reg}(F;\bar{x}\,|\,\bar{y})$.

If we fix $y$ in (1.2) at its reference value $\bar{y}$, we obtain the property of *metric subregularity*, which we sometimes call simply *subregularity*. Specifically, a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be metrically subregular at $\bar{x}$ for $\bar{y}$ if $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and there exists a number $\kappa \in [0,+\infty)$ together with a neighborhood $U$ of $\bar{x}$ such that

$$d(x,F^{-1}(\bar{y})) \le \kappa d(\bar{y},F(x)) \text{ for all } x \in U. \tag{1.3}$$

The infimum of the set of values $\kappa$ for which (1.3) holds is called the *modulus of metric subregularity*, denoted by $\operatorname{subreg}(F;\bar{x}\,|\,\bar{y})$. A mapping $F$ is metrically subregular at $\bar{x}$ for $\bar{y}$ if and only if its inverse $F^{-1}$ is *calm* at $\bar{y}$ for $\bar{x}$, a property which corresponds to the Aubin continuity with one of the variables fixed.

A mapping $F$ is said to be *strongly metrically subregular*, or simply *strongly subregular* at $\bar{x}$ for $\bar{y}$ if $F$ is metrically subregular at $\bar{x}$ for $\bar{y}$ and in addition $\bar{x}$ is an isolated point in $F^{-1}(\bar{y}) \cap U$. In this case, $F^{-1}$ has the *isolated calmness* property at $\bar{y}$ for $\bar{x}$.

If $f$ is a (single-valued) function, we write, with some abuse of notation, $\mathrm{reg}(f;\bar{x})$ and $\mathrm{subreg}(f;\bar{x})$ instead of $\mathrm{reg}(f;\bar{x}\,|\,f(\bar{x}))$ and $\mathrm{subreg}(f;\bar{x}\,|\,f(\bar{x}))$, respectively.

Clearly, the above definitions of regularity properties can be extended in a straightforward manner to general metric spaces.

All the above concepts have been well studied. They are discussed in detail in [5, 15, 17, 25, 30]. The metric subregularity, which is the main object of study in the current paper, is implicitly present already in the pioneering work by Graves [13], as shown in [5, Section 5D]. This property plays a major role in deriving the Lagrange multiplier rule in its various forms, see e.g. [17, Section 2.1]. For the most recent developments in research on metric subregularity, we refer the readers to [2, 6, 15, 18, 19, 22, 23, 26, 27, 31–33].

It turns out that the Eckard–Young equality (1.1) is a special case of a general paradigm which can be described as

$$\mathrm{rad} = \frac{1}{\mathrm{reg}}, \tag{1.4}$$

where rad is the appropriately defined radius of the considered regularity property, and reg is the modulus of this property. This paradigm was first established in [3] for the property of metric regularity. Specifically, it was established that if a mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is metrically regular at $\bar{x}$ for $\bar{y}$, then

$$\mathrm{rad}[\mathrm{MR}]F(\bar{x}\,|\,\bar{y}) := \inf_{B\in L(\mathbb{R}^n,\mathbb{R}^m)} \big\{ \|B\| \;|\; F+B \text{ is not metrically regular at } \bar{x} \text{ for } \bar{y}+B\bar{x} \big\} = \frac{1}{\mathrm{reg}(F;\bar{x}\,|\,\bar{y})}. \tag{1.5}$$

Moreover, the equality remains true if the infimum is taken with respect to all matrices $B$ of rank one, or the class of perturbations is enlarged to the family of functions $h:\mathbb{R}^n \to \mathbb{R}^m$ that are Lipschitz continuous around $\bar{x}$, with $\|B\|$ replaced by the Lipschitz modulus $\mathrm{lip}(h;\bar{x})$. That is, the radius of metric regularity is the same for all perturbations $h$ ranging from Lipschitz continuous functions to linear mappings of rank one.

Subsequently, in [4] this radius equality was shown to hold in the same form for the properties of strong regularity and strong metric subregularity. Specifically, if a mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is strongly regular or strongly subregular at $\bar{x}$ for $\bar{y}$, respectively, then the equality (1.5) holds with “not metrically regular” replaced by “not strongly regular” or “not strongly subregular”, respectively, and in the second case $\mathrm{reg}(F;\bar{x}\,|\,\bar{y})$ on the right side is replaced by $\mathrm{subreg}(F;\bar{x}\,|\,\bar{y})$.

In some situations it is more convenient to work with the reciprocal of the regularity modulus reg. We denote this reciprocal by rg and then equality (1.4) becomes

$$\mathrm{rad} = \mathrm{rg}. \tag{1.6}$$

In the case of the conventional metric regularity, rg corresponds to the *modulus of surjection* ‘sur’ used by Ioffe [15]; see also other examples in [19–21]. This notation is in agreement with the natural convention, which we adopt here, that if a mapping does not possess a certain regularity property, then the regularity modulus equals $+\infty$ and the corresponding radius equals 0.

It turns out, however, that the (not strong) metric subregularity does not obey the radius paradigm, at least in the form (1.4) or (1.6). This effect was first noted in [4] and also discussed in [5, Section 6A].

*Example 1.1* By a fundamental result of Robinson [29], every polyhedral mapping, that is, a mapping whose graph is the union of finitely many polyhedral convex sets, is outer Lipschitz continuous around every point in its domain. Hence, inasmuch outer Lipschitz continuity of the inverse implies metric subregularity, every polyhedral mapping $F$ is metrically subregular at any $\bar{x}$ for any $\bar{y}$ such that $(\bar{x},\bar{y}) \in \mathrm{gph}\,F$. It is elementary to observe that the sum of any polyhedral mapping and a linear mapping is again polyhedral. Hence, if $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is a polyhedral mapping and $(\bar{x},\bar{y}) \in \operatorname{gph} F$, then

$$\inf_{B\in L(\mathbb{R}^n,\mathbb{R}^m)} \left\{\|B\| \mid F+B \text{ is not metrically subregular at } \bar{x} \text{ for } \bar{y}+B\bar{x}\right\} = +\infty. \tag{1.7}$$

Clearly, the quantity $\operatorname{subreg}(F;\bar{x}\,|\,\bar{y})$ could be anything; thus the equality (1.4) does not hold in general for polyhedral mappings. □

*Example 1.2* Consider the zero function $f:\mathbb{R}\to\mathbb{R}$, that is $f(x)=0$ for all $x\in\mathbb{R}$. Then $f^{-1}(0)=\mathbb{R}$ and $f^{-1}(y)=\emptyset$ for all $y\neq 0$. Thus, the zero mapping is metrically subregular at any $\bar{x}$ for 0, and the subregularity modulus is of course zero. The function $h(x)=x^2$ is Lipschitz continuous around $\bar{x}=0$ with Lipschitz modulus zero, but the mapping $(f+h)(x)=x^2$ is not metrically subregular at 0 for 0. Hence, the radius of metric subregularity of the zero mapping with respect to smooth perturbations is zero, but this does not fall into the pattern of (1.4). Also note that the zero function is a polyhedral mapping, hence, in the light of the preceding example, its radius for linear perturbations is $+\infty$, while when we change to quadratic perturbations and use the Lipschitz modulus to measure the radius, it becomes zero. □

Note that there are four components involved in a radius equality (1.5): a regularity property, the basic underlying mapping $F$, the mapping $B$ representing the perturbations, and the "size" of the perturbation, which in this case is measured by the norm of $B$. In this paper we consider the metric subregularity property, for which the basic mapping $F$ will be a set-valued mapping with closed graph. The perturbations will be represented by the following three classes of functions: Lipschitz continuous functions, semismooth functions and continuously differentiable ($C^1$) functions, all around/at the reference point. For all the three classes we will use the Lipschitz modulus at the reference point as a measure of the size of the perturbation. Note that for the second class the Lipschitz modulus can be expressed in terms of Clarke's generalized Jacobian, while for $C^1$ functions this would be the norm of the derivative at the reference point.

The next Section 2 provides some preliminary material used throughout the paper. This includes basic notation and general conventions, definitions of the three classes of perturbations studied in the paper and corresponding radii, and a certain new primal-dual derivative which gives rise to a collection of 'regularity constants' used in the radius estimates. In Section 3, we establish lower and upper bounds for the radius of metric subregularity for Lipschitzian perturbations and the exact radius formula for the other classes of perturbations. The case when the size of the perturbation is measured by the Frobenius norm on the space of matrices is also discussed. Section 4 is devoted to applications to constraint systems, while the last Section 5 identifies possible directions for future research.

## 2 Preliminaries

### 2.1 Notation and general conventions

Throughout we consider mappings acting between finite dimensional spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. The spaces are assumed equipped with arbitrary norms denoted by the same symbol $\|\cdot\|$. We usually keep the same notation for the duals of $\mathbb{R}^n$ and $\mathbb{R}^m$. However, in some situations when this can cause confusion, we write explicitly $(\mathbb{R}^n)^*$ and $(\mathbb{R}^m)^*$. The corresponding dual norms are denoted $\|\cdot\|_*$. Given an $m\times n$ matrix $B$, the symbol $B^T$ stands for the transposed matrix, and both $B$ and $B^T$ are identified with the corresponding linear operators acting between $\mathbb{R}^m$ and $\mathbb{R}^n$ or their duals.

We denote by $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ a *set-valued* mapping acting from $\mathbb{R}^n$ to the subsets of $\mathbb{R}^m$. If $F$ is a function, that is, for each $x\in\mathbb{R}^n$ the set of values $F(x)$ consists of no more than one element, then we use a small letter $f$ and write $f:\mathbb{R}^n\to\mathbb{R}^m$. The graph of a mapping $F$ is defined as $\operatorname{gph} F := \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m \mid y\in F(x)\}$ and its domain is $\operatorname{dom} F := \{x\in\mathbb{R}^n \mid F(x)\neq\emptyset\}$. The inverse of a mapping $F$ is the mapping $y\mapsto F^{-1}(y) := \{x\in\mathbb{R}^m \mid y\in F(x)\}$. In this paper we consider mappings with *closed graph*.

The Lipschitz modulus of a function $f:\mathbb{R}^n\to\mathbb{R}^m$ around a point $\bar{x}$ is defined by

$$\mathrm{lip}(f;\bar{x}) := \limsup_{x,x'\to\bar{x},\,x\neq x'} \frac{\|f(x)-f(x')\|}{\|x-x'\|}.$$

Having $\mathrm{lip}(f;\bar{x})<l$ corresponds to having a neighborhood $U$ of $\bar{x}$ such that $f$ is Lipschitz continuous on $U$ with Lipschitz constant $l$. Conversely, if $f$ is Lipschitz continuous around $\bar{x}$ with Lipschitz constant $l$ then we have $\mathrm{lip}(f;\bar{x})\le l$. If $f$ is not Lipschitz continuous around $\bar{x}$ then $\mathrm{lip}(f;\bar{x})=+\infty$.

Given a closed set $A\subset\mathbb{R}^n$ and a point $\bar{x}\in A$, we define

(i) the *tangent (Bouligand) cone* to $A$ at $\bar{x}$:

$$T_A(\bar{x}) := \{u\in\mathbb{R}^n \mid \exists u_i\to u,\ t_i\searrow 0 \quad \text{such that} \quad \bar{x}+t_iu_i\in A,\ \forall i\in\mathbb{N}\};$$

(ii) the *Fréchet normal cone* to $A$ at $\bar{x}$ as the (negative) *polar* cone to $T_A(\bar{x})$:

$$N_A(\bar{x}) := (T_A(\bar{x}))^\circ = \{x^*\in\mathbb{R}^n \mid \langle x^*,u\rangle\le 0 \quad \text{for all} \quad u\in T_A(\bar{x})\};$$

(iii) the *limiting normal cone* to $A$ at $\bar{x}$:

$$\overline{N}_A(\bar{x}) := \{x^*\in\mathbb{R}^n \mid \exists x_i \xrightarrow{A} \bar{x},\ x_i^*\to x^* \quad \text{such that} \quad x_i^*\in N_A(x_i),\ \forall i\in\mathbb{N}\}.$$

If $\bar{x}\notin A$, we use the convention that the three cones above are empty.

Given an extended-real-valued function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ and a point $\bar{x}\in\mathrm{dom}\,f$, its *limiting subdifferential* at $\bar{x}$ can be defined by

$$\overline{\partial} f(\bar{x}) := \{x^*\in\mathbb{R}^n \mid (x^*,-1)\in\overline{N}_{\mathrm{epi}\,f}(\bar{x},f(\bar{x}))\},$$

where $\mathrm{epi}\,f := \{(x,\mu)\in\mathbb{R}^n\times\mathbb{R} \mid f(x)\le\mu\}$ is the *epigraph* of $f$. Given a function $f:\mathbb{R}^n\to\mathbb{R}^m$, Lipschitz continuous around a point $\bar{x}\in\mathbb{R}^n$, its *Clarke generalized Jacobian* at $\bar{x}$ is defined by

$$\partial_C f(\bar{x}) := \mathrm{co}\left\{\lim_{k\to+\infty}\nabla f(x_k) \mid x_k\to\bar{x},\ f \text{ is differentiable at } x_k\right\},$$

where co stands for the *convex hull*.

Given a set-valued mapping $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^m$ and a point $(\bar{x},\bar{y})\in\mathrm{gph}\,F$, the cones defined above give rise to the following *generalized derivatives*:

(i) the set-valued mapping $DF(\bar{x},\bar{y}):\mathbb{R}^n\rightrightarrows\mathbb{R}^m$, defined by

$$DF(\bar{x},\bar{y})(u) := \{v\in\mathbb{R}^m \mid (u,v)\in T_{\mathrm{gph}\,F}(\bar{x},\bar{y})\}, \quad u\in\mathbb{R}^n,$$

is called the *graphical derivative* of $F$ at $(\bar{x},\bar{y})$;

(ii) the set-valued mapping $D^*F(\bar{x},\bar{y}):\mathbb{R}^m\rightrightarrows\mathbb{R}^n$, defined by

$$D^*F(\bar{x},\bar{y})(v^*) := \{u^*\in\mathbb{R}^n \mid (u^*,-v^*)\in N_{\mathrm{gph}\,F}(\bar{x},\bar{y})\}, \quad v^*\in\mathbb{R}^m,$$

is called the *Fréchet coderivative* of $F$ at $(\bar{x},\bar{y})$.

(iii) the set-valued mapping $\overline{D}^*F(\bar{x},\bar{y}):\mathbb{R}^m\rightrightarrows\mathbb{R}^n$, defined by

$$\overline{D}^*F(\bar{x},\bar{y})(v^*) := \{u^*\in\mathbb{R}^n \mid (u^*,-v^*)\in\overline{N}_{\mathrm{gph}\,F}(\bar{x},\bar{y})\}, \quad v^*\in\mathbb{R}^m,$$

is called the *limiting coderivative* of $F$ at $(\bar{x},\bar{y})$.

Recently, a finer, directionally dependent notion of a limiting normal cone has been introduced, cf. [8, 9, 12]. In addition to a set $A$ and a point $\bar{x} \in A$, one specifies also a direction $u \in \mathbb{R}^n$. The cone

$$\overline{N}_A(\bar{x};u) := \{x^* \in \mathbb{R}^n \mid \exists t_i \searrow 0, u_i \to u, x_i^* \to x^* \text{ such that } x_i^* \in N_A(\bar{x}+t_i u_i), \forall i \in \mathbb{N}\}$$

is then called the *directional limiting normal cone* to $A$ at $\bar{x}$ in the direction $u$.

It is easy to see that $\overline{N}_A(\bar{x};u) = \emptyset$ when $u \notin T_A(\bar{x})$ and

$$\overline{N}_A(\bar{x}) = \bigcup_{\|u\|=1} \overline{N}_A(\bar{x};u) \cup N_A(\bar{x}). \tag{2.1}$$

Relation (2.1) plays an important role in various conditions relaxing the standard criteria (sufficient conditions) for various Lipschitzian properties of set-valued mappings; see, e.g., [8, 11].

A set $A$ is called *directionally regular* [**?**] at $\bar{x} \in A$ in the direction $u$ if

$$\overline{N}_A(\bar{x};u) = \{x^* \in \mathbb{R}^n \mid \forall t_i \searrow 0, \exists u_i \to u, x_i^* \to x^* \text{ such that } x_i^* \in N_A(\bar{x}+t_i u_i), \forall i \in \mathbb{N}\},$$

and simply *directionally regular* at $\bar{x}$ if it is directionally regular at $\bar{x}$ in all directions.

Given a set-valued mapping $F$, a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$ and a pair of directions $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, the set-valued mapping $\overline{D}^* F((\bar{x},\bar{y});(u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$\overline{D}^* F((\bar{x},\bar{y});(u,v))(v^*) := \{u^* \in \mathbb{R}^n \mid (u^*,-v^*) \in \overline{N}_{\operatorname{gph} F}((\bar{x},\bar{y});(u,v))\}, \quad v^* \in \mathbb{R}^m,$$

is called the *directional limiting coderivative of* $F$ at $(\bar{x},\bar{y})$ in the direction $(u,v)$.

With $F$ and $(\bar{x},\bar{y})$ as above, the *limit set, critical for metric subregularity*, denoted by $\mathrm{Cr}_0 F(\bar{x},\bar{y})$, is the collection of all elements $(v,u^*) \in \mathbb{R}^m \times \mathbb{R}^n$ such that there are sequences $t_i \searrow 0$, $(u_i),(u_i^*) \subset \mathbb{R}^n$, $(v_i),(v_i^*) \subset \mathbb{R}^m$ with $v_i \to v$, $u_i^* \to u^*$,

$$(-u_i^*, v_i^*) \in N_{\operatorname{gph} F}(\bar{u}+t_i u_i, \bar{v}+t_i v_i) \quad \text{and} \quad \|u\| = \|v^*\|_* = 1.$$

As proved in [8, Theorem 3.2], the condition $(0,0) \notin \mathrm{Cr}_0 F(\bar{x},\bar{y})$ is sufficient for metric subregularity of $F$ at $\bar{u}$ for $\bar{v}$.

In our analysis we make use also of a generalization of the semismoothness property, introduced by Mifflin in [24]. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is (weakly) *semismooth* at $\bar{x}$, provided it is Lipschitz continuous around $\bar{x}$ and the limit

$$\lim\{Vu' \mid V \in \partial_C f(\bar{x}+tu'),\ u' \to u,\ t \searrow 0\} \tag{2.2}$$

exists for all $u \in \mathbb{R}^n$; here $\partial_C f$ stands for the *Clarke generalized Jacobian* of $f$. It is easy to verify that this property implies directional differentiability of $f$ at $\bar{x}$ and limit (2.2) amounts to $f'(\bar{x};u)$ (the *Hadamard directional derivative* of $f$ at $\bar{x}$ in the direction $u$).

## 2.2 Classes of perturbations and definitions of the radii

As discussed in Section 1, the radius of subregularity depends on the choice of the class of functions that are used as perturbations. We consider three such classes: Lipschitz continuous, semismooth and $C^1$ functions.

$$\begin{aligned} \mathscr{F}_{Lip} &:= \{h : \mathbb{R}^n \to \mathbb{R}^m \mid h \text{ is Lipschitz continuous around } \bar{x}\}, \\ \mathscr{F}_{ss} &:= \{h : \mathbb{R}^n \to \mathbb{R}^m \mid h \text{ is semismooth at } \bar{x}\}, \\ \mathscr{F}_{C^1} &:= \{h : \mathbb{R}^n \to \mathbb{R}^m \mid h \text{ is } C^1 \text{ around } \bar{x}\}. \end{aligned}$$

Without loss of generality, we will assume that perturbation functions $h$ in all three definitions satisfy $h(\bar{x}) = 0$.

The corresponding radii are defined as follows:

$$\mathrm{rad}[\mathrm{SR}]_{\mathscr{P}} F(\bar{x}\,|\,\bar{y}) := \inf_{h \in \mathscr{F}_{\mathscr{P}}} \{\mathrm{lip}(h;\bar{x}) \mid F+h \text{ is not metrically subregular at } \bar{x} \text{ for } \bar{y}\},$$

where $\mathscr{P}$ stands for $Lip$, $ss$ or $C^1$. Note that, for every $h \in \mathscr{F}_{Lip}$, in view of [30, Theorem 9.62] it holds

$$\mathrm{lip}(h;\bar{x}) = \sup\{\|B\| \mid B \in \partial_C h(\bar{x})\},$$

where $\partial_C h(x)$ stands for the Clarke generalized Jacobian of $h$ at $x$.

If $\mathrm{rad}[\mathrm{SR}]_{\mathscr{P}} F(\bar{x}|\bar{y}) > 0$, then $F$ is necessarily subregular at $\bar{x}$ for $\bar{y}$ since $0 \in \mathscr{F}_{\mathscr{P}}$ whenever $\mathscr{P}$ stands for any of the three classes considered in this paper. In the degenerate case when $F$ is not subregular at $\bar{x}$ for $\bar{y}$, the above definition of the radius automatically gives $\mathrm{rad}[\mathrm{SR}]_{\mathscr{P}} F(\bar{x}|\bar{y}) = 0$.

We obviously have $\mathscr{F}_{C^1} \subset \mathscr{F}_{ss} \subset \mathscr{F}_{Lip}$ and

$$\mathrm{rad}[\mathrm{SR}]_{Lip} F(\bar{x}|\bar{y}) \le \mathrm{rad}[\mathrm{SR}]_{ss} F(\bar{x}|\bar{y}) \le \mathrm{rad}[\mathrm{SR}]_{C^1} F(\bar{x}|\bar{y}). \tag{2.3}$$

## 2.3 Primal-dual derivative and regularity constants

Given $(\bar{x},\bar{y}) \in \mathrm{gph}\, F$, we define the *primal-dual derivative* $\widehat{D}F(\bar{x},\bar{y}) : \mathbb{R}^n \times (\mathbb{R}^m)^* \rightrightarrows (\mathbb{R}^n)^* \times \mathbb{R}^m$ of $F$ at $(\bar{x},\bar{y})$ as follows: for all $(u,v^*) \in \mathbb{R}^n \times (\mathbb{R}^m)^*$,

$$\widehat{D}F(\bar{x},\bar{y})(u,v^*) := \left\{(u^*,v) \in (\mathbb{R}^n)^* \times \mathbb{R}^m \mid (u^*,-v^*) \in \overline{N}_{\mathrm{gph}\, F}((\bar{x},\bar{y});(u,v))\right\}. \tag{2.4}$$

In other words,

$$\widehat{D}F(\bar{x},\bar{y})(u,v^*) = \left\{(u^*,v) \in (\mathbb{R}^n)^* \times \mathbb{R}^m \mid u^* \in \overline{D}^* F((\bar{x},\bar{y});(u,v))(v^*)\right\}.$$

The next proposition, which follows directly from the definitions, shows that the mapping $\widehat{D}F(\bar{x},\bar{y})$ combines features of the graphical derivative and the limiting coderivative: tangents (related to the graphical derivative) are linked with limiting normals (related to the coderivative) to the graph of $F$ in a suitable way.

**Proposition 2.1** $\widehat{D}F(\bar{x},\bar{y})(u,v^*) \subset \overline{D}^* F(\bar{x},\bar{y})(v^*) \times DF(\bar{x},\bar{y})(u)$ *for all* $(u,v^*) \in \mathbb{R}^n \times (\mathbb{R}^m)^*$.

Using (2.4) we define two image sets under $\widehat{D}F(\bar{x},\bar{y})$:

$$\widehat{\mathfrak{D}}F(\bar{x},\bar{y}) := \left\{(u^*,v) \in \widehat{D}F(\bar{x},\bar{y})(u,v^*) \mid \|u\| = \|v^*\|_* = 1\right\}, \tag{2.5}$$

$$\widehat{\mathfrak{D}}^\circ F(\bar{x},\bar{y}) := \left\{(u^*,v) \in \widehat{D}F(\bar{x},\bar{y})(u,v^*) \mid \|u\| = \|v^*\|_* = 1,\ {u^*}^T u = {v^*}^T v\right\}. \tag{2.6}$$

Observe that the set (2.5) is a small modification of the limit set $\mathrm{Cr}_0 F(\bar{x},\bar{y})$ [8]: $(u^*,v) \in \widehat{\mathfrak{D}}F(\bar{x},\bar{y})$ if and only if $(v,-u^*) \in \mathrm{Cr}_0 F(\bar{x},\bar{y})$.

**Proposition 2.2** *The image set* (2.6) *admits an equivalent representation involving an* $m \times n$ *matrix:*

$$\widehat{\mathfrak{D}}^\circ F(\bar{x},\bar{y}) = \left\{(u^*,v) \in \widehat{D}F(\bar{x},\bar{y})(u,v^*) \mid \|u\| = \|v^*\|_* = 1,\right.$$
$$\left. B^T v^* = u^*,\ Bu = v,\ B \in L(\mathbb{R}^n,\mathbb{R}^m)\right\}. \tag{2.7}$$

*Proof* Let $u,u^* \in \mathbb{R}^n$, $v,v^* \in \mathbb{R}^m$ and $\|u\| = \|v^*\|_* = 1$. We need to check the equivalence of the condition ${u^*}^T u = {v^*}^T v$ to the pair of conditions $Bu = v$ and $B^T v^* = u^*$ for some $m \times n$ matrix $B$.

Suppose that ${u^*}^T u = {v^*}^T v$. Choose vectors $z^* \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ such that $\|w\| = \|z^*\|_* = {z^*}^T u = {v^*}^T w = 1$, and set

$$B := v{z^*}^T + w{u^*}^T - ({u^*}^T u)w{z^*}^T. \tag{2.8}$$

Then $Bu = v + ({u^*}^T u)w - ({u^*}^T u)w = v$ and $B^T v^* = (v^T v^*)z^* + u^* - ({u^*}^T u)z^* = u^*$.

Conversely, suppose that $Bu = v$ and $B^T v^* = u^*$ for some $m \times n$ matrix $B$. Then ${u^*}^T u = u^T u^* = u^T B^T v^* = {v^*}^T Bu = {v^*}^T v$.

*Remark 2.3* The above proof of Proposition 2.2 is constructive. In the first part, it not only establishes the existence of a matrix $B$ with required properties; it provides the formula (2.8) for constructing such a matrix.

The following quantities are instrumental in deriving bounds for the radius of metric subregularity:

$$\mathrm{rg}[F](\bar{x},\bar{y}) := \inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid (u^*,v)\in \widehat{\mathfrak{D}}F(\bar{x},\bar{y})\big\}, \tag{2.9}$$

$$\mathrm{rg}^\circ[F](\bar{x},\bar{y}) := \inf\big\{\|B\| \mid B\in L(\mathbb{R}^n,\mathbb{R}^m), \\ (B^Tv^*,Bu)\in \widehat{\mathfrak{D}}^\circ F(\bar{x},\bar{y}),\ \|u\|=\|v^*\|_*=1\big\}. \tag{2.10}$$

The next two modifications of (2.9) and (2.10) can also be useful:

$$\overline{\mathrm{rg}}[F](\bar{x},\bar{y}) := \inf\big\{\|u^*\|_*+\|v\| \mid (u^*,v)\in \widehat{\mathfrak{D}}F(\bar{x},\bar{y})\big\}, \tag{2.11}$$

$$\underline{\mathrm{rg}}^\circ[F](\bar{x},\bar{y}) := \inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid (u^*,v)\in \widehat{\mathfrak{D}}^\circ F(\bar{x},\bar{y})\big\}. \tag{2.12}$$

They provide, respectively, an upper bound for (2.9) and a lower bound for (2.10). This explains their notations. Note that (2.12) is also an upper bound for (2.9).

**Proposition 2.4** (i) $\mathrm{rg}[F](\bar{x},\bar{y}) \le \underline{\mathrm{rg}}^\circ[F](\bar{x},\bar{y}) \le \mathrm{rg}^\circ[F](\bar{x},\bar{y})$*;*
(ii) $\mathrm{rg}[F](\bar{x},\bar{y}) \le \overline{\mathrm{rg}}[F](\bar{x},\bar{y}) \le 2\mathrm{rg}[F](\bar{x},\bar{y})$*;*
(iii) $\mathrm{rg}[F](\bar{x},\bar{y}) \ge \inf\{\|u^*\|_* \mid u^*\in \overline{D}^*F(\bar{x},\bar{y})(v^*),\ \|v^*\|_*=1\}$*;*
(iv) $\mathrm{rg}[F](\bar{x},\bar{y}) \ge \inf\{\|v\| \mid v\in DF(\bar{x},\bar{y})(u),\ \|u\|=1\}$*.*

*Proof* (i) In view of the definitions (2.9) and (2.12), the first inequality is immediate from comparing (2.5) and (2.6). In view of the definitions (2.10) and (2.12), the second inequality follows from Proposition 2.2.

The inequalities in (ii) are immediate from comparing the definitions (2.9) and (2.11).

Inequalities (iii) and (iv) are consequences of the definitions (2.5) and (2.6), and Proposition 2.1.

The quantities in the right-hand sides of the inequalities in parts (iii) and (iv) of Proposition 2.4 equal to the reciprocals of the moduli of the metric regularity and strong metric subregularity, respectively; cf. [5, Theorems 4C.2 and 4E.1], and in view of [5, Theorems 6A.7 and 6A.9], are exactly the radii of the corresponding properties. Thus, the value of $\mathrm{rg}[F](\bar{x},\bar{y})$ is an upper bound for both these radii.

## 3 The radius theorem

In this section we present the main results of this paper.

We start with a lemma which is a consequence of [30, Theorem 10.41 and Exercise 10.43].

**Lemma 3.1** *Consider a mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graph, a function $h:\mathbb{R}^n\to\mathbb{R}^m$ and a point $(u,v)\in \mathrm{gph}\,(F+h)$ such that h is Lipschitz continuous around u. Then*

$$\overline{D}^*(h+F)(u,v)(v^*) \subset \overline{D}^*h(u)(v^*)+\overline{D}^*F(u,v-h(u))(v^*) \quad \textit{for all } v^*\in\mathbb{R}^m. \tag{3.1}$$

*As a consequence,*

$$\overline{N}_{\mathrm{gph}\,(h+F)}(u,v) \subset \{(u_h^*+u_F^*,v^*) \mid \\ (u_h^*,v^*)\in \overline{N}_{\mathrm{gph}\,h}(u,h(u)),\ (u_F^*,v^*)\in \overline{N}_{\mathrm{gph}\,F}(u,v-h(u))\}.$$

*If, additionally, h is strictly differentiable at u, then*

$$\overline{N}_{\mathrm{gph}\,(h+F)}(u,v) = \{(u^*,v^*)\in\mathbb{R}^n\times\mathbb{R}^m \mid (u^*+\nabla h(u)^Tv^*,v^*)\in \overline{N}_{\mathrm{gph}\,F}(u,v-h(u))\}.$$

Our main result given next provides lower and upper bounds for the radius of metric subregularity for Lipschitzian perturbations and the exact radius formula for the other classes of perturbations.

**Theorem 3.2** *Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graph and a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Then*

$$\operatorname{rg}[F](\bar{x},\bar{y}) \le \operatorname{rad}[\mathrm{SR}]_{Lip} F(\bar{x}\,|\,\bar{y}) \le \overline{\operatorname{rg}}[F](\bar{x},\bar{y}), \tag{3.2}$$

$$\operatorname{rad}[\mathrm{SR}]_{ss} F(\bar{x}\,|\,\bar{y}) = \operatorname{rad}[\mathrm{SR}]_{C^1} F(\bar{x}\,|\,\bar{y}) = \operatorname{rg}^\circ[F](\bar{x},\bar{y}). \tag{3.3}$$

*Proof* Step 1: $\operatorname{rg}[F](\bar{x},\bar{y}) \le \operatorname{rad}[\mathrm{SR}]_{Lip} F(\bar{x}\,|\,\bar{y})$. Let $h \in \mathscr{F}_{Lip}$ be such that $h+F$ is not metrically subregular at $\bar{x}$ for $\bar{y}$. From [8, Theorem 3.2] we obtain that $(0,0) \in \mathrm{Cr}_0(h+F)(\bar{x},\bar{y})$. This implies that there exist sequences $t_k$, $u_k$, $v_k$, $u_k^*$, $v_k^*$ such that

$$t_k \searrow 0,\ v_k \to 0,\ u_k^* \to 0,\ \|u_k\| = \|v_k^*\|_* = 1\ (k=1,2,\ldots), \tag{3.4}$$
$$(-u_k^*, v_k^*) \in N_{\operatorname{gph}(h+F)}(\bar{x}+t_k u_k, \bar{y}+t_k v_k)\ (k=1,2,\ldots).$$

By Lemma 3.1, there are elements $u_{h,k}^*$ such that

$$(u_{h,k}^*, v_k^*) \in \overline{N}_{\operatorname{gph} h}(\bar{x}+t_k u_k, h(\bar{x}+t_k u_k)), \tag{3.5}$$

$$(-u_k^* - u_{h,k}^*, v_k^*) \in \overline{N}_{\operatorname{gph} F}(\bar{x}+t_k u_k, \bar{y}+t_k \tilde{v}_k), \tag{3.6}$$

where

$$\tilde{v}_k := v_k - h(\bar{x}+t_k u_k)/t_k. \tag{3.7}$$

Let $\gamma > \operatorname{lip}(h;\bar{x})$. Then, for all $k$ sufficiently large, in view of [30, Proposition 9.24(b)],

$$\left\|u_{h,k}^*\right\|_* \le \gamma \|v_k^*\|_* = \gamma, \quad \text{and}$$
$$\|\tilde{v}_k\| = \|v_k - (h(\bar{x}+t_k u_k) - h(\bar{x}))/t_k\| \le \|v_k\| + \gamma\|u_k\| = \|v_k\| + \gamma.$$

Without loss of generality, we can assume that

$$u_k \to u,\ v_k^* \to v^*,\ u_{h,k}^* \to u_h^*,\ \tilde{v}_k \to \tilde{v},\ \|u\| = \|v^*\|_* = 1. \tag{3.8}$$

We conclude that $\|\tilde{v}\| \le \gamma$, $\left\|u_h^*\right\|_* \le \gamma$ and $(-u_h^*, v^*) \in \overline{N}_{\operatorname{gph} F}((\bar{x},\bar{y}),(u,\tilde{v}))$, i.e. $(-u_h^*, \tilde{v}) \in \widehat{D}F(\bar{x},\bar{y})(u,-v^*)$. Thus, $\operatorname{rg}[F](\bar{x},\bar{y}) \le \max\{\left\|u_h^*\right\|_*, \|\tilde{v}\|\} \le \gamma$. Taking infimum in the last inequality over all $\gamma > \operatorname{lip}(h;\bar{x})$ and then over all $h \in \mathscr{F}_{Lip}$ such that $h+F$ is not metrically subregular at $\bar{x}$ for $\bar{y}$, we arrive at $\operatorname{rg}[F](\bar{x},\bar{y}) \le \operatorname{rad}[\mathrm{SR}]_{Lip} F(\bar{x}\,|\,\bar{y})$.

Step 2: $\operatorname{rad}[\mathrm{SR}]_{Lip} F(\bar{x}\,|\,\bar{y}) \le \overline{\operatorname{rg}}[F](\bar{x},\bar{y})$. To show this inequality, we construct a special Lipschitz continuous perturbation $h$. Given a strictly decreasing sequence $\tau = (\tau_k)$ of positive real numbers converging to 0, set for every $k = 1,2,\ldots$,

$$a_{k+1} := \tau_{k+1} + \frac{\tau_k - \tau_{k+1}}{2(k+1)}, \quad b_k := \tau_k - \frac{\tau_k - \tau_{k+1}}{2(k+1)}.$$

Then $\tau_{k+1} < a_{k+1} < b_k < \tau_k$. Define a function $\chi_\tau : \mathbb{R} \to \mathbb{R}$ recursively as follows:

$$\chi_\tau(t) := \begin{cases} 0 & \text{if } t = 0, \\ \chi_\tau(a_{k+1}) - a_{k+1} + t & \text{if } a_{k+1} < t \le b_k, \\ \chi_\tau(b_k) & \text{if } b_k < t \le a_k, \\ -\chi_\tau(-t) & \text{if } t < 0. \end{cases}$$

Thus, $\chi_\tau$ is linear on every interval $[a_{k+1}, b_k]$ and $[-b_k, -a_{k+1}]$ with slope 1 and constant on every interval $[b_k, a_k]$ and $[-a_k, -b_k]$. In particular, $\chi_\tau$ is Lipschitz continuous on $\mathbb{R}$ with modulus 1 and continuously differentiable at $\tau_k$ with the derivative equal 0. Moreover, for all $t \in (b_k, a_k)$, we have

$$\chi_\tau(t) = \sum_{j=k}^{\infty} (b_j - a_{j+1}) = \sum_{j=k}^{\infty} (\tau_j - \tau_{j+1}) \left(1 - \frac{1}{j+1}\right) = \tau_k - \sum_{j=k}^{\infty} \frac{\tau_j - \tau_{j+1}}{j+1},$$

and consequently,

$$\tau_k > \chi_\tau(\tau_k) > \tau_k - \frac{1}{k+1}\sum_{j=k}^{\infty}(\tau_j - \tau_{j+1}) = \tau_k\left(1 - \frac{1}{k+1}\right),$$

showing

$$\lim_{k\to+\infty}\frac{\chi_\tau(\tau_k)}{\tau_k} = \lim_{k\to+\infty}\frac{\chi_\tau(-\tau_k)}{-\tau_k} = 1.$$

Next, consider $(u^*,v) \in \widehat{D}F(\bar{x},\bar{y})(u,v^*)$ with $\|u\| = \|v^*\|_* = 1$ and choose elements $\hat{u}^* \in (\mathbb{R}^n)^*$ and $\hat{v} \in \mathbb{R}^m$ with $\|\hat{u}^*\|_* = \|\hat{v}\| = 1$ such that

$$\hat{u}^{*T}u = \|u\| = 1, \quad v^{*T}\hat{v} = \|v^*\|_* = 1.$$

By the definition of $\widehat{D}F(\bar{x},\bar{y})$, there exist sequences $t_k \searrow 0$, $u_k \to u$, $v_k \to v$, $u_k^* \to u^*$ and $v_k^* \to v^*$ such that

$$(u_k^*, -v_k^*) \in N_{\operatorname{gph} F}(\bar{x}+t_k u_k, \bar{y}+t_k v_k).$$

By passing to a subsequence if necessary, we can assume that the sequence $\tau_u := (t_k\hat{u}^{*T}u_k)$ is strictly decreasing. If $u^{*T}u \neq 0$, we can also assume that the sequence $\tau_{u^*} := (t_k|u^{*T}u_k|)$ is strictly decreasing; in this case we set $\zeta(x) := u^{*T}x - \chi_{\tau_{u^*}}(u^{*T}x)$, and observe that

$$\lim_{k\to+\infty}\frac{\zeta(t_k u_k)}{t_k} = (u^{*T}u)\lim_{k\to+\infty}\frac{\zeta(t_k u_k)}{t_k u^{*T}u_k} = (u^{*T}u)\left(1 - \lim_{k\to+\infty}\frac{\chi_{\tau_{u^*}}(t_k u^{*T}u_k)}{t_k u^{*T}u_k}\right) = 0.$$

When $u^{*T}u = 0$, we set $\zeta(x) := u^{*T}x$ and observe that $\zeta(t_k u_k)/t_k = u^{*T}u_k \to 0$ as $k \to +\infty$. In both cases, $\zeta$ is Lipschitz continuous on $\mathbb{R}^n$ with modulus $\|u^*\|_*$ and continuously differentiable at $t_k u_k$ with the derivative $\nabla\zeta(t_k u_k) = u^*$. Next, consider the mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$h(x) := \chi_{\tau_u}(\hat{u}^{*T}(x-\bar{x}))v + \zeta(x-\bar{x})\hat{v}, \quad x \in \mathbb{R}^n.$$

We have

$$\lim_{k\to+\infty}\frac{h(\bar{x}+t_k u_k)}{t_k} = \lim_{k\to+\infty}\left(\frac{\chi_{\tau_u}(t_k\hat{u}^{*T}u_k)}{t_k}v + \frac{\zeta(t_k u_k)}{t_k}\hat{v}\right) = v.$$

Further, $h$ is continuously differentiable at $\bar{x}+t_k u_k$ with the derivative $\nabla h(\bar{x}+t_k u_k) = \hat{v}u^{*T}$, implying $\nabla h(\bar{x}+t_k u_k)^T v_k^* = u^*(\hat{v}^T v_k^*) \to u^*$. By virtue of Lemma 3.1, we obtain

$$(u_k^* - u^*(\hat{v}^T v_k^*), -v_k^*) \in N_{\operatorname{gph}(F-h)}(\bar{x}+t_k u_k, \bar{y}+t_k v_k - h(\bar{x}+t_k u_k)).$$

Since $u_k^* - u^*(\hat{v}^T v_k^*) \to 0$ and $v_k - h(\bar{x}+t_k u_k)/t_k \to 0$ as $k \to 0$, we obtain that $(0,0) \in \operatorname{Cr}_0(F-h)(\bar{x},\bar{y})$. By [8, Theorem 3.2(2)], we can now find a $C^1$ perturbation $\tilde{h}$ with $\tilde{h}(\bar{x}) = 0$ and $\|\nabla\tilde{h}(\bar{x})\| = 0$ such that $F - h + \tilde{h}$ is not metrically subregular at $(\bar{x},\bar{y})$. We now want to estimate $\operatorname{lip}(h;\bar{x})$. Taking any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\|h(x_1) - h(x_2)\| \le (\|v\|\,\|\hat{u}^*\|_* + \|\hat{v}\|\,\|u^*\|_*)\,\|x_1 - x_2\| = (\|v\| + \|u^*\|_*)\,\|x_1 - x_2\|.$$

Hence, $\operatorname{lip}(h-\tilde{h};\bar{x}) = \operatorname{lip}(h;\bar{x}) \le \|v\| + \|u^*\|_*$ and, since $(h-\tilde{h})(\bar{x}) = 0$, we conclude that $\operatorname{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) \le \|v\| + \|u^*\|_*$. The inequality $\operatorname{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) \le \overline{\operatorname{rg}}[F](\bar{x},\bar{y})$ follows. This completes the proof of (3.2).

Step 3: $\operatorname{rg}^\circ[F](\bar{x},\bar{y}) \le \operatorname{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y})$. Let $h \in \mathscr{F}_{ss}$ be such that $h+F$ is not metrically subregular at $\bar{x}$ for $\bar{y}$. Then $h \in \mathscr{F}_{Lip}$ and, as shown above, there exist sequences $t_k$, $u_k$, $v_k$, $\tilde{v}_k$, $u_k^*$, $u_{h,k}^*$, $v_k^*$ and vectors $u$, $\tilde{v}$, $v^*$, $u_h^*$ such that conditions (3.4), (3.5), (3.6), (3.7) and (3.8) hold true. Thanks to the Lipschitz continuity of $h$, it follows from (3.5) that $-u_{h,k}^* \in \overline{\partial}\left\langle v_k^*, h\right\rangle(\bar{x}+t_k u_k)$ (cf. e.g. [30, Proposition 9.24]), and consequently, $u_{h,k}^* \in B_k^T v_k^*$ where $-B_k \in \partial_C h(\bar{x}+t_k u_k)$. From the Lipschitz continuity of $h$, the sequence of matrices $B_k$ is bounded. Without loss of generality, we can assume that $B_k \to B \in -\partial_C h(\bar{x})$. Note that $\|B\| \le \operatorname{lip}(h;\bar{x})$. Thus, $u_h^* = B^T v^*$. Since $h$ is semismooth, it is directionally differentiable in all directions, and, in view of (3.7) $\tilde{v}_k \to -h'(\bar{x};u) = Bu$. It now follows from (3.6) that $(-B^T v^*, v^*) \in \overline{N}_{\operatorname{gph} F}((\bar{x},\bar{y});(u,Bu))$, i.e. $(B^T(-v^*), Bu) \in \widehat{D}F(\bar{x},\bar{y})(u,-v^*)$. Since $\|u\| = \|-v^*\|_* = 1$, we have $\operatorname{rg}^\circ[F](\bar{x},\bar{y}) \le \|B\| \le \operatorname{lip}(h;\bar{x})$. Taking infimum in the last inequality over all $h \in \mathscr{F}_{ss}$ such that $h+F$ is not metrically subregular at $\bar{x}$ for $\bar{y}$, we arrive at $\text{rg}^\circ[F](\bar{x},\bar{y}) \le \text{rad}[\text{SR}]_{ss}F(\bar{x}|\bar{y})$.

Step 4: $\text{rad}[\text{SR}]_{C^1}F(\bar{x}|\bar{y}) \le \text{rg}^\circ[F](\bar{x},\bar{y})$. Suppose $\text{rg}^\circ[F](\bar{x},\bar{y}) < +\infty$; otherwise there is nothing to prove. Let $\varepsilon > 0$. It follows from the definition of $\text{rg}^\circ[F](\bar{x},\bar{y})$ that there exists a matrix $B \in \mathbb{R}^{m\times n}$ with $\|B\| < \text{rg}^\circ[F](\bar{x},\bar{y}) + \varepsilon$ and vectors $u \in \mathbb{R}^n$ and $v^* \in \mathbb{R}^m$ with $\|u\| = \|v^*\|_* = 1$ such that $(B^Tv^*, -v^*) \in \overline{N}_{\text{gph}\,F}((\bar{x},\bar{y});(u,Bu))$. Hence, there exist sequences $t_k \searrow 0$, $u_k \to u$, $v_k \to Bu$, $u_k^* \to B^Tv^*$ and $v_k^* \to v^*$ such that

$$(u_k^*, -v_k^*) \in N_{\text{gph}\,F}(\bar{x}+t_ku_k, \bar{y}+t_kv_k) \tag{3.9}$$

for all $k = 1,2,\ldots$. Set $h(x) := -B(x-\bar{x})$ $(x \in \mathbb{R}^n)$. Obviously, $h(\bar{x}) = 0$, $h$ is $C^1$ and $\nabla h(x) = -B$ for any $x \in \mathbb{R}^n$. Invoking Lemma 3.1, with $h+F$ and $-h$ in place of $F$ and $h$, respectively, we obtain from (3.9) that

$$(\hat{u}_k^*, -v_k^*) \in N_{\text{gph}\,(h+F)}(\bar{x}+t_ku_k, \bar{y}+t_k\hat{v}_k),$$

where $\hat{v}_k := v_k - Bu_k$, $\hat{u}_k^* := u_k^* - B^Tv_k^*$. Observe that $\hat{v}_k \to 0$ and $\hat{u}_k^* \to 0$ as $k \to +\infty$, which implies that $(0,0) \in \text{Cr}_0(h+F)(\bar{x},\bar{y})$. By [8, Theorem 3.2(2)], we can now find a $C^1$ perturbation $\tilde{h}$ with $\tilde{h}(\bar{x}) = 0$ and $\|\nabla\tilde{h}(\bar{x})\| = 0$ such that $F+h+\tilde{h}$ is not metrically subregular at $\bar{x}$ for $\bar{y}$. Since $(h+\tilde{h})(\bar{x}) = 0$ and $\text{lip}(h+\tilde{h};\bar{x}) = \|\nabla(h+\tilde{h})(\bar{x})\| = \|B\|$, we conclude that $\text{rad}[\text{SR}]_{C^1}F(\bar{x}|\bar{y}) \le \|B\| < \text{rg}^\circ[F](\bar{x},\bar{y}) + \varepsilon$. Taking infimum in the last inequality over all $\varepsilon > 0$, we arrive at $\text{rad}[\text{SR}]_{C^1}F(\bar{x}|\bar{y}) \le \text{rg}^\circ[F](\bar{x},\bar{y})$. In view of (2.3), this completes the proof of (3.3).

*Remark 3.3* Unlike the case of semismooth and $C^1$ perturbations, where Theorem 3.2 establishes the exact formula for the radius, in the case of more general Lipschitz perturbations the theorem gives only lower and upper bounds for the respective radius, which, in view of Proposition 2.4(ii), differ by a factor of at most 2. We do not know if these bounds are sharp. Obtaining sharp bounds is an interesting problem for future research

By using the first inequality in (2.3) and Proposition 2.4(i), one obtains additional bounds for the radii of subregularity, as stated in the following corollary.

**Corollary 3.4** *Consider a mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with closed graph and a point $(\bar{x},\bar{y}) \in \text{gph}\,F$. Then*

(i) $\text{rad}[\text{SR}]_{Lip}F(\bar{x}|\bar{y}) \le \text{rg}^\circ[F](\bar{x},\bar{y})$;
(ii) $\text{rad}[\text{SR}]_{ss}F(\bar{x}|\bar{y}) = \text{rad}[\text{SR}]_{C^1}F(\bar{x}|\bar{y}) \ge \underline{\text{rg}}^\circ[F](\bar{x},\bar{y}) \ge \text{rg}[F](\bar{x},\bar{y})$.

In accordance with Theorem 3.2, condition $\text{rg}[F](\bar{x},\bar{y}) > 0$ guarantees that $F$ is metrically subregular at $\bar{x}$ for $\bar{y}$ together with all its perturbations by Lipschitz continuous functions with small Lipschitz modulus, while condition $\text{rg}^\circ[F](\bar{x},\bar{y}) > 0$ plays a similar role with respect to semismooth and $C^1$ perturbations of $F$. In fact, both conditions correspond to certain regularity properties of $F$ at $\bar{x}$ for $\bar{y}$ being stronger than conventional metric subregularity and, in view of Proposition 2.4(iv) and the well-known graphical derivative criterion for strong metric subregularity [5, Theorem 4E.1], weaker than strong metric subregularity. Formula (3.3) agrees with the pattern of (1.6) with $\text{rg}^\circ[F](\bar{x},\bar{y})$ playing the role of the regularity 'modulus' rg. Note that the mentioned regularity properties, despite possessing certain stability with respect to small perturbations, are not 'robust': they can be violated in a neighbourhood of the reference point $(\bar{x},\bar{y})$; see the example in Section 4.

Computing $\text{rg}^\circ[F](\bar{x},\bar{y})$ using (2.10) and (2.7) involves minimization over five parameters: four vectors $u, v, u^*, v^*$ and a matrix $B$. The number of parameters could be reduced by eliminating the matrix if for given $u, v, u^*, v^*$, satisfying ${u^*}^Tu = {v^*}^Tv$ and $\|u\| = \|v^*\|_* = 1$, we were able to solve analytically the problem

$$\min_{B\in L(\mathbb{R}^n,\mathbb{R}^m)} \|B\| \text{ subject to } B^Tv^* = u^*,\ Bu = v,$$

where $\|B\|$ denotes the operator norm.

Currently we know the explicit solution to this problem only for the Frobenius norm $\|B\|_F$ in the case when $\mathbb{R}^n$ and $\mathbb{R}^m$ are considered with the Euclidean norms. Specifically, the next proposition deals with the convex constrained optimization problem

$$\min \frac{1}{2}\|B\|_F^2 \text{ subject to } B^T v^* = u^*,\ Bu = v. \tag{3.10}$$

**Proposition 3.5** *Let vectors $u, u^* \in \mathbb{R}^n$ and $v, v^* \in \mathbb{R}^m$ satisfy conditions*

$$u^{*T} u = v^{*T} v, \quad \|u\|_2 = \|v^*\|_2 = 1, \tag{3.11}$$

*where $\|\cdot\|_2$ denotes the Euclidean norm. The unique minimizer of the problem* (3.10) *is given by the matrix*

$$\overline{B} := v^* u^{*T} + vu^T - (u^{*T} u) v^* u^T,$$

*and in this case,*

$$\|\overline{B}\|_F^2 = \|u^*\|_2^2 + \|v\|_2^2 - \left(u^{*T} u\right)^2. \tag{3.12}$$

*Proof* The feasibility of $\overline{B}$ in the problem (3.10) can be shown by straightforward calculations:

$$\overline{B}^T v^* = \left(u^* v^{*T} + uv^T - (u^{*T} u) uv^{*T}\right) v^* = \|v^*\|_2^2 u^* + \left(v^T v^* - (u^{*T} u)\,\|v^*\|_2^2\right) u = u^*,$$

$$\overline{B} u = (v^* u^{*T} + vu^T - (u^{*T} u) v^* u^T) u = (u^{*T} u) v^* + \|u\|_2^2 v - (u^{*T} u)\,\|u\|_2^2 v^* = v.$$

Furthermore, $\overline{B}$ satisfies the first-order KKT condition for problem (3.10) with multipliers $\eta = -u^*$ and $\eta^* = (u^{*T} u) v^* - v$; indeed:

$$\overline{B} + v^* \eta^T + \eta^* u^T = v^* u^{*T} + vu^T - (u^{*T} u) v^* u^T - v^* u^{*T} + ((u^{*T} u) v^* - v) u^T = 0.$$

Since (3.10) is a strictly convex program, our claim about the optimality of $\overline{B}$ is verified. Next we show (3.12).

$$\begin{aligned}
\|\overline{B}\|_F^2 &= \operatorname{tr}\left(\overline{B}^T \overline{B}\right)\\
&= \operatorname{tr}\Big(\left(u^* v^{*T} + uv^T - (u^{*T} u) uv^{*T}\right)\left(v^* u^{*T} + vu^T - (u^{*T} u) v^* u^T\right)\Big)\\
&= \operatorname{tr}\big(\|v^*\|_2^2 u^* u^{*T} + (v^{*T} v) uu^{*T} - \|v^*\|_2^2 (u^{*T} u) uu^{*T}\\
&\quad + (v^{*T} v) u^* u^T + \|v\|_2^2 uu^T - (u^{*T} u)(v^{*T} v) uu^T\\
&\quad - \|v^*\|_2^2 (u^{*T} u) uu^{*T} - (u^{*T} u)(v^{*T} v) uu^T + (u^{*T} u)^2 \|v^*\|_2^2 uu^T\big)\\
&= \operatorname{tr}\Big(\|v^*\|_2^2 u^* u^{*T} + \big((v^{*T} v) - 2\|v^*\|_2^2 (u^{*T} u)\big) uu^{*T}\\
&\quad + (v^{*T} v) u^* u^T + \big(\|v\|_2^2 - (u^{*T} u)\big(2(v^{*T} v) - (u^{*T} u)\,\|v^*\|_2^2\big)\big) uu^T\Big)\\
&= \|v^*\|_2^2 \|u^*\|_2^2 + \big((v^{*T} v) - 2\|v^*\|_2^2 (u^{*T} u)\big)(u^{*T} u)\\
&\quad + (v^{*T} v)(u^{*T} u) + \big(\|v\|_2^2 - (u^{*T} u)\big(2(v^{*T} v) - (u^{*T} u)\,\|v^*\|_2^2\big)\big)\,\|u\|_2^2.
\end{aligned}$$

In view of (3.11), we get

$$\begin{aligned}
\|\overline{B}\|_F^2 &= \|u^*\|_2^2 - (u^{*T} u)^2 + (v^{*T} v)(u^{*T} u) + \|v\|_2^2 - (u^{*T} u)(v^{*T} v)\\
&= \|u^*\|_2^2 - (u^{*T} u)^2 + \|v\|_2^2.
\end{aligned}$$

The proof is complete.

In view of the above proposition, in the Euclidean space setting the following analogue (upper bound) of the quantity (2.10) can be used for estimating the radius of subregularity:

$$\begin{aligned}
\overline{\operatorname{rg}}^\circ[F](\bar{x}, \bar{y}) := \inf\Big\{\sqrt{\|u^*\|_2^2 + \|v\|_2^2 - (u^{*T} u)^2} \mid (u^*, v) \in \widehat{D}F(\bar{x}, \bar{y})(u, v^*),&\\
u^{*T} u = v^{*T} v,\ \|u\|_2 = \|v^*\|_2 = 1\Big\}.&
\end{aligned} \tag{3.13}$$

The next proposition provides relationships between this new quantity and (2.10).

**Proposition 3.6** *Consider a mapping $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$. If both $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with the Euclidean norm, then*

$$\operatorname{rg}^\circ[F](\bar{x},\bar{y}) \le \overline{\operatorname{rg}}^\circ[F](\bar{x},\bar{y}) \le \sqrt{2}\,\operatorname{rg}^\circ[F](\bar{x},\bar{y}).$$

*Proof* The first inequality is a consequence of Proposition 3.5 since the spectral norm of a matrix, i.e. the operator norm with respect to the Euclidean norm, is always less than or equal to the Frobenius norm. The second inequality follows immediately from the estimate $\sqrt{\|u^*\|_2^2+\|v\|_2^2-(u^{*T}u)^2} \le \sqrt{2}\max\{\|u^*\|_2, \|v\|_2\}$.

**Corollary 3.7** *If both $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with the Euclidean norm, then*

(i) $\operatorname{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) \le \overline{\operatorname{rg}}^\circ[F](\bar{x},\bar{y})$;
(ii) $\frac{1}{\sqrt{2}}\overline{\operatorname{rg}}^\circ[F](\bar{x},\bar{y}) \le \operatorname{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y}) = \operatorname{rad}[\mathrm{SR}]_{C^1}F(\bar{x}\,|\,\bar{y}) \le \overline{\operatorname{rg}}^\circ[F](\bar{x},\bar{y})$.

## 4 Applications to constraint systems

Consider the constraint system

$$x \in D, \quad g(x) \in K, \tag{4.1}$$

where $D \subset \mathbb{R}^n$, $K \subset \mathbb{R}^m$, $g:\mathbb{R}^n \to \mathbb{R}^m$, $\bar{x} \in D$ and $g(\bar{x}) \in K$. The inclusions (4.1) can be equivalently written as $0 \in F(x)$, where

$$F(x) := \begin{cases} K - g(x) & \text{if } x \in D, \\ \emptyset & \text{otherwise.} \end{cases} \tag{4.2}$$

Observe that $\bar{y} := 0 \in F(\bar{x})$.

Before we apply our theory to the set-valued mapping $F$ given by (4.2), we recall two facts used in the proof of Proposition 4.3 below. The first one comes from [**?**, Proposition 3.2].

**Lemma 4.1** *Given two sets $A_1 \subset \mathbb{R}^n$ and $A_2 \subset \mathbb{R}^m$, a point $\bar{x} = (\bar{x}_1,\bar{x}_2) \in A_1 \times A_2$ and a direction $u = (u_1,u_2) \in \mathbb{R}^n \times \mathbb{R}^m$, one has the inclusion*

$$\overline{N}_{A_1\times A_2}(\bar{x};u) \subset \overline{N}_{A_1}(\bar{x}_1;u_1) \times \overline{N}_{A_2}(\bar{x}_2;u_2).$$

*This inclusion becomes equality provided that either $A_1$ is directionally regular at $\bar{x}_1$ in the direction $u_1$ or $A_2$ is directionally regular at $\bar{x}_2$ in the direction $u_2$.*

Next we need [11, formula (2.4)] for computing the directional limiting coderivative.

**Lemma 4.2** *Consider the mapping $F := f_1 + F_2$, where $F_2:\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has closed graph and $f_1:\mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable at $\bar{x} \in \operatorname{dom} F_2$. Given a $\bar{y} \in F(\bar{x})$, a pair $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$ and a $y^* \in \mathbb{R}^m$, it holds*

$$\overline{D}^*F((\bar{x},\bar{y});(u,v))(y^*) = \nabla f_1(\bar{x})^T y^* + \overline{D}^*F_2((\bar{x},\bar{y}-f_1(\bar{x}));(u,v-\nabla f_1(\bar{x})u))(y^*). \tag{4.3}$$

Since formula (4.3) was given in [11] without proof, we provide here its short proof for completeness.

*Proof* In view of the differentiability of $f_1$ near $\bar{x}$, we have

$$D^*F(x,y)(y^{*\prime}) = \nabla f_1(x)^T y^{*\prime} + D^*F_2(x,y-f_1(x))(y^{*\prime})$$

for all $x$ near $\bar{x}$ and all $y \in F(x)$ and $y^{*\prime} \in \mathbb{R}^m$. By the definition of the directional limiting coderivative and using the above equality and continuous differentiability of $f_1$, we have

$$\begin{aligned}
\overline{D}^* F((\bar{x},\bar{y});(u,v))(y^*) &= \operatorname*{Lim\,sup}_{\substack{u'\to u,\, v'\to v\\ y^{*\prime}\to y^*,\, t\searrow 0}} D^*F(\bar{x}+tu',\bar{y}+tv')(y^{*\prime})\\
&= \operatorname*{Lim\,sup}_{\substack{u'\to u,\, v'\to v\\ y^{*\prime}\to y^*,\, t\searrow 0}} \left[\nabla f_1(\bar{x}+tu')^T y^{*\prime} + D^*F_2(\bar{x}+tu',\bar{y}+tv'-f_1(\bar{x}+tu'))(y^{*\prime})\right]\\
&= \nabla f_1(\bar{x})^T y^* + \operatorname*{Lim\,sup}_{\substack{u'\to u,\, v'\to v\\ y^{*\prime}\to y^*,\, t\searrow 0}} D^*F_2(\bar{x}+tu',\bar{y}-f_1(\bar{x})+t(v'-\nabla f_1(\bar{x})u'-o(t)/t))(y^{*\prime})\\
&= \nabla f_1(\bar{x})^T y^* + \overline{D}^* F_2((\bar{x},\bar{y}-f_1(\bar{x}));(u,v-\nabla f_1(\bar{x})u))(y^*).
\end{aligned}$$

The proof is complete.

Below we compute the quantities crucial for determining estimates for the radii of metric subregularity of $F$.

**Proposition 4.3** *Suppose that the sets $D$ and $K$ are closed, and either $D$ is directionally regular at $\bar{x}$ or $K$ is directionally regular at $g(\bar{x})$. Suppose also that $g$ is continuously differentiable near $\bar{x}$. Then*

$$\begin{aligned}
\operatorname{rg}[F](\bar{x},\bar{y}) = \inf\big\{&\max\{\|u^*\|_*, \|v\|\} \mid u^* + \nabla g(\bar{x})^T v^* \in \overline{N}_D(\bar{x};u),\\
&-v^* \in \overline{N}_K(g(\bar{x}); v+\nabla g(\bar{x})u),\ \|u\| = \|v^*\|_* = 1\big\}, \qquad (4.4)\\
\operatorname{rg}^\circ[F](\bar{x},\bar{y}) = \inf\big\{&\|B\| \mid B \in L(\mathbb{R}^n,\mathbb{R}^m),\ (B+\nabla g(\bar{x}))^T v^* \in \overline{N}_D(\bar{x};u),\\
&-v^* \in \overline{N}_K(g(\bar{x});(B+\nabla g(\bar{x}))u),\ \|u\| = \|v^*\|_* = 1\big\}, \qquad (4.5)\\
\overline{\operatorname{rg}}[F](\bar{x},\bar{y}) = \inf\big\{&\|u^*\|_* + \|v\| \mid u^* + \nabla g(\bar{x})^T v^* \in \overline{N}_D(\bar{x};u),\\
&-v^* \in \overline{N}_K(g(\bar{x}); v+\nabla g(\bar{x})u),\ \|u\| = \|v^*\|_* = 1\big\}. \qquad (4.6)
\end{aligned}$$

*Proof* Observe that $F(x) = H(x) - g(x)$ $(x \in \mathbb{R}^n)$, where $\operatorname{gph} H = D \times K$. By Lemma 4.1,

$$\overline{N}_{D\times K}((\bar{x},g(\bar{x}));(u,v)) = \overline{N}_D(\bar{x};u) \times \overline{N}_K(g(\bar{x});v)$$

for all $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$. Hence, by virtue of Lemma 4.2, we obtain

$$\begin{aligned}
\overline{D}^* F((\bar{x},0);(u,v))(v^*) &= \overline{D}^* H((\bar{x},g(\bar{x}));(u,v+\nabla g(\bar{x})u))(v^*) - \nabla g(\bar{x})^T v^*\\
&= \begin{cases} \overline{N}_D(\bar{x};u) - \nabla g(\bar{x})^T v^* & \text{if } -v^* \in \overline{N}_K(g(\bar{x}); v+\nabla g(\bar{x})u),\\ \emptyset & \text{otherwise.}\end{cases}
\end{aligned}$$

It follows from the representations (2.4), (2.5) and (2.7) that

$$\begin{aligned}
\widehat{D}F(\bar{x},0)(u,v^*) = &\big(\overline{N}_D(\bar{x};u) - \nabla g(\bar{x})^T v^*\big) \times \big\{v \in \mathbb{R}^m \mid\\
&-v^* \in \overline{N}_K(g(\bar{x}); v+\nabla g(\bar{x})u)\big\},\ (u,v^*) \in \mathbb{R}^n \times (\mathbb{R}^m)^*,\\
\widehat{\mathfrak{D}}F(\bar{x},0) = &\big\{(u^*,v) \in (\mathbb{R}^n)^* \times \mathbb{R}^m \mid u^* + \nabla g(\bar{x})^T v^* \in \overline{N}_D(\bar{x};u),\\
&-v^* \in \overline{N}_K(g(\bar{x}); v+\nabla g(\bar{x})u),\ \|u\| = \|v^*\|_* = 1\big\},\\
\widehat{\mathfrak{D}}^\circ F(\bar{x},0) = &\big\{(B^T v^*, Bu) \mid (B+\nabla g(\bar{x}))^T v^* \in \overline{N}_D(\bar{x};u),\\
&-v^* \in \overline{N}_K(g(\bar{x});(B+\nabla g(\bar{x}))u),\ \|u\| = \|v^*\|_* = 1\big\}.
\end{aligned}$$

Substituting the last two expressions into the definitions (2.9), (2.10) and (2.11) leads to the representations (4.4), (4.5) and (4.6), respectively.

The next corollary is a consequence of Proposition 4.3 and Theorem 3.2. It gives estimates for the radii of metric subregularity of $F$ at $\bar{x}$ for 0, or equivalently, of calmness of the corresponding solution mapping

$$S(y) := \{x \in D \mid g(x) + y \in K\}, \quad y \in \mathbb{R}^m$$

at 0 for $\bar{x}$.

**Corollary 4.4** *Under the assumptions of Proposition 4.3,*

$$\begin{aligned}
&\inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid u^*+\nabla g(\bar x)^T v^*\in \overline{N}_D(\bar x;u),\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\}\\
\le \operatorname{rad}[\mathrm{SR}]_{Lip}F(\bar x|0) \le &\inf\big\{\|u^*\|_*+\|v\| \mid u^*+\nabla g(\bar x)^T v^*\in \overline{N}_D(\bar x;u),\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\},\\
\operatorname{rad}[\mathrm{SR}]_{ss}F(\bar x|0) =&\operatorname{rad}[\mathrm{SR}]_{C^1}F(\bar x|0)\\
=&\inf\big\{\|B\| \mid B\in L(\mathbb{R}^n,\mathbb{R}^m),(B+\nabla g(\bar x))^T v^*\in \overline{N}_D(\bar x;u),\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);(B+\nabla g(\bar x))u),\ \|u\|=\|v^*\|_*=1\big\}.
\end{aligned}$$

The particular case of the constraint system

$$g(x)\in K$$

corresponds to taking $D=\mathbb{R}^n$ in (4.1), while the "feasibility" mapping takes the form

$$F(x):=K-g(x),\quad x\in\mathbb{R}^n. \tag{4.7}$$

Assuming that $g(\bar x)\in K$, we again have $\bar y:=0\in F(\bar x)$. Note that the set $D=\mathbb{R}^n$ is automatically directionally regular at any point.

**Corollary 4.5** *Suppose that the set $K$ is closed, $g$ is continuously differentiable near $\bar x$ and $F$ is given by* (4.7)*. Then*

$$\begin{aligned}
\operatorname{rg}[F](\bar x,\bar y) =&\inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid u^*+\nabla g(\bar x)^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\},\\
\operatorname{rg}^\circ[F](\bar x,\bar y) =&\inf\big\{\|B\| \mid B\in L(\mathbb{R}^n,\mathbb{R}^m),\ (B+\nabla g(\bar x))^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);(B+\nabla g(\bar x))u),\ \|u\|=\|v^*\|_*=1\big\},\\
\overline{\operatorname{rg}}[F](\bar x,\bar y) =&\inf\big\{\|u^*\|_*+\|v\| \mid u^*+\nabla g(\bar x)^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\}.
\end{aligned}$$

**Corollary 4.6** *Under the assumptions of Corollary 4.5,*

$$\begin{aligned}
&\inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid u^*+\nabla g(\bar x)^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\}\\
\le \operatorname{rad}[\mathrm{SR}]_{Lip}F(\bar x|0) \le &\inf\big\{\|u^*\|_*+\|v\| \mid u^*+\nabla g(\bar x)^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);v+\nabla g(\bar x)u),\ \|u\|=\|v^*\|_*=1\big\},\\
\operatorname{rad}[\mathrm{SR}]_{ss}F(\bar x|0) =&\operatorname{rad}[\mathrm{SR}]_{C^1}F(\bar x|0)\\
=&\inf\big\{\|B\| \mid B\in L(\mathbb{R}^n,\mathbb{R}^m),(B+\nabla g(\bar x))^T v^*=0,\\
&\qquad -v^*\in \overline{N}_K(g(\bar x);(B+\nabla g(\bar x))u),\ \|u\|=\|v^*\|_*=1\big\}.
\end{aligned}$$

Now we illustrate the above results by examples.

*Example 4.7* It is easy to check by direct computation that, for the zero mapping $f:\mathbb{R}\to\mathbb{R}$ (that is $f(x)=0$ for all $x\in\mathbb{R}$) considered in Example 1.2, it holds $\operatorname{rg}f(\bar x,0)=\operatorname{rg}^\circ f(\bar x,0)=0$ for any $\bar x\in\mathbb{R}$. Hence, by Theorem 3.2,

$$\operatorname{rad}[\mathrm{SR}]_{Lip}f(\bar x|0)=\operatorname{rad}[\mathrm{SR}]_{ss}f(\bar x|0)=\operatorname{rad}[\mathrm{SR}]_{C^1}f(\bar x|0)=0,$$

which of course agrees with the observation made in Example 1.2. □

Next we consider a couple of more involved examples.

*Example 4.8* Let the mapping $F:\mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined as follows:

$$F(x)=\begin{cases} x-K & \text{if } x\in D,\\ \emptyset & \text{otherwise,}\end{cases} \tag{4.8}$$

where $D=\{(x_1,x_2)\in\mathbb{R}^2 \mid |x_2|\le x_1\}$ and $K=\{(x_1,x_2)\in\mathbb{R}^2_+ \mid x_1x_2=0\}$ is the "complementary angle". The mapping (4.8) can be considered as a special case of (4.2) with $g$ being the identity mapping. We have $(\bar{x},\bar{y})\in \operatorname{gph} F$ with $\bar{x}=\bar{y}=0\in\mathbb{R}^2$.

Since $F$ is polyhedral, it is metrically subregular at $\bar{x}$ for $\bar{y}$. At the same time, it is not strongly subregular at $\bar{x}=0$ for $\bar{y}=0$ as 0 is not an isolated point of $F^{-1}(0)=D\cap K=\mathbb{R}_+\times\{0\}$. Next we employ the tools of Section 3 to demonstrate that the metric subregularity of $F$ is preserved if it is perturbed by functions from the classes $\mathscr{F}_{Lip}$, $\mathscr{F}_{ss}$, $\mathscr{F}_{C^1}$ with sufficiently small Lipschitz moduli at $\bar{x}$, and compute the respective radii.

In the current setting, formulas (4.4), (4.6) and (4.5) take, respectively, the following form:

$$\begin{aligned} \operatorname{rg}[F](\bar{x},\bar{y}) =& \inf\big\{\max\{\|u^*\|_*,\|v\|\} \mid u^*+v^*\in \overline{N}_D(\bar{x};u),\\ & -v^*\in\overline{N}_K(\bar{x};u+v),\ \|u\|=\|v^*\|_*=1\big\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \overline{\operatorname{rg}}[F](\bar{x},\bar{y}) =& \inf\big\{\|u^*\|_*+\|v\| \mid u^*+v^*\in \overline{N}_D(\bar{x};u),\\ & -v^*\in\overline{N}_K(\bar{x};u+v),\ \|u\|=\|v^*\|_*=1\big\}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \operatorname{rg}^\circ[F](\bar{x},\bar{y}) =& \inf\big\{\|B\| \mid B\in L(\mathbb{R}^n,\mathbb{R}^m),\ (B+I)^Tv^*\in \overline{N}_D(\bar{x};u),\\ & -v^*\in\overline{N}_K(\bar{x};(B+I)u),\ \|u\|=\|v^*\|_*=1\big\}, \end{aligned} \tag{4.11}$$

where $I$ denotes the identity mapping.

The directional limiting normal cones to $D$ and $K$ involved in (4.9), (4.10) and (4.11) can be easily computed. For any $u=(u_1,u_2)\in\mathbb{R}^2$, we have

$$\overline{N}_D(\bar{x};u)=N_D(u)=\begin{cases} \{(\xi_1,\xi_2)\mid \xi_1+|\xi_2|\le 0\} & \text{if } u_1=u_2=0,\\ \{(\xi_1,\xi_2)\mid \xi_1=-\xi_2\le 0\} & \text{if } u_1=u_2>0,\\ \{(\xi_1,\xi_2)\mid \xi_1=\xi_2\le 0\} & \text{if } u_1=-u_2>0,\\ \{(0,0)\} & \text{if } |u_2|<u_1,\\ \emptyset & \text{otherwise,}\end{cases} \tag{4.12}$$

$$\overline{N}_K(\bar{x};u)=N_K(u)=\begin{cases} \mathbb{R}^2_- & \text{if } u_1=u_2=0,\\ \{0\}\times\mathbb{R} & \text{if } u_1>0,\ u_2=0,\\ \mathbb{R}\times\{0\} & \text{if } u_1=0,\ u_2>0,\\ \emptyset & \text{otherwise.}\end{cases} \tag{4.13}$$

Of course, only the points producing nonempty cones are of interest. Besides, in accordance with (4.9), (4.10) and (4.11), one only needs to compute normals to $D$ at nonzero points; thus, the first case in (4.12) can be excluded. These observations leave us with three cases in (4.12) (cases 2–4) and three cases in (4.13) (cases 1–3), which produce 9 combinations.

Let vectors $u=(u_1,u_2)$, $v=(v_1,v_2)$, $u^*=(u_1^*,u_2^*)$ and $v^*=(v_1^*,v_2^*)$ be such that

$$u^*+v^*\in\overline{N}_D(u),\quad -v^*\in\overline{N}_K(u+v),\quad \|u\|=\|v^*\|_*=1. \tag{4.14}$$

Case 4 in (4.12) leads to $u^*+v^*=0$, and consequently, $\|u^*\|_*=\|v^*\|_*=1$. Similarly, case 1 in (4.13) leads to $u+v=0$, and consequently, $\|v\|=\|u\|=1$. Thus, in each of these two cases, $\max\{\|u^*\|_*,\|v\|\}\ge 1$.

In all four combinations of the remaining cases 2 and 3 in (4.12) and cases 2 and 3 in (4.13), we have $|u_1|=|u_2|$, $|u_1^*+v_1^*|=|u_2^*+v_2^*|$, and either $|v_1|=|u_1|$ and $v_2^*=0$, or $|v_2|=|u_2|$ and $v_1^*=0$. Further analysis of these combinations depends on the type of the norm on $\mathbb{R}^2$ used in the above relations. Let $\mathbb{R}^2$ be equipped with the $l_p$ ($1\le p\le+\infty$) norm: $\|(u_1,u_2)\|_p=(|u_1|^p+|u_2|^p)^{\frac{1}{p}}$ for all $(u_1,u_2)\in\mathbb{R}^2$. Recall the usual convention: $\|(u_1,u_2)\|_\infty=\max\{|u_1|,|u_2|\}$.

Since $\|u\| = 1$, we have $|u_1| = |u_2| = 2^{-\frac{1}{p}}$ Since $\|v^*\|_* = 1$, we also have either $|v_1| = 2^{-\frac{1}{p}}$, $|v_1^*| = 1$ and $|u_1^* + v_1^*| = |u_2^*|$, or $|v_2| = 2^{-\frac{1}{p}}$, $|v_2^*| = 1$ and $|u_2^* + v_2^*| = |u_1^*|$. In both cases, we obtain $\|v\| \geq 2^{-\frac{1}{p}}$, $|u_1^*| + |u_2^*| \geq 1$, and consequently, using the standard relationship between $l_q$ and $l_1$ norms, $\|u^*\|_* = \|u^*\|_q \geq 2^{-\frac{1}{p}}\,\|u^*\|_1 = 2^{-\frac{1}{p}}(|u_1^*| + |u_2^*|) \geq 2^{-\frac{1}{p}}$, where $q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus, $\max\{\|u^*\|_*, \|v\|\} \geq 2^{-\frac{1}{p}}$. Since $2^{-\frac{1}{p}} \leq 1$, taking into account the estimates for case 4 in (4.12) and case 1 in (4.13), we conclude that $\mathrm{rg}[F](\bar{x},\bar{y}) \geq 2^{-\frac{1}{p}}$.

Moreover, the above estimate is attained. Indeed, take $u := \left(2^{-\frac{1}{p}}, 2^{-\frac{1}{p}}\right)$, $v := \left(0, -2^{-\frac{1}{p}}\right)$, $u^* := \left(-\frac{1}{2}, -\frac{1}{2}\right)$ and $v^* := (0,1)$ to satisfy all the conditions in (4.14). Then $\|v\|_p = \|u^*\|_q = 2^{-\frac{1}{p}}$. It follows that $\mathrm{rg}[F](\bar{x},\bar{y}) = 2^{-\frac{1}{p}}$. Observe that $v = Bu$ and $u^* = B^T v^*$, where $B = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Obviously, $\|B\| = 2^{-\frac{1}{p}}$. Comparing formulas (4.9) and (4.11) and taking into account Proposition 2.4(i), we conclude that $\mathrm{rg}[F](\bar{x},\bar{y}) = \mathrm{rg}^{\circ}[F](\bar{x},\bar{y}) = \underline{\mathrm{rg}}^{\circ}[F](\bar{x},\bar{y}) = 2^{-\frac{1}{p}}$. At the same time, by (4.10), $\overline{\mathrm{rg}}[F](\bar{x},\bar{y}) \leq 2^{\frac{1}{q}}$. In accordance with Theorem 3.2,

$$\mathrm{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{C^1}F(\bar{x}\,|\,\bar{y}) = 2^{-\frac{1}{p}}.$$

In the particular cases of interest, we have the following values for the radii:

- $p = 1$: $\mathrm{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{C^1}F(\bar{x}\,|\,\bar{y}) = \frac{1}{2}$;
- $p = 2$: $\mathrm{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{C^1}F(\bar{x}\,|\,\bar{y}) = \frac{1}{\sqrt{2}}$;
- $p = +\infty$: $\mathrm{rad}[\mathrm{SR}]_{Lip}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{ss}F(\bar{x}\,|\,\bar{y}) = \mathrm{rad}[\mathrm{SR}]_{C^1}F(\bar{x}\,|\,\bar{y}) = 1$.

Observe that in the case of the Euclidean norm ($p = 2$), the vectors in the above example, which insure that the estimate for the regularity constant is attained, satisfy also $u^{*T}u = v^{*T}v = -\frac{1}{\sqrt{2}}$ and $\|u^*\|_2^2 + \|v\|_2^2 - (u^{*T}u)^2 = \frac{1}{2}$. Hence, by (3.13) and Proposition 3.6, $\overline{\mathrm{rg}}^{\circ}[F](\bar{x},\bar{y}) = \mathrm{rg}[F](\bar{x},\bar{y}) = \mathrm{rg}^{\circ}[F](\bar{x},\bar{y}) = \underline{\mathrm{rg}}^{\circ}[F](\bar{x},\bar{y})$. □

*Example 4.9* When dealing with more complicated constraint systems than the one considered above, analyzing multiple individual cases may not be practical. It can be more convenient to compute the needed regularity constants by solving appropriate optimization problems. For instance, in the above example, when $p = 2$ for the constant (3.13), we have:

$$(\overline{\mathrm{rg}}^{\circ}[F](\bar{x},\bar{y}))^2 = \inf\Big\{\|u^*\|_2^2 + \|v\|_2^2 - (u^{*T}u)^2 \mid u^* + v^* \in \overline{N}_D(u), \\ -v^* \in \overline{N}_K(u+v),\ u^{*T}u = v^{*T}v,\ \|u\|_2 = \|v^*\|_2 = 1\Big\}. \quad (4.15)$$

As discussed above, when computing regularity constants, only four very similar combinations of two cases in (4.12) and two cases in (4.13) are of interest, and it is sufficient to consider only one of them. For instance, the combination of the second case in (4.12) and the second case in (4.13) gives us $u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $v^* = (0, \pm 1)$, $v \in \left\{\left(x - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \mid x \geq 0\right\}$ and $u^* \in \{(-y, y \mp 1) \mid y \geq 0\}$. The objective function of the respective minimization problem in the right-hand side of (4.15) amounts to

$$y^2 + (y \mp 1)^2 + \left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2} - \frac{1}{2}(-y + y \mp 1)^2 = x^2 - \sqrt{2}x + 2y^2 \mp 2y + \frac{3}{2},$$

and the compatibility constraint is fulfilled:

$$u^{*T}u = \frac{1}{\sqrt{2}}(-y + y \mp 1) = \mp\frac{1}{\sqrt{2}}, \quad v^{*T}v = (\pm 1)\left(-\frac{1}{\sqrt{2}}\right) = \mp\frac{1}{\sqrt{2}}.$$

The respective subproblem of (4.15) reduces, thus, to choosing the second component of the vector $v^*$: either 1 or $-1$, and two one-dimensional convex minimization problems, the second one depending on the choice:

$$\min_{x \geq 0}\left(x^2 - \sqrt{2}x\right) \quad \text{and} \quad \min_{y \geq 0}\left(y^2 \mp y\right).$$

Since $y^2 - y \le y^2 + y$ for all $y \ge 0$, one has to choose $v^* = (0,1)$, which leads to considering $u^* \in \{(-y, y-1) \mid y \ge 0\}$ and choosing the minus sign in the second minimization problem. The solutions $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{2}$ of the above problems provide us with the same "optimal" vectors $v = \left(0, -\frac{1}{\sqrt{2}}\right)$ and $u^* = \left(-\frac{1}{2}, -\frac{1}{2}\right)$, and the value of the constant (3.13): $\overline{\mathrm{rg}}^\circ[F](\bar{x},\bar{y}) = \frac{1}{\sqrt{2}}$. □

In general, computation of $\overline{\mathrm{rg}}^\circ[F](\bar{x},\bar{y})$ in the case of the constraint system (4.1) with Euclidean norms and polyhedral sets $D$ and $K$ amounts to solving a disjunctive program with a smooth objective function. Computing the other regularity constants may be more demanding because of the nonsmoothness of their objective functions.

At the end of the paper, we present an example, which demonstrates lack of robustness of metric subregularity.

*Example 4.10* Let two sequences $\{a_k\}$ and $\{b_k\}$ of positive numbers be given, such that $a_{k+1} < b_k < a_k$ $(k = 1,2,\ldots)$, $a_k \to 0$ (and consequently $b_k \to 0$) and $\frac{b_k - a_{k+1}}{a_k - b_k} \to 0$ as $k \to +\infty$. For all $k = 1,2,\ldots$, set

$$\varphi(t) := \begin{cases} t - a_{k+1} & \text{if } a_{k+1} \le t < b_k, \\ 1 & \text{if } b_k \le t < a_k, \end{cases}$$

and define a real-valued function $f$ on $(-a_1, a_1)$ by $f(x) := \int_0^{|x|} \varphi(t)dt$. Thus, the graph of $f$ consists of linear pieces with slope 1 (when $b_k < |x| < a_k$) and parabolic pieces (when $a_{k+1} < |x| < b_k$), with the contribution of the latter diminishing as $x$ approaching 0.

Observe that $\lim_{t \uparrow b_k} \varphi(t) = b_k - a_{k+1} < 1$ for all $k$ large enough, and consequently, $f(x) < |x|$ when $|x|$ is small enough. On the other hand, $f(0) = 0$ and, for any nonzero $x \in (-a_1, a_1)$ and with $n$ being the smallest natural number such that $a_n \le |x|$, we have

$$\begin{aligned} f(x) &> |x| - \sum_{k=n}^{\infty}(b_k - a_{k+1}) \ge |x| - \left(\max_{k \ge n} \frac{b_k - a_{k+1}}{a_k - b_k}\right) \sum_{k=n}^{\infty}(a_k - b_k) \\ &> |x| - \left(\max_{k \ge n} \frac{b_k - a_{k+1}}{a_k - b_k}\right) \sum_{k=n}^{\infty}(a_k - a_{k+1}) \ge |x| \left(1 - \max_{k \ge n} \frac{b_k - a_{k+1}}{a_k - b_k}\right). \end{aligned}$$

Hence, $\lim_{x \to 0} \frac{f(x)}{|x|} = 1$, and consequently, $Df(0,0)(u) = |u|$ for all $u \in \mathbb{R}$. It follows from Proposition 2.4(iv) that $\mathrm{rg}[f](0,0) \ge 1$. (It is not difficult to show that $\mathrm{rg}[f](0,0) = 1$.) Thus, $f$ is metrically subregular (in fact, strongly subregular) at 0 together with all its perturbations by Lipschitz continuous functions with Lipschitz modulus 1. At the same time, $f$ is not metrically subregular at any $a_k$ $(k = 1,2,\ldots)$. □

Thus, for $F$ given by (4.2), the positiveness of the radius $\mathrm{rad}[\mathrm{SR}]_{Lip} F(\bar{x},0)$ does not imply the existence of neighbourhoods of $\bar{x}$ and 0 where the subregularity is preserved. In fact, this holds only at points $x$ and 0 for $x$ close to $\bar{x}$ [10].

## 5 Further research

In this paper we obtain expressions and bounds for the radius of metric subregularity of mappings, in various settings, based on generalized derivatives. In the last section we specify these expressions/bounds for a mapping describing a system of constraints typically appearing in optimization. We do not discuss here how to efficiently compute these quantities; this remains an open task for further research. On a broader level, one may ask what would be the aim for having these quantities computed.

In the Introduction we mentioned that the radius of nonsingularity of matrices is ultimately related to their condition number. The concept of conditioning plays a major role in numerical linear algebra, and preconditioning is a highly efficient tool for enhancing computations in numerical linear algebra. Then we come to the natural question whether the expressions for the radius of regularity properties (not only subregularity) could be utilized in procedures for conditioning of problems of feasibility and optimization. Although there is a bulk of studies in those directions, see the monograph [1], the results in the whole area seem to be scattered and lacking unifying ideas. We believe that the radius theorems could serve as a basis for such a unification. In any case, developing techniques for conditioning of optimization problems is a challenging avenue for further research.

In this paper we consider mappings acting in finite dimensions which is essential for the proofs. Could (some of) the results be extended to infinite-dimensional spaces? As for the other regularity properties, there is a partial progress on that for metric regularity. Most notably, Ioffe constructed in [14] a Lipschitz continuous and weakly continuously Fréchet differentiable mapping acting in a separable Hilbert space for which the the radius equality (1.4) is violated. On the positive side, Ioffe and Sekiguchi [16] showed that this equality holds in infinite dimensions for certain classes of mappings with convex graphs, including in particular semi-infinite inequality systems.

In another direction, the existing radius theorems are quite general but cannot be applied to situations where the perturbed mapping has a specific form, that is, in the case of *structured* perturbations; for an earlier work, see [28].

For instance, there are apparently no radius theorems for the Karush-Kuhn-Tucker (KKT) conditions in nonlinear programming, because the perturbed mapping there ought to have the form corresponding to a KKT condition. It is an open question whether one might find a radius theorem, for various regularity properties, even for the standard nonlinear programming problem.

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