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Domain Decomposition Preconditioning for Well Models for Reservoir Problems

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Domain Decomposition Preconditioning for Well Models for Reservoir Problems

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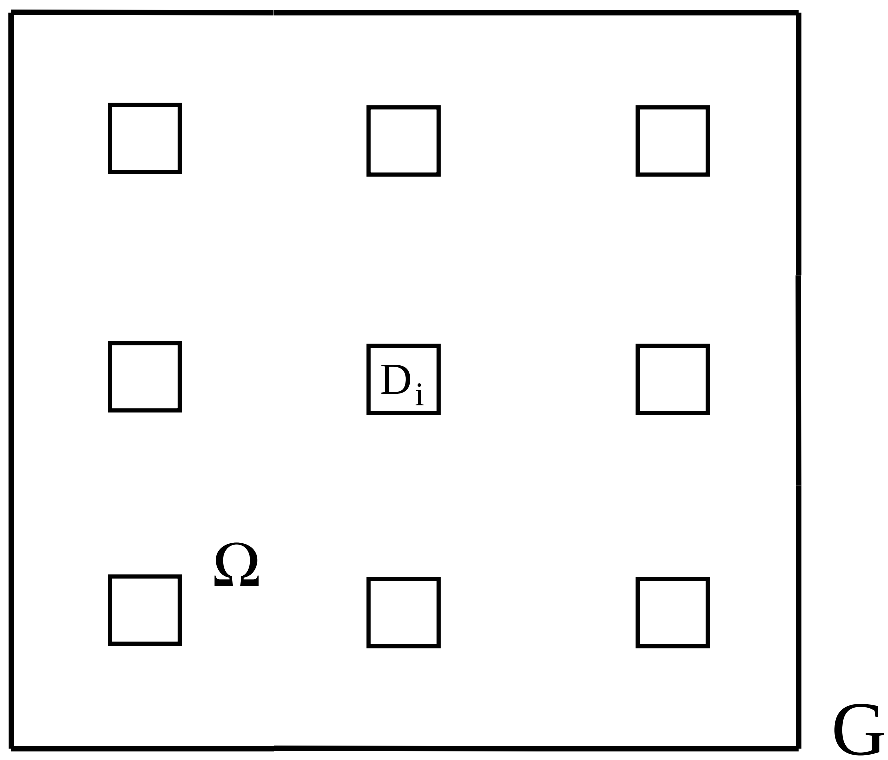
Abstract. Well models for reservoir simulation play very important role. A special property of the considered problem is that an operator of this problem inside small subdomains, which correspond to wells, is unknown. The purpose is the design of an effective preconditioner for the iterative solution of the problem. The construction of the preconditioner consists of two steps. The first step is auxiliary. In this step we suggest a preconditioning operator for a problem with small holes. The holes correspond to subdomains with wells. The construction of a preconditioner for the first step is based on the multilevel decomposition and the fictitious space method. The second step of the construction is more important. The design of the preconditioning operator is based on few domain decomposition methods using the additive Schwarz method. Domain decomposition methods with using algebraic multigrid methods are considered in [2]. Trace theorem in Sobolev spaces $H^1(\Omega)$ is a very important in a construction of domain decomposition preconditioners. Corresponding theorems were presented for small domains for the cases of Sobolev spaces and its finite element subspaces in [6]. Using these theorems, optimal preconditioners with respect to condition numbers and arithmetic cost for the case of small coefficients in holes were suggested [6] (see also [1]). An error analysis of the finite element method for well models was presented in [7]. The trace theorems for the Sobolev space $H^1(\Omega)$ in bad domains also was presented in [3].

Section 1: ***Preconditioning for Homogeneous Problems with Small Holes***

Consider the following problem

$$\begin{cases} -p\Delta u + qu = f \quad \text{in} \quad \Omega \\ u(x) = 0, \quad x \in \partial\Omega \end{cases} \tag{1}$$



Let G be a union of n+1 nonoverlapping subdomains

$$\bar{G} = \bar{\Omega} + \bigcup_{i=1}^{n} D_i,$$

D_i, $i = 1, 2, \ldots, n$ be a hole.
Assume

$$\text{diam}\, D_i = O(H_i), \quad 0 < H_i \le 1, \quad i = 1, 2, \ldots, n$$

And if D_i, D_j be neighbor subdomains, then

$$\mathrm{dist}(D_i, D_j) = O(\max\{H_i, H_j\})$$

Let

$$p = \mathrm{const} > 0$$
$$q = \mathrm{const} \geq 0$$

and let us consider the following weak form of the elliptic boundary value problem (1): Find $u \in H_0^1(\Omega)$:

$$a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u,v) = \int_\Omega (p(\nabla u, \nabla v) + q\,u\,v)\,d\Omega,$$

$$l(v) = \int_\Omega f\,v\,d\Omega.$$

F.E. Discretization and BPX-like Preconditioner

Hierarchical triangulations:

$$G_0^h, G_1^h, \ldots, G_J^h.$$

Hierarchical F.E. spaces:

$$W_0^h, W_1^h, \ldots, W_J^h,$$

which correspond to hierarchical F.E. spaces on $G_0^h, G_1^h, \ldots, G_J^h$ in G and

$$W_k = \{\varphi_i^{(k)}\}_{i=1}^{N_k},$$

$\varphi_i^{(k)}$ be a nodal basis function in W_k , h_k be a mesh size in G_k, $h_J = h$.

Multilevel preconditioner:

$$B^{-1}u = B_D^{-1}u = BPX_D(p,q)u = \sum_{k=k_0}^{J} \delta_k \sum_{\mathrm{supp}\,\varphi_i^{(k)} \subset \overset{\circ}{\Omega}} (u, \varphi_i^{(k)})\varphi_i^{(k)},$$

or

$$B^{-1}u = B_N^{-1}u = BPX_N(p,q)u = \sum_{k=k_0}^{J} \delta_k \sum_{\operatorname{supp}\overset{\circ}{\varphi}_i^{(k)} \cap \Omega \neq \emptyset} (u, \varphi_i^{(k)})_\Omega \tilde{\varphi}_i^{(k)},$$

$$\tilde{\varphi}_i^k = \varphi_i^{(k)}\Big|_\Omega,$$

$$\delta_k = 0, \ k = 0,1,\dots,k_0 - 1,$$

$$\delta_k = (p + \frac{q}{2^k})^{-1}, \ k = k_0, k_0 + 1, \dots, J.$$

where

$$0 \le k_0 \le J, \quad k_0 = \max : p \le q h_{k_0}^2 .$$

Theorem 1 $\exists c_1, c_2 \neq c(h, p, q, H_i):$

$$c_1(Bv, v) \le (Av, v) \le c_2(Bv, v) \quad \forall v \in W_J$$

Remark For the case of Dirichlet boundary conditions we do not need the assumption

$$\operatorname{dist}(D_i, D_j) = O(\max\{H_i, H_j\})$$

Numerical experiments for BPX for domains with holes

The problem

$$\begin{cases} -\Delta u(x,y) = f(x,y), & (x,y) \in \Omega, \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

is approximated by bilinear finite element functions on uniform grids with a mesh size h

$$A_{\Omega^h,\Omega^h} u_{\Omega^h} + C_{\Omega^h,\partial\Omega^h} g_{\partial\Omega^h} = f_{\Omega^h} .$$

Here the matrices A_{Ω^h,Ω^h} and $C_{\Omega^h,\partial\Omega^h}$ are the blocks from the matrix

$$A_h = \begin{bmatrix} A_{\Omega^h,\Omega^h} & C_{\Omega^h,\partial\Omega^h} \\ C^T_{\Omega^h,\partial\Omega^h} & A_{\partial\Omega^h,\partial\Omega^h} \end{bmatrix}$$

which corresponds to F.E. approximation of $a(u,v)$ in $H^1(\Omega)$.

The system is solved by CG with BPX preconditioner up to the condition

$\dfrac{\| u_h^k - u \|_{A_h}}{\| u_h^0 - u \|_{A_h}} \le \varepsilon$. The initial guess is $u_h^0 = \begin{bmatrix} 0_{\Omega^h} \\ g_{\partial\Omega^h} \end{bmatrix}$.

<u>Experiment 1: unit square with one hole</u>

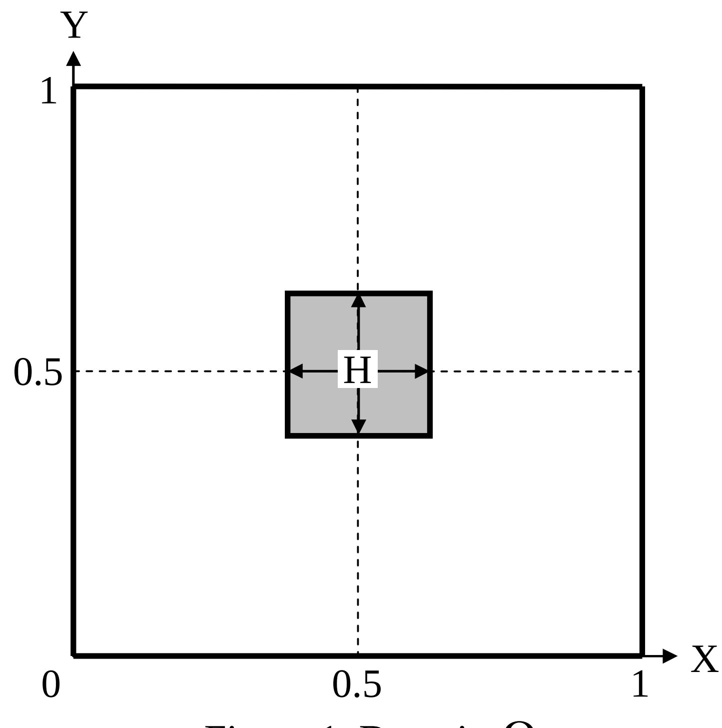


Figure 1. Domain Ω

$u(x,y) = x(1-x)y(1-y) + |x-0{,}5| \cdot |y-0{,}4|$,
$\varepsilon = 0.001$.

Table 1

N	No hole	$H = \frac{1}{2}$	$H = \frac{1}{4}$	$H = \frac{1}{8}$	$H = \frac{1}{16}$	$H = \frac{1}{32}$	One mesh point
32	**7**	**7**	**7**	**8**	**9**	**8**	**8**
64	**7**	**7**	**7**	**8**	**9**	**10**	**9**
128	**7**	**7**	**7**	**8**	**9**	**10**	**9**
256	**7**	**7**	**7**	**8**	**9**	**10**	**9**
512	**6**	**7**	**7**	**8**	**9**	**10**	**9**

Experiment 2: unit square with four holes

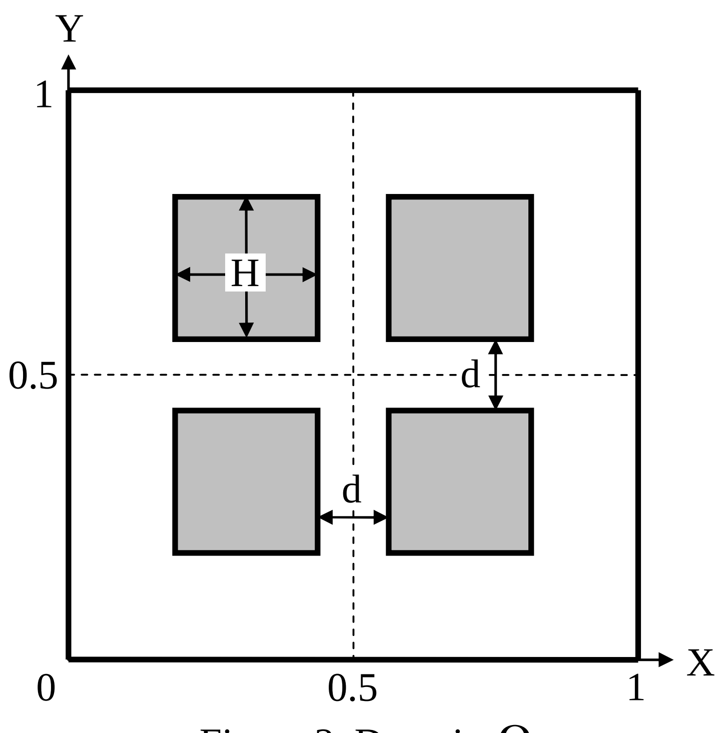


Figure 2. Domain Ω

$$u(x,y) = x(1-x)y(1-y) + |x-0,5| \cdot |y-0,4|$$
$\varepsilon = 0.001$.

Table 2

	H = 1/4, d =				H = 1/8, d =				H = 1/16, d =				H = 1/32, d =			
N	1/4	1/8	1/16	1/32	1/4	1/8	1/16	1/32	1/4	1/8	1/16	1/32	1/4	1/8	1/16	1/32
32	**7**	**7**	**8**		**7**	**8**	**9**		**8**	**7**	**8**		**9**	**8**	**8**	
64	**7**	**7**	**8**	**9**	**7**	**8**	**9**	**10**	**8**	**8**	**8**	**9**	**9**	**8**	**8**	**9**
128	**7**	**7**	**8**	**9**	**8**	**8**	**9**	**10**	**8**	**8**	**8**	**9**	**9**	**9**	**8**	**9**
256	**7**	**7**	**8**	**9**	**8**	**8**	**9**	**9**	**8**	**8**	**9**	**9**	**9**	**9**	**8**	**9**
512	**7**	**7**	**8**	**9**	**8**	**8**	**9**	**9**	**8**	**8**	**8**	**9**	**9**	**8**	**8**	**9**

Section 2: *Domain Decomposition for Well Models*

Let Ω be a bounded and polygonal domain and Γ is its boundary. In Ω let us consider the following boundary value problem

$$\begin{cases} Lu(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \Gamma. \end{cases} \qquad (1)$$

Here L is a second order elliptic operator. For simplicity of presentation of main idea how to construct preconditioning operators, here we consider Dirichlet boundary condition, but other kind of boundary conditions can be considered.

Let $a(u,v)$ is the bilinear for which corresponds to the problem (1), the weak formulation is:
find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = \int_\Omega f(x)v(x)dx \quad \forall v \in H_0^1(\Omega).$$

We suppose that $a(u,v)$ is a symmetric

$$a(u,v) = a(v,u) \quad \forall u, v \in H_0^1(\Omega)$$

and the inequalities

$$\alpha \|\Omega\|^2_{H^1(\Omega)} \le a(v,v) \le \beta \|\Omega\|^2_{H^1(\Omega)} \quad \forall v \in H_0^1(\Omega),$$

hold, where α, β are some positive constants.
Let Ω be a union of n+1 nonoverlapping subdomains such that

$$\bar{\Omega} = \bigcup_{i=0}^{n} \bar{\Omega}_i, \; \Omega_i \cap \Omega_j = 0, \; i \ne j$$

holds.

Here we assume that $\Omega_i, \; i = 1, 2, \ldots, n$ corresponds to wells, it means that $\mathrm{meas}(\Omega_i)$ is small, Ω_0 is a multiconnected subdomain and

$$Lu(x) = -\Delta u(x), \quad x \in \Omega_o.$$

Denote by $\tilde{\Omega}_i$ an extension of Ω_i such that

$$\Omega_i \subset \tilde{\Omega}_i, \; \mathrm{dist}(\partial\Omega_i, \partial\tilde{\Omega}_i) = O(\tilde{h}), \quad i = 1, 2, \ldots, n$$

for some parameter $\tilde{h}$.

Let $\Omega^h = \bigcup_{i=0}^{n} \Omega_i^h$ be a quasiuniform triangulation of the domain Ω, which can be characterized by the parameter h.

Here Ω_i^h corresponds Ω_i, and let us denote by $\tilde{\Omega}_i^h$ the triangulation of $\tilde{\Omega}_i$.
Introduce a piecewise linear finite element space $H \subset H_0^1(\Omega)$ on the triangulation Ω^h and piecewise linear finite element spaces $H_i \subset H_0^1(\Omega_i)$ and $\tilde{H}_i \subset H_0^1(\tilde{\Omega}_i)$ on the triangulations Ω_i^h and $\tilde{\Omega}_i^h$, respectively, $i = 1, 2, \ldots, n$

and H_0 on Ω_0^h with "Dirichlet boundary conditions", $H_{0,N}$ on the triangulation Ω_0^h with "Neumann boundary conditions". Define the operator A by

$$(Au^h, v^h) = a(u^h, v^h) \quad \forall u^h, v^h \in H.$$

Variant 1

To construct a domain decomposition preconditioner for A, we use Additive Schwarz Method (ASM) [3],[4].

Lemma1(ASM):

Let the Hilbert space H *with the scalar product* $(\bullet,\bullet)$ *be decomposed into a vector sum of subspace*

$$H = H_1 + H_2 + ... + H_m,$$

$A : H \to H$ *be a linear, self-adjoint, bounded, and positive definite operator. Let* $P_i, i = 1, 2, ..., m$, *be operators of orthogonal projection of* H *onto* H_i *with respect to the scalar product* $(\bullet,\bullet)_A$ *generated by the operator* A

$$a(u, v) = (A u, v) = (u, v)_A .$$

Assume that positive constants α *and* β *exist such that for any element* $u \in H$ *there exists* $u_i \in H_i$, *such that*

$$\begin{aligned} &u_1 + u_2 + ... + u_m = u, \\ &\alpha((u_1, u_1)_A + (u_2, u_2)_A + ... + (u_m, u_m)_A) \le (u, u)_A, \\ &((P_1 + P_2 + ... + P_m)u, u)_A \le \beta(u, u)_A . \end{aligned}$$

Also, let operators $B_i : H \to H,\ \text{Im}\, B_i = H_i,\ i = 1, 2, ..., m$, *self-adjoint in* H *be determined such that*

$$\check{c}(B_i\, u, u) \le (A\, u, u) \le \hat{c}(B_i\, u, u),\ \forall u \in H_i .$$

Then,

$$\alpha\, \check{c}(A^{-1}\, u, u) \le (B^{-1}\, u, u) \le \beta\, \hat{c}(A^{-1}\, u, u),\ \forall u \in H$$

where

$$B^{-1} = B_1^{+} + B_2^{+} + ... + B_m^{+},$$

and B_i^{+} *is a pseudo-inverse operator for* B_i.

To implement ASM, consider the following decomposition of the space H

$$H = H_0 + \tilde{H}_1 + ... + \tilde{H}_n .$$

Let B_0 be a preconditioner in the space H_0 such that

$$c_1 \left\| u^h \right\|^2_{H^1(\Omega_0)} \leq (B_0 u^h, u^h) \leq c_2 \left\| u^h \right\|^2_{H^1(\Omega_0)} \quad \forall u^h \in H_o ,$$

where c_1, c_2 are constants. Let P_i, $i = 1, 2, ..., n$ the ortoprojectors

$$P_i : H \to \tilde{H}_i$$

with respect to the scalar product $a(u, v)$

$$(P_i u^h, v^h) = a(u^h, v^h) \quad \forall v^h \in \tilde{H}_i .$$

It means that P_i is defined by a direct solution in the subspace $\tilde{H}_i$, i.e. for given $u^h \in H$ we solve the problem:
Find $u_i^h = P_i u^h$ as a solution of the following problem

$$u_i^h \in \tilde{H}_i : \; a(u_i^h, v^h) = a(u^h, v^h) \quad \forall v^h \in \tilde{H}_i .$$

Define

$$B_{v_1}^{-1} = B_0^+ + P_1 + ... + P_n ,$$

where B_0^+ is the pseudo-inverse operator for B_0.
To analyze the decomposition (2), we use the following

Lemma 2

There exist a positive constant c_3*, which is independent of the parameters* $h,\tilde{h}$*, and for any* u^h *the exist* $u_0^h \in H_0$, $u_i^h \in \tilde{H}_i$

$$u^h = \sum_{i=0}^{n} u_i^h ,$$

$$c_3 \tilde{h} \sum_{i=0}^{n} a(u_i^h, u_i^h) \le a(u^h, u^h).$$

Then from ASM, the following theorem holds.

Theorem 2

$$c_4 (A^{-1} u^h, u^h) \le (B_{v_1}^{-1} u^h, u^h) \le c_5 (A^{-1} u^h, u^h) \quad \forall u^h \in H.$$

Here

$$c_4 = \tilde{h} c_3 \min\{c_1, 1\},$$

$$c_5 = 2 \max\{c_2, 1\}.$$

Note that the constants c_4, c_5 are independent of the constant α, β. It means that the operator L in (1) can be arbitrary in Ω_i, $i = 1, 2, ..., n$ and only an existence of the constants α, β is important.

Variant 2

For this variant we need the following lemma.

Lemma 3

Let V *and* W *be two Hilbert spaces with scalar products* $(\bullet,\bullet)_V$ *and* $(\bullet,\bullet)_W$. *Moreover, let* Σ *and* A *be self-adjoint, positive definite operators in* V *and* W, *respectively. We denote by*

$$(\varphi,\psi)_\Sigma = (\Sigma\varphi,\psi)_V \text{ and } (u,v)_A = (Au,v)_W$$

the scalar products in V *and* W *generated by the operators* Σ *and* A. *Let* $t: V \to W$ *be a linear operator such that*

$$\alpha(\varphi,\varphi)_\Sigma \le (t\varphi,t\varphi)_S \le \beta(\varphi,\varphi)_\Sigma \quad \forall\varphi\in V.$$

Finally, we set

$$C^+ = t\Sigma^{-1}t^*,$$

where t^* *is the adjoint to the operator with respect to scalar products* $(\bullet,\bullet)_V$ *and* $(\bullet,\bullet)_W$. *Then,*

$$\alpha(Cu,u)_W \le (Au,u)_W \le \beta(Cu,u)_W,$$
$$\forall u\in \mathrm{Im}(t) := \{u\in W;\ \exists v\in V : u = tv\}.$$

Denote by $\gamma = \partial\Omega_0 \setminus \partial\Omega$ the union of boundaries of the subdomains Ω_i, $i = 1,2,...,n$ and by $H_h(\gamma)$ the trace space of functions from H on γ. To define a precondotioner operator, let us consider an extension operator

$$t : H_{0,N} \to H$$

be an extension operator such that for a given function $u_0^h \in H_{0,N}$ define the trace function $\varphi^h \in H_h(\gamma)$ on γ

$$\varphi^h(x) = u_0^h(x),\ x\in\gamma.$$

Then define the extension operator t as

$$tu_0^h = u^h \in H:$$
$$\begin{cases} a(u^h, v^h) = 0 \ \ \forall v^h \in H_i, \ i = 1, 2, ..., n \\ u^h(x) = \varphi^h(x), \ \ x \in \gamma \\ u^h(x) = u_0^h(x), \ \ x \in \Omega_o. \end{cases}$$

Since the operator t is an energetic harmonic operator, then

$$a(tu_0^h, tu_0^h) = \inf_{\substack{w^h \in H, \\ w^h(x) = u_0^h(x), x \in \Omega_0}} a(w^h, w^h).$$

To give a matrix representation of the operator t, split of vertices of functions $u^h \in H$ into three groups:

- the first group corresponds to vertices from Ω_i, $i = 1, 2, ..., n$ (vector $\overline{u}'$);
- the second group corresponds to vertices from γ (vector $\overline{\varphi}$);
- the third group corresponds to vertices from Ω_0 (vector $\overline{u}_0$).

$$u^h \in H \leftrightarrow \overline{u} = \begin{bmatrix} \overline{u}' \\ \overline{\varphi} \\ \overline{u}_0 \end{bmatrix}$$

Let H' be a finite element space on $\bigcup_{i=1}^{n} \overline{\Omega}_i$ (including vertices on γ).

Define the matrix A' such that

$$(A' \begin{bmatrix} \overline{u}' \\ \overline{\varphi} \end{bmatrix}, \begin{bmatrix} \overline{v}' \\ \overline{\psi} \end{bmatrix}) = a(u^h{}', v^h{}'), \ \ \forall u^h{}', v^h{}' \in H'$$

and consider the block representation of the matrix A'

$$A' = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then the matrix representation $\overline{t}$ of the operator t is the following

$$\overline{t}\begin{bmatrix} \overline{\varphi} \\ \overline{u}_0 \end{bmatrix} = \begin{bmatrix} -A_{11}^{-1} A_{12}\, \overline{\varphi} \\ \overline{\varphi}_0 \\ \overline{u}_0 \end{bmatrix} = \begin{bmatrix} -A_{11}^{-1} A_{12} & O_{1\gamma} \\ I_\gamma & O_{\gamma 0} \\ O_{0\gamma} & I_0 \end{bmatrix} \begin{bmatrix} \overline{\varphi} \\ \overline{u}_0 \end{bmatrix},$$

where I_γ, I_0 are identity matrices corresponding to γ, Ω_0, respectively, and $O_{\bullet\bullet}$ are corresponding null-matrices.

Now introduce the operator A_N

$$(A_N u^h, v^h) = a_{\Omega_0}(u^h, v^h) \quad \forall u^h, v^h \in H_{0,N}$$

where $a_{\Omega_0}(\bullet,\bullet)$ is the restriction of $a(\bullet,\bullet)$ on Ω_0.

Define the preconditioning operator $B_{V_2} = tA_N^{-1}t^T + P_1 + ... + P_n$.

Then from ASM and Lemma 3 the following theorem holds.

Theorem 3

Assume that

$$a(tu_0^h, tu_0^h) \le \alpha_t a(u_0^h, u_0^h) \quad \forall u^h \in H_{0,N},$$

then

$$\beta_t (A^{-1}u^h, u^h) \le (B_{V_2}^{-1}u^h, u^h) \le 2(A^{-1}u^h, u^h) \quad \forall u^h \in H_{0,N},$$

where

$$\beta_t = 1/(2 + 3\alpha_t).$$

Remark

Instead of $A_N, P_i \ \ i = 1, 2, ..., n$, i.e. direct solvers in $H_{0,N}, H_1, ..., H_n$ any spectrally equivalent operators can be used.

Numerical experiments for Variant 1

The problem

$$\begin{cases} -\nabla(p\nabla u) = f(x,y), & (x,y) \in \Omega = \Omega_0 \bigcup \Omega_1, \\ \quad u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases}$$

is approximated by bilinear finite element functions on uniform grids with a mesh size h

$$A_{\Omega^h,\Omega^h} u_{\Omega^h} + C_{\Omega^h,\partial\Omega^h} g_{\partial\Omega^h} = f_{\Omega^h} .$$

Here the matrices A_{Ω^h,Ω^h} and $C_{\Omega^h,\partial\Omega^h}$ are defined as in Section 1.

Coefficient $p(x,y)$:

$$p(x,y) = 1 \text{ in } \Omega_0,$$

$$p(x,y) = p_1 = \text{const} \text{ in } \Omega_1.$$

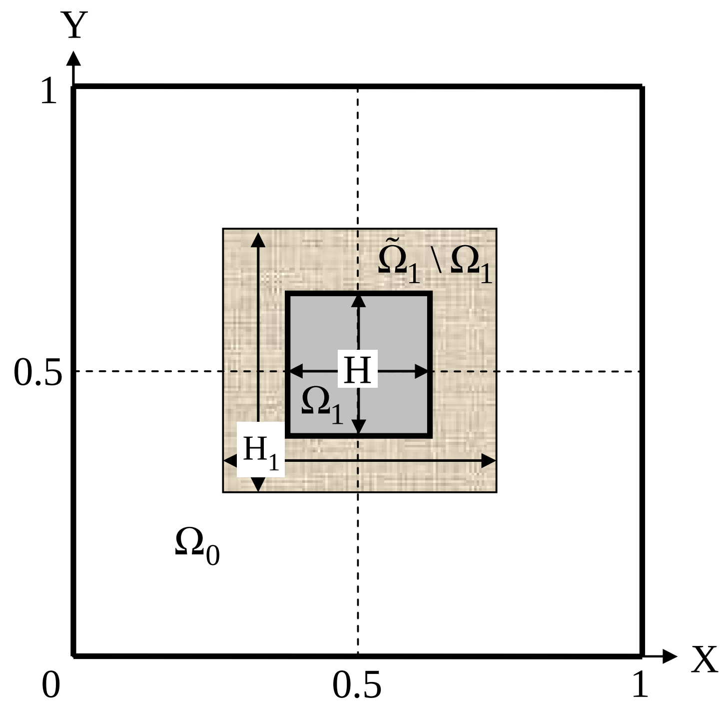


Figure 1. Domain Ω.

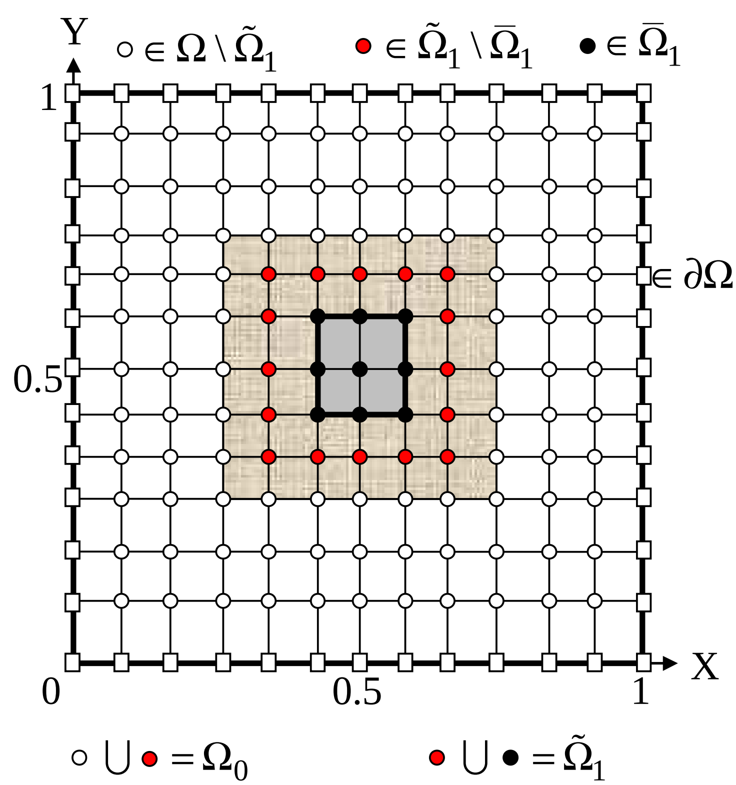


Figure 2. Mesh nodes.

$$u(x,y) = x(1-x)y(1-y) + |x-0,5|\cdot|y-0,4|,$$
$$\varepsilon = 0.001.$$

Define the preconditioner

$$B^{-1} = B_{px} + B_{\tilde{\Omega}_1}$$

$$B_{px} \approx \begin{bmatrix} A_{\Omega\setminus\tilde{\Omega}_1} & A_{\Omega\setminus\tilde{\Omega}_1;\tilde{\Omega}_1\setminus\bar{\Omega}_1} & 0 \\ A_{\tilde{\Omega}_1\setminus\bar{\Omega}_1;\Omega\setminus\tilde{\Omega}_1} & A_{\tilde{\Omega}_1\setminus\bar{\Omega}_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}^+, \quad B_{\tilde{\Omega}_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{\tilde{\Omega}_1\setminus\bar{\Omega}_1} & A_{\tilde{\Omega}_1\setminus\bar{\Omega}_1;\bar{\Omega}_1} \\ 0 & A_{\bar{\Omega}_1;\tilde{\Omega}_1\setminus\bar{\Omega}_1} & A_{\bar{\Omega}_1} \end{bmatrix}^+$$

The steepest-descent method

Table 3

h^{-1}	$H=1/4,\ H_1=3\cdot H$							$H=1/8,\ H_1=3\cdot H$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	31	31	31	**31**	31	31	31	54	54	54	**54**	54	54	53
64	31	31	31	**31**	31	31	31	53	53	53	**53**	53	53	53
128	30	30	30	**30**	30	30	31	51	51	51	**51**			
256				**29**							**49**			
512				**27**							**46**			
1024				**25**							**42**			

h^{-1}	$H=1/16,\ H_1=3\cdot H$							$H=1/32,\ H_1=3\cdot H$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	91	91	91	**91**	91	91	91	нет узлов во внутренней области						
64	86	86	86	**86**	86	86	86	131	131	131	**131**	131	131	131
128				**83**							**122**			
256				**79**							**115**			
512				**73**							**107**			
1024				**67**							**97**			

The CG method

Table 4

h^{-1}	$H=1/4,\ H_1=3\cdot H$							$H=1/8,\ H_1=3\cdot H$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	**10**	**10**	**10**	**11**	**11**	**11**	**11**	**12**	**12**	**12**	**12**	**12**	**12**	**12**
64	**10**	**10**	**10**	**10**	**11**	**11**	**11**	**12**	**12**	**12**	**12**	**12**	**12**	**12**
128	**10**	**10**	**10**	**10**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**12**	**12**	**12**
256	**10**	**10**	**10**	**10**	**10**	**11**	**11**	**11**	**11**	**11**	**11**	**11**	**12**	**12**
512	**9**	**9**	**9**	**9**	**10**	**11**	**11**	**10**	**10**	**10**	**10**	**11**	**12**	**12**

h^{-1}	$H=1/16,\ H_1=3\cdot H$							$H=1/32,\ H_1=3\cdot H$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	**14**	**14**	**14**	**13**	**13**	**13**	**13**	**14**	**14**	**14**	**14**	**13**	**13**	**13**
64	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**14**	**14**	**14**	**13**	**13**	**13**	**14**
128	**12**	**12**	**12**	**12**	**12**	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**14**
256	**12**	**12**	**12**	**12**	**12**	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**13**
512	**12**	**12**	**12**	**12**	**12**	**12**	**13**	**13**	**13**	**13**	**13**	**13**	**13**	**13**

Numerical experiments for Variant 2

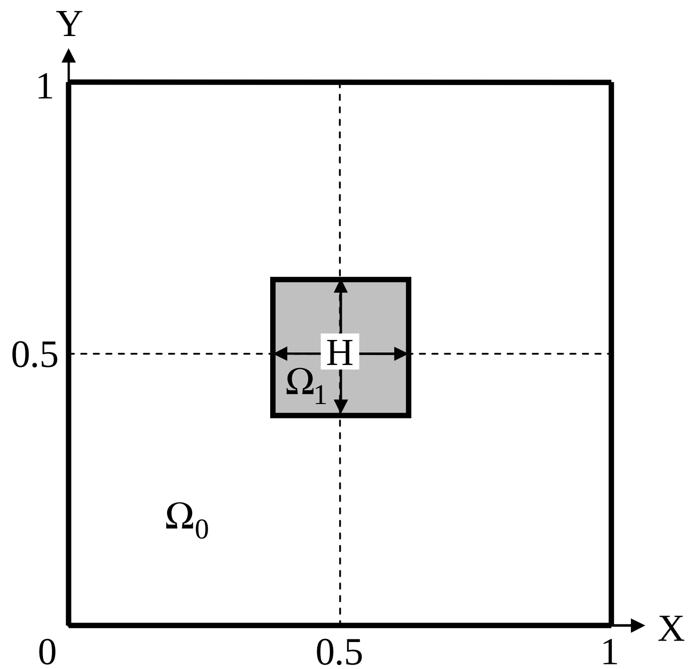


Figure 6. Domain Ω.

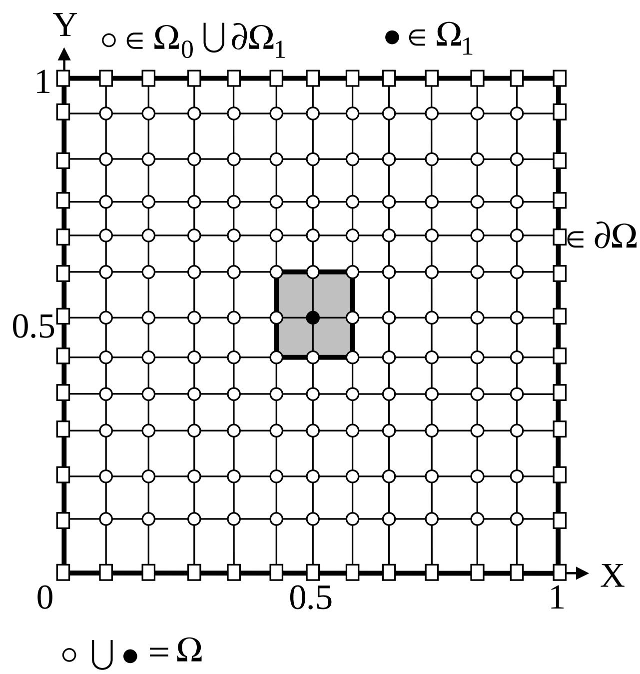


Figure 7. Mesh nodes.

Define the preconditioner

$$B^{-1} = P_d B_{px} P_d^* + B_{\Omega_1},\ B_{px} \approx \Delta^{-1} = \begin{bmatrix} \Delta_{\Omega_0 \cup \partial\Omega_1} & \Delta_{\Omega_0 \cup \partial\Omega_1; \tilde{\Omega}_1 \setminus \bar{\Omega}_1} \\ \Delta_{\Omega_1; \Omega_0 \cup \partial\Omega_1} & \Delta_{\Omega_1} \end{bmatrix}^{-1},$$

$$\Delta - \text{mesh-operator for } \begin{cases} -\Delta u = f \text{ in } \Omega_0, \\ u = 0 \text{ on } \partial\Omega,\ \partial u / \partial \vec{n} = 0 \text{ on } \partial\Omega_1 \end{cases}$$

P_d – extension-operator of mesh-function from $\Omega_0 \cup \partial\Omega_1$ to Ω_1

$$B_{\Omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & A_{\Omega_1} \end{bmatrix}^+, \quad \text{or } B_{\Omega_1} = B_{px,\Omega_1} \approx \begin{bmatrix} 0 & 0 \\ 0 & A_{\Omega_1} \end{bmatrix}^+$$

Table 5

h^{-1}	$H = 1/4$							$H = 1/8$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	8	8	8	8	17	25	37	7	8	9	7	14	16	25
	7	**7**	**7**	**7**	**14**	**21**	**25**	**7**	**7**	**7**	**7**	**13**	**15**	**19**
64	8	8	9	7	17	30	43	7	8	9	7	15	25	34
	7	**7**	**7**	**7**	**14**	**23**	**34**	**7**	**7**	**7**	**7**	**13**	**21**	**24**
128	7	8	8	7	17	31	46	7	8	9	7	14	29	42
	7	**7**	**7**	**7**	**14**	**24**	**36**	**7**	**7**	**7**	**7**	**13**	**23**	**33**
256	7	7	8	7	16	31	47	7	7	8	7	14	30	44
	7	**7**	**7**	**7**	**13**	**24**	**37**	**7**	**7**	**7**	**7**	**13**	**23**	**35**
512	7	7	8	7	15	31	45	7	7	8	6	12	29	42
	6	**6**	**6**	**6**	**13**	**24**	**37**	**6**	**6**	**6**	**6**	**12**	**23**	**32**

h^{-1}	$H=1/16$							$H=1/32$						
	p_1							p_1						
	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
32	7	8	9	7	11	14	18	7	7	7	7	9	10	13
	7	**7**	**7**	**7**	**10**	**11**	**15**	**7**	**7**	**7**	**7**	**9**	**10**	**13**
64	7	8	9	7	12	16	26	7	7	9	7	10	13	18
	7	**7**	**7**	**7**	**12**	**15**	**19**	**7**	**7**	**7**	**7**	**10**	**11**	**13**
128	7	8	9	7	13	25	33	7	8	9	7	12	15	26
	7	**7**	**7**	**7**	**12**	**21**	**24**	**7**	**7**	**7**	**7**	**12**	**15**	**18**
256	7	7	8	7	12	25	40	7	7	8	7	12	22	28
	7	**7**	**7**	**7**	**12**	**22**	**32**	**7**	**7**	**7**	**7**	**12**	**20**	**24**
512	7	7	8	6	12	25	40	7	7	8	6	11	21	36
	6	**6**	**7**	**6**	**12**	**22**	**30**	**6**	**6**	**7**	**6**	**11**	**21**	**28**

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