JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

A Robust FEM-BEM Solver for Time-Harmonic Eddy Current Problems

Michael Kolmbauer
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

Ulrich Langer
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

NuMa-Report No. 2011-05 May 2011

A–4040 LINZ, Altenbergerstraße 69, Austria

Technical Reports before 1998:

1995

95-1 Hedwig Brandstetter
Was ist neu in Fortran 90? March 1995

95-2 G. Haase, B. Heise, M. Kuhn, U. Langer
Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations. August 1995

95-3 Joachim Schöberl
An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions. August 1995

1996

96-1 Ferdinand Kickinger
Automatic Mesh Generation for 3D Objects. February 1996

96-2 Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn
Preprocessing in BE/FE Domain Decomposition Methods. February 1996

96-3 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. February 1996

96-4 Bodo Heise und Michael Jung
Robust Parallel Newton-Multilevel Methods. February 1996

96-5 Ferdinand Kickinger
Algebraic Multigrid for Discrete Elliptic Second Order Problems. February 1996

96-6 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. May 1996

96-7 Michael Kuhn
Benchmarking for Boundary Element Methods. June 1996

1997

97-1 Bodo Heise, Michael Kuhn and Ulrich Langer
A Mixed Variational Formulation for 3D Magnetostatics in the Space $H(rot) \cap H(div)$ February 1997

97-2 Joachim Schöberl
Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case. June 1997

97-3 Ferdinand Kickinger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl
Numerical Estimates of Inequalities in $H^{\frac{1}{2}}$. August 1997

97-4 Joachim Schöberl
Programmbeschreibung NAOMI 2D und Algebraic Multigrid. September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`

A Robust FEM-BEM Solver for Time-Harmonic Eddy Current Problems

Michael Kolmbauer and Ulrich Langer

Abstract This paper is devoted to the construction and analysis of robust solution techniques for time-harmonic eddy current problems in unbounded domains. We discretize the time-harmonic eddy current equation by means of a symmetrically coupled finite and boundary element method, taking care of the different physical behavior in conducting and non-conducting subdomains, respectively. We construct and analyse a block-diagonal preconditioner for the system of coupled finite and boundary element equations that is robust with respect to the space discretization parameter as well as all involved "bad" parameters like the frequency, the conductivity and the reluctivity. Block-diagonal preconditioners can be used for accelerating iterative solution methods such like the Minimal Residual Method.

1 Introduction

In many practical applications, the excitation is time-harmonic. Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation. Following this strategy, Copeland et al. [2011], Kolmbauer and Langer [2011a] and Bachinger et al. [2005, 2006] applied harmonic and multiharmonic approaches to parabolic initial-boundary value problems and the eddy current problem, respectively. Indeed, in Kolmbauer and Langer [2011a], a preconditioned MinRes solver for the solution of the eddy current

Michael Kolmbauer
Institute of Computational Mathematics, Johannes Kepler University, Linz, Austria, kolmbauer@numa.uni-linz.ac.at

Ulrich Langer
RICAM, Austrian Academy of Sciences, Linz, Austria, ulanger@numa.uni-linz.ac.at

problem in bounded domains was constructed that is robust with respect to both the discretization parameter h and the frequency ω. The key point of this parameter-robust solver is the construction of a block-diagonal preconditioner, where standard $\mathbf{H}(\mathbf{curl})$ FEM magneto-static problems have to be solved or preconditioned. The aim of this contribution is to generalize these ideas to the case of unbounded domains in terms of a coupled Finite Element (FEM) - Boundary Element (BEM) Method. In this case we are also able to construct a block-diagonal preconditioner, where now standard coupled FEM-BEM $\mathbf{H}(\mathbf{curl})$ problems, as arising in the magneto-static case, have to be solved or preconditioned.

The paper is now organized as follow. We introduce the frequency domain equations in Section 2. In the same section, we provide the symmetrically coupled FEM-BEM discretization of these equations. In Section 3, we construct and analyse our parameter-robust block-diagonal preconditioner used in a MinRes setting for solving the resulting system of linear algebraic equations. Finally, we discuss the practical realization of our preconditioner.

2 Frequency domain FEM-BEM

As a model problem, we consider the following eddy current problem:

$$\begin{cases} \sigma \frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu_1(|\mathbf{curl}\,\mathbf{u}|)\,\mathbf{curl}\,\mathbf{u}) = \mathbf{f} & \text{in } \Omega_1 \times (0,T), \\ \mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}) = \mathbf{0} & \text{in } \Omega_2 \times (0,T), \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega_2 \times (0,T), \\ \mathbf{u} = \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{curl}\,\mathbf{u} = \mathcal{O}(|\mathbf{x}|^{-1}) & \text{for } |\mathbf{x}| \to \infty, \\ \mathbf{u} = \mathbf{u_0} & \text{on } \Omega_1 \times \{0\}, \\ \mathbf{u_1} \times \mathbf{n} = \mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T), \\ \nu_1(|\mathbf{curl}\,\mathbf{u_1}|)\mathbf{curl}\,\mathbf{u_1} \times \mathbf{n} = \mathbf{curl}\,\mathbf{u_2} \times \mathbf{n} & \text{on } \Gamma \times (0,T), \end{cases} \tag{1}$$

where the computational domain $\Omega = \mathbb{R}^3$ is split into the two non-overlapping subdomains Ω_1 and Ω_2. The conducting subdomain Ω_1 is assumed to be a simply connected Lipschitz polyhedron, whereas the non-conducting subdomain Ω_2 is the complement of Ω_1 in $\mathbb{R}^3$, i.e $\mathbb{R}^3 \backslash \overline{\Omega}_1$. Furthermore, we denote by Γ the interface between the two subdomains, i.e. $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. The exterior unit normal vector of Ω_1 on Γ is denoted by $\mathbf{n}$, i.e. $\mathbf{n}$ points from Ω_1 to Ω_2. The reluctivity ν_1 is supposed to be independent of $|\mathbf{curl}\,\mathbf{u}|$, i.e. we assume the eddy current problem (1) to be linear. The conductivity σ is zero in Ω_2, and piecewise constant and uniformly positive in Ω_1.

We assume that the source $\mathbf{f}$ is given by a time-harmonic excitation with the frequency $\omega > 0$ and amplitudes $\mathbf{f^c}$ and $\mathbf{f^s}$ in the conducting domain Ω_1. Therefore, the solution $\mathbf{u}$ is time-harmonic as well, with the same base frequency ω, i.e.

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u^c}(\mathbf{x})\cos(\omega t) + \mathbf{u^s}(\mathbf{x})\sin(\omega t). \tag{2}$$

Using the time-harmonic representation (2) of the solution, we can state the eddy current problem (1) in the frequency domain as follows:

$$\text{Find } \mathbf{u} = (\mathbf{u^c}, \mathbf{u^s}) : \quad \begin{cases} \omega\,\sigma\,\mathbf{u^s} + \mathbf{curl}\,(\nu_1\,\mathbf{curl}\,\mathbf{u^c}) = \mathbf{f^c} & \text{in } \Omega_1, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u^c} = \mathbf{0} & \text{in } \Omega_2, \\ -\omega\,\sigma\,\mathbf{u^c} + \mathbf{curl}\,(\nu_1\,\mathbf{curl}\,\mathbf{u^s}) = \mathbf{f^s} & \text{in } \Omega_1, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u^s} = \mathbf{0} & \text{in } \Omega_2, \end{cases} \tag{3}$$

with the corresponding decay and interface conditions from (1). Deriving the variational formulation and integrating by parts once more in the exterior domain yields: Find $(\mathbf{u^c}, \mathbf{u^s}) \in \mathbf{H}(\mathbf{curl}, \Omega_1)^2$ such that

$$\begin{cases} \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^c}, \mathbf{curl}\,\mathbf{v^c})_{L_2(\Omega_1)} - \langle\gamma_N\mathbf{u^c}, \gamma_D\mathbf{v^c}\rangle_\tau = \langle\mathbf{f^c}, \mathbf{v^c}\rangle, \\ -\omega(\sigma\mathbf{u^c}, \mathbf{v^s})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^s}, \mathbf{curl}\,\mathbf{v^s})_{L_2(\Omega_1)} - \langle\gamma_N\mathbf{u^s}, \gamma_D\mathbf{v^s}\rangle_\tau = \langle\mathbf{f^s}, \mathbf{v^s}\rangle, \end{cases}$$

for all $(\mathbf{v^c}, \mathbf{v^s}) \in \mathbf{H}(\mathbf{curl}, \Omega_1)^2$. Here γ_D and γ_N denote the Dirichlet trace $\gamma_D := \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ and the Neumann trace $\gamma_N := \mathbf{curl}\mathbf{u} \times \mathbf{n}$ on the interface Γ. $\langle\cdot,\cdot\rangle_\tau$ denotes the $L_2(\Gamma)$-based duality product. In order to deal with the expression on the interface Γ, we use the framework of the symmetric FEM-BEM coupling for eddy current problems (see Hiptmair [2002]). So, using the boundary integral operators $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{N}$, as defined in Hiptmair [2002], we end up with the weak formulation of the time-harmonic eddy current problem: Find $(\mathbf{u^c}, \mathbf{u^s}) \in \mathbf{H}(\mathbf{curl}, \Omega_1)^2$ and $(\boldsymbol{\lambda}^\mathbf{c}, \boldsymbol{\lambda}^\mathbf{s}) \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)^2$ such that

$$\begin{cases} \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^c}, \mathbf{curl}\,\mathbf{v^c})_{L_2(\Omega_1)}, \\ \qquad -\langle\mathbf{N}(\gamma_D\mathbf{u^c}), \gamma_D\mathbf{v^c}\rangle_\tau + \langle\mathbf{B}(\boldsymbol{\lambda}^\mathbf{c}), \gamma_D\mathbf{v^c}\rangle_\tau = \langle\mathbf{f^c}, \mathbf{v^c}\rangle, \\ \qquad \langle\boldsymbol{\mu}^\mathbf{c}, (\mathbf{C} - \mathbf{Id})(\gamma_D\mathbf{u^c})\rangle_\tau - \langle\boldsymbol{\mu}^\mathbf{c}, \mathbf{A}(\boldsymbol{\lambda}^\mathbf{c})\rangle_\tau = 0, \\ -\omega(\sigma\mathbf{u^c}, \mathbf{v^s})_{L_2(\Omega_1)} + (\nu_1\mathbf{curl}\,\mathbf{u^s}, \mathbf{curl}\,\mathbf{v^s})_{L_2(\Omega_1)}, \\ \qquad -\langle\mathbf{N}(\gamma_D\mathbf{u^s}), \gamma_D\mathbf{v^s}\rangle_\tau + \langle\mathbf{B}(\boldsymbol{\lambda}^\mathbf{s}), \gamma_D\mathbf{v^s}\rangle_\tau = \langle\mathbf{f^s}, \mathbf{v^s}\rangle, \\ \qquad \langle\boldsymbol{\mu}^\mathbf{s}, (\mathbf{C} - \mathbf{Id})(\gamma_D\mathbf{u^s})\rangle_\tau - \langle\boldsymbol{\mu}^\mathbf{s}, \mathbf{A}(\boldsymbol{\lambda}^\mathbf{s})\rangle_\tau = 0, \end{cases} \tag{4}$$

for all $(\mathbf{v^c}, \mathbf{v^s}) \in \mathbf{H}(\mathbf{curl}, \Omega_1)^2$ and $(\boldsymbol{\mu}^\mathbf{c}, \boldsymbol{\mu}^\mathbf{s}) \in \mathbf{H}_\parallel^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)^2$. This variational form is the starting point of the discretization in space. Therefore, we use a regular triangulation $\mathcal{T}_h$, with mesh size $h > 0$, of the domain Ω_1 with tetrahedral elements. $\mathcal{T}_h$ induces a mesh $\mathcal{K}_h$ of triangles on the boundary Γ. On these meshes, we consider Nédélec basis functions of order p yielding the conforming finite element subspace $\mathcal{ND}_p(\mathcal{T}_h)$ of $\mathbf{H}(\mathbf{curl}, \Omega_1)$, see Nédélec [1986]. Further, we use the space of divergence free Raviart-Thomas basis functions $\mathcal{RT}_p^0(\mathcal{K}_h) := \{\lambda_h \in \mathcal{RT}_p(\mathcal{K}_h), \mathrm{div}_\Gamma\lambda_h = 0\}$ being a conforming finite element subspace of $\mathbf{H}_\parallel^{-\frac{1}{2}}(\mathrm{div}_\Gamma 0, \Gamma)$. Let $\{\varphi_i\}$ denote the basis of

$\mathcal{ND}_p(\mathcal{T}_h)$, and let $\{\psi_i\}$ denote the basis of $\mathcal{RT}^0_p(\mathcal{K}_h)$. Then the matrix entries corresponding to the operators in (4) are given by the formulas

$$
\begin{aligned}
(\mathbf{K})_{ij} &:= (\nu\, \mathbf{curl}\, \boldsymbol{\varphi}_\mathbf{i}, \mathbf{curl}\, \boldsymbol{\varphi}_\mathbf{j})_{\mathbf{L_2}(\Omega_1)} - \langle \mathbf{N}(\gamma_D \boldsymbol{\varphi}_\mathbf{i}), \gamma_D \boldsymbol{\varphi}_\mathbf{j} \rangle_\tau, \\
(\mathbf{M})_{ij} &:= \omega(\sigma \boldsymbol{\varphi}_\mathbf{i}, \boldsymbol{\varphi}_\mathbf{j})_{\mathbf{L_2}(\Omega_1)}, \\
(\mathbf{A})_{ij} &:= \langle \boldsymbol{\psi}_\mathbf{i}, \mathbf{A}(\boldsymbol{\psi}_\mathbf{j}) \rangle_\tau, \\
(\mathbf{B})_{ij} &:= \langle \boldsymbol{\psi}_\mathbf{i}, (\mathbf{C} - \mathbf{Id})(\gamma_D \boldsymbol{\varphi}_\mathbf{j}) \rangle_\tau.
\end{aligned}
$$

The entries of the right-hand side vector are given by the the formulas $(\mathbf{f^c})_i := (\mathbf{f^c}, \boldsymbol{\varphi}_\mathbf{i})_{\mathbf{L_2}(\Omega_1)}$ and $(\mathbf{f^s})_i := (\mathbf{f^s}, \boldsymbol{\varphi}_\mathbf{i})_{\mathbf{L_2}(\Omega_1)}$. The resulting system $\mathcal{A}\mathbf{x} = \mathbf{f}$ of the coupled finite and boundary element equations has now the following structure:

$$
\begin{pmatrix}
\mathbf{M} & 0 & \mathbf{K} & \mathbf{B}^T \\
0 & 0 & \mathbf{B} & -\mathbf{A} \\
\mathbf{K} & \mathbf{B}^T & -\mathbf{M} & 0 \\
\mathbf{B} & -\mathbf{A} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{u^s} \\ \boldsymbol{\lambda}^\mathbf{s} \\ \mathbf{u^c} \\ \boldsymbol{\lambda}^\mathbf{c}
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{f^c} \\ 0 \\ \mathbf{f^s} \\ 0
\end{pmatrix}. \tag{5}
$$

In fact, the system matrix $\mathcal{A}$ is symmetric and indefinite and obtains a double saddle-point structure. Since $\mathcal{A}$ is symmetric, the system can be solved by a MinRes method, see, e.g., Paige and Saunders [1975]. Anyhow, the convergence rate of any iterative method deteriorates with respect to the meshsize h and the "bad" parameters ω, ν and σ, if applied to the unpreconditioned system (5). Therefore, preconditioning is a challenging topic.

3 A parameter-robust preconditioning technique

In this section, we investigate a preconditioning technique for double saddle-point equations with the block-structure (5). Due to the symmetry and coercivity properties of the underlying operators, the blocks fulfill the following properties: $\mathbf{K} = \mathbf{K}^T \geq 0$, $\mathbf{M} = \mathbf{M}^T > 0$ and $\mathbf{A} = \mathbf{A}^T > 0$.

Zulehner [2010] constructed a parameter-robust block-diagonal preconditioner for the distributed optimal control of the Stokes equations. The structural similarities to that preconditioner gives us a hint how to choose the block-diagonal preconditioner in our case. Therefore, we propose the following preconditioner

$$
\mathcal{C} = \operatorname{diag}\ (\mathcal{I}_{FEM}, \mathcal{I}_{BEM}, \mathcal{I}_{FEM}, \mathcal{I}_{BEM}),
$$

where the diagonal blocks are given by $\mathcal{I}_{FEM} = \mathbf{M} + \mathbf{K}$ and $\mathcal{I}_{BEM} = \mathbf{A} + \mathbf{B}\mathcal{I}_{FEM}^{-1}\mathbf{B}^T$. Being aware that $\mathcal{I}_{FEM}$ and $\mathcal{I}_{BEM}$ are symmetric and positive definite, we conclude that $\mathcal{C}$ is also symmetric and positive definite. Therefore, $\mathcal{C}$ induces the energy norm $\|\mathbf{u}\|_\mathcal{C} = \sqrt{\mathbf{u}^T \mathcal{C} \mathbf{u}}$. Using this special norm, we can apply the Theorem of Babuška-Aziz Babuška [1971] to the variational problem:

$$\text{Find } \mathbf{x} \in \mathbb{R}^N : \quad \mathbf{w}^T \mathcal{A}\mathbf{x} = \mathbf{w}^T \mathbf{f}, \quad \forall\, \mathbf{w} \in \mathbb{R}^N.$$

The main result is now summarized in the following lemma.

Lemma 1. *The matrix $\mathcal{A}$ satisfies the following norm equivalence inequalities:*

$$\frac{1}{\sqrt{7}} \|\mathbf{x}\|_{\mathcal{C}} \leq \sup_{\mathbf{w}\neq 0} \frac{\mathbf{w}^T \mathcal{A}\mathbf{x}}{\|\mathbf{w}\|_{\mathcal{C}}} \leq 2\, \|\mathbf{x}\|_{\mathcal{C}} \quad \forall\, \mathbf{x} \in \mathbb{R}^N.$$

Proof. Throughout the proof, we use the following notation: $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4})^T$ and $\mathbf{y} = (\mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}, \mathbf{y_4})^T$. The upper bound follows by reapplication of Cauchy's inequality several time. The expressions corresponding to the Schur complement can be derived in the following way:

$$\mathbf{y_1}^T \mathbf{B}^T \mathbf{x_4} = \mathbf{y_1} \mathcal{I}_{FEM}^{1/2} \mathcal{I}_{FEM}^{-1/2} \mathbf{B}^T \mathbf{x_4} \leq \|\mathcal{I}_{FEM}^{1/2} \mathbf{y_1}\|_{l_2} \|\mathcal{I}_{FEM}^{-1/2} \mathbf{B}^T \mathbf{x_4}\|_{l_2}$$

Therefore, we end up with an upper bound with constant 2.

In order to compute the lower bound, we use a linear combination of special test vectors. For the choice $\mathbf{w_1} = (\mathbf{x_1}, \mathbf{x_2}, -\mathbf{x_3}, -\mathbf{x_4})^T$, we obtain

$$\mathbf{w_1}^T \mathcal{A}\mathbf{x} = \mathbf{x_1}^T \mathbf{M}\mathbf{x_1} + \mathbf{x_3}^T \mathbf{M}\mathbf{x_3};$$

for $\mathbf{w_2} = (\mathbf{x_3}, -\mathbf{x_4}, \mathbf{x_1}, -\mathbf{x_2})^T$, we get

$$\mathbf{w_2}^T \mathcal{A}\mathbf{x} = \mathbf{x_1}^T \mathbf{K}\mathbf{x_1} + \mathbf{x_3}^T \mathbf{K}\mathbf{x_3} + \mathbf{x_2}^T \mathbf{A}\mathbf{x_2} + \mathbf{x_4}^T \mathbf{A}\mathbf{x_4};$$

for $\mathbf{w_3} = ((\mathbf{x_4}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0}, (\mathbf{x_2}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0})^T$, we have

$$\begin{aligned}\mathbf{w_3}^T \mathcal{A}\mathbf{x} = {} & \mathbf{x_4}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_4} + \mathbf{x_2}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_2} \\ & + \mathbf{x_4}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} + \mathbf{x_4}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3} \\ & + \mathbf{x_2}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1} - \mathbf{x_2}^T \mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3};\end{aligned}$$

for $\mathbf{w_4} = (-(\mathbf{x_3}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0}, -(\mathbf{x_1}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0})^T$, we get

$$\begin{aligned}\mathbf{w_4}^T \mathcal{A}\mathbf{x} = {} & -\mathbf{x_3}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} - \mathbf{x_3}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3} \\ & - \mathbf{x_3}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_4} - \mathbf{x_1}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1} \\ & - \mathbf{x_1}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_2} + \mathbf{x_1}^T \mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3};\end{aligned}$$

and, finally, for the choice $\mathbf{w_5} = (-(\mathbf{x_1}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0}, (\mathbf{x_3}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1})^T, \mathbf{0})^T$, we obtain

$$\begin{aligned}\mathbf{w_5}^T \mathcal{A}\mathbf{x} = {} & -\mathbf{x_1}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} - \mathbf{x_1}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3} \\ & - \mathbf{x_1}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_4} + \mathbf{x_3}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1} \\ & + \mathbf{x_3}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T \mathbf{x_2} - \mathbf{x_3}^T \mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.\end{aligned}$$

Therefore, we end up with the following expression

$$\begin{aligned}(\mathbf{w_1}+\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T\mathcal{A}\mathbf{x} &= \mathbf{x_1}^T\mathbf{M}\mathbf{x_1}+\mathbf{x_3}^T\mathbf{M}\mathbf{x_3}\\ &+\mathbf{x_1}^T\mathbf{K}\mathbf{x_1}+\mathbf{x_3}^T\mathbf{K}\mathbf{x_3}+\mathbf{x_2}^T\mathbf{A}\mathbf{x_2}+\mathbf{x_4}^T\mathbf{A}\mathbf{x_4}\\ &+\mathbf{x_4}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_4}+\mathbf{x_2}^T\mathbf{B}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{B}^T\mathbf{x_2}\\ &-\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ &-\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}\\ &-2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}+2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.\end{aligned}$$

For estimating the non-symmetric terms, we use the following result:

$$\begin{aligned}-2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1} &\geq -2\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{K}\mathbf{x_3}\|_{l_2}\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{M}\mathbf{x_1}\|_{l_2}\\ &\geq -\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{K}\mathbf{x_3}\|_{l_2}^2-\|(\mathbf{K}+\mathbf{M})^{-1/2}\mathbf{M}\mathbf{x_1}\|_{l_2}^2\\ &= -\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}.\end{aligned}$$

Analogously, we obtain

$$2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3} \geq -\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}-\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}.$$

Hence, putting all terms together, we have

$$\begin{aligned}(\mathbf{w_1}+\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T\mathcal{A}\mathbf{x} &= \mathbf{x}^T\mathcal{C}\mathbf{x}\\ &-2\mathbf{x_3}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_3}-2\mathbf{x_1}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_1}\\ &-2\mathbf{x_3}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_3}-2\mathbf{x_1}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_1}.\end{aligned}$$

In order to get rid of the four remaining terms, we use, for $i=1,3$,

$$\mathbf{x_i}^T\mathbf{K}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{K}\mathbf{x_i} \leq \mathbf{x_i}^T\mathbf{K}\mathbf{x_i} \quad \text{and} \quad \mathbf{x_i}^T\mathbf{M}(\mathbf{K}+\mathbf{M})^{-1}\mathbf{M}\mathbf{x_i} \leq \mathbf{x_i}^T\mathbf{M}\mathbf{x_i}.$$

Hence by adding $\mathbf{w_1}$ and $\mathbf{w_2}$ twice more, we end up with the desired result

$$\underbrace{(3\mathbf{w_1}+3\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5})^T}_{:=\mathbf{w}^T}\mathcal{A}\mathbf{x} \geq \mathbf{x}^T\mathcal{C}\mathbf{x}+\mathbf{x_2}^T\mathbf{A}\mathbf{x_2}+\mathbf{x_4}^T\mathbf{A}\mathbf{x_4} \geq \mathbf{x}^T\mathcal{C}\mathbf{x}.$$

The next step is to compute (and estimate) the $\mathcal{C}$ norm of the special test vector. Straightforward estimations yield

$$\|\mathbf{w}\|_{\mathcal{C}}^2 = \|3\mathbf{w_1}+3\mathbf{w_2}+\mathbf{w_3}+\mathbf{w_4}+\mathbf{w_5}\|_{\mathcal{C}}^2 \leq 7\,\|\mathbf{x}\|_{\mathcal{C}}^2.$$

This completes the proof.

Now, from Lemma 1, we obtain that the condition number of the preconditioned system can be estimated by the constant $c = 2\sqrt{7}$ that is obviously independent of the meshsize h and all involved parameters ω, ν and σ, i.e.

$$\kappa_{\mathcal{C}}(\mathcal{C}^{-1}\mathcal{A}) := \|\mathcal{C}^{-1}\mathcal{A}\|_{\mathcal{C}}\|\mathcal{A}^{-1}\mathcal{C}\|_{\mathcal{C}} \leq 2\sqrt{7}. \tag{6}$$

The condition number defines the convergence behaviour of the MinRes method applied to the preconditioned system (see e.g. Greenbaum [1997]), as stated in the following theorem:

Theorem 1 (Robust solver). *The MinRes method applied to the preconditioned system* $\mathcal{C}^{-1}\mathcal{A}\mathbf{u} = \mathcal{C}^{-1}\mathbf{f}$ *converges. At the 2m-th iteration, the preconditioned residual* $\mathbf{r^m} = \mathcal{C}^{-1}\mathbf{f} - \mathcal{C}^{-1}\mathcal{A}\mathbf{u^m}$ *is bounded as*

$$\left\|\mathbf{r^{2m}}\right\|_{\mathcal{C}} \leq \frac{2q^m}{1+q^{2m}}\left\|\mathbf{r^0}\right\|_{\mathcal{C}}, \quad \textit{where} \quad q = \frac{2\sqrt{7}-1}{2\sqrt{7}+1}. \tag{7}$$

4 Conclusion, Outlook and Acknowledgments

The method developed in this work shows great potential for solving time-harmonic eddy current problems in an unbounded domain in a robust way. The solution of a fully coupled 4×4 block-system can be reduced to the solution of a block-diagonal matrix, where each block corresponds to standard problems. We mention, that by analogous procedure, we can state another robust block-diagonal preconditioner $\tilde{\mathcal{C}} = \text{diag}\,(\tilde{\mathcal{I}}_{FEM}, \tilde{\mathcal{I}}_{BEM}, \tilde{\mathcal{I}}_{FEM}, \tilde{\mathcal{I}}_{BEM})$, with $\tilde{\mathcal{I}}_{FEM} = \mathbf{M} + \mathbf{K} + \mathbf{B}^T\tilde{\mathcal{I}}_{BEM}^{-1}\mathbf{B}$ and $\tilde{\mathcal{I}}_{BEM} = \mathbf{A}$.

Of course this block-diagonal preconditioner is only a theoretical one, since the exact solution of the diagonal blocks corresponding to a standard FEM discretized stationary problem and the Schur-complement of a standard FEM-BEM discretized stationary problem are still prohibitively expensive. Nevertheless, as for the FEM discretized version in Kolmbauer and Langer [2011a], this theoretical preconditioner allows us replace the solution of a time-dependent problem by the solution of a sequence of time-independent problems in a robust way, i.e. independent of the the space and time discretization parameters h and ω and all additional "bad" parameters. Therefore, the issue of finding robust solvers for the fully coupled time-harmonic system matrix $\mathcal{A}$ can be reduced to finding robust solvers for the blocks $\mathcal{I}_{FEM}$ and $\mathcal{I}_{BEM}$, or $\tilde{\mathcal{I}}_{FEM}$ and $\tilde{\mathcal{I}}_{BEM}$. By replacing these diagonal blocks by standard preconditioners, it is straight-forward to derive mesh-independent convergence rates, see, e.g., Funken and Stephan [2009]. Unfortunately, the construction of fully robust preconditioners for the diagonal blocks is not straight forward and has to be studied. Candidates are $\mathcal{H}$ matrix, multigrid multigrid and domain decomposition preconditioners, see, e.g. Bebendorf [2008], Arnold et al. [2000] and Hu and Zou [2003], respectively.

The preconditioned MinRes solver presented in this paper can also be generalized to eddy current optimal control problems studied in Kolmbauer and Langer [2011b] for the pure FEM case in bounded domains.

The authors gratefully acknowledge the financial support by the Austrian Science Fund (FWF) under the grants P19255 and W1214-N15, project DK04. We also thank the Austria Center of Competence in Mechatronics (ACCM), which is a part of the COMET K2 program of the Austrian Government, for supporting our work on eddy current problems.

References

D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$. *Numer. Math.*, 85(2):197–217, 2000.

I. Babuška. Error-bounds for finite element method. *Numer. Math.*, 16(4): 322–333, 1971.

F. Bachinger, U. Langer, and J. Schöberl. Numerical analysis of nonlinear multiharmonic eddy current problems. *Numer. Math.*, 100(4):593–616, 2005.

F. Bachinger, U. Langer, and J. Schöberl. Efficient solvers for nonlinear time-periodic eddy current problems. *Comput. Vis. Sci.*, 9(4):197–207, 2006.

M. Bebendorf. *Hierarchical Matrices*. Springer, 2008.

D. Copeland, M. Kolmbauer, and U. Langer. Domain decomposition solvers for frequency-domain finite element equation. In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *LNCSE*, pages 301–308, Heidelberg, 2011. Springer.

S. A. Funken and E. P. Stephan. Fast solvers with block-diagonal preconditioners for linear FEM-BEM coupling. *Numer. Linear Algebra Appl.*, 16 (5):365–395, 2009. ISSN 1070-5325.

A. Greenbaum. *Iterative methods for solving linear systems*, volume 17 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.

R. Hiptmair. Symmetric coupling for eddy current problems. *SIAM J. Numer. Anal.*, 40(1):41–65 (electronic), 2002.

Q. Hu and J. Zou. A nonoverlapping domain decomposition method for Maxwell's equations in three dimensions. *SIAM J. Numer. Anal.*, 41(5): 1682–1708, 2003.

M. Kolmbauer and U. Langer. A frequency-robust solver for the time-harmonic eddy current problem. In *Scientific Computing in Electrical Engineering SCEE 2010*, 2011a. (accepted).

M. Kolmbauer and U. Langer. A robust preconditioned Minres-solver for distributed time-periodic eddy current optimal control problems. DK-Report 2011-07, Doctoral Program Computational Mathematics, Linz, May 2011b.

J.-C. Nédélec. A new family of mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 50(1):57–81, 1986.

C. C. Paige and M. A. Saunders. Solutions of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 12(4):617–629, 1975.

W. Zulehner. Non-standard norms and robust estimates for saddle point problems. NuMa-Report 2010-07, Institute of Computational Mathematics, Linz, November 2010.

Latest Reports in this series

2009

[..]

2010

2010-01	Joachim Schöberl, René Simon and Walter Zulehner *A Robust Multigrid Method for Elliptic Optimal Control Problems*	Januray 2010
2010-02	Peter G. Gruber *Adaptive Strategies for High Order FEM in Elastoplasticity*	March 2010
2010-03	Sven Beuchler, Clemens Pechstein and Daniel Wachsmuth *Boundary Concentrated Finite Elements for Optimal Boundary Control Problems of Elliptic PDEs*	June 2010
2010-04	Clemens Hofreither, Ulrich Langer and Clemens Pechstein *Analysis of a Non-standard Finite Element Method Based on Boundary Integral Operators*	June 2010
2010-05	Helmut Gfrerer *First-Order Characterizations of Metric Subregularity and Calmness of Constraint Set Mappings*	July 2010
2010-06	Helmut Gfrerer *Second Order Conditions for Metric Subregularity of Smooth Constraint Systems*	September 2010
2010-07	Walter Zulehner *Non-standard Norms and Robust Estimates for Saddle Point Problems*	November 2010
2010-08	Clemens Hofreither L_2 *Error Estimates for a Nonstandard Finite Element Method on Polyhedral Meshes*	December 2010
2010-09	Michael Kolmbauer and Ulrich Langer *A frequency-robust solver for the time-harmonic eddy current problem*	December 2010
2010-10	Clemens Pechstein and Robert Scheichl *Weighted Poincaré inequalities*	December 2010

2011

2011-01	Huidong Yang and Walter Zulehner *Numerical Simulation of Fluid-Structure Interaction Problems on Hybrid Meshes with Algebraic Multigrid Methods*	February 2011
2011-02	Stefan Takacs and Walter Zulehner *Convergence Analysis of Multigrid Methods with Collective Point Smoothers for Optimal Control Problems*	February 2011
2011-03	Michael Kolmbauer *Existance and Uniqueness of Eddy Current Problems in Bounded and Unbounded Domains*	May 2011
2011-04	Michael Kolmbauer and Ulrich Langer *A Robust Preconditioned-MinRes-Solver for Distributed Time-Periodic Eddy Current Optimal Control*	May 2011
2011-05	Michael Kolmbauer and Ulrich Langer *A Robust FEM-BEM Solver for Time-Harmonic Eddy Current Problems*	May 2011

From 1998 to 2008 reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`