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Introduction to Magnetohydrodynamics

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Abstract

Magnetohydrodynamics denotes the study of the dynamics of electrically conducting fluids. It establishes a coupling between the Navier-Stokes equations for fluid dynamics and Maxwell's equations for electromagnetism. The main concept behind Magnetohydrodynamics is that magnetic fields can induce currents in a moving conductive fluid, which in turn create forces on the fluid and influence the magnetic field itself.

This Bachelor thesis is concerned with the mathematical modelling of Magnetohydrodynamics. After introducing and deriving Maxwell's Equations and the Navier-Stokes equations, their coupling is described and the governing equations for Magnetohydrodynamics are obtained. Eventually some applications of Magnetohydrodynamics are described, i.e., in industrial processes, geophysics and astrophysics.

Contents

Chapter 1

Introduction

1.1 What is Magnetohydrodynamics?

In nature as in industrial processes, we can observe magnetic fields influencing the behaviour of fluids and flows. In the metallurgical industry, magnetic fields are used to stir, pump, levitate and heat liquid metals. The earth's magnetic field, protecting the surface from deadly radiation, is generated by the motion of the earth's liquid core. Sunspots and solar flares are generated by the solar magnetic field and the galactic magnetic field which influences the formation of stars from interstellar gas clouds. We use the word *Magnetohydrodynamics (MHD)* for all of these phenomena, where the magnetic field $\boldsymbol{B}$ and the velocity field $\mathbf{u}$ are coupled, given there is an electrically conducting and non-magnetic fluid, e.g. liquid metals, hot ionised gases (plasmas) or strong electrolytes. The magnetic field can induce currents into such a moving fluid and this creates forces acting on the fluid and altering the magnetic field itself.

1.2 Notation

In this short section of the thesis all facts about notation are presented. First of all, if not specified otherwise, a normal letter x represents a scalar value, whereas a bold letter $\boldsymbol{x} = (x_1, x_2, x_3)^T$ indicates a three dimensional vector. This holds for variables, constants and functions. Functions are either of static or dynamic nature. Static functions depend only on space coordinates, e.g., $T(\boldsymbol{x})$ gives the temperature in the point $\boldsymbol{x}$, whereas dynamic functions additionally have a time-dependent component t, e.g., $T(\boldsymbol{x}, t)$ gives the temperature in the point $\boldsymbol{x}$ at time t. The symbol $\cdot$ represents the euclidean inner product and $\times$ stands for the cross product. With $|\,.\,|$ we denote either the absolute value of a scalar or the euclidean norm for vectors. Throughout the thesis the following differential operators will occur:

Definition 1.1. *Let* $\mathbf{u} : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ *and* $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ *be sufficiently smooth. Then the* gradient ∇*, the* divergence $\nabla\cdot$*, the* curl $\nabla\times$ *and the* Laplacian Δ *are defined as*

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} \qquad \nabla \times \mathbf{u} = \begin{pmatrix} \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} \\ -\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix}$$

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \qquad \Delta \mathbf{u} = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \Delta u_3 \end{pmatrix}$$

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Remark 1.2. *As we can see in Definition 1.1, the differential operators are only using the partial derivatives of the space coordinates, but not the time coordinate. Usually the operators are defined for every variable of a function.*

Apart from differential operators, we will encounter different types of integrals, which are also summarized in the following definition.

Definition 1.3. *Let* $\mathbf{u} : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ *be a sufficiently smooth vector field. We define the line integral as*

$$\int_C \mathbf{u} \cdot \mathrm{d}\boldsymbol{l} := \int_I \mathbf{u}(\phi(\tau)) \cdot \phi'(\tau)\, \mathrm{d}\tau, \tag{1.1}$$

with $\phi : I \to \mathbb{R}^3$ *a parametrisation of the curve* $C \subset \mathbb{R}^3$ *and* $I = [a,b] \subset \mathbb{R}$.
Furthermore we define the surface integral

$$\int_S \mathbf{u} \cdot \vec{n}\, \mathrm{d}\boldsymbol{S} := \int_K \mathbf{u}(\psi(\sigma,\tau)) \cdot \left(\frac{\partial \psi}{\partial \sigma} \times \frac{\partial \psi}{\partial \tau} \right) \mathrm{d}(\sigma,\tau), \tag{1.2}$$

where $\psi : K \to \mathbb{R}^3$ *is a parametrisation of the surface* $S \subset \mathbb{R}^3$ *and* $K \subset \mathbb{R}^2$ *is a reference domain.*
The volume integral is defined as

$$\int_V \mathbf{u}\, d\boldsymbol{x} = \int_V \mathbf{u}(x_1,x_2,x_3,t)\, \mathrm{d}(x_1,x_2,x_3), \tag{1.3}$$

where $V \subset \mathbb{R}^3$ *is a volume.*

With these definitions, we can formulate two important theorems, which relate the different types of integrals to each other.

Theorem 1.4 (Stokes' Theorem)**.** *Let* S *be a surface in* $\mathbb{R}^3$*, parametrised by* $\varphi : M \to \mathbb{R}^3$*, where* M *is a superset of a normal range* K *in* $\mathbb{R}^2$*, with* $\varphi(K) = S$*. Let* $\varphi \in C^2(M, \mathbb{R}^3)$ *and* ∂S *be the boundary curve of* S*. Let* $\mathbf{u} \in C^1(\varphi(M), \mathbb{R}^3)$*. Then it holds that*

$$\int_S \nabla \times \mathbf{u} \cdot \vec{n}\, \mathrm{d}\boldsymbol{S} = \int_{\partial S} \mathbf{u} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.4}$$

Proof. See Thm. 8.50 in [2]. □

Theorem 1.5 (Divergence Theorem)**.** *Let V be a sufficiently smooth subset of $\mathbb{R}^3$ and let ∂V its boundary. Let $M \supseteq V$ and $\mathbf{u} \in C^1(M, \mathbb{R}^3)$. Then it holds that*

$$\int_{\partial V} \mathbf{u} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \mathbf{u} \, d\boldsymbol{x}. \tag{1.5}$$

Proof. See Thm. 8.58 and Remark 8.59 in [2] □

Chapter 2

Electromagnetism - Maxwell's Equations

2.1 Introduction

2.1.1 Electromagnetic quantities

Throughout this thesis, we will deal with a set of physical quantities to describe electromagnetic processes. These are summarized in Table 2.1. However, some quantities need further explanation. The electric field $\boldsymbol{E} = \boldsymbol{E}_s + \boldsymbol{E}_i$ consists of two parts, the electrostatic field $\boldsymbol{E}_s$ and the induced electric field $\boldsymbol{E}_i$ which have a similar effect but a different structure. The electric current density $\boldsymbol{J} = \boldsymbol{J}_c + \boldsymbol{J}_i$ consists of the conduct current density $\boldsymbol{J}_c$ and the induced current density $\boldsymbol{J}_i$. The magnetisation $\boldsymbol{M}$ quantifies the strength of a permanent magnet and the electric polarisation $\boldsymbol{P}$ is its electric counterpart.

Variable	Physical quantity	Unit
$\boldsymbol{E}$	electric field intensity	$[V/m]$
$\boldsymbol{D}$	electric flux density	$[As/m^2]$
$\boldsymbol{H}$	magnetic field density	$[A/m]$
$\boldsymbol{B}$	magnetic flux density	$[Vs/m^2]$
$\boldsymbol{J}$	electric current density	$[A/m^2]$
ρ_c	electric charge density	$[As/m^2]$
$\boldsymbol{M}$	magnetisation	$[Vs/m^2]$
$\boldsymbol{P}$	electric polarisation	$[As/m^2]$

Table 2.1: Overview of electromagnetic quantities

2.2 Maxwell's Equations

We continue now by deriving Maxwell's Equations. We start with the integral form of each equation and will then extract the differential form. Both forms are commonly used, but for the following chapters, the latter will be used. The content of this section is based on [3] and [4].

2.2.1 The Ampère-Maxwell law

Ampère's law, or more correctly Ampère-Maxwell's law describes the relation of a magnetic field and an electric current.
In terms of mathematics, we can write the relation as

$$\oint_C \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot \vec{n} \, \mathrm{d}\boldsymbol{S}, \tag{2.1}$$

where C is a closed curve bounding a surface S.
Now applying Stokes' theorem to the left hand side of (2.1) gives us

$$\int_S \nabla \times \boldsymbol{H} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot \vec{n} \, \mathrm{d}\boldsymbol{S},$$

and as this holds for any surface S, we obtain

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{2.2}$$

We call (2.2) the Ampère-Maxwell law in differential form. In the static case, (2.2) reduces to $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, which is commonly known as Ampère's law.

2.2.2 The magnetic Gauss law

Gauss's law for magnetic fields gives information about the magnetic flux. It states that the total magnetic flux through any closed surface is zero [3]. We can write this in terms of mathematics as

$$\oint_S \boldsymbol{B} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = 0, \tag{2.3}$$

with S a closed surface surrounding a volume V. Using the divergence theorem we obtain

$$0 = \oint_S \boldsymbol{B} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{B} \, \mathrm{d}\boldsymbol{x},$$

which holds for arbitrary volumes V with closed surface S. From this we can derive

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.4}$$

which is called the magnetic Gauss law in differential form.

2.2.3 Faraday's law

Now we derive Faraday's law, which describes the relation between an electric field and a changing magnetic field. The notion behind this law is that changing magnetic flux through a surface induces an electromotive force in any boundary path of that surface [3]. Mathematically, it states

$$\oint_C \boldsymbol{E} \cdot \, \mathrm{d}\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S}, \tag{2.5}$$

with S a surface bounded by a closed curve C and $\vec{n}$ the unit normal vector of S. We again apply Stokes' theorem on the line integral,

$$\oint_C \boldsymbol{E} \cdot \, \mathrm{d}\boldsymbol{l} = \int_S \nabla \times \boldsymbol{E} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S},$$

and (2.5) turns into

$$\int_S \nabla \times \boldsymbol{E} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S}.$$

This equation holds for any surface S, therefore we can derive Faraday's law in differential form:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2.6}$$

2.2.4 Gauss's law for electric fields

The electric Gauss law relates the electric flux through a closed surface with the total charge within that surface and these are, in fact, proportional. We can formulate this in a mathematical way as follows:

$$\oint_S \boldsymbol{D} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = \int_V \rho_c \, \mathrm{d}\boldsymbol{x}. \tag{2.7}$$

Applying the divergence theorem gives us

$$\int_V \nabla \cdot \boldsymbol{D} \, \mathrm{d}\boldsymbol{x} = \oint_S \boldsymbol{D} \cdot \vec{n} \, \mathrm{d}\boldsymbol{S} = \int_V \rho_c \, \mathrm{d}\boldsymbol{x},$$

and as this holds for any volume V, we obtain

$$\nabla \cdot \boldsymbol{D} = \rho_c, \tag{2.8}$$

which is called the electric Gauss law in differential form.

Material	σ
Distilled water	$\approx 10^{-4}$
Weak electrolytes	10^{-4} to 10^{-2}
Strong electrolytes	10^{-2} to 10^{2}
Molten glass ($1400°C$)	10 to 10^{-2}
"Cold" plasmas ($\sim 10^4 K$)	$\approx 10^{3}$
"Hot" Plasmas ($\sim 10^6 K$)	$\approx 10^{6}$
Steel ($1500°C$)	$0.7 * 10^{6}$
Aluminium ($700°C$)	$5 * 10^{6}$
Sodium ($400°C$)	$6 * 10^{6}$

Table 2.2: Typical values for σ in MHD [6]

2.2.5 Constitutive equations and material laws

In addition to Maxwell's equations there exist some constitutive equations. First we have the equation

$$\boldsymbol{B} = \mu \boldsymbol{H} + \mu_0 \boldsymbol{M}, \tag{2.9}$$

which relates the magnetic flux density with the magnetic field intensity and the permanent magnetisation. We have to be careful with the parameters μ and μ_0. The parameter μ is called the *magnetic permeability* and it varies with each material and can depend on the magnetic field in a nonlinear way, i.e., $\mu = \mu(|\boldsymbol{H}|)$. By μ_0 we denote the *vacuum permeability* and it is a natural constant of magnitude $\mu_0 = 4\pi 10^{-7}\,[N/A^2]$. Hence we can rewrite the permeability of any material as $\mu = \mu_r \mu_0$, with μ_r the *relative permeability*.

The second equation we consider is

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \boldsymbol{P}, \tag{2.10}$$

and it is the electric counterpart of (2.9). The parameter ε is the *electric permittivity*, which varies with each material. As before, we know the *vacuum permittivity* $\varepsilon_0 = 8.854187\ldots 10^{-12}\,[As/(Vm)]$, so we can assign each material its *relative permittivity* ε_r, with $\varepsilon = \varepsilon_0 \varepsilon_r$.

Both natural constants are closely related to another natural constant, the speed of light c, as they fulfil

$$\mu_0 \varepsilon_0 c^2 = 1,$$

or, in short, $(\mu_0 \varepsilon_0)^{-1/2} = c$.

Remark 2.1. *For MHD, we are interested in isotropic liquid electrical conductors, such as liquid metals, molten salts and electrolytes. For these materials, the permittivity and permeability are constant and equal to their vacuum values, i.e., $\mu_r = 1 = \varepsilon_r$, so we can safely assume that $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ [6].*

Finally, there is a law of great significance, Ohm's law. Here, we relate the conduct current density $\boldsymbol{J}_c$ with the electric field $\boldsymbol{E}$ and some material parameter σ, the *conductivity*. Mathematically, the following equation holds:

$$\boldsymbol{J}_c = \sigma \boldsymbol{E}. \tag{2.11}$$

However, if the medium is moving along a velocity field $\mathbf{v}$, the conduct current density $\boldsymbol{J}_c$ is additionally related to the magnetic field $\boldsymbol{B}$. Thus (2.11) becomes

$$\boldsymbol{J}_c = \sigma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}). \tag{2.12}$$

2.2.6 The Lorentz force

Let us consider a moving particle with velocity $\mathbf{v}$ and carrying a charge q. The force acting on this particle is

$$\mathbf{f} = q\boldsymbol{E}_s + q\boldsymbol{E}_i + q(\mathbf{v} \times \boldsymbol{B})$$

with $\boldsymbol{E}_s$ the electrostatic field, $\boldsymbol{E}_i$ the induced electric field and $\mathbf{v} \times \boldsymbol{B}$ the magnetic force. We combine $\boldsymbol{E}_s$ and $\boldsymbol{E}_i$ to the electric field $\boldsymbol{E}$ and note that the quantity $q\boldsymbol{E}$ is called electric force. We call

$$\mathbf{f} = q\left(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}\right) \tag{2.13}$$

the Lorentz force.

2.2.7 The conservation of charge

The conservation of charge is a strong, local statement about the relation of current and charge. A change in charge in a medium, or more generally, a fixed region, cannot occur unless charge is transported trough the surface of said region. Let V be a fixed volume with closed surface $S = \partial V$. The total amount of charge in this volume is

$$\mathcal{Q}(t) = \int_V \rho_c(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{V}, \tag{2.14}$$

and the amount of electric current flowing through S is $\oint_S \boldsymbol{J} \cdot \vec{n}\, \mathrm{d}\boldsymbol{S}$. Thus the conservation of charge says

$$\frac{\mathrm{d}\mathcal{Q}}{\mathrm{d}t} = -\oint_S \boldsymbol{J} \cdot \vec{n}\, \mathrm{d}\boldsymbol{S}. \tag{2.15}$$

We insert (2.14) into (2.15) and apply the divergence theorem to the right hand side and obtain

$$\int_V \frac{\partial \rho_c}{\partial t}\, \mathrm{d}\boldsymbol{V} = -\int_V \nabla \cdot \boldsymbol{J}\, \mathrm{d}\boldsymbol{V}. \tag{2.16}$$

Equation (2.16) holds for any volume V, so we obtain the differential form of the conservation of charge

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_c}{\partial t}. \tag{2.17}$$

Note that the conservation of charge is not an independent assumption, it can be derived from Maxwell's equations and is therefore built into the laws of electromagnetism. To do so, we start with the Ampere-Maxwell law, take the divergence and swap the time-derivative with the divergence operator. Applying Gauss's law for electric fields yields then the conservation of charge equation.

2.2.8 Summary of Maxwell's Equations

Let us recapitulate what we have derived by now. First we have the governing equations, consisting of the Ampére-Maxwell law, the magnetic Gauss law, Faraday's law and Gauss's law for electric fields,

$$\begin{aligned}
\nabla \times \boldsymbol{H} &= \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \\
\nabla \cdot \boldsymbol{B} &= 0, \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\
\nabla \cdot \boldsymbol{D} &= \rho_c.
\end{aligned}$$

Then we continue with the constitutive equations, namely the $\boldsymbol{B}$-$\boldsymbol{H}$-$\boldsymbol{M}$ relation, the $\boldsymbol{D}$-$\boldsymbol{E}$-$\boldsymbol{P}$ relation and Ohm's law for fields in motion,

$$\begin{aligned}
\boldsymbol{B} &= \mu \boldsymbol{H} + \mu_0 \boldsymbol{M}, \\
\boldsymbol{D} &= \varepsilon \boldsymbol{E} + \boldsymbol{P}, \\
\boldsymbol{J}_c &= \sigma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}).
\end{aligned}$$

Finally, we have two electromagnetic phenomena, the Lorentz force per particle and the conservation of charge,

$$\begin{aligned}
\mathbf{f} &= q\left(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}\right), \\
\nabla \cdot \boldsymbol{J} &= -\frac{\partial \rho_c}{\partial t}.
\end{aligned}$$

2.3 Simplifications for MHD

2.3.1 Maxwell's Equations, constitutive laws and the Laplace force

So far we have discussed the full Maxwell Equations. However for MHD some simplifications can be made. First, let U be a characteristic speed, L a characteristic length scale, B a characteristic magnetic field strength, H a characteristic magnetic field intensity, J a characteristic electric current density and E a characteristic electric field strength. From this, we derive a characteristic time scale $T = U/L$. Then our key quantities and differential operators can be written as

$$\begin{aligned} &\mathbf{v} = U\mathbf{v}^*, \quad \boldsymbol{x} = L\boldsymbol{x}^*, \quad \boldsymbol{E} = E\boldsymbol{E}^*, \quad \boldsymbol{J} = J\boldsymbol{J}^*, \quad \nabla = \frac{1}{L}\nabla^*, \\ &t = Tt^*, \quad \boldsymbol{B} = B\boldsymbol{B}^*, \quad \boldsymbol{H} = H\boldsymbol{H}^*, \quad \frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t^*}, \quad \Delta = \frac{1}{L^2}\Delta^*. \end{aligned} \tag{2.18}$$

Let us take a look at the constitutive relations. In MHD, we consider only materials which are conducting and are free of electric polarisation, magnetisation and impressed currents and have a linear $\boldsymbol{B}$-$\boldsymbol{H}$-Relation, i.e., $\boldsymbol{J}_i = 0$, $\boldsymbol{P} = 0$, $\boldsymbol{M} = 0$ and $\mu \neq \mu(|\boldsymbol{H}|)$. Moreover, from Remark 2.1 we know that μ and ε are constant and equal to their vacuum values, i.e., $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. Thus our material laws (2.9) and (2.10) become

$$\boldsymbol{B} = \mu\boldsymbol{H}, \tag{2.19}$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E}, \tag{2.20}$$

and with our extended form of Ohm's law (2.12), the current density $\boldsymbol{J}$ can be expressed as

$$\boldsymbol{J} = \boldsymbol{J}_c = \sigma\left(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}\right). \tag{2.21}$$

We non-dimensionalise Maxwell's equations, beginning with Faraday's law (2.6)

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \Leftrightarrow \qquad \frac{E}{L}\nabla^* \times \boldsymbol{E}^* = -\frac{B}{T}\frac{\partial \boldsymbol{B}^*}{\partial t^*}.$$

Hence, we can deduce

$$\frac{E}{L} \sim \frac{B}{T} \qquad \Leftrightarrow \qquad \frac{E}{B} \sim \frac{L}{T} = U.$$

Next we consider Ampère-Maxwell's equation (2.2),

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \Leftrightarrow \qquad \frac{H}{L}\nabla \times \boldsymbol{H}^* = J\boldsymbol{J}^* + \frac{D}{T}\frac{\partial \boldsymbol{D}^*}{\partial t^*}.$$

On the one hand, the contribution of the time derivative of $\boldsymbol{D}$ to the curl of $\boldsymbol{H}$ is of order

$$\frac{D/T}{H/L} \sim \frac{\mu\varepsilon E L}{BT} \sim \frac{U^2}{c^2}.$$

In MHD, we consider only speeds of magnitude $|\mathbf{v}| \ll c$, which means that the contribution is very small. On the other hand, if we compare the magnitudes of current density $\boldsymbol{J}$ with $\frac{\partial \boldsymbol{D}}{\partial t}$, we obtain

$$\frac{J}{D/T} \sim \frac{\sigma E T}{\varepsilon E} = \frac{\sigma}{\varepsilon}T,$$

which is, as we can see in Table 2.2, much larger than U^2/c^2. Therefore we can omit the dependence of the Ampère-Maxwell law on $\frac{\partial \boldsymbol{D}}{\partial t}$ and use the pre-Maxwell form, $\nabla \times \boldsymbol{H} = \boldsymbol{J}$, or, in terms of $\boldsymbol{B}$,

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}. \tag{2.22}$$

Let us recall the Lorentz force (2.13), which acts on a single particle. In MHD we are interested in the whole force acting on a medium, so we sum (2.13) up over a unit volume. The sum of charges $\sum q$ becomes the charge density ρ_c and $\sum q\mathbf{v}$ transforms into the electric current density $\boldsymbol{J}$.
Hence, (2.13) becomes

$$\mathbf{F} = \rho_c \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B}, \tag{2.23}$$

with $\mathbf{F}$ being the Lorentz force acting on a unit volume of our medium (volumetric Lorentz force). We now survey the contributions to the Lorentz force in their magnitude and follow the procedure suggested in [1]. First of all, let us recall the conservation of charge in differential form (2.17),

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_c}{\partial t}.$$

Taking the divergence of (2.21) and plugging in (2.17), (2.8) and (2.20), we obtain

$$0 = -\nabla \cdot \boldsymbol{J} + \sigma \nabla \cdot \boldsymbol{E} + \sigma \nabla \cdot (\mathbf{v} \times \boldsymbol{B}) = \frac{\partial \rho_c}{\partial t} + \frac{\sigma}{\varepsilon} \rho_c + \sigma \nabla \cdot (\mathbf{v} \times \boldsymbol{B}), \tag{2.24}$$

with σ the conductivity of our medium. The quantity $\tau_e = \varepsilon/\sigma$ is called charge relaxation time and is, as we can deduce with Remark 2.1 and Table 2.2, around $10^{-18}\,[s]$ for the materials typically used in MHD. In MHD we are interested in events on a much larger time-scale, so we neglect the contribution of $\frac{\partial \rho_c}{\partial t}$ in comparison with ρ_c/τ_e. Now non-dimensionalising the reduced form of (2.24) gives us

$$\rho_c = -\varepsilon \frac{U}{L} B \nabla^* \cdot (\mathbf{v}^* \times \boldsymbol{B}^*) \qquad \text{or} \qquad \rho_c \sim \varepsilon \frac{UB}{L}$$

and from Ohm's law follows $E \sim J/\sigma$. Hence $\rho_c \boldsymbol{E}$ is of magnitude

$$\rho_c E \sim \varepsilon \frac{UB}{L} \frac{J}{\sigma} \sim \tau_e \frac{UJB}{L},$$

and because τ_e is very small, the contribution of the electric force is negligible in comparison with the magnetic force. Thus, (2.23) reduces to

$$\mathbf{F} = \boldsymbol{J} \times \boldsymbol{B}. \tag{2.25}$$

This reduced form of the volumetric Lorentz force is also called the *Laplace force* [6]. To sum up, we have seen that the displacement currents in (2.2) are negligible and the charge density ρ_c is of small significance, so we set it zero and drop the electric Gauss law (2.8). Maxwell's equations for MHD are therefore

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{J} = 0, \tag{2.26}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{2.27}$$

with our extended Ohm's law and the Laplace force

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}), \qquad \mathbf{F} = \boldsymbol{J} \times \boldsymbol{B}. \tag{2.28}$$

2.3.2 The induction equation

We can combine our extension of Ohm's law (2.21), Faraday's law (2.6) and Ampère's law (2.22) to eliminate $\boldsymbol{E}$ from our equations and relate $\boldsymbol{B}$ to $\mathbf{v}$. We start with Faraday's law (2.6) and insert (2.21)

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} = -\nabla \times \left(\frac{\boldsymbol{J}}{\sigma} - \mathbf{v} \times \boldsymbol{B}\right) = -\frac{1}{\sigma}\nabla \times \boldsymbol{J} + \nabla \times (\mathbf{v} \times \boldsymbol{B}).$$

Using Ampère's law (2.22), we obtain

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{B}) - \frac{1}{\sigma\mu}\nabla \times (\nabla \times \boldsymbol{B})\,.$$

Using the vector identity $\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \Delta \boldsymbol{B}$ and the magnetic Gauss law $\nabla \cdot \boldsymbol{B} = 0$, we can formulate the *induction equation* as

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{B}) + \lambda \Delta \boldsymbol{B}, \tag{2.29}$$

with $\lambda = (\sigma\mu)^{-1}$ the *magnetic diffusivity*. This equation is, as we will see, one the most important ones for MHD.
Finally, let us non-dimensionalise the induction equation:

$$\begin{aligned}
&\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{B}) + \lambda \Delta \boldsymbol{B} \\
\Leftrightarrow \quad &\frac{B}{T}\frac{\partial \boldsymbol{B}^*}{\partial t^*} = \frac{VB}{L}\nabla^* \times (\mathbf{v}^* \times \boldsymbol{B}^*) + \frac{B}{L^2\sigma\mu}\Delta^* \boldsymbol{B}^* \\
\Leftrightarrow \quad &\frac{\partial \boldsymbol{B}^*}{\partial t^*} = \underbrace{\frac{VT}{L}}_{=1}\nabla^* \times (\mathbf{v}^* \times \boldsymbol{B}^*) + \underbrace{\frac{T}{L^2\sigma\mu}}_{=\frac{1}{Re_m}}\Delta^* \boldsymbol{B}^*
\end{aligned}$$

For simplicity, we will now drop the asterisk for the dimensionless parameters and instead use the normal variables. The dimensionless induction equation then reads

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{B}) + \frac{1}{Re_m}\Delta \boldsymbol{B}, \tag{2.30}$$

with $Re_m = VL/\lambda$ and $\lambda = (\sigma\mu)^{-1}$. We call Re_m the *magnetic Reynolds number* and it is a measure for conductivity and, in terms of mathematics, for the relative strengths of advection and diffusion in (2.30). Small and high magnetic Reynolds numbers result in a very different behaviour of the medium. This concludes, for now, our derivation of the (electro)magnetic part of MHD.

Chapter 3

Fluid dynamics - The Navier-Stokes equations

3.1 Introduction

We will now derive the second half of MHD, the fluid dynamics. For now, we consider fluids under the influence of a generic body force. In fluid dynamics, we try to find the velocity field $\mathbf{v}$ and the pressure p, as these quantities describe the motion and its properties, e.g. laminar/turbulent, compressible/incompressible, stationary/dynamic. However, before we can write down the governing equations of fluid dynamics, we need some preparations and tools. The content of the following sections is based on [5].

3.2 Representation of a flow

There is more than one way to describe and represent a flow. In particular, two different, yet directly connected approaches are used, the Lagrangian and the Eulerian representation. We will discuss them both.

3.2.1 Lagrangian representation

In the Lagrangian representation the motion of the fluid is described by following the position of each particle.
Let $(T_1, T_2) \subset \mathbb{R}$ be a non-empty time interval in which we consider the flow. Let $\Omega(t) \subset \mathbb{R}^3$ be the domain which is occupied by the fluid at time $t \in (T_1, T_2)$. Let $t_0 \in (T_1, T_2)$ be a fixed reference time. Then each fluid particle in $\Omega(t_0)$ can be identified by its position $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T \in \Omega(t_0)$. The motion of a fluid particle is now described by a vector field $\hat{\boldsymbol{T}} : \Omega(t_0) \times (T_1, T_2) \to \Omega(t)$, $t \in (T_1, T_2)$. $\hat{\boldsymbol{T}}(\hat{\boldsymbol{x}}, t)$ is called the trajectory of a fluid particle and returns the position of the particle $\hat{\boldsymbol{x}}$ at the time t.

Hence for the velocity field $\hat{\mathbf{v}}$ and the acceleration field $\hat{\mathbf{a}}$, the following identities hold

$$\hat{\mathbf{v}}(\hat{\boldsymbol{x}},t) = \frac{\partial \hat{\boldsymbol{T}}}{\partial t}(\hat{\boldsymbol{x}},t) \quad \text{and} \quad \hat{\mathbf{a}}(\hat{\boldsymbol{x}},t) = \frac{\partial^2 \hat{\boldsymbol{T}}}{\partial t^2}(\hat{\boldsymbol{x}},t). \tag{3.1}$$

The tupel $(\hat{\boldsymbol{x}},t) \in \Omega(t_0) \times (T_1,T_2)$ is called *Lagrangian coordinates.*

3.2.2 Eulerian representation

The Eulerian representation uses a more intuitive way to describe the flow. Here the motion of the fluid is described via the velocity field $\mathbf{v} : D \to \mathbb{R}^3$, with $D := \{(\boldsymbol{x},t) \in \mathbb{R} : x \in \Omega(t),\ t \in (T_1,T_2)\}$ a space-time cylinder. The vector field $\mathbf{v}(\boldsymbol{x},t)$ describes the velocity of the particle $\boldsymbol{x}$ at time t,

$$\mathbf{v}(\boldsymbol{x},t) = \hat{\mathbf{v}}(\hat{\boldsymbol{x}},t) = \frac{\partial \hat{\boldsymbol{T}}}{\partial t}(\hat{\boldsymbol{x}},t), \text{ with } \boldsymbol{x} = \hat{\boldsymbol{T}}(\hat{\boldsymbol{x}},t). \tag{3.2}$$

For the acceleration $\mathbf{a} = \mathbf{a}(\boldsymbol{x},t)$ of fluid particle $\boldsymbol{x}$ at time t, it holds that

$$\begin{aligned}\mathbf{a}(\boldsymbol{x},t) = \hat{\mathbf{a}}(\hat{\boldsymbol{x}},t) = \frac{\partial^2 \hat{\boldsymbol{T}}}{\partial t^2}(\hat{\boldsymbol{x}},t) &= \frac{\partial}{\partial t}\left(\frac{\partial \hat{\boldsymbol{T}}}{\partial t}(\hat{\boldsymbol{x}},t)\right) = \frac{\partial}{\partial t}\left(\mathbf{v}(\boldsymbol{x},t)\right) = \frac{\partial}{\partial t}\Big(\mathbf{v}(\hat{\boldsymbol{T}}(\hat{\boldsymbol{x}},t),t)\Big) \\ &= \sum_{i=1}^{3} \frac{\partial \mathbf{v}}{\partial x_i}(\boldsymbol{x},t)\frac{\partial \hat{\boldsymbol{T}}_i}{\partial t}(\hat{\boldsymbol{x}},t) + \frac{\partial \mathbf{v}}{\partial t} = \sum_{i=1}^{3} \frac{\partial \mathbf{v}}{\partial x_i} v_i + \frac{\partial \mathbf{v}}{\partial t} \\ &= (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}.\end{aligned}$$

The tupel $(\boldsymbol{x},t) \in D$ is called *Eulerian coordinates.*

Definition 3.1. *Let* $\mathbf{v}$ *be a vector field defined as in* (3.2). *Then the differential operators*

$$\mathbf{v}\cdot\nabla := \sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i}, \tag{3.3}$$

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla), \tag{3.4}$$

are called convective derivative *and* total derivative *(or* material derivative*), respectively.*

Suppose we have a given velocity field $\mathbf{v} : D \to \mathbb{R}^3,\ (\boldsymbol{x},t) \mapsto \mathbf{v}(\boldsymbol{x},t)$. To calculate the trajectory $\hat{\boldsymbol{T}}(\hat{\boldsymbol{x}},t)$ of a fluid particle $\hat{\boldsymbol{x}}$, we simply have to solve the initial value problem

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \mathbf{v}(\boldsymbol{x},t), \\ \boldsymbol{x}(t_0) &= \hat{\boldsymbol{x}},\end{aligned} \tag{3.5}$$

with $\boldsymbol{x}(t) = \hat{\boldsymbol{T}}(\hat{\boldsymbol{x}},t)$. This is useful to visualize the flow of a fluid, e.g. water in a turbine.

3.3 The transport theorem

Let $\omega(t) \subset \Omega(t)$ be a sufficiently smooth, simply connected domain, which is occupied by a fixed set of fluid particles at the time $t \in (T_1, T_2)$. Hence,

$$\omega(t) = \{\hat{\boldsymbol{T}}(\hat{\boldsymbol{x}}, t) : \hat{\boldsymbol{x}} \in \omega(t_0)\}. \tag{3.6}$$

Let $F : \mathbb{R}^3 \times (T_1, T_2) \to \mathbb{R}^3$ be a prescribed function, called *property density.* Integration over $\omega(t)$ provides us with the *property*

$$\mathcal{F}(t) := \int_{\omega(t)} F(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}. \tag{3.7}$$

The transportation theorem describes the change of such a property over time.

Theorem 3.2 (Reynolds transportation theorem)**.** *Let* $t_0 \in (T_1, T_2)$*,* $\omega(t_0) \subset \Omega(t_0) \subset \mathbb{R}^d$ *a bounded, sufficiently smooth domain with* $\overline{\omega(t_0)} \subset \Omega(t_0)$*,* $D := \{(\boldsymbol{x}, t) \in \mathbb{R}^{d+1} : \boldsymbol{x} \in \Omega(t),\ t \in (T_1, T_2)\}$ *a space-time cylinder,* $\mathbf{v} : D \to \mathbb{R}^d$*,* $F : D \to \mathbb{R}$ *be continuously differentiable.*
Then $\mathcal{F}$ *is well-defined by* (3.5)*,* (3.6) *and* (3.7) *on a time-scale* $(t_1, t_2) \subset (T_1, T_2)$ *and*

$$\frac{d}{dt}\mathcal{F}(t) = \int_{\omega(t)} \left[\frac{\partial F}{\partial t}(\boldsymbol{x}, t) + \nabla \cdot (F\mathbf{v})(\boldsymbol{x}, t)\right] d\boldsymbol{x}, \tag{3.8}$$

with $\nabla \cdot (F\,\mathbf{v}) = (\mathbf{v} \cdot \nabla)\, F + F\,\nabla \cdot \mathbf{v} = \nabla F \cdot \mathbf{v} + F\,\nabla \cdot \mathbf{v}$.

Proof. We will only prove the 1-dimensional case, as the proof for higher dimensions is far more technical.
Let $\mathcal{F}$ be defined as in (3.7). Application of the substitution rule yields

$$\mathcal{F}(t) = \int_{\omega(t)} F(x, t) \, \mathrm{d}x = \int_{\omega(t_0)} F(\hat{\boldsymbol{T}}(\hat{x}, t), t) \frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x}, t) \, \mathrm{d}\hat{x}$$

Plugging this into (3.8) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t_0)} F(\hat{\boldsymbol{T}}(\hat{x}, t), t) \frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x}, t) \, \mathrm{d}\hat{x}$$

There is no time dependence in our integration domain, hence we can switch integration and differentiation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) &= \int_{\omega(t_0)} \frac{\mathrm{d}}{\mathrm{d}t}\left(F(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t)\right)\,\mathrm{d}\hat{x} \\
&= \int_{\omega(t_0)} \frac{\partial F}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial t}(\hat{x},t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) + \frac{\partial F}{\partial t}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) \\
&\qquad + \frac{\partial^2 \hat{\boldsymbol{T}}}{\partial t \partial \hat{x}}(X,t)F(\hat{\boldsymbol{T}}(\hat{x},t),t)\,\mathrm{d}\hat{x} \\
&= \int_{\omega(t_0)} \frac{\partial F}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)\hat{v}(\hat{x},t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) + \frac{\partial F}{\partial t}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) \\
&\qquad + \frac{\partial}{\partial \hat{x}}(\hat{v}(\hat{x},t))F(\hat{\boldsymbol{T}}(\hat{x},t),t)\,\mathrm{d}\hat{x}.
\end{aligned}$$

Here we used identity (3.1) and in the next step, we apply (3.2). Therefore

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) &= \int_{\omega(t_0)} \frac{\partial F}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)v(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) + \frac{\partial F}{\partial t}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) \\
&\qquad + \frac{\partial}{\partial \hat{x}}(v(\hat{\boldsymbol{T}}(\hat{x},t),t))F(\hat{\boldsymbol{T}}(\hat{x},t),t)\,\mathrm{d}\hat{x} \\
&= \int_{\omega(t_0)} \frac{\partial F}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)v(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) + \frac{\partial F}{\partial t}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t) \\
&\qquad + \frac{\partial v}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t)F(\hat{\boldsymbol{T}}(\hat{x},t),t)\,\mathrm{d}\hat{x} \\
&= \int_{\omega(t_0)} \left[\frac{\partial F}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)v(\hat{\boldsymbol{T}}(\hat{x},t),t) + \frac{\partial F}{\partial t}(\hat{\boldsymbol{T}}(\hat{x},t),t)\right. \\
&\qquad \left. + \frac{\partial v}{\partial \hat{\boldsymbol{T}}}(\hat{\boldsymbol{T}}(\hat{x},t),t)F(\hat{\boldsymbol{T}}(\hat{x},t),t)\right]\frac{\partial \hat{\boldsymbol{T}}}{\partial \hat{x}}(\hat{x},t)\,\mathrm{d}\hat{x}.
\end{aligned}$$

Back substitution and reordering results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) = \int_{\omega(t)} \frac{\partial F}{\partial t}(x,t) + \frac{\partial F}{\partial x}(x,t)\,v(x,t) + \frac{\partial v}{\partial x}(x,t)\,F(x,t)\,\mathrm{d}x,$$

which is, in fact, the 1D-version of (3.8). □

Remark 3.3. *The difficulty in the proof for higher dimensions is that the Jacobian of* $\hat{x}$ *is now a* 3×3*-Matrix, so differentiating the determinant is far more technical.*

3.4 The continutiy equation

Let $\rho : D \to \mathbb{R}^+, (\boldsymbol{x},t) \mapsto \rho(\boldsymbol{x},t)$ and $\mathcal{M}(t) := \int_{\omega(t)} \rho(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}$. The function $\rho(\cdot,.)$ is called *mass density* $[kg/m^3]$ and describes the density of the fluid at the position

$\boldsymbol{x}$ at time t, whereas $\mathcal{M}(t)$ is the mass $[kg]$ of a control domain $\omega(t)$. The principle of conservation of mass now states, that no mass is created or annihilated over time, thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = 0, \quad \text{for all } t \in (T_1, T_2). \tag{3.9}$$

Application of Theorem 3.2 on (3.9) gives us

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega(t)} \rho(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x} \overset{(3.8)}{=} \int_{\omega(t)} \frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \nabla \cdot (\rho\, \mathbf{v})(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}. \tag{3.10}$$

Equation (3.10) is called the *continuity equation in integral form.* To obtain a partial differential equation (PDE), we notice that (3.10) holds for any domain $\omega(t)$ satisfying the conditions of Theorem 3.2. Hence, the *continuity equation in classical form* for compressible fluids is

$$\frac{\partial \rho}{\partial t}(\boldsymbol{x}, t) + \nabla \cdot (\rho\, \mathbf{v})(\boldsymbol{x}, t) = 0, \quad \text{for all } (\boldsymbol{x}, t) \in D. \tag{3.11}$$

In the special case of incompressible fluids, the mass density is constant, $\rho(\boldsymbol{x}, t) = \text{const} > 0$, so all derivatives of ρ vanish. Hence, the continuity equation for incompressible fluids is

$$\nabla \cdot \mathbf{v} = 0. \tag{3.12}$$

3.5 Equations of motion

From Newton's laws of motion, the law of conservation of momentum follows. It states that the change over time of momentum of a closed domain must be equal to the external forces on the domain. Let $\omega(t)$ be defined as before and act as our closed domain. Let

$$\mathcal{I}(t) = \int_{\omega(t)} \rho(\boldsymbol{x}, t)\, \mathbf{v}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \tag{3.13}$$

be the momentum of $\omega(t)$ and $F(\omega(t))$ the external force. The external force can be separated into the body force $F_V(\omega(t))$ and the surface force $F_S(\omega(t))$, so $F(\omega(t)) = F_V(\omega(t)) + F_S(\omega(t))$. The body force can be represented as

$$F_V(\omega(t)) = \int_{\omega(t)} \mathbf{f}(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{x}, \tag{3.14}$$

with the *body force density* $\mathbf{f}$. For any body force density $\mathbf{f}$, we can define a vector field $\mathbf{b}$, such that $\mathbf{f} = \rho\, \mathbf{b}$. The surface force is given by

$$F_S(\omega(t)) = \int_{\partial\omega(t)} (\sigma \cdot \vec{n})(\boldsymbol{x}, t)\, \mathrm{d}\boldsymbol{S}, \tag{3.15}$$

where $\sigma = (\sigma_{ji})_{i,j=\overline{1,3}}$ is *Cauchy's stress tensor* and $\vec{n}$ is the outer unit normal vector. Using the divergence theorem on (3.15), we obtain

$$F_S(\omega(t)) = \int_{\omega(t)} \nabla \cdot \sigma(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}. \tag{3.16}$$

The law of conservation of momentum states that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}(t) = F(\omega(t)) \tag{3.17}$$

is fulfilled. Plugging (3.13), (3.14) and (3.16) into (3.17) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega(t)} \rho(\boldsymbol{x}, t)\, \mathbf{v}(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = \int_{\omega(t)} \rho(\boldsymbol{x}, t)\, \mathbf{b}(\boldsymbol{x}, t) + \nabla \cdot \sigma(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}$$

If our domain fulfils the requirements for Theorem 3.2, we can apply the transport theorem component-wise:

$$\int_{\omega(t)} \frac{\partial(\rho\, v_i)}{\partial t}(\boldsymbol{x}, t) + \nabla \cdot ((\rho\, v_i)\mathbf{v})(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x} = \int_{\omega(t)} f_i(\boldsymbol{x}, t) + [\nabla \cdot \sigma]_i(\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x},$$

for $i = 1, 2, 3$. For simplicity, we will now drop the dependence on $\boldsymbol{x}$ and t in the notation. This holds for arbitrary $\omega(t)$, we can therefore extract the equation of motion component-wise,

$$\frac{\partial \rho}{\partial t}\, v_i + \rho\, \frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla)\, (\rho\, v_i)(\boldsymbol{x}, t) + \rho\, v_i\, \nabla \cdot \mathbf{v} = f_i + [\nabla \cdot \sigma]_i, \tag{3.18}$$

for $i = 1, 2, 3$.

3.6 The Navier-Stokes equations

Now we have the equations of motion, depending on the velocity $\mathbf{v}$, the mass density ρ, the pressure p and the stress tensor σ, which depend all on $\boldsymbol{x}$ and t. Our goal is now to rewrite σ in terms of $\mathbf{v}$. For simplicity, we only consider Newtonian fluids with constant viscosity, i.e., $\mu_f = const$. With these given properties, the following constitutive law for the stress tensor applies:

$$\sigma = -pI + \lambda\, \mathrm{tr}(\mathcal{E})I + 2\mu_f \mathcal{E}. \tag{3.19}$$

Here, μ_f is the *(first) viscosity* of the flow, λ is a material parameter related to μ_f and

$$\mathcal{E} = \frac{1}{2}\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right) \tag{3.20}$$

is the *symmetric part of the velocity gradient*. Physics suggests $\mu_f \geq 0$ and $\lambda \sim \frac{2}{3}\mu_f$. Combining (3.19) and (3.20) gives us for the divergence of σ

$$[\nabla \cdot \sigma]_i = \sum_{j=1}^{3} \frac{\partial \sigma_{ji}}{\partial x_j} = -\frac{\partial}{\partial x_i}\left(p + \frac{2}{3}\mu_f \nabla \cdot \mathbf{v}\right) + \sum_{j=1}^{3} \mu_f \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{3.21}$$

in the i-th component, for $i = 1, 2, 3$. We can rewrite the second term in (3.21) as

$$\mu_f \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \mu_f \left(\Delta v_i + \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{v}) \right),$$

so (3.21) becomes

$$[\nabla \cdot \sigma]_i = -\frac{\partial}{\partial x_i} \left(p - \frac{1}{3} \mu_f (\nabla \cdot \mathbf{v}) \right) + \mu_f \Delta v_i. \tag{3.22}$$

Plugging (3.22) into (3.18) yields for the i-th component

$$\frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla)(\rho v_i) + \rho v_i \nabla \cdot \mathbf{v} = f_i - \frac{\partial p}{\partial x_i} + \frac{1}{3} \mu_f \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{v}) + \mu_f \Delta v_i.$$

Now let us examine this equation in detail. The convective term can be expanded to

$$(\mathbf{v} \cdot \nabla)(\rho v_i) = \mathbf{v} \cdot \nabla(\rho v_i) = \mathbf{v} \cdot (\nabla \rho) v_i + \mathbf{v} \cdot (\nabla v_i) \rho,$$

so we can rewrite the i-th component as follows:

$$\begin{aligned} v_i\Big(\frac{\partial \rho}{\partial t} &+ \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot (\nabla \rho)\Big) \\ &+ \rho \frac{\partial v_i}{\partial t} + \mathbf{v} \cdot (\nabla v_i) \rho = f_i - \frac{\partial p}{\partial x_i} + \frac{1}{3} \mu_f \frac{\partial}{\partial x_i} (\nabla \cdot \mathbf{v}) + \mu_f \Delta v_i. \end{aligned} \tag{3.23}$$

Due to (3.11), the first line of (3.23) is equal to 0, thus we obtain the *Navier-Stokes equations (NSE) for compressible (Newtonian) fluids*

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} - \frac{1}{3} \mu_f \nabla (\nabla \cdot \mathbf{v}) - \mu_f \Delta \mathbf{v} + \nabla p &= \mathbf{f}, \\ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \mathbf{v}) &= 0. \end{aligned} \tag{3.24}$$

If we consider the incompressible case, where $\rho = \text{const} > 0$ and then divide the equation by ρ, the compressible NSE (3.24) simplifies to

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \nu \Delta \mathbf{v} + \frac{1}{\rho} \nabla p &= \frac{1}{\rho} \mathbf{f}, \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned} \tag{3.25}$$

with $\nu = \frac{\mu_f}{\rho} > 0$ the kinematic viscosity. We call (3.25) the *incompressible Navier-Stokes equations.*

3.6.1 Non-dimensionalisation and the Reynolds number *Re*

In the rest of this chapter we will non-dimensionalise the Navier-Stokes equations and for simplicity, we assume incompressibility, i.e., $\rho(\boldsymbol{x}, t) = \text{const} > 0$. Let L a characteristic length-scale, T a characteristic time-scale, U be a characteristic velocity satisfying $U = L/T$, and $P = \rho U^2$ a characteristic pressure. Then, with the same definitions as in (2.18),

$$\begin{aligned} & \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left(\mathbf{v} \cdot \nabla\right) \mathbf{v} - \mu_f \Delta \mathbf{v} + \nabla p = \mathbf{f} \\ \Leftrightarrow \quad & \rho \frac{U}{T} \frac{\partial \mathbf{v}^*}{\partial t^*} + \rho \frac{U^2}{L} \left(\mathbf{v}^* \cdot \nabla^*\right) \mathbf{v}^* - \mu_f \frac{U}{L^2} \Delta^* \mathbf{v}^* + \frac{P}{L} \nabla^* p^* = \mathbf{f}. \end{aligned}$$

Multiplying this equation with $L/(\rho U^2)$ gives us

$$\frac{L}{UT} \frac{\partial \mathbf{v}^*}{\partial t^*} + 1 \left(\mathbf{v}^* \cdot \nabla^*\right) \mathbf{v}^* - \frac{\mu_f}{\rho U L} \Delta^* \mathbf{v}^* + \frac{P}{\rho U^2} \nabla p^* = \frac{L}{\rho U^2} \mathbf{f}$$

We can simply confirm that $L/(UT) = 1$ and, because of the way we chose P, that $P/(\rho U^2) = 1$. We additionally set $(L/(\rho U^2))\mathbf{f} = \mathbf{f}^*$ and then omit the asterisks, to finally obtain the dimensionless equation

$$\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla\right) \mathbf{v} - \frac{1}{Re} \Delta \mathbf{v} + \nabla p = \mathbf{f} \tag{3.26}$$

with the dimensionless parameter $Re = UL/\nu$, the Reynolds number. For $Re \ll 1$ the flow is dominated by the viscosity term, whereas for $Re \gg 1$ the convective term dominates.
This concludes our consideration of fluid dynamics with a generic body force.

Chapter 4

Magnetohydrodynamics (MHD)

4.1 The governing equations

In the previous chapters we have gained knowledge about electromagnetism and fluid dynamics. We will now couple these two topics to derive the governing equations for MHD. We will consider only non-magnetic, conducting, Newtonian fluids, with uniform kinematic viscosity, i.e., $\nu = \text{const}$, and incompressible flow. The following content is based on the findings of [1] and [6].

First we consider the reduced Maxwell's equations (2.26)-(2.27), i.e.,

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{J} = 0, \tag{4.1}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0, \tag{4.2}$$

with Ohm's law and the Laplace force

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \mathbf{u} \times \boldsymbol{B}), \qquad \mathbf{F} = \boldsymbol{J} \times \boldsymbol{B}. \tag{4.3}$$

If we combine these, we gain a transportation equation for $\boldsymbol{B}$, the induction equation (2.29), i.e.,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{B}) + \lambda \Delta \boldsymbol{B}, \tag{4.4}$$

with $\lambda = (\sigma\mu)^{-1}$ the magnetic diffusivity.

On the other hand, the equations of motion give us the Navier-Stokes equations for incompressible flow (3.25),

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p &= \frac{1}{\rho} \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned}$$

Now we substitute the Lorentz force (4.3b) for $\mathbf{f}$ and obtain

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p &= \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned} \tag{4.5}$$

with $\nu = \mu_f/\rho$ the kinematic viscosity.
For MHD, it might be useful to introduce another form of the Navier-Stokes equations. Let $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ denote the *vorticity field*. Using the vector identity $\nabla(\mathbf{u}^2/2) = (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u})$ and neglecting the other forces $\mathbf{f}$ we can rewrite the first equation of (3.25) as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla(\mathbf{u}^2/2) - \mathbf{u}\times\boldsymbol{\omega} - \nu\Delta\mathbf{u} + \frac{1}{\rho}\nabla p = 0,$$

or

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega} - \nabla\left(\frac{p}{\rho} + \frac{\mathbf{u}^2}{2}\right) + \nu\Delta\mathbf{u}. \tag{4.6}$$

We now take the curl of (4.6) and switch the order of the differential operators. We know that the curl of a gradient of a scalar function is zero, so

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) + \nu\Delta\boldsymbol{\omega}. \tag{4.7}$$

We call (4.7) the *vorticity equation*.

4.2 Non-dimensionalisation

In this section, we will discuss four dimensionless quantities which appear regularly in MHD. First, we have the *Reynolds number*, $Re = UL/\nu$, with U a typical velocity and L a characteristic length-scale. This number represents the ratio of viscous forces and inertia. It is obtained by bringing the Navier-Stokes equations in absence of other forces to a dimensionless form. Let us redo the procedure, but this time starting from (4.5). Let us further assume that $\boldsymbol{J}$ is primarily driven by $\mathbf{u}\times\boldsymbol{B}$, then Ohm's law gives us

$$J \sim \sigma B V.$$

Knowing this, we non-dimensionalise (4.5) as before,

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B} \\ \Leftrightarrow\ &\frac{U}{T}\frac{\partial \mathbf{u}^*}{\partial t^*} + \frac{U^2}{L}(\mathbf{u}^*\cdot\nabla^*)\,\mathbf{u}^* - \frac{\nu U}{L^2}\Delta^*\mathbf{u}^* + \frac{P}{\rho L}\nabla^* p^* = \frac{\sigma B^2 U}{\rho}\boldsymbol{J}^*\times\boldsymbol{B}^*. \end{aligned}$$

We multiply this with L/U^2 and drop the asterisks,

$$\frac{L}{UT}\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \frac{\nu}{UL}\Delta\mathbf{u} + \frac{P}{\rho U^2}\nabla p = \frac{\sigma B^2 L}{\rho U}\boldsymbol{J}\times\boldsymbol{B}$$

As before, $LU/T = 1$, $P/(\rho U^2) = 1$ and $Re = UL/\nu$. The dimensionless quantity in front of the Lorentz force is called the *interaction parameter* $N = \sigma B^2 L/(\rho U)$ and it represents the ratio of Lorentz force to inertia. The third dimensionless quantity we observe is the *Hartmann number* $Ha = (ReN)^{1/2}$. It is a hybrid of Reynolds number and interaction parameter. Here Ha^2 serves as a representation of the ratio of Lorentz force to viscous forces. The final dimensionless parameter, the magnetic Reynolds number $Re_m = UL/\lambda$, is not derived from the Navier-Stokes equations but from the induction equation (4.4). It represents the ratio between advection and diffusion of the magnetic field.

4.2.1 Summary

So far we have non-dimensionalised the induction equation (4.4) and the NSE for MHD (4.5). The equations in dimensionless form read as follows:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{B}) + \frac{1}{Re_m} \Delta \boldsymbol{B}$$

and

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \frac{1}{Re} \Delta \mathbf{u} + \nabla p = N\, \boldsymbol{J} \times \boldsymbol{B}.$$

We call $Re = VL/\nu$ the Reynolds number and $N = \sigma B^2 L/\rho V$ the interaction parameter, with V a characteristic velocity and L a characteristic length scale. These two parameters are linked by the Hartmann number $Ha = (N\,Re)^{1/2}$. And the fourth parameter is called the magnetic Reynolds number $Re_m = VL/\lambda = VL\mu\sigma$.

4.3 Diffusion and Convection

We have now established a coupling between electromagnetism and fluid dynamics. We will, at least for the rest of this chapter, only examine one half of the coupling. In particular, we take the velocity field $\mathbf{u}$ as prescribed and neglect the NSE entirely. In other words, we take a look at the interaction between $\mathbf{u}$ and $\boldsymbol{B}$, without taking the origin of $\mathbf{u}$ or the back interaction of the Laplace force into account.
Let us recall the induction equation (4.4), which we obtained from Maxwell's equations,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{B}) + \lambda \Delta \boldsymbol{B},$$

with $\lambda = (\sigma\mu)^{-1}$, and, obtained by taking the curl of the NSE, the vorticity equation (4.7),

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla \boldsymbol{\omega},$$

where $\nu = \mu_f/\rho$.
If we compare (4.4) and (4.7), we might expect a perfect analogon. This is not entirely correct, as $\mathbf{u}$ is obviously related to $\boldsymbol{\omega}$ in a different way than its related to $\boldsymbol{B}$. But as the governing equations consist of the same differential operators, we have analogous results for MHD as in classical vortex dynamics.

We now discuss the parts of the induction equation. First, taking into account that $\nabla \cdot \mathbf{u} = 0$, we use the vector identity $\nabla \times (\mathbf{u} \times \boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{B}$ to rewrite (4.4), which yields

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\mathbf{u} + \lambda \Delta \boldsymbol{B}. \tag{4.8}$$

The terms on the left hand side represent the total derivative (cf. Definition 3.1) of $\boldsymbol{B}$, denoted by $\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t}$. Here, the term $(\mathbf{u} \cdot \nabla)\boldsymbol{B}$ represents the change in the magnetic field caused by fluid particles entering or leaving an infinitesimal volume. It is only considered important if the velocity field $\mathbf{u}$ is parallel to the direction of the greatest change in $\boldsymbol{B}$.
The first term on the right hand side of (4.8) reflects *the field production by stretching of the field lines.* The term vanishes in simple cases, as planar flow with the magnetic field perpendicular to the flow, but reaches its maximum near stagnation points.
Finally, the second term on the right hand side of (4.8) stands for the *diffusion of the magnetic field*, it describes the transport of the magnetic field via diffusion, just like, for instance, heat.

4.3.1 Diffusion of the magnetic field

Now let $\mathbf{u} = 0$. Then the induction equation (4.4) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \lambda \Delta \boldsymbol{B}, \tag{4.9}$$

with $\lambda = (\sigma\mu)^{-1}$. This equation may look familiar, if we compare it to the diffusion equation for heat,

$$\frac{\partial T}{\partial t} = \alpha \Delta T, \tag{4.10}$$

with α the thermal conductivity and T the temperature. As both phenomena can be described by the same governing equation, the results of the theory of diffusion of heat can be used to describe the diffusion of the magnetic field. Just as with heat, we cannot suddenly force a distribution of $\boldsymbol{B}$ on a medium, instead we can only prescribe boundary conditions and wait for the field to diffuse inward.
Let us emphasise this with an example which can be verified in a laboratory experiment. Consider a very long cylinder $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq R^2, a \leq x_3 \leq b\}$ with $a, b, R \in \mathbb{R}^+$ and $|b - a| \gg R$. We will therefore use cylindrical coordinates (r, ϑ, z). We create the initial magnetic field by using a solenoid through which a constant electric current is flowing. Then we know that $\boldsymbol{B}$ has the form

$$\boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ B(r,t) \end{pmatrix}.$$

At time $t = 0$, we have an uniform magnetic field B_0 inside the cylinder and zero outside,

$$B(r, 0) = B_0 \quad \text{for } r \leq R.$$

Furthermore we have the boundary conditions

$$\left.\begin{aligned} B(R,t) &= 0 \\ \frac{\partial B}{\partial r}(0,t) &= 0 \end{aligned}\right\} \forall t.$$

Taking the special form of $\boldsymbol{B}$ into account and transforming (4.9) into cylindrical coordinates ($\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2}$), we obtain a scalar partial differential equation (PDE) for B,

$$\frac{\partial B}{\partial t} = \frac{\lambda}{r}\frac{\partial}{\partial r}(r\frac{\partial B}{\partial r}), \tag{4.11a}$$

with the initial condition

$$B(r,0) = B_0, \quad \text{for } r \in [0,R] \tag{4.11b}$$

and the boundary conditions

$$\left.\begin{aligned} B(R,t) =&0 \\ \frac{\partial B}{\partial r}(0,t) =&0 \end{aligned}\right\} \text{for } t \in [0,\infty). \tag{4.11c}$$

To solve this PDE, we use a special ansatz function for the solution B, namely a separation of variables. We assume B can be separated as

$$B(r,t) = f(t)\, g(r). \tag{4.12}$$

We apply the derivatives occurring in (4.11) to our ansatz and obtain

$$\begin{aligned} \frac{\partial B}{\partial t} &= \frac{\partial}{\partial t}(f(t)\, g(r)) = f'(t)\, g(r), \\ \frac{\partial B}{\partial r} &= \frac{\partial}{\partial r}(f(t)\, g(r)) = f(t)\, g'(r), \\ \frac{\partial}{\partial r}(r\,\frac{\partial B}{\partial r}) &= \frac{\partial}{\partial r}(r\, f(t)\, g'(r)) = f(t)(g'(r) + r\, g''(r)). \end{aligned} \tag{4.13}$$

If we require that $B(r,t) \neq 0$, then we know that also $f(t) \neq 0$ and $g(r) \neq 0$. We insert the derivatives (4.13) into (4.11) and obtain

$$f'(t)\, g(r) = \frac{\lambda}{r}\left(f(t)(g'(r) + r\, g''(r))\right).$$

Dividing through $\lambda\, f(t)\, g(r)$ yields

$$\frac{1}{\lambda}\frac{f'(t)}{f(t)} = \frac{1}{r}\frac{g'(r) + r\, g''(r)}{g(r)}.$$

This equation must hold for all $(r,t) \in [0,R] \times [0,\infty)$, so both sides have to be equal to a real constant, say $\gamma^2 \in \mathbb{R}$. Knowing this, we obtain two ordinary differential equations (ODEs)

$$f'(t) + \lambda\,\gamma^2\,f(t) = 0, \tag{4.14}$$

$$r\,g''(r) + g'(r) + r\,\gamma^2\,g(r) = 0, \tag{4.15}$$

where the constant γ^2 has to be real and positive, as otherwise the magnetic energy would not dissipate. The ODE (4.15) has a special form, it is called a *zero-index Bessel equation of first order*, whose solution is the *first order zero-index Bessel function* $g(r) = J_0(\gamma\,r)$. This function has a countable set of roots $\{\beta_n\}_{n\in\mathbb{N}}$. The function $g(r)$ furthermore has to match the boundary condition,

$$B(R,t) = 0 \Leftrightarrow J_0(\gamma\,R)f(t) = 0 \Leftrightarrow J_0(\gamma\,R) = 0, \tag{4.16}$$

from which we deduce that $\gamma\,R$ must be a value in the set of roots $\{\beta_n\}_{n\in\mathbb{N}}$. So there is a countable set of eigenvalues $\gamma_n = \beta_n/R$ satisfying (4.16) with β_n the roots of the first order zero-index Bessel function J_0. For each of these γ_n, we can solve the other ODE (4.14) directly, obtaining

$$f_n(t) = a_n\,\exp(-\gamma_n^2\,\lambda\,t) = a_n\,\exp(-\beta_n^2\,\lambda\,t/R^2),$$

as a solution, with $a_n = \text{const} \in \mathbb{R}$ for all $n \in \mathbb{N}$. As each $f_n(t)$ is a solution of (4.14), the final solution is a series

$$B(r,t) = \sum_{n=0}^{\infty} A_n\,\exp(-\frac{\beta_n^2\,\lambda\,t}{R^2})J_0(\beta_n\frac{r}{R}). \tag{4.17}$$

The coefficients A_n can be determined by using the initial condition (4.11b), which now reads

$$\sum_{n=0}^{\infty} A_n\,J_0(\beta_n\frac{r}{R}) = B_0, \quad \text{for } r \in [0,R],$$

and the orthogonality of Bessel functions. Let us examine the solution in more detail. The roots β_n of the first order zero index Bessel function are

$$\beta_0 \approx 2.40, \qquad \beta_1 \approx 5.52, \qquad \beta_2 \approx 8.65, \qquad \beta_3 \approx 11.79, ...$$

so we see that with increasing n, the β_n^2 are growing rapidly. We can use this fact if we want to determine the characteristic time for the extinction of an initial magnetic field. In this case, it is sufficient to use only the 0-th term of the series, as the exponential function decreases very fast for growing β_n. This simplified solution is now reduced to a single term,

$$B(r,t) = A_0\exp(-\frac{\beta_0^2\,\lambda\,t}{R^2})J_0(\beta_0\frac{r}{R}). \tag{4.18}$$

We obtain the characteristic time in the same way as for the heat equation, which is as follows:

$$\tau = \frac{R^2}{\lambda\,\beta_0^2} \approx \frac{\mu\,\sigma\,R^2}{5.78}. \tag{4.19}$$

The idea behind this formula, at least for its heat counterpart, is that τ represents the time, which is needed for the heat to travel a distance R.
We can apply this formula to some real life examples. If we consider a cylinder of magnetic iron, i.e., $\lambda = (\mu\,\sigma)^{-1} \sim 10^3[S/m^2]$, with a radius of $R = 10[cm]$, we obtain $\tau \approx \frac{10^3(10^{-1})^2}{5.78} \approx 1.7[s]$ for the extinction time. We can also use this formula, leaving aside the fact of different geometries, with magnitudes which apply for Earth. Here we have $\lambda \sim 10^{-1}[S/m^2]$ and $R = 6 * 10^6[m]$, so the extinction time is $\tau \approx 2 * 10^4[y]$. This would mean that the Earth's magnetic field would have become extinct by now, as the evolution of Earth's magnetic field can be traced back about $10^9[y]$. As the magnetic field is still active today, there must be some kind of effect which generates the magnetic field. This leads us to the so called *Geodynamo theory*, which is beyond the scope of this thesis (see [1, p.166-199]).

4.3.2 Convection of the magnetic field

Let us assume now the case where λ is negligibly small, i.e., $\lambda \approx 0$. This means we have no diffusion. The induction equation now reads

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{B}). \tag{4.20}$$

This equation is identical to the vorticity equation for inviscid fluids. Therefore two important results of vortex theory, namely Helmholtz's first law and Kelvin's theorem, have their analoga in MHD. These two theorems are gathered into a single theorem, which is called *Alfvén's Theorem*.

Theorem 4.1 (Alfvén's Theorem)**.**

1. *The fluid elements that lie on a magnetic field line at some initial instant continue to lie on that field line for all time, i.e., the field lines are frozen into the fluid.*
2. *The magnetic flux linking any loop moving with the fluid is constant, i.e.,*

$$\frac{d}{dt}\int_{S(t)} \boldsymbol{B} \cdot \vec{n}\, \mathrm{d}\boldsymbol{S} = 0, \tag{4.21}$$

with $S(t)$ a surface bounded by a closed curve $C(t)$.

Proof. See [1]. □

The assumption $\lambda = 0$ requires fairly large magnetic Reynolds numbers. This is typically the case in astrophysics, where, due to the enormous length scales, the magnetic Reynolds number can exceed values of $\sim 10^8$. A particular example for the frozen-in behaviour of magnetic field lines in astrophysics is the phenomenon of sunspots.
To get a glimpse at the mechanics behind sunspots, we first have to look at the inner structure of the sun. We start with the surface of the sun. We know that it is not

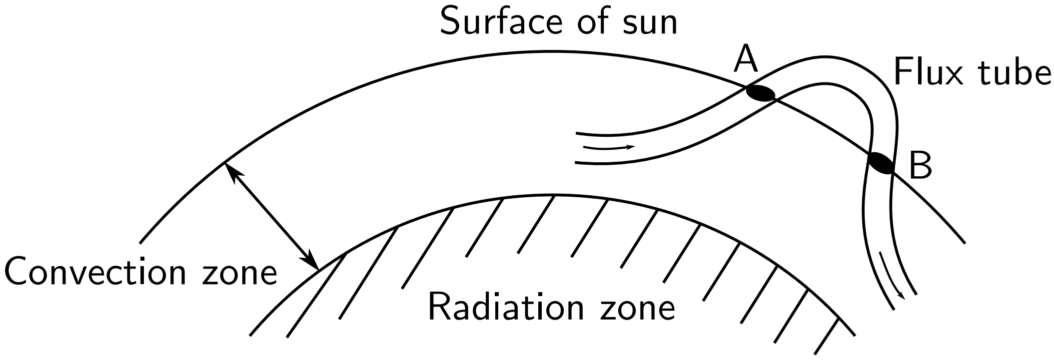

Figure 4.1: Schematic representation of the formation of sunspots

uniformly bright, but has, as a result to the convective turbulence occurring in the outer layer of the sun, a granular structure. The convective layer, as the outer layer is called, has an average thickness of $2 * 10^5[km]$. It consists of convection cells, which form a pattern that is gradually evolving. The granular pattern of the sun's surface now arises as hot, bright cells rise to the surface and colder, darker cells sink back into the interior. The typical velocity for this convection is around $1[km/s]$, from which we deduce estimates for the (magnetic) Reynolds numbers, i.e., $Re \sim 10^{11}$ and $Re_m \sim 10^8$, which is very large. The sun itself has an average magnetic field of a few *Gauss* $[Gs]$, which is similar to the earth's one (note that $1[Gs] = 10^{-4}[T] = 10^{-4}[Vs/m^2]$). As the magnetic Reynolds number is high, this magnetic field is most likely to be frozen into the fluid. Due to differential rotation, the magnetic field is stretched and intensified, until the field strength rises in horizontal flux tubes. As the Lorentz force points radially outward for these tubes, the pressure and density inside these tubes is less than the surroundings, which results in a buoyancy force. For very thick tubes, this force is strong enough to partially destabilise the convection, so these parts tend to drift towards the surface. From time to time, flux tubes of diameter $\sim 10^4[km]$ emerge through the surface into the sun's atmosphere. Now sunspots are the areas where the flux tube leaves and re-enters the surface (A and B in Figure 4.1). As the magnetic field in the flux tubes is of enormous field strength ($\sim 3000[Gs]$), it cools the surface in these areas through suppression of fluid motion and convection.

Chapter 5

Applications of MHD at low magnetic Reynolds numbers

In this chapter, we will discuss some applications of MHD under the restriction of low magnetic Reynolds numbers, i.e., $Re_m \ll 1$. In other words, the magnetic field $\boldsymbol{B}$ influences the velocity field $\mathbf{u}$ but there is no significant influence of $\mathbf{u}$ on $\boldsymbol{B}$. Such is the case in liquid metal MHD, which are common in industrial processes. The contents of this chapter are based on [1].
Before we consider such applications, we have to take a look at the governing equations for low Re_m.

5.1 Simplifications for low Re_m

The core issue of the low Re_m approximation is that, compared to the imposed magnetic field, the magnetic field generated by induced currents is of little significance.
Now let the magnetic field be steady, i.e., independent of time. Let $\boldsymbol{E}_0$, $\boldsymbol{J}_0$ and $\boldsymbol{B}_0$ be the fields if $\mathbf{u} = 0$ and let $\mathbf{e}$, $\mathbf{j}$ and $\mathbf{b}$ be the infinitesimal perturbation in $\boldsymbol{E}$, $\boldsymbol{J}$ and $\boldsymbol{B}$ which are generated by a vanishingly small velocity field $\mathbf{u}$. These lead to the following equations:

$$\nabla \times \boldsymbol{E}_0 = -\frac{\partial \boldsymbol{B}_0}{\partial t} = 0, \qquad \boldsymbol{J}_0 = \sigma \boldsymbol{E}_0, \tag{5.1}$$

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t}, \qquad \mathbf{j} = \sigma(\mathbf{e} + \mathbf{u} \times \boldsymbol{B}_0). \tag{5.2}$$

Note that we have skipped the contribution of $\mathbf{u} \times \mathbf{b}$ in (5.2). From Faraday's law (5.2) we can deduce that $\mathbf{e} \sim u\mathbf{b}$, as we can easily confirm:

$$\nabla \times \mathbf{e} = -\frac{\partial \mathbf{b}}{\partial t} \quad \Leftrightarrow \quad \frac{\mathbf{e}}{l} \sim \frac{\mathbf{b}}{t} \quad \Leftrightarrow \quad \mathbf{e} \sim \frac{l}{t}\mathbf{b}.$$

Again, we choose the characteristic length and time scale so that $l/t = u$ with u the characteristic velocity. Hence, we can neglect the contribution of $\mathbf{e}$ in (5.2) and Ohm's law reduces to

$$\boldsymbol{J} = \boldsymbol{J}_0 + \mathbf{j} = \sigma(\boldsymbol{E}_0 + \mathbf{u} \times \boldsymbol{B}_0).$$

We know from (5.1) that $\boldsymbol{E}_0$ is irrotational, so we can write it as $\boldsymbol{E}_0 = -\nabla V$ with V an electrostatic potential. The final form of Ohm's law is therefore

$$\boldsymbol{J} = \sigma\left(-\nabla V + \mathbf{u} \times \boldsymbol{B}_0\right), \tag{5.3}$$

and for the Laplace force we obtain

$$\mathbf{F} = \boldsymbol{J} \times \boldsymbol{B}_0. \tag{5.4}$$

These two equations fully define the Laplace force. Also, $\boldsymbol{J}$ is uniquely determined by (5.3) as both the curl and the divergence are known [1].

5.2 MHD generators and pumps

When we want to understand the concept behind MHD generators (and pumps), we have to look at the influence of the magnetic field at the boundary layers. In fact, we will discuss the so called *Hartmann boundary layer*. One of the key differences to conventional boundary layers is the influence of a steady magnetic field which is perpendicular to the boundary.

5.2.1 Theory of Hartmann layers

Let us consider a rectilinear shear flow adjacent to a plane and stationary surface, i.e.,

$$\mathbf{u} = \begin{pmatrix} u(x_2) \\ 0 \\ 0 \end{pmatrix}. \tag{5.5}$$

Note that $\nabla \cdot \mathbf{u} = 0$ is satisfied. Far from the wall we assume the flow to be uniform and equal to $u_\infty > 0$, but due to the no-slip boundary condition, i.e. $u = 0$ at the

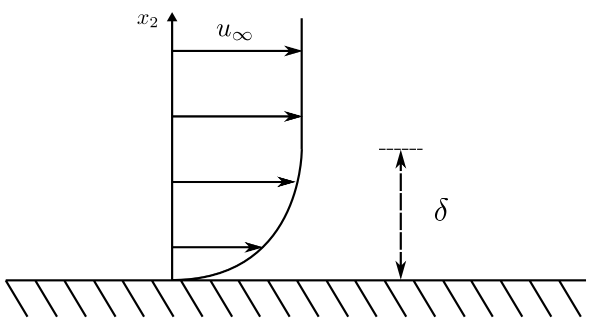

Figure 5.1: Velocity profile and the Hartmann layer

boundary, there will be some sort of boundary layer close to said surface. Furthermore, we have a uniform imposed magnetic field,

$$\boldsymbol{B}_0 = \boldsymbol{B} = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}. \tag{5.6}$$

Moreover, we assume that there is no imposed electric field, i.e., $\boldsymbol{E}_0 = \nabla V = 0$. From (5.5) we deduce that $\boldsymbol{\omega} = (0, 0, u'(x_2))^T$, hence

$$\boldsymbol{B} \cdot \boldsymbol{\omega} = 0.$$

Now we take the divergence of (5.3) and obtain with some reordering (noting that $\nabla \cdot \boldsymbol{J} = 0$)

$$\nabla \cdot (\nabla V) = \nabla \cdot (\mathbf{u} \times \boldsymbol{B}_0) = \boldsymbol{B} \cdot (\nabla \times \mathbf{u}) = \boldsymbol{B} \cdot \boldsymbol{\omega} = 0.$$

This means that $\Delta V = 0$. Furthermore, we deduce from (5.3) that

$$\boldsymbol{J} = \sigma(\mathbf{u} \times \boldsymbol{B}) = \sigma(\begin{pmatrix} u(x_2) \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix}) = \sigma \begin{pmatrix} 0 \\ 0 \\ u(x_2)B \end{pmatrix},$$

and with (5.4) we obtain

$$\mathbf{F} = \boldsymbol{J} \times \boldsymbol{B} = \begin{pmatrix} 0 \\ 0 \\ \sigma\, u(x_2)B \end{pmatrix} \times \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} = \begin{pmatrix} -\sigma\, u(x_2)B^2 \\ 0 \\ 0 \end{pmatrix}.$$

Let us recall the NSE (4.5),

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p = \frac{1}{\rho} \boldsymbol{J} \times \boldsymbol{B},$$

which reduces to

$$-\nu\rho u''(x_2) + \frac{\partial p}{\partial x_1} = -\sigma\, u(x_2)B^2, \tag{5.7}$$

$$\frac{\partial p}{\partial x_2} = 0, \tag{5.8}$$

$$\frac{\partial p}{\partial x_3} = 0. \tag{5.9}$$

We can transform (5.7) into

$$\frac{\partial^2}{\partial x_2^2}(u(x_2) - u_\infty) - \frac{u(x_2) - u_\infty}{\delta^2} = 0, \quad \text{with } \delta = \left(\frac{\rho\nu}{\sigma B^2}\right)^{\frac{1}{2}}.$$

To obtain this result, we simply insert the special velocity $u = u_\infty/(\rho\nu)$ into (5.7) and solve for the pressure derivative, which is constant with regard to x_2. This we deduce from (5.8). Now we set $w = u - u_\infty$ and solve the resulting 2nd order ODE. We obtain the solution

$$w(x_2) = c_1\, e^{\frac{x_2}{\delta}} + c_2\, e^{-\frac{x_2}{\delta}} \quad \leftrightarrow \quad u(x_2) = u_\infty + c_1\, e^{\frac{x_2}{\delta}} + c_2\, e^{-\frac{x_2}{\delta}}.$$

The velocity u additionally has to match the boundary conditions

$$u(0) = 0$$
$$\lim_{x_2 \to \infty} u(x_2) = u_\infty$$

from which we deduce the final solution

$$u(x_2) = u_\infty(1 - e^{-\frac{x_2}{\delta}}). \tag{5.10}$$

We see that the velocity will increase exponentially over a short distance from the boundary (see Figure 5.1). The boundary layer formed by this is called the *Hartmann layer* and its characteristic thickness $\sim \delta$ is usually different to the thickness of a conventional boundary layer.
Now we extend our experiment a bit. Let us consider the same flow as above, but this time between two resting, parallel plates at $x_2 = \pm h$. Additionally we allow an imposed electric field in x_3-direction, i.e., $\boldsymbol{E}_0 = (0, 0, E_0)^T$. Hence, $\boldsymbol{J} = (0, 0, \sigma(E_0 + u(x_2)B))^T$ and for the Laplace force we obtain

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ \sigma E_0 + \sigma\, u(x_2)B \end{pmatrix} \times \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} = \sigma \begin{pmatrix} -E_0 B - u(x_2)B^2 \\ 0 \\ 0 \end{pmatrix}.$$

As before, the NSE reduces to

$$\nu\rho u''(x_2) - \sigma\, u(x_2)B^2 = \frac{\partial p}{\partial x_1} + \sigma E_0 B.$$

It can be easily shown that the solution to this equation is

$$u(x_2) = u_0 \left[1 - \frac{\cosh(x_2/\delta)}{\cosh(h/\delta)}\right], \tag{5.11}$$

where the constant u_0 is given by

$$\sigma B^2 u_0 = -\frac{\partial p}{\partial x_1} - \sigma E_0 B. \tag{5.12}$$

Now recall the *Hartmann number* Ha, introduced in Section 4.2, but this time with h our characteristic length,

$$Ha = Bh(\sigma/\rho\nu)^{\frac{1}{2}} = \frac{h}{\delta},$$

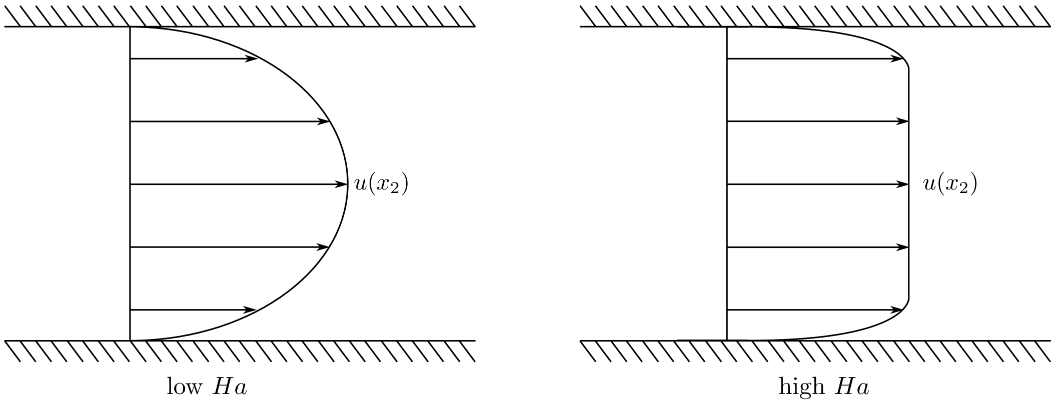

Figure 5.2: Velocity profile at low and high Hartmann numbers

which explains where this dimensionless parameter comes from. With this notation, we obtain the final solution

$$u(x_2) = u_0 \left[1 - \frac{\cosh(Ha(x_2/h))}{\cosh(Ha)} \right], \tag{5.13}$$

with u_0 as in (5.11).
Let us take a look at the limits of the Hartmann number. For $Ha \to 0$, the solution takes the form

$$u(x_2) = u_0 \left(1 - \left(\frac{x_2}{h} \right) \right), \tag{5.14}$$

which is a parabolic velocity profile. For the other limit, i.e., $Ha \to \infty$, we observe the formation of Hartmann layers at the boundaries with an exponential velocity profile in said boundary layers and constant flow in between (see Figure 5.2).

5.2.2 Applications of Hartmann layers

Real life applications of Hartmann layers now consist of, among others, MHD generators and pumps. It also requires a large Hartmann number Ha, so we have the relations

$$\begin{aligned} u &\approx u_0, \\ J &\approx \sigma(E_0 + u_0 B), \\ u_0 B &= -E_0 - \frac{1}{\sigma B}\frac{\partial p}{\partial x_1}. \end{aligned} \tag{5.15}$$

We are now free to choose E_0, the external electric field. Depending on our choice we obtain three different cases. First we consider the case that E_0 is zero or $\ll 1$. Then we obtain $J \approx \sigma u_0 B$ and $\frac{\partial p}{\partial x_1} \approx -\sigma B^2 u_0$. This means we induce a current, but at the same time we also observe a pressure drop. In other words, we convert mechanical energy into electrical energy plus heat. Such a device is commonly known as *generator*. For the second case, we have no current, i.e., we choose $\boldsymbol{J} = 0$. Hence we know that

$E_0 = -u_0 B$ and moreover, that the Laplace force is zero and the pressure gradient vanishes. The pressure therefore remains constant and we can measure E_0 to obtain u_0. This device is called a *MHD flow meter*.
Eventually we consider the case of negative E_0, but of higher magnitude than $-u_0 B$. Then the direction of $\boldsymbol{J}$ and, of course, $\boldsymbol{J} \times \boldsymbol{B}$ is reversed. We deduce that the pressure gradient $\frac{\partial p}{\partial x_1}$ is positive, which means that electrical energy is converted into mechanical energy and heat. Such devices are called *pumps*. These MHD pumps are commonly used in the metallurgical and the nuclear industry, as they contain no moving parts and are therefore, at least theoretically, very resistant against abrasion. For instance, in *fast breeder nuclear reactors*, the coolant is liquid sodium, which is moved via the MHD pump principle.

Chapter 6

Conclusion

We have started with the electromagnetic part of MHD, namely Maxwell's equations. There we first derived the full equations and constitutive equations and then, with the help of some model assumptions, reduced the number of unknowns and equations. By this, we obtained the induction equation, or, in other words, a transport equation for the magnetic field $\boldsymbol{B}$. We continued with fluid dynamics, where we derived the continuity equation and the equations of motion. These equations together with the model assumptions of *Newtonian fluids* and incompressibility resulted in the Navier-Stokes equation for incompressible fluids (NSE). The coupling between the two equations happens because of the presence of a velocity field $\mathbf{u}$ in the induction equation and the presence of the Lorentz (or Laplace) force in the NSE. From non-dimensionalising these equations we received four dimensionless parameters, which measure the relative strength of different physical phenomena, e.g., the ratio between viscous forces and inertia. We continued with analysing the parts of the induction equation. We concluded the thesis with a particular application of MHD, namely MHD pumps and generators and the theory behind, the Hartmann layer.
We have derived, under some assumptions, a model for MHD. The next possible steps could be an analysis of the existence and uniqueness of a solution and the derivation of a weak formulation with appropriate spaces. And from this weak formulation, we could survey a Finite Element Method (FEM) for MHD. And eventually, as the governing equations are both non-linear (convective term in the NSE and curl in the induction equation) and time-dependent, we could survey efficient solvers for the discretised problem.

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