JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

A Robust Preconditioned MinRes-Solver for Time-Periodic Eddy Current Problems

Michael Kolmbauer
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

Ulrich Langer
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

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A ROBUST PRECONDITIONED MINRES-SOLVER FOR TIME-PERIODIC EDDY CURRENT PROBLEMS

MICHAEL KOLMBAUER AND ULRICH LANGER

Abstract. This work is devoted to fast and parameter-robust iterative solvers for frequency domain finite element equations, approximating the eddy current problem with harmonic or multiharmonic excitations in time. We construct a preconditioned MinRes solver for the frequency domain equations, that is robust with respect to the discretization parameters as well as all involved "bad" parameters like the conductivity, the reluctivity and possible regularization parameters.

1. Introduction

The multiharmonic finite element method or harmonic-balanced finite element method has been used by many authors in different applications (e.g. [3, 16, 20, 45, 52]). Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation.

Following this strategy, Copeland et al. [10, 11], Bachinger et al. [4, 5], and Kolmbauer and Langer [30] applied harmonic and multiharmonic approaches to scalar parabolic and eddy current problems. Indeed, in [30] a preconditioned MinRes solver for the solution of time-harmonic eddy current problems is constructed, that is robust with respect to both, the discretization parameter h and all involved parameters like frequency, conductivity and reluctivity. This block-diagonal preconditioning technique has already been proposed in [10] for a simple scalar parabolic problem, and have been used for eddy current problem in [30]. There a rescaling of the unknowns was necessary. Now this rescaling is not necessary anymore. We mention that block-diagonal preconditioners similar to that in [30] have also been used and analysed in [5, 9, 12] and in terms of operator preconditioning in [23, 31, 38]. In this work we extend the results obtained in [30] to various primal and mixed formulations of the eddy current problem. We analyse various types of regularization techniques and even extend the theory to the case of vanishing conductivity.

One technique of construction and analysis of parameter-robust preconditioners for saddle point problems was introduced by Schöberl and Zulehner in [48], and then generalized to a more constructive framework by Zulehner in [53]. We mention that there is also a recent work on this topic by Mardal and Winther [38].

This approach of constructive block-diagonal preconditioning allows us to break down the inversion of fully coupled block-matrices to the inversion of several simple

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problems, that can be replaced by parameter-robust and (almost) optimal preconditioners for standard $\mathbf{H}(\mathbf{curl})$ or H^1 problems like

$$(\alpha\,\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + (\beta\mathbf{u}, \mathbf{v})_0 \quad \text{and} \quad (\alpha\nabla p, \nabla q)_0 + (\beta p, q)_0,$$

respectively. Here $(\cdot,\cdot)_0$ denotes the usual inner product in L_2. The coefficient α and β are positive, piecewise constant functions, that may have large jumps. These standard problems can be handled by well-known preconditioners like multigrid preconditioners [1, 21], auxiliary space preconditioners [24, 51], and domain decomposition (DD) preconditioners [25, 49, 50] in the $\mathbf{H}(\mathbf{curl})$ setting and multigrid or multilevel preconditioners [7, 17, 32, 33, 34, 43], and domain decomposition preconditioners [50] in the H^1 setting, respectively.

As a model problem we consider an eddy current problem with homogeneous Dirichlet boundary conditions and periodicity conditions in time: For given $\mathbf{f}$, find $\mathbf{u}$, such that

$$\left\{ \begin{aligned} \sigma\frac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{u}) &= \mathbf{f} && \text{in } \Omega\times(0,T], \\ \operatorname{div}(\sigma\mathbf{u}) &= 0 && \text{in } \Omega\times(0,T], \\ \mathbf{u}\times\mathbf{n} &= \mathbf{0} && \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(0) &= \mathbf{u}(T) && \text{in } \overline{\Omega}, \end{aligned} \right. \tag{1}$$

where $\Omega\subset\mathbb{R}^3$ is a simply connected, bounded Lipschitz domain. The reluctivity $\nu=\nu(\mathbf{x})$ is supposed to be independent of $|\,\mathbf{curl}\,\mathbf{u}|$, i.e. we assume that the eddy current problem (1) is linear. The conductivity σ is piecewise constant. Additionally, we assume, that $\sigma\in L^\infty(\Omega)$, and $\sigma\geq\underline{\sigma}>0$, i.e. the conductivity is strictly positive in the whole computational domain. However, in many practical applications, the computational domains consist of conducting ($\sigma>0$) and non-conducting ($\sigma=0$) regions. At the end of this work, we also treat this general case in § 6.1. Throughout this paper, we assume that the given right-hand side $\mathbf{f}$ is weakly divergence-free, i.e.,

$$(\mathbf{f}, \nabla p)_0 =, 0 \quad \forall p\in H_0^1(\Omega). \tag{2}$$

Moreover, for simplicity, we only consider time-harmonic and multiharmonic, weakly divergence-free excitations $\mathbf{f}$. The multiharmonic excitations can also be seen as an approximation of a more general excitations in time. Since we are interested in the construction and analysis of fast and robust solvers, it is obviously enough to consider the multiharmonic case.

The outline of this work is the following. In Section 2, we firstly consider our model problem (1) with a time-harmonic excitation $\mathbf{f}$ on the right-hand side. For the system of frequency domain equations, appropriate primal and mixed variational formulations are derived, wherein the gauging condition is incorporated implicitly or explicitly. The section closes with the discretization in space in terms of a finite element discretization. Section 3 is devoted to the robust iterative solution of the resulting linear systems of equations by means of a preconditioned MinRes solver. For all the different formulations, we construct parameter-robust block-diagonal preconditioners and provide quantitative estimates of the condition numbers yielding the corresponding convergence rate estimates for the preconditioned MinRes solver. In Section 4, we generalize our preconditioners to the more general time-periodic eddy current problem (1) with a multiharmonic excitation $\mathbf{f}$ in time. In Section 5, we present numerical results confirming the rate estimates given in Sections 3 and 4. Finally, in Section 6, we discuss the application of the block-diagonal preconditioning method to the case of inexact regularization techniques and the case of non-symmetric formulations which are important for solving non-linear eddy current problems.

2. The time-harmonic case

We assume that $\mathbf{f}$ is given by some time-harmonic excitation with frequency $\omega > 0$ and the amplitudes $\mathbf{f^c}$ and $\mathbf{f^s}$, i.e. $\mathbf{f}(\mathbf{x},t) = \mathbf{f^c}(\mathbf{x})\cos(\omega t) + \mathbf{f^s}(\mathbf{x})\sin(\omega t)$. Therefore, the solution $\mathbf{u}$ is time-harmonic as well, with the same base frequency ω and amplitudes $\mathbf{u^c}(\mathbf{x})$ and $\mathbf{u^s}(\mathbf{x})$:

$$\mathbf{u}(\mathbf{x},t) = \mathbf{u^c}(\mathbf{x})\cos(\omega t) + \mathbf{u^s}(\mathbf{x})\sin(\omega t). \tag{3}$$

In fact, (3) is the real reformulation of a complex time-harmonic approach, where the source $\mathbf{f}$ is given by the real part of a complex time-harmonic excitation, i.e. $\mathbf{f}(\mathbf{x},t) = \mathrm{Re}[\hat{\mathbf{f}}(\mathbf{x})e^{i\omega t}]$, and consequently the solution has the form $\mathbf{u}(\mathbf{x},t) = \hat{\mathbf{u}}(\mathbf{x})e^{i\omega t}$, with the complex-valued amplitude $\hat{\mathbf{u}} = \mathbf{u^c} - i\mathbf{u^s}$ (see e.g. [4, 45]). Using the real-valued time-harmonic representation of the solution (3), we can rewrite the eddy current problem (1) in the frequency domain as follows:

$$\text{Find } (\mathbf{u^c},\mathbf{u^s}) \text{:} \quad \begin{cases} \omega\sigma\mathbf{u^s} + \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{u^c}) = \mathbf{f^c}, & \text{in } \Omega,\\ \omega\sigma\mathbf{u^c} - \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{u^s}) = -\mathbf{f^s}, & \text{in } \Omega,\\ \omega\,\mathrm{div}(\sigma\mathbf{u^c}) = 0, & \text{in } \Omega,\\ \omega\,\mathrm{div}(\sigma\mathbf{u^s}) = 0, & \text{in } \Omega,\\ \mathbf{u^c}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega,\\ \mathbf{u^s}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where we have multiplied the gauging equations $\mathrm{div}(\sigma\mathbf{u^j}) = 0$, $j \in \{c,s\}$, by the base frequency ω. This system of partial differential equations is the starting point for the derivation of a variational formulation and the discretization in space by means of the finite element method.

2.1. **Variational formulations.** We investigate both, primal and mixed variational formulations, wherein the Coulomb gauging condition is incorporated implicitly or explicitly, respectively. All the presented formulations are equivalent in the sense, that they have the same unique solution for the amplitudes $\mathbf{u^c}$ and $\mathbf{u^s}$. In the variational framework we work on the well-known Hilbert spaces $\mathbf{H}(\mathbf{curl})$ and $H^1(\Omega)$, as well as their subspaces with vanishing Dirichlet traces, i.e.,

$$\begin{aligned} \mathbf{H_0}(\mathbf{curl}) &:= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}) : \mathbf{u}\times\mathbf{n} = \mathbf{0} \text{ on } \partial\Omega\},\\ H^1_0(\Omega) &:= \{p \in H^1(\Omega) : p = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Formulation 1: If we assume, that the source $\mathbf{f}$ is weakly divergence free (see (2)), we observe that the gauging condition $\mathrm{div}(\sigma\mathbf{u^j}) = 0$ ($j \in \{c,s\}$) in (4) is fulfilled naturally. Therefore, it is not necessary to incorporated the gauging condition into the system (4) explicitly. Therefore, the corresponding variational problem reads as: Find $(\mathbf{u^c},\mathbf{u^s}) \in \mathbf{H_0}(\mathbf{curl})^2$, such that

$$\mathcal{A}_1\left((\mathbf{u^c},\mathbf{u^s}),(\mathbf{v^c},\mathbf{v^s})\right) = \int_\Omega [\mathbf{f^c}\cdot\mathbf{v^c} - \mathbf{f^s}\cdot\mathbf{v^s}]\,\mathrm{d}\mathbf{x}, \tag{5}$$

for all test functions $(\mathbf{v^c},\mathbf{v^s}) \in \mathbf{H_0}(\mathbf{curl})^2$. Here the symmetric and indefinite bilinear form $\mathcal{A}$ is given by

$$\begin{aligned} \mathcal{A}_1\left((\mathbf{u^c},\mathbf{u^s}),(\mathbf{v^c},\mathbf{v^s})\right) :&= (\nu\,\mathbf{curl}\,\mathbf{u^c},\mathbf{curl}\,\mathbf{v^c})_0 + \omega(\sigma\mathbf{u^s},\mathbf{v^c})_0\\ &\quad - (\nu\,\mathbf{curl}\,\mathbf{u^s},\mathbf{curl}\,\mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c},\mathbf{v^s})_0. \end{aligned} \tag{6}$$

This variational problem has a unique solution (cf. Lemma 2), and, for a weakly divergence-free right-hand side $\mathbf{f}$, the solution fulfills the gauge condition $\mathrm{div}(\sigma\mathbf{u^j}) = 0$ for $j \in \{c,s\}$.

Formulation 2: In many cases it is very convenient, to incorporate the gauging conditions $\operatorname{div}(\sigma\mathbf{u^j}) = 0$ ($j \in \{c, s\}$) in a mixed variational framework. The corresponding mixed variational problem reads as: Find $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$, such that

$$\mathcal{B}_1\left((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\right) = \int_\Omega [\mathbf{f^c} \cdot \mathbf{v^c} - \mathbf{f^s} \cdot \mathbf{v^s}]\, d\mathbf{x}, \tag{7}$$

for all test functions $(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$. Here the symmetric and indefinite bilinear form $\mathcal{B}_1$ is given by

$$\begin{aligned} &\mathcal{B}_1\left((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\right) := \\ &\quad (\nu\, \mathbf{curl}\, \mathbf{u^c}, \mathbf{curl}\, \mathbf{v^c})_0 + \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_0 + \omega(\sigma\mathbf{v^c}, \nabla p^c)_0 - \omega(\sigma\mathbf{v^s}, \nabla p^s)_0 \\ &\quad -(\nu\, \mathbf{curl}\, \mathbf{u^s}, \mathbf{curl}\, \mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c}, \mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c}, \nabla q^c)_0 - \omega(\sigma\mathbf{u^s}, \nabla q^s)_0. \end{aligned} \tag{8}$$

There exists a unique solution $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)$ (cf. Lemma 3). Additionally, if we assume the right-hand side to be weakly divergence free, the Lagrange parameters p^c and p^s vanish at the solution, i.e. $p^c = 0$ and $p^s = 0$.
Formulation 3: Since at the solution of (7), the Lagrange parameters vanish, i.e. $p^c = 0$ and $p^s = 0$, we can add a suitable bilinear form to $\mathcal{B}_1$ in *Formulation 2*. The resulting variational problem reads as: Find $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$, such that

$$\mathcal{B}_2\left((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\right) = \int_\Omega [\mathbf{f^c} \cdot \mathbf{v^c} - \mathbf{f^s} \cdot \mathbf{v^s}]\, d\mathbf{x}, \tag{9}$$

for all test functions $(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$. Here the symmetric and indefinite bilinear form $\mathcal{B}_2$ is given by

$$\begin{aligned} &\mathcal{B}_2((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)) := \\ &\quad (\nu\, \mathbf{curl}\, \mathbf{u^c}, \mathbf{curl}\, \mathbf{v^c})_0 + \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_0 + \omega(\sigma\mathbf{v^c}, \nabla p^c)_0 - \omega(\sigma\mathbf{v^s}, \nabla p^s)_0 - \omega(\sigma\nabla p^c, \nabla q^c)_0 \\ &\quad -(\nu\, \mathbf{curl}\, \mathbf{u^s}, \mathbf{curl}\, \mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c}, \mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c}, \nabla q^c)_0 - \omega(\sigma\mathbf{u^s}, \nabla q^s)_0 + \omega(\sigma\nabla p^s, \nabla q^s)_0. \end{aligned} \tag{10}$$

Again, the mixed variational problem has a unique solution (cf. Lemma 4) . Additionally, the solution of *Formulation 2* also solves *Formulation 3*, and vice versa. Therefore these two formulations are equivalent.
Formulation 4: Finally, we give a primal version of *Formulation 3*. The variational problem reads: Find $(\mathbf{u^c}, \mathbf{u^s}) \in \mathbf{H_0}(\mathbf{curl})^2$, such that

$$\mathcal{A}_2\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right) = \int_\Omega [\mathbf{f^c} \cdot \mathbf{v^c} - \mathbf{f^s} \cdot \mathbf{v^s}]\, d\mathbf{x}, \tag{11}$$

for all test functions $(\mathbf{v^c}, \mathbf{v^s}) \in \mathbf{H_0}(\mathbf{curl})^2$. Here the symmetric and indefinite bilinear form $\mathcal{A}_2$ is given by

$$\begin{aligned} \mathcal{A}_2\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right) :&= (\nu\, \mathbf{curl}\, \mathbf{u^c}, \mathbf{curl}\, \mathbf{v^c})_0 + \omega(\sigma\nabla\mathbf{P}(\mathbf{u^c}), \nabla\mathbf{P}(\mathbf{v^c}))_0 + \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_0 \\ &\quad - ((\nu\, \mathbf{curl}\, \mathbf{u^s}, \mathbf{curl}\, \mathbf{v^s})_0 - \omega(\sigma\nabla\mathbf{P}(\mathbf{u^s}), \nabla\mathbf{P}(\mathbf{v^s}))_0) + \omega(\sigma\mathbf{u^c}, \mathbf{v^s})_0. \end{aligned} \tag{12}$$

Therein we use the weighted Helmholtz projection $\mathbf{P}$, where for given $\mathbf{u} \in \mathbf{H_0}(\mathbf{curl})$, $p := \mathbf{P}(\mathbf{u}) \in H_0^1(\Omega)$ is the unique solution of the variational form:

$$\omega \int_\Omega \sigma\nabla p \cdot \nabla q\, d\mathbf{x} = \omega \int_\Omega \sigma\mathbf{u} \cdot \nabla q\, d\mathbf{x}, \quad \forall q \in H_0^1(\Omega). \tag{13}$$

The solution $\mathbf{P}(\mathbf{u})$ fulfills the estimate $\|\sigma^{1/2}\nabla\mathbf{P}(\mathbf{u})\|_{\mathbf{L_2}(\Omega)} \le \|\sigma^{1/2}\mathbf{u}\|_{\mathbf{L_2}(\Omega)}$. The additional expression is chosen in such a way, that it does not vanish on the kernel of the **curl** operator, and on the other hand $\mathbf{P}(\mathbf{u^c})$ and $\mathbf{P}(\mathbf{u^s})$ vanish at the solution, i.e. $\mathbf{P}(\mathbf{u^c}) = 0$ and $\mathbf{P}(\mathbf{u^s}) = 0$ (see [30, 35]). It can be shown, that *Formulation 4* is nothing else, than an equivalent primal formulation of *Formulation 3*. Indeed, from (9), we obtain for $j \in \{c, s\}$ by setting the test functions equal to zero, i.e., $\mathbf{v^j} = \mathbf{0}$:

$$\omega(\sigma\nabla p^j, \nabla q^j)_0 = \omega(\sigma\mathbf{u^j}, \nabla q^j)_0, \quad \forall q^j \in H_0^1(\Omega),$$

and therefore by (13) $p^j = \mathbf{P}(\mathbf{u^j})$. Furthermore, for $q^j = 0$, we obtain

$$\begin{aligned}
&\mathcal{B}_2((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, 0, 0)) = \\
&\quad = \mathcal{A}_1\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right) + \omega(\sigma\mathbf{v^c}, \nabla p^c)_0 - \omega(\sigma\mathbf{v^s}, \nabla p^s)_0 \\
&\quad = \mathcal{A}_1\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right) + \omega(\sigma\nabla\mathbf{P}(\mathbf{v^c}), \nabla p^c)_0 - \omega(\sigma\nabla\mathbf{P}(\mathbf{v^s}), \nabla p^s)_0 \\
&\quad = \mathcal{A}_1\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right) + \omega(\sigma\nabla\mathbf{P}(\mathbf{v^c}), \nabla\mathbf{P}(\mathbf{u^c}))_0 - \omega(\sigma\nabla\mathbf{P}(\mathbf{v^s}), \nabla\mathbf{P}(\mathbf{u^s}))_0 \\
&\quad = \mathcal{A}_2\left((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})\right).
\end{aligned}$$

The first part $(\mathbf{u^c}, \mathbf{u^s})$ of the solution $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)$ of *Formulation 3* solves *Formulation 4*, and vice versa. The relation $p^j = \mathbf{P}(\mathbf{u^j}) = 0$ is nothing else than the weakly divergence-freeness property of the solution $(\mathbf{u^c}, \mathbf{u^s})$.

In that sense all the four formulations are equivalent and have the same unique solution for the amplitudes $\mathbf{u^c}$ and $\mathbf{u^s}$.

2.2. **Finite element discretization in space.** The variational forms $\mathcal{A}_1$, $\mathcal{B}_1$, $\mathcal{B}_1$ and $\mathcal{A}_2$ are the starting points of a discretization in space. Therefore, we use a regular triangulation $\mathcal{T}_h$, with mesh size $h > 0$, of the computational domain Ω with tetrahedral elements. On this mesh we consider Nédélec basis functions of lowest order $\mathcal{ND}_0(\mathcal{T}_h)$, see [41, 42], a conforming finite element subspace of $\mathbf{H_0}(\mathbf{curl})$. Furthermore, we use the space of continuous piecewise linear functions $\mathcal{S}^1(\mathcal{T}_h)$ as the conforming finite element subspace of $H_0^1(\Omega)$. Let $\{\boldsymbol{\varphi}_\mathbf{i}\}_{i=1,N_h}$ denote the usual nodal basis of $\mathcal{ND}_0(\mathcal{T}_h)$, and let $\{\psi_i\}_{i=1,M_h}$ denote the basis of $\mathcal{S}^1(\mathcal{T}_h)$, respectively. We define the following matrices:

$$\begin{aligned}
(\mathbf{K_h})_{ij} &= (\nu\,\mathbf{curl}\,\boldsymbol{\varphi}_\mathbf{i}, \mathbf{curl}\,\boldsymbol{\varphi}_\mathbf{j})_0, \\
(\mathbf{K_{r,h}})_{ij} &= (\nu\,\mathbf{curl}\,\boldsymbol{\varphi}_\mathbf{i}, \mathbf{curl}\,\boldsymbol{\varphi}_\mathbf{j})_0 + \omega(\sigma\nabla\mathbf{P}(\boldsymbol{\varphi}_\mathbf{i}), \nabla\mathbf{P}(\boldsymbol{\varphi}_\mathbf{j}))_0, \\
(\mathbf{M}_{\omega\sigma,\mathbf{h}})_{ij} &= \omega(\sigma\boldsymbol{\varphi}_\mathbf{i}, \boldsymbol{\varphi}_\mathbf{j})_0, \\
(\mathbf{D}_{\omega\sigma,\mathbf{h}})_{ij} &= \omega(\sigma\boldsymbol{\varphi}_\mathbf{i}, \nabla\psi_j)_0, \\
(\mathbf{L}_{\omega\sigma,\mathbf{h}})_{ij} &= \omega(\sigma\nabla\psi_i, \nabla\psi_j)_0.
\end{aligned}$$

The entries of the right-hand side vector are given by the formulas $(\mathbf{f_h^c})_i = (\mathbf{f^c}, \boldsymbol{\varphi}_\mathbf{i})_0$ and $(\mathbf{f_h^s})_i = (\mathbf{f^s}, \boldsymbol{\varphi}_\mathbf{i})_0$. The resulting systems of the finite element equations have the following structure:
Formulation 1:

$$\underbrace{\begin{pmatrix} \mathbf{K_h} & \mathbf{M}_{\omega\sigma,\mathbf{h}} \\ \mathbf{M}_{\omega\sigma,\mathbf{h}} & -\mathbf{K_h} \end{pmatrix}}_{=:\mathbf{A_{h,1}}} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ -\mathbf{f_h^s} \end{pmatrix}$$

Formulation 2:

$$\underbrace{\begin{pmatrix} \mathbf{K_h} & \mathbf{M}_{\omega\sigma,\mathbf{h}} & {\mathbf{D}_{\omega\sigma,\mathbf{h}}}^T & \mathbf{0} \\ \mathbf{M}_{\omega\sigma,\mathbf{h}} & -\mathbf{K_h} & \mathbf{0} & -{\mathbf{D}_{\omega\sigma,\mathbf{h}}}^T \\ \mathbf{D}_{\omega\sigma,\mathbf{h}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D}_{\omega\sigma,\mathbf{h}} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{=:\mathbf{B_{h,1}}} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \\ \mathbf{p_h^c} \\ \mathbf{p_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ -\mathbf{f_h^s} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

Formulation 3:

$$\underbrace{\begin{pmatrix} \mathbf{K_h} & \mathbf{M}_{\omega\sigma,\mathbf{h}} & {\mathbf{D}_{\omega\sigma,\mathbf{h}}}^T & \mathbf{0} \\ \mathbf{M}_{\omega\sigma,\mathbf{h}} & -\mathbf{K_h} & \mathbf{0} & -{\mathbf{D}_{\omega\sigma,\mathbf{h}}}^T \\ \mathbf{D}_{\omega\sigma,\mathbf{h}} & \mathbf{0} & -\mathbf{L}_{\omega\sigma,\mathbf{h}} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D}_{\omega\sigma,\mathbf{h}} & \mathbf{0} & \mathbf{L}_{\omega\sigma,\mathbf{h}} \end{pmatrix}}_{=:\mathbf{B_{h,2}}} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \\ \mathbf{p_h^c} \\ \mathbf{p_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ -\mathbf{f_h^s} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

Formulation 4:

$$\underbrace{\begin{pmatrix} \mathbf{K_{r,h}} & \mathbf{M}_{\omega\sigma,\mathbf{h}} \\ \mathbf{M}_{\omega\sigma,\mathbf{h}} & -\mathbf{K_{r,h}} \end{pmatrix}}_{=:\mathbf{A_{h,2}}} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ -\mathbf{f_h^s} \end{pmatrix}$$

In fact, the system matrices $\mathbf{A_{h,i}}$ and $\mathbf{B_{h,i}}$ $(i = 1, 2)$ of these systems of algebraic equations are symmetric and indefinite, and have a saddle-point or double saddle-point structure, respectively. Since the system matrices are symmetric, the systems can be solved by a MinRes method, see, e.g. [44]. Anyhow, the convergence rate of any iterative method deteriorates with respect to the meshsize h and the "bad" parameters ω, ν and σ, if applied to the unpreconditioned systems. Therefore, preconditioning is a challenging topic.

3. Block-diagonal preconditioning

It is well known, that block-diagonal preconditioners, where each block represents the inner product of the Hilbert space, where the corresponding functions are leaving, lead to mesh independent convergence rates, if used in an iterative method. This result can be extended to obtain even parameter independent convergence rates, by introducing appropriate scaled inner products in the individual spaces. Therefore, our ingredients for the construction of parameter-robust preconditioners are, on the one hand, a constructive preconditioning strategy based on space interpolation proposed by Zulehner [53], and, on the other hand, the introduction of non-standard norms, inspired by the space interpolation technique, and the inf-sup and sup-sup condition in the theorem of Babuška-Aziz [2]. We mention, that block-diagonal preconditioners for *Formulation 1* have also been used and analysed in [9, 12, 30] and in the terms of operator preconditioning in [23, 31]. Nevertheless, we repeat the analysis, since our framework allows to use a constructive approach that even can be generalized to more involving problems straight forward as done in [26, 27, 28, 29, 30].

3.1. Abstract preconditioning theory by operator interpolation.

For constructing block-diagonal preconditioners, we want to use the general space and operator interpolation framework proposed by Zulehner for saddle point problems [53]. Let us consider a symmetric and indefinite matrix $\mathbf{G}$, reformulated as a block matrix of the form

$$\mathbf{G} = \begin{pmatrix} \mathbf{A} & \mathbf{B^T} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}, \tag{14}$$

where $\mathbf{A}$ and $\mathbf{C}$ are symmetric and positive $n \times n$ matrices. It is well-known (see e.g. [36, 40]), that the block-diagonal preconditioners

$$\mathbf{P_1} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{pmatrix} \quad \text{and} \quad \mathbf{P_2} = \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix},$$

where $\mathbf{S} = \mathbf{C} + \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$ and $\mathbf{R} = \mathbf{A} + \mathbf{B}^T\mathbf{C}^{-1}\mathbf{B}$ denote the negative Schur complements of (14), yield uniformly bounded spectra of the preconditioned systems ${\mathbf{P_1}}^{-1}\mathbf{G}$ and ${\mathbf{P_2}}^{-1}\mathbf{G}$, i.e., all eigenvalues of the preconditioned system matrices are located in the set $(-1, \frac{1-\sqrt{5}}{2}] \cup \{1\} \cup (1, \frac{1+\sqrt{5}}{2}]$. Therefore, the following norm equivalences follow

$$c_1\|\mathbf{x}\|_{\mathbf{P_1}} \leq \|\mathbf{G}\mathbf{x}\|_{\mathbf{P_1}^{-1}} \leq c_2\|\mathbf{x}\|_{\mathbf{P_1}} \quad \text{and} \quad c_1\|\mathbf{x}\|_{\mathbf{P_2}} \leq \|\mathbf{G}\mathbf{x}\|_{\mathbf{P_2}^{-1}} \leq c_2\|\mathbf{x}\|_{\mathbf{P_2}}, \tag{15}$$

with the constants $c_1 = (\sqrt{5}-1)/2$ and $c_2 = (\sqrt{5}+1)/2$. Here $\|\cdot\|_{\mathbf{P_i}}$ and $\|\cdot\|_{\mathbf{P_i}^{-1}}$ $(i = 1, 2)$ denote the norms in the Euclidean vector space $\mathbb{R}^{2n}$, induced by the symmetric and positive definite matrices $\mathbf{P_i}$ and $\mathbf{P_i}^{-1}$, respectively. Indeed, from these two Schur complement preconditioners, we can derive further block-diagonal preconditioners by the use of space and operator interpolation theory (see e.g. [6]). Additionally, the resulting spectral bounds only depend on c_1 and c_2 and therefore yield uniformly bounded constants. This approach is summarized in the following result, for details see [53, Section 3].

Theorem 1. *For the interpolation matrix*

$$\mathbf{P_I} = [\mathbf{P_1}, \mathbf{P_2}]_{\frac{1}{2}} = \begin{pmatrix} [\mathbf{A}, \mathbf{R}]_{\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & [\mathbf{S}, \mathbf{C}]_{\frac{1}{2}} \end{pmatrix},$$

we have the norm equivalence

$$c_1\|\mathbf{x}\|_{\mathbf{P_I}} \leq \|\mathbf{G}\mathbf{x}\|_{\mathbf{P_I}^{-1}} \leq c_2\|\mathbf{x}\|_{\mathbf{P_I}}, \tag{16}$$

where c_1 and c_2 are the constants of (15), that are obviously independent of any involved model and discretization parameters.

In the previous theorem, the interpolation of matrices $[\cdot,\cdot]_{\frac{1}{2}}$ is defined by the relation $[\mathbf{M}, \mathbf{N}]_{\frac{1}{2}} := \mathbf{M}^{\frac{1}{2}}(\mathbf{M}^{-\frac{1}{2}}\mathbf{N}\mathbf{M}^{-\frac{1}{2}})^{\frac{1}{2}}\mathbf{M}^{\frac{1}{2}}$. The norm equivalence (16), immediately imply a bound for the condition number of the preconditioned system $\mathbf{P_I}^{-1}\mathbf{G}$, i.e.

$$\kappa_{\mathbf{P_I}}(\mathbf{P_I}^{-1}\mathbf{G}) := \|\mathbf{P_I}^{-1}\mathbf{G}\|_{\mathbf{P_I}}\|\mathbf{G}^{-1}\mathbf{P_I}\|_{\mathbf{P_I}} \leq \frac{c_2}{c_1} = \frac{\sqrt{5}+1}{\sqrt{5}-1},$$

that plays a crucial role for providing convergence rates for iterative methods.

Furthermore, (16) is heavily related the inf-sup and sup-sup conditions in the well-known theorem of Babuška-Aziz [2]. This observation provides the bridge between finding robust block diagonal preconditioners for saddle point problems and providing well-posedness results of the corresponding problem in some non-standard norm.

3.2. **Preconditioning the primal problems.** The requirement, that the blocks $\mathbf{A}$ and $\mathbf{C}$ have to be positive definite, limits the application of Theorem 1 to the different formulations $(1, 2, 3, 4)$ in the previous section. In fact, due to the non-trivial kernel of the **curl** operator, it can only be applied to *Formulation 4*. Nevertheless, from there, we learn how to construct the block-diagonal preconditioners in the remaining cases.
Formulation 4: We explore the 2×2 block-structure of our system matrix

$$\mathbf{A_{h,2}} = \begin{pmatrix} \mathbf{K_{r,h}} & \mathbf{M_{\omega\sigma,h}} \\ \mathbf{M_{\omega\sigma,h}} & -\mathbf{K_{r,h}} \end{pmatrix}.$$

Due to the exact regularization, the matrix $\mathbf{K_{r,h}}$ is positive definite. Hence we can build the Schur complements given by $\mathbf{S_h} = \mathbf{R_h} = \mathbf{K_{r,h}} + \mathbf{M_{\omega\sigma,h}}\mathbf{K_{r,h}^{-1}}\mathbf{M_{\omega\sigma,h}}$. Due to [53] a candidate for a parameter-independent block-diagonal preconditioners is an interpolant of the previous standard Schur complement preconditioners. Hence we interpolate to obtain a new preconditioner

$$\operatorname{diag}\left([\mathbf{K_{r,h}}, \mathbf{S_h}]_{\frac{1}{2}}, [\mathbf{S_h}, \mathbf{K_{r,h}}]_{\frac{1}{2}}\right).$$

The following computations are straight-forward, using the simple spectral inequality

$$\frac{1}{\sqrt{2}}(1+\sqrt{x}) \leq \sqrt{1+x} \leq 1+\sqrt{x}, \quad \forall x \in \mathbb{R}^+.$$

in the context of matrix functions, abbreviated by the notation $(1+\sqrt{x}) \sim \sqrt{1+x}$. Consequently, we obtain

$$\begin{aligned}
[\mathbf{K_{r,h}}, \mathbf{S_h}]_{\frac{1}{2}} = [\mathbf{S_h}, \mathbf{K_{r,h}}]_{\frac{1}{2}} &= \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} \left(\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}(\mathbf{K_{r,h}} + \mathbf{M}_{\omega\sigma,\mathbf{h}}\mathbf{K}_{\mathbf{r,h}}^{-1}\mathbf{M}_{\omega\sigma,\mathbf{h}})\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} \\
&\sim \mathbf{K_{r,h}} + \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} \left(\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}\mathbf{M}_{\omega\sigma,\mathbf{h}}\mathbf{K}_{\mathbf{r,h}}^{-1}\mathbf{M}_{\omega\sigma,\mathbf{h}}\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}\right)^{\frac{1}{2}} \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} \\
&= \mathbf{K_{r,h}} + \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} \left(\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}\mathbf{M}_{\omega\sigma,\mathbf{h}}\mathbf{K}_{\mathbf{r,h}}^{-\frac{1}{2}}\right) \mathbf{K}_{\mathbf{r,h}}^{\frac{1}{2}} = \mathbf{K_{r,h}} + \mathbf{M}_{\omega\sigma,\mathbf{h}}.
\end{aligned}$$

Hence, we have, that the preconditioner $\mathbf{P_{h,2}}$, given by the block-diagonal matrix

$$\mathbf{P_{h,2}} = \text{diag}\ (\mathbf{K_{r,h}} + \mathbf{M}_{\omega\sigma,\mathbf{h}}, \mathbf{K_{r,h}} + \mathbf{M}_{\omega\sigma,\mathbf{h}}), \tag{17}$$

fulfills the norm equivalence

$$c_1\|\mathbf{u_h}\|_{\mathbf{P_{h,2}}} \leq \|\mathbf{A_{h,2}u_h}\|_{\mathbf{P_{h,2}}^{-1}} \leq c_2\|\mathbf{u_h}\|_{\mathbf{P_{h,2}}}, \quad \forall \mathbf{u_h} \in \mathbb{R}^{2n}.$$

where the constants c_1 and c_2 are independent of the meshsize and the involved parameters. In the next step we improve the quantitative estimate of the condition number. Inspired by the structure of the preconditioner (17), obtained by a constructive approach, we introduce the non-standard norm $\|\cdot\|_{\mathcal{P}_2}$ in $\mathbf{H_0(curl)}^2$ by

$$\|(\mathbf{u^s}, \mathbf{u^c})\|_{\mathcal{P}_2}^2 = \sum_{j\in\{c,s\}} (\nu\, \mathbf{curl}\, \mathbf{u^j}, \mathbf{curl}\, \mathbf{u^j})_0 + \omega(\sigma\nabla\mathbf{P}(\mathbf{u^j}), \nabla\mathbf{P}(\mathbf{u^j}))_0 + \omega(\sigma\mathbf{u^j}, \mathbf{u^j})_0.$$

The main result is summarized in the following lemma, that claims that an inf-sup condition (estimate from below) and a sup-sup condition (estimate from above) are fulfilled with parameter-independent constants, namely $1/\sqrt{2}$ and 1.

Lemma 1. *We have*

$$\frac{1}{\sqrt{2}}\|(\mathbf{u^c}, \mathbf{u^s})\|_{\mathcal{P}_2} \leq \sup_{0\neq(\mathbf{v^c},\mathbf{v^s})\in\mathbf{H_0(curl)}^2} \frac{\mathcal{A}_2((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s}))}{\|(\mathbf{v^c}, \mathbf{v^s})\|_{\mathcal{P}_2}} \leq \|(\mathbf{u^c}, \mathbf{u^s})\|_{\mathcal{P}_2}, \tag{18}$$

for all $(\mathbf{u^c}, \mathbf{u^s}) \in \mathbf{H_0(curl)}^2$.

Proof. Boundedness follows from reapplication of Cauchy's inequality. The lower estimate can be attained by choosing $\mathbf{v^c} = \mathbf{u^s} + \mathbf{u^c}$ and $\mathbf{v^s} = \mathbf{u^c} - \mathbf{u^s}$. Note that, for this special choice of the test function, we have $\|(\mathbf{v^c}, \mathbf{v^s})\|_{\mathcal{P}_2} = \sqrt{2}\|(\mathbf{u^c}, \mathbf{u^s})\|_{\mathcal{P}_2}$. □

In general, an inf-sup bound for $\mathbf{H_0(curl)}^2$ does not imply such a lower bound on a subspace $\mathcal{ND}_0(\mathcal{T}_h)^2$. However, in this case the inequalities (18) remain also valid for the Nédélec finite element subspace $\mathcal{ND}_0(\mathcal{T}_h)^2$, since the proof can be repeated for the finite element functions step by step. Therefore, the weighted Helmholtz-projection $\mathbf{P}$ defined in (13) has to be replaced by a discrete weighted Helmholtz-projection $\mathbf{P_h}$, solving the discrete version of (13). Hence, it follows by the theorem of Babuška-Aziz, that there exists a unique solution of the corresponding variational problem (5), and that the solution continuously depends on the data, uniformly in all involved parameters.

Formulation 1: In order to construct a robust preconditioner for *Formulation 1*, we can not use Theorem 1, since the $(1,1)$-bock $\mathbf{K_h}$ is not positive definite and therefore not invertible. Nevertheless, we can repeat the procedure of the previous paragraph of constructing non-standard norms in $\mathbf{H_0}(\mathbf{curl})^2$ and proving the inf-sup and sup-sup conditions, appearing in the theorem of Babuška-Aziz. Inspired by the structure of the preconditioner (17), obtained by a constructive approach for *Formulation 4*, we introduce the non-standard norm $\|\cdot\|_{\mathcal{P}_1}$ in $\mathbf{H_0}(\mathbf{curl})^2$:

$$\|(\mathbf{u^s},\mathbf{u^c})\|^2_{\mathcal{P}_1} = \sum_{j\in\{c,s\}} (\nu\,\mathbf{curl}\,\mathbf{u^j},\mathbf{curl}\,\mathbf{u^j})_0 + \omega(\sigma\mathbf{u^j},\mathbf{u^j})_0.$$

Since σ is assumed to be strictly positive, this expression is really a norm. Additionally, the $\mathcal{P}_1$- and the $\mathcal{P}_2$-norm are equivalent, with constants independent of the involved parameters, i.e., $\|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_1} \le \|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_2} \le 2\|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_1}$. The main result is summarized in the following lemma, that claims that an inf-sup condition and a sup-sup condition are fulfilled with parameter-independent constants, namely $1/\sqrt{2}$ and 1.

Lemma 2. *We have*

$$\frac{1}{\sqrt{2}}\|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_1} \le \sup_{0\neq(\mathbf{v^c},\mathbf{v^s})\in\mathbf{H_0}(\mathbf{curl})^2} \frac{\mathcal{A}_1((\mathbf{u^c},\mathbf{u^s}),(\mathbf{v^c},\mathbf{v^s}))}{\|(\mathbf{v^c},\mathbf{v^s})\|_{\mathcal{P}_1}} \le \|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_1}, \tag{19}$$

for all $(\mathbf{u^c},\mathbf{u^s})\in\mathbf{H_0}(\mathbf{curl})^2$.

Proof. Boundedness follows from reapplication of Cauchy's inequality. The lower estimate can be attained by choosing $\mathbf{v^c}=\mathbf{u^s}+\mathbf{u^c}$ and $\mathbf{v^s}=\mathbf{u^c}-\mathbf{u^s}$. Note that, for this special choice of the test function, we have $\|(\mathbf{v^c},\mathbf{v^s})\|_{\mathcal{P}_1}=\sqrt{2}\|(\mathbf{u^c},\mathbf{u^s})\|_{\mathcal{P}_1}$. □

Furthermore, the inequalities (19) remain valid for the Nédélec finite element subspace $\mathcal{ND}_0(\mathcal{T}_h)^2$, since the proof can be repeated for the finite element functions step by step. This result gives rise to the block-diagonal preconditioner

$$\mathbf{P_{h,1}} = \text{diag}\,(\mathbf{K_h}+\mathbf{M}_{\omega\sigma,\mathbf{h}},\mathbf{K_h}+\mathbf{M}_{\omega\sigma,\mathbf{h}}). \tag{20}$$

Remark 1. *Using this special norm* $\|\cdot\|_{\mathcal{P}_1}$*, an optimal discretization error estimate can be obtained, meaning, that the discretization error can be estimated by the approximation error, uniformly in the involved parameters, i.e.*

$$\|(\mathbf{u^c},\mathbf{u^s})-(\mathbf{u^c_h},\mathbf{u^s_h})\|_{\mathcal{P}_1} \le (1+\sqrt{2}) \inf_{(\mathbf{v^c_h},\mathbf{v^s_h})\in\mathcal{ND}_0(\mathcal{T}_h)^2} \|(\mathbf{u^c},\mathbf{u^s})-(\mathbf{v^c_h},\mathbf{v^s_h})\|_{\mathcal{P}_1}.$$

The discretization error follows by the approximation properties of the finite element space $\mathcal{ND}_0(\mathcal{T}_h)$ *in* $\mathbf{H_0}(\mathbf{curl})$ *(see e.g.* [39]*).*

3.3. **Preconditioning the mixed problems.** In order to construct robust preconditioners for the mixed problems, we heavily take advantage, that we already have constructed a robust preconditioner for the upper left two-times-two block of the four-times-four block matrices.

Formulation 2: We propose the following block-diagonal preconditioner

$$\mathbf{Q_h} = \text{diag}\,(\mathbf{H_h},\mathbf{H_h},\mathbf{D}_{\omega\sigma,\mathbf{h}}\mathbf{H_h}^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T,\mathbf{D}_{\omega\sigma,\mathbf{h}}\mathbf{H_h}^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T), \tag{21}$$

where we use the abbreviation $\mathbf{H_h} := \mathbf{K_h}+\mathbf{M}_{\omega\sigma,\mathbf{h}}$. Indeed, this block-diagonal preconditioner exhibits high structural similarities to a standard Schur complement preconditioner. Since the matrix $\mathbf{D}_{\omega\sigma,\mathbf{h}}$ has full rank, the block $\mathbf{D}_{\omega\sigma,\mathbf{h}}\mathbf{H_h}^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T$ is positive definite and therefore the whole preconditioner $\mathbf{Q_h}$ is positive definite. According to the choice of the block-diagonal preconditioner (21), we introduce the non-standard norm $\|\cdot\|_{\mathcal{Q}}$ in the product space $\mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$:

$$\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{Q}}^2 := \sum_{j\in\{c,s\}} (\nu\, \mathbf{curl}\, \mathbf{u^j}, \mathbf{curl}\, \mathbf{u^j})_0 + \omega(\sigma \mathbf{u^j}, \mathbf{u^j})_0 + \sup_{\mathbf{v^j}\in \mathbf{H_0}(\mathbf{curl})} \frac{\omega^2(\sigma\mathbf{v^j}, \nabla p^j)_0^2}{(\nu\, \mathbf{curl}\, \mathbf{v^j}, \mathbf{curl}\, \mathbf{v^j})_0 + \omega(\sigma\mathbf{v^j}, \mathbf{v^j})_0}.$$

Therein, the sup-expression, is nothing else than the continuous representation of the Schur complement in (21). The main result is summarized in the following lemma, that claims that an inf-sup condition and a sup-sup condition are fulfilled with parameter-independent constants, namely $1/(3\sqrt{2})$ and $(1+\sqrt{5})/2$.

Lemma 3. *We have*

(22)
$$\frac{1}{3\sqrt{2}}\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{Q}} \le \sup_{0\neq(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)} \frac{\mathcal{B}_1((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s))}{\|(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\|_{\mathcal{Q}}}$$
$$\frac{1+\sqrt{5}}{2}\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{Q}} \ge \sup_{0\neq(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)} \frac{\mathcal{B}_1((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s))}{\|(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\|_{\mathcal{Q}}},$$

for all $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$.

Proof. For the proof, let us split the bilinear form $\mathcal{B}_1$ as follows:

$$\mathcal{B}_1((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s) = a((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})) + b((\mathbf{v^c}, \mathbf{v^s}), (p^c, p^s)) + b((\mathbf{u^c}, \mathbf{u^s}), (q^c, q^s))$$

with

$$\begin{aligned} a((\mathbf{u^c}, \mathbf{u^s}), (\mathbf{v^c}, \mathbf{v^s})) :&= (\nu\, \mathbf{curl}\, \mathbf{u^c}, \mathbf{curl}\, \mathbf{v^c})_0 + \omega(\sigma\mathbf{u^s}, \mathbf{v^c})_0 \\ &\quad - (\nu\, \mathbf{curl}\, \mathbf{u^s}, \mathbf{curl}\, \mathbf{v^s})_0 + \omega(\sigma\mathbf{u^c}, \mathbf{v^s})_0 \quad \text{and} \\ b((\mathbf{v^c}, \mathbf{v^s}), (p^c, p^s)) &:= \omega(\sigma\mathbf{v^c}, \nabla p^c)_0 - \omega(\sigma\mathbf{v^s}, \nabla p^s)_0, \end{aligned}$$

and verify the conditions in the theorem of Brezzi (cf. [8]). The bilinear form $a(\cdot,\cdot)$ is bounded with constant 1 and fulfills an inf-sup condition with constant $1/\sqrt{2}$ (see proof of Lemma 2). Boundedness of $b(\cdot,\cdot)$ follows by Cauchy's inequality and the expression of the norm as a supremum

$$\begin{aligned} \omega(\sigma\mathbf{u^j}, \nabla p^j)_0 &\le \|\sqrt{\omega\sigma}\mathbf{u^j}\|_0\, \|\sqrt{\omega\sigma}\nabla p^j\|_0 \\ &\le \|\sqrt{\omega\sigma}\mathbf{u^j}\|_0 \sup_{\mathbf{v^j}\in\mathbf{H_0}(\mathbf{curl})} \frac{\omega(\sigma\nabla p^j, \mathbf{v^j})_0}{\sqrt{\omega\|\sqrt{\sigma}\mathbf{v^j}\|_0^2 + \|\sqrt{\nu}\,\mathbf{curl}\,\mathbf{v^j}\|_0^2}}. \end{aligned}$$

Therefore, boundedness of $b(\cdot,\cdot)$ follows with constant 1. Finally, the bilinear form $b(\cdot,\cdot)$, satisfies an inf-sup condition with constant $1/2$:

$$\begin{aligned} &\sup_{(\mathbf{v^c}, \mathbf{v^s})} \frac{b((\mathbf{v^c}, \mathbf{v^s}), (p^c, p^s))}{\sqrt{\sum_{j\in\{c,s\}}(\nu\,\mathbf{curl}\,\mathbf{v^j}, \mathbf{curl}\,\mathbf{v^j})_0 + \omega(\sigma\mathbf{v^j}, \mathbf{v^j})_0}} \\ &\qquad \ge \frac{1}{2}\sum_{j\in\{c,s\}} \sup_{\mathbf{v^j}} \frac{\omega(\sigma\mathbf{v^j}, \nabla p^j)_0}{\sqrt{(\nu\,\mathbf{curl}\,\mathbf{v^j}, \mathbf{curl}\,\mathbf{v^j})_0 + \omega(\sigma\mathbf{v^j}, \mathbf{v^j})_0}}. \end{aligned}$$

Consequently, the inf-sup and sup-sup condition can be derived by combining the estimates. □

Furthermore, the inequalities (22) remain valid for the finite element subspace $\mathcal{ND}_0(\mathcal{T}_h)^2 \times \mathcal{S}^1(\mathcal{T}_h)^2$, since the proof can be repeated for the finite element functions step by step.

Formulation 3: We propose the following block-diagonal preconditioner

$$\mathbf{R_h} = \text{diag}\,(\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}, \mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}, \mathbf{L}_{\omega\sigma,\mathbf{h}}, \mathbf{L}_{\omega\sigma,\mathbf{h}}). \tag{23}$$

The big advantage of this preconditioner is, that instead of the inversion of a Schur complement, a scaled H_0^1 stiffness matrix has to be inverted. According to the choice of the block-diagonal preconditioner (23), we introduce the non-standard norm $\|\cdot\|_{\mathcal{R}}$ in the product space $\mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$:

$$\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{R}}^2 := \sum_{j\in\{c,s\}} (\nu\,\mathbf{curl}\,\mathbf{u^j}, \mathbf{curl}\,\mathbf{u^j})_0 + \omega(\sigma\mathbf{u^j}, \mathbf{u^j})_0 + \omega(\sigma\nabla p^j, \nabla p^j)_0.$$

The main result is summarized in the following lemma, that claims that an inf-sup and a sup-sup condition are fulfilled with parameter-independent constants, namely $\frac{1}{\sqrt{2}}$ and 2.

Lemma 4. *We have*

$$\frac{1}{\sqrt{2}}\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{R}} \le \sup_{0\neq(\mathbf{v^c},\mathbf{v^s},q^c,q^s)} \frac{\mathcal{B}_2((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s))}{\|(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\|_{\mathcal{R}}} \tag{24}$$

$$2\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{R}} \ge \sup_{0\neq(\mathbf{v^c},\mathbf{v^s},q^c,q^s)} \frac{\mathcal{B}_2((\mathbf{u^c}, \mathbf{u^s}, p^c, p^s), (\mathbf{v^c}, \mathbf{v^s}, q^c, q^s))}{\|(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\|_{\mathcal{R}}}, \tag{25}$$

forall $(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) \in \mathbf{H_0}(\mathbf{curl})^2 \times H_0^1(\Omega)^2$.

Proof. Boundedness follows from reapplication of Cauchy's inequality. The lower estimate can be attained by choosing $\mathbf{v^c} = \mathbf{u^c} - \mathbf{u^s}$, $\mathbf{v^s} = \mathbf{u^s} + \mathbf{u^c}$, $q^c = p^s - p^c$ and $q^s = p^s + p^c$. Note that, for this special choice of the test function, we have $\|(\mathbf{v^c}, \mathbf{v^s}, q^c, q^s)\|_{\mathcal{R}} = \sqrt{2}\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s)\|_{\mathcal{R}}$. □

Furthermore, the inequalities (24) remain valid for the finite element subspace $\mathcal{ND}_0(\mathcal{T}_h)^2 \times \mathcal{S}^1(\mathcal{T}_h)^2$, since the proof can be repeated for the finite element functions step by step.

Remark 2. *Using this special norm* $\|\cdot\|_{\mathcal{R}}$*, an optimal discretization error estimate can be obtained, meaning, that the discretization error can be estimated by the approximation error, uniformly in the involved parameters, i.e.*

$$\begin{aligned}&\|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) - (\mathbf{u^c_h}, \mathbf{u^s_h}, p^c_h, p^s_h)\|_{\mathcal{R}}\\ &\le (1 + 2\sqrt{2}) \inf_{(\mathbf{v^c_h},\mathbf{v^s_h},q^c_h,q^s_h)\in\mathcal{ND}_0(\mathcal{T}_h)^2\times\mathcal{S}^1(\mathcal{T}_h)^2} \|(\mathbf{u^c}, \mathbf{u^s}, p^c, p^s) - (\mathbf{v^c_h}, \mathbf{v^s_h}, q^c_h, g^s_h)\|_{\mathcal{R}}.\end{aligned}$$

The discretization error follows by the approximation properties of the finite element space $\mathcal{ND}_0(\mathcal{T}_h)$ *in* $\mathbf{H_0}(\mathbf{curl})$ *and* $\mathcal{S}^1(\mathcal{T}_h)$ *in* $H_0^1(\Omega)$ *(see e.g.* [39]*).*

3.4. MinRes convergence analysis.

The next theorem collects the key results of this work, that the spectral condition number of the preconditioned system matrices can be estimated by constants independent of any involved model or discretization parameters.

Theorem 2. *The following condition number estimates are valid:*

$$\kappa_{\mathbf{P_{h,1}}}(\mathbf{P_{h,1}}^{-1}\mathbf{A_{h,1}}) := \|\mathbf{P_{h,1}}^{-1}\mathbf{A_{h,1}}\|_{\mathbf{P_{h,1}}}\|\mathbf{A_{h,1}}^{-1}\mathbf{P_{h,1}}\|_{\mathbf{P_{h,1}}} \le \sqrt{2} \approx 1.41421,$$

$$\kappa_{\mathbf{Q_h}}(\mathbf{Q_h}^{-1}\mathbf{B_{h,1}}) := \|\mathbf{Q_h}^{-1}\mathbf{B_{h,1}}\|_{\mathbf{Q_h}}\|\mathbf{B_{h,1}}^{-1}\mathbf{Q_h}\|_{\mathbf{Q_h}} \le \frac{3\sqrt{2}(1+\sqrt{5})}{2} \approx 6.86474,$$

$$\kappa_{\mathbf{R_h}}(\mathbf{R_h}^{-1}\mathbf{B_{h,2}}) := \|\mathbf{R_h}^{-1}\mathbf{B_{h,2}}\|_{\mathbf{R_h}}\|\mathbf{B_{h,2}}^{-1}\mathbf{R_h}\|_{\mathbf{R_h}} \le 2\sqrt{2} \approx 2.82843,$$

$$\kappa_{\mathbf{P_{h,2}}}(\mathbf{P_{h,2}}^{-1}\mathbf{A_{h,2}}) := \|\mathbf{P_{h,2}}^{-1}\mathbf{A_{h,2}}\|_{\mathbf{P_{h,2}}}\|\mathbf{A_{h,2}}^{-1}\mathbf{P_{h,2}}\|_{\mathbf{P_{h,2}}} \le \sqrt{2} \approx 1.41421,$$

where the constants are obviously independent of the space and time-discretization parameters h *and* ω*, as well as the involved model parameters* ν *and* σ*.*

Proof. The proof immediately follows from Lemma 2, Lemma 3, Lemma 4 and Lemma 1. □

The condition number estimates of the preconditioned systems immediately yield the convergence rate estimate of the MinRes method (see e.g. [19]). Therefore, the number of MinRes iterations required for reducing the initial error by some fixed factor $\delta > 0$ is independent of the space and time discretization parameters h and ω and the involved model parameters ν and σ. The MinRes convergence analysis is summarized for *Formulation 1, 2, 3* and *4*, i.e. for

$$(\mathbf{A}, \mathbf{P}) \in \{(\mathbf{A_{h,1}}, \mathbf{P_{h,1}}), (\mathbf{B_{h,1}}, \mathbf{Q_h}), (\mathbf{B_{h,2}}, \mathbf{R_h}), (\mathbf{A_{h,2}}, \mathbf{P_{h,2}})\},$$

in the following corollary.

Corollary 1. *The MinRes method applied to the preconditioned systems converges for arbitrary initial guess* $\mathbf{w^0}$. *At the m-th iteration, the preconditioned residual* $\mathbf{r^m} := \mathbf{P}^{-1}(\mathbf{f} - \mathbf{A}\mathbf{w^m})$ *is bounded as*

$$\left\|\mathbf{r^{2m}}\right\|_{\mathbf{P}} \leq \frac{2q^m}{1+q^{2m}} \left\|\mathbf{r^0}\right\|_{\mathbf{P}} \quad \textit{where} \quad q = \frac{\kappa_{\mathbf{P}}(\mathbf{P}^{-1}\mathbf{A}) - 1}{\kappa_{\mathbf{P}}(\mathbf{P}^{-1}\mathbf{A}) + 1}. \tag{26}$$

The estimates of the condition numbers $\kappa_{\mathbf{P}}(\mathbf{P}^{-1}\mathbf{A})$ *are according to Theorem 2.*

3.5. **Practical block-diagonal preconditioning.** The application of the proposed block diagonal preconditioners involves the solution of systems with the diagonal blocks of (17), (20), (21) and (23). However, in large-scale computations, these diagonal blocks have to be replaced by easy "invertible" symmetric and positive definite preconditioners. Therefore, we introduce a common notation for the preconditioners (17), (20), (21) and (23). Let us define $\mathbf{P} \in \mathbb{R}^{n\times n}$ by

$$\mathbf{P} = \text{diag}\ (\mathbf{D_i})_{i=1,\ldots,m},$$

with $\mathbf{D_i} \in \mathbb{R}^{n_i\times n_i}$, such that $n = n_1 + \ldots + n_m$. Here $m = 2$ for *Formulation 1* and *4* and $m = 4$ for *Formulation 2* and *3*. Depending on the preconditioner, $\mathbf{D_i}$ is either $\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}$ or $\mathbf{L}_{\omega\sigma,\mathbf{h}}$ or the Schur complement expression $\mathbf{D}_{\omega\sigma,\mathbf{h}}(\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}})^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T$. If the diagonal blocks $\mathbf{D_i}$ are replaced by spectral equivalent preconditioners $\tilde{\mathbf{D}}_i$, i.e.

$$\underline{c}_i\, \mathbf{x}^T\tilde{\mathbf{D}}_i\mathbf{x} \leq \mathbf{x}^T\mathbf{D}_i\mathbf{x} \leq \overline{c}_i\, \mathbf{x}^T\tilde{\mathbf{D}}_i\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{n_i},$$

the block-diagonal matrix $\tilde{\mathbf{P}} := \text{diag}\ (\tilde{\mathbf{D}}_\mathbf{i})_{i=1,\ldots,m}$ obviously fulfills the spectral equivalence inequality

$$\min\{\underline{c}_1, \ldots, \underline{c}_m\}\, \mathbf{x}^T\tilde{\mathbf{P}}\mathbf{x} \leq \mathbf{x}^T\mathbf{P}\mathbf{x} \leq \max\{\overline{c}_1, \ldots, \overline{c}_m\}\, \mathbf{x}^T\tilde{\mathbf{P}}\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Consequently, a bound for the condition number of the new preconditioned system matrix $\tilde{\mathbf{P}}^{-1}\mathbf{A}$ is given by

$$\kappa_{\tilde{\mathbf{P}}}(\tilde{\mathbf{P}}^{-1}\mathbf{A}) \leq \kappa_{\mathbf{P}}(\mathbf{P}^{-1}\mathbf{A}) \frac{\max\{\overline{c}_1, \ldots, \overline{c}_m\}}{\min\{\underline{c}_1, \ldots, \underline{c}_m\}}.$$

Therefore, it remains to discuss possible parameter-robust and optimal or almost-optimal preconditioners for the diagonal blocks $\mathbf{D_i}$. Basically, we are dealing with three different types of problems:

- $\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}$**:** The solution of the system with this block corresponds to the solution of a standard $\mathbf{H}(\mathbf{curl})$ problem. Depending on the parameter setting, candidates for robust and (almost) optimal preconditioners are multigrid preconditioners [1, 21], auxiliary space preconditioners [24, 51], and domain decomposition preconditioners [25, 50, 49].

$\mathbf{L}_{\omega\sigma,\mathbf{h}}$**:** The solution of the system with this block corresponds to the solution of a standard H^1 problem. Depending on the parameter setting, candidates for robust and (almost) optimal preconditioners are multigrid or multilevel preconditioners [7, 17, 32, 33, 34, 43], and domain decomposition preconditioners [50].

$\mathbf{D}_{\omega\sigma,\mathbf{h}}(\mathbf{K_h}+\mathbf{M}_{\omega\sigma,\mathbf{h}})^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T$**:** In practical applications it is not very convenient to work with this Schur complement preconditioner. Anyhow, it is possible to derive a closed expression for the Schur complement. Using Cauchy's inequality, the following estimate follows:

$$\sup_{\mathbf{v^j}\in\mathbf{H_0}(\mathbf{curl})}\frac{\omega(\sigma\mathbf{v^j},\nabla p^j)_0}{\sqrt{(\nu\,\mathbf{curl}\,\mathbf{v^j},\mathbf{curl}\,\mathbf{v^j})_0+\omega(\sigma\mathbf{v^j},\mathbf{v^j})_0}}\le\sqrt{\omega(\sigma\nabla p^j,\nabla p^j)_0}.$$

Furthermore, for the choice $\mathbf{v^j}=\nabla p^j\in\mathbf{H_0}(\mathbf{curl})$, there holds

$$\sup_{\mathbf{v^j}\in\mathbf{H_0}(\mathbf{curl})}\frac{\omega(\sigma\mathbf{v^j},\nabla p^j)_0}{\sqrt{(\nu\,\mathbf{curl}\,\mathbf{v^j},\mathbf{curl}\,\mathbf{v^j})_0+\omega(\sigma\mathbf{v^j},\mathbf{v^j})_0}}\ge\sqrt{\omega(\sigma\nabla p^j,\nabla p^j)_0}.$$

Consequently, we obtain the identity

$$\sup_{\mathbf{v^j}\in\mathbf{H_0}(\mathbf{curl})}\frac{\omega(\sigma\mathbf{v^j},\nabla p^j)_0}{\sqrt{(\nu\,\mathbf{curl}\,\mathbf{v^j},\mathbf{curl}\,\mathbf{v^j})_0+\omega(\sigma\mathbf{v^j},\mathbf{v^j})_0}}=\sqrt{\omega(\sigma\nabla p^j,\nabla p^j)_0}. \tag{27}$$

Furthermore, the equality (27) is also valid for the finite element spaces $\mathcal{ND}_0(\mathcal{T}_h)$ and $\mathcal{S}^1(\mathcal{T}_h)$, since the estimates can be repeated for the finite functions step by step. Consequently, we obtain the identity

$$\mathbf{D}_{\omega\sigma,\mathbf{h}}(\mathbf{K_h}+\mathbf{M}_{\omega\sigma,\mathbf{h}})^{-1}\mathbf{D}_{\omega\sigma,\mathbf{h}}{}^T=\mathbf{L}_{\omega\sigma,\mathbf{h}}.$$

Hence, we can use the same preconditioners as for $\mathbf{L}_{\omega\sigma,\mathbf{h}}$.

Now it is clear, that the results of Corollary 1 remain valid with $\mathbf{P}$ replaced by $\tilde{\mathbf{P}}$. Depending on the robustness and optimality conditions of the chosen preconditioners for the diagonal blocks, we obtain a robust and optimal solver.

4. The multiharmonic case

Let us now assume that the right-hand side $\mathbf{f}$ is multiharmonic, i.e. $\mathbf{f}$ has the form

$$\mathbf{f}(\mathbf{x},t)=\sum_{k=0}^{N}\mathbf{f}_\mathbf{k}^\mathbf{c}(\mathbf{x})\cos(k\omega t)+\mathbf{f}_\mathbf{k}^\mathbf{s}(\mathbf{x})\sin(k\omega t),$$

with some given natural number N. We mention that the multiharmonic representation (4) can also be seen as an approximation of a general periodic right-hand side $\mathbf{f}$ by a truncated Fourier series. Due to the linearity of (1), the solution has the same structure, i.e.

$$\mathbf{u_N}(\mathbf{x},t)=\sum_{k=0}^{N}\mathbf{u}_\mathbf{k}^\mathbf{c}(\mathbf{x})\cos(k\omega t)+\mathbf{u}_\mathbf{k}^\mathbf{s}(\mathbf{x})\sin(k\omega t).$$

Again, due to the linearity, the huge $(2N+1)\times(2N+1)$ system decouples into N 2×2 systems of partial differential equations for the two Fourier coefficients of $\mathbf{u}$ belonging to the mode k, and a 1×1 system of partial differential equations for the mode $k=0$. Clearly, we do not have to solve for the $\mathbf{u}_\mathbf{0}^\mathbf{s}$, since $\sin(0\,\omega t)=0$. Hence we have to solve the following decoupled system of partial differential equations in the frequency domain: Find $\mathbf{u} = (\mathbf{u_1^c}, \mathbf{u_1^s}, \ldots, \mathbf{u_N^c}, \mathbf{u_N^s})$, such that

$$\begin{cases} \omega k \sigma \mathbf{u_k^s} + \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{u_k^c}) = \mathbf{f_k^c}, & \text{in } \Omega, \\ \omega k \sigma \mathbf{u_k^c} - \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{u_k^s}) = -\mathbf{f_k^s}, & \text{in } \Omega, \\ \omega k \operatorname{div}(\sigma \mathbf{u_k^c}) = 0, & \text{in } \Omega, \\ \omega k \operatorname{div}(\sigma \mathbf{u_k^s}) = 0, & \text{in } \Omega, \\ \mathbf{u_k^c} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega, \\ \mathbf{u_k^s} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases} \tag{28}$$

The finite element discretization of each block ($k = 1, \ldots, N$) leads to a 2×2 block-matrix $\mathbf{A_{h,i}^k}$ or a 4×4 block-matrix $\mathbf{B_{h,i}^k}$, for the primal or mixed formulations of Section 2, respectively, that formally have the same structure as $\mathbf{A_{h,i}}$ and $\mathbf{B_{h,i}}$ in Section 2 with ω replaced by $k\omega$. For the resulting system of linear equations, we obtain the same condition number estimates as in Theorem 2, where the condition numbers are additionally independent of the modes k and the total number of modes N. The case $k = 0$ has to be treated separately. Find $\mathbf{u_0^c}$, such that

$$\begin{cases} \mathbf{curl}(\nu\ \mathbf{curl}\, \mathbf{u_0^c}) = \mathbf{f_0^c}, & \text{in } \Omega, \\ \operatorname{div}(\sigma \mathbf{u_0^c}) = 0, & \text{in } \Omega, \\ \mathbf{u_0^c} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega. \end{cases}$$

Again, a similar analysis as in Section 2 can be done. The finite element discretization of the resulting mixed problem leads to the following system of linear equations, given by

$$\begin{pmatrix} \mathbf{K_h} & \mathbf{D_{\sigma,h}}^T \\ \mathbf{D_{\sigma,h}} & -\mathbf{L_{\sigma,h}} \end{pmatrix} \begin{pmatrix} \mathbf{u_{h,0}^c} \\ \mathbf{p_{h,0}^c} \end{pmatrix} = \begin{pmatrix} \mathbf{f_{h,0}^c} \\ \mathbf{0} \end{pmatrix}.$$

This system can directly be tackled, e.g., by using a domain decomposition preconditioner, cf. [25].

Remark 3. *By approximating a general right-hand side* $\mathbf{f}$ *in terms of a Fourier series, i.e.,*

$$\mathbf{f}(\mathbf{x}, t) = \sum_{k=0}^{\infty} \mathbf{f_k^c}(\mathbf{x}) \cos(k\omega t) + \mathbf{f_k^s}(\mathbf{x}) \sin(k\omega t),$$

it follows, that the solution $\mathbf{u}$ *has the same structure. For numerical approximation, the infinite series is truncated at a finite number* N*, i.e.,*

$$\mathbf{u_N}(\mathbf{x}, t) = \sum_{k=0}^{N} \mathbf{u_k^c}(\mathbf{x}) \cos(k\omega t) + \mathbf{u_k^s}(\mathbf{x}) \sin(k\omega t).$$

The error due to the truncation of the Fourier series has been analyzed in [4]*, where it is shown, that under certain regularity assumptions on* $\mathbf{f}$*, the error* $\|\mathbf{u} - \mathbf{u_N}\|$ *behaves like* $\mathcal{O}(N^{-1})$.

5. Numerical results

In order to confirm our theoretical results numerically, we report on our first numerical tests for an academic example, namely for the simple time-harmonic case. The numerical results presented in this section were attained using ParMax[1]. First, we demonstrate the robustness of the block-diagonal preconditioners with respect to the frequency ω and the conductivity σ. Therefore, for the solution of the preconditioning equations arising from the diagonal blocks, we use the sparse direct solver UMFPACK that is very efficient for several thousand unknowns in the

[1] `http://www.numa.uni-linz.ac.at/P19255/software.shtml`

TABLE 1. Formulation 1, parameter setting: $\nu = 1$, $\sigma = 1$.

	$\log_{10}\omega$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
38	4	4	2	3	3	9	16	8	4	4	2	0.001007	0.000259
196	6	4	2	3	3	9	16	10	6	4	4	0.007135	0.000509
1208	6	4	4	2	3	9	19	14	6	4	4	0.065675	0.004476
16736	6	4	4	2	3	9	20	20	8	4	4	0.645854	0.10463
62048	8	4	4	2	3	7	19	22	8	4	4	7.11876	3.46829

TABLE 2. Formulation 1, parameter setting: $\nu = 1$, $\sigma_1 = 1$, $\omega = 1$.

	$\log_{10}\sigma_2$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
38	7	7	7	7	7	9	18	14	6	4	4	0.001269	0.000218
196	4	5	5	7	7	9	17	9	9	9	9	0.006714	0.000576
1208	6	5	5	7	7	9	19	12	9	9	9	0.055624	0.004189
16736	6	5	5	5	7	9	18	17	9	9	9	0.496736	0.109205
62048	8	4	5	5	7	7	17	18	8	7	7	6.17143	3.53488

case of three-dimensional problems [13, 14, 15]. We provide academic test cases for *Formulation 1*, *Formulation 3* and *Formulation 4*.

5.1. **Formulation 1.** Table 1 and Table 2 provide the number of MinRes iterations needed for reducing the initial residual by a factor 10^{-8} for different ω, σ and h for *Formulation 1*. These numerical experiments were performed for a three-dimensional linear problem on the unit cube $\Omega = (0,1)^3$, discretized by tetrahedra for the case $\nu = 1$ (Due to scaling arguments, it can always be achieved, that $\nu = 1$). Furthermore the piecewise constant conductivity σ is given by

$$\sigma = \left\{ \begin{array}{ll} \sigma_1 & \text{in } \Omega_1 = \{(x,y,z)^T \in [0,1]^3 : z > 0.5\} \\ \sigma_2 & \text{in } \Omega_2 = \{(x,y,z)^T \in [0,1]^3 : z \leq 0.5\} \end{array} \right. . \tag{29}$$

These experiments demonstrate the independence of the MinRes convergence rate on the parameters ω and σ, and the mesh size h since the number of iterations is bounded by 22 for all computed constellations. Furthermore, we table the CPU time of MinRes solver (CPU) and the factorization time for the preconditioner (FAC) in seconds.

5.2. **Formulation 3.** Table 3 and Table 4 provide the same experiments for *Formulation 3*. Again, the numerical results show robustness of our preconditioner, since the number of iterations is bounded by 31 for all computed constellations..

5.3. **Formulation 4.** Table 5 and Table 6 provide the same experiments for *Formulation 4*. Instead of the preconditioner $\mathbf{P_{h,2}}$, we use the spectral equivalent preconditioner $\mathbf{P_{h,1}}$. Furthermore, the application of the Helmholtz projector $\mathbf{P_h}$ is realized via the Schur complement $\mathbf{D}^T_{\omega\sigma,\mathbf{h}}\mathbf{L}^{-1}_{\omega\sigma,\mathbf{h}}\mathbf{D}_{\omega\sigma,\mathbf{h}}$. Again, the numerical results show robustness of our preconditioner, since the number of iterations is bounded by 27 for all computed constellations.

6. Generalizations

6.1. **The case of vanishing conductivity.** Eddy current problems are essentially different in conducting ($\sigma > 0$) and non-conducting regions ($\sigma = 0$). In order to

Table 3. Formulation 3, parameter setting: $\nu = 1$, $\sigma = 1$.

	$\log_{10}\omega$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
54	13	13	14	14	14	16	23	17	15	13	13	0.003093	0.000215
250	25	23	23	21	21	22	27	25	25	25	25	0.021179	0.000596
1458	19	19	19	21	21	21	27	25	23	21	21	0.152970	0.005349
9826	25	25	25	25	25	25	30	31	29	29	29	1.40394	0.115046
71874	25	19	19	19	19	20	26	29	25	23	23	13.6531	3.51824

Table 4. Formulation 3, parameter setting: $\nu = 1$, $\sigma_1 = 1$, $\omega = 1$.

	$\log_{10}\sigma_2$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
54	31	27	23	22	18	16	23	23	19	19	17	0.004050	0.000298
250	11	12	15	17	19	22	25	22	22	20	18	0.019550	0.000662
1458	19	15	15	16	20	21	26	21	18	16	15	0.140093	0.005013
9826	23	19	19	21	22	25	28	25	22	22	21	1.25811	0.112381
71874	25	17	14	14	18	20	25	25	22	22	20	12.0554	3.55502

Table 5. Formulation 4, parameter setting: $\nu = 1$, $\sigma = 1$.

	$\log_{10}\omega$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
38	5	5	5	5	6	10	17	8	6	6	6	0.002435	0.000217
196	9	8	8	8	8	12	22	13	10	10	10	0.018868	0.000375
1208	11	9	9	9	10	14	24	17	11	11	11	0.148059	0.004035
16736	12	10	10	10	10	14	27	25	13	13	13	1.38511	0.102779
62048	13	9	8	8	8	12	24	25	13	9	9	13.0067	3.35732

Table 6. Formulation 4, parameter setting: $\nu = 1$, $\sigma_1 = 1$, $\omega = 1$.

	$\log_{10}\sigma_2$											CPU	FAC
DOF	-10	-8	-6	-4	-2	0	2	4	6	8	10		
38	22	20	16	14	10	10	20	14	9	8	8	0.003126	0.000235
196	6	7	8	10	10	12	21	14	13	12	12	0.017918	0.000370
1208	9	8	10	10	12	14	24	16	12	12	12	0.148247	0.004143
16736	11	8	10	12	12	14	24	22	12	12	12	1.25063	0.104136
62048	13	8	9	10	10	12	22	22	12	12	12	11.6058	3.38669

gain uniqueness in the non-conducting regions, we introduce some regularization. Candidates are elliptic and parabolic regularization. Since, for preconditioning purpose, both of them can be handled in the same framework, we start with introducing formal regularization parameters ($i = 1, 2$).

$$\mathcal{R}_i(\sigma) := \begin{cases} \sigma_\varepsilon := \max(\sigma, \varepsilon), & i = 1 \\ \sigma & i = 2 \end{cases}, \quad \mathcal{Q}_i(\mathbf{u}) := \begin{cases} \mathbf{0}, & i = 1 \quad \text{(parabolic)} \\ \varepsilon \mathbf{u}, & i = 2 \quad \text{(elliptic)} \end{cases}$$

Here $\varepsilon > 0$ is a small regularization parameter. We mention, that for the parabolic and elliptic regularization technique, we have to deal with an additional error of order $\mathcal{O}(\varepsilon)$ (see [4] and [46]). Therefore, we are dealing with the following perturbed problem:

$$\begin{cases} \mathcal{R}_i(\sigma)\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu\,\mathbf{curl}\,\mathbf{u}) + \mathcal{Q}_i(\mathbf{u}) = \mathbf{f} & \text{in } \Omega\times(0,T], \\ \mathbf{u}\times\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\times(0,T), \\ \mathbf{u}(0) = \mathbf{u}(T) & \text{in } \Omega. \end{cases}$$

Performing the same time-harmonic finite element discretization as in Section 2, for $i = 1, 2$, we end up with the system matrices

$$\mathbf{A}_{\mathbf{h},\varepsilon} := \begin{pmatrix} \mathbf{K_h} + \varepsilon\mathbf{M_h} & \mathbf{M}_{\omega\sigma,\mathbf{h}} \\ \mathbf{M}_{\omega\sigma,\mathbf{h}} & -(\mathbf{K_h} + \varepsilon\mathbf{M_h}) \end{pmatrix} \quad \text{and} \quad \mathbf{A}_{\mathbf{h},\sigma_\varepsilon} := \begin{pmatrix} \mathbf{K_h} & \mathbf{M}_{\omega\sigma_\varepsilon,\mathbf{h}} \\ \mathbf{M}_{\omega\sigma_\varepsilon,\mathbf{h}} & -\mathbf{K_h} \end{pmatrix}.$$

By a similar procedure, we can show, that the block-diagonal preconditioners

$$\begin{aligned} \mathbf{P}_{\mathbf{h},\varepsilon} &:= \operatorname{diag}\,(\mathbf{K_h} + \varepsilon\mathbf{M_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}, \mathbf{K_h} + \varepsilon\mathbf{M_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}) \quad \text{and} \\ \mathbf{P}_{\mathbf{h},\sigma_\varepsilon} &:= \operatorname{diag}\,(\mathbf{K_h} + \mathbf{M}_{\omega\sigma_\varepsilon,\mathbf{h}}, \mathbf{K_h} + \mathbf{M}_{\omega\sigma_\varepsilon,\mathbf{h}}), \end{aligned}$$

lead to parameter-independent condition number estimates:

$$\begin{aligned} \kappa_{\mathbf{P}_{\mathbf{h},\varepsilon}}(\mathbf{P}_{\mathbf{h},\varepsilon}{}^{-1}\mathbf{A}_{\mathbf{h},\varepsilon}) &:= \|\mathbf{P}_{\mathbf{h},\varepsilon}{}^{-1}\mathbf{A}_{\mathbf{h},\varepsilon}\|_{\mathbf{P}_{\mathbf{h},\varepsilon}} \|\mathbf{A}_{\mathbf{h},\varepsilon}{}^{-1}\mathbf{P}_{\mathbf{h},\varepsilon}\|_{\mathbf{P}_{\mathbf{h},\varepsilon}} \leq \sqrt{2} \approx 1.41421, \\ \kappa_{\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}}(\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}{}^{-1}\mathbf{A}_{\mathbf{h},\sigma_\varepsilon}) &:= \|\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}{}^{-1}\mathbf{A}_{\mathbf{h},\sigma_\varepsilon}\|_{\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}} \|\mathbf{A}_{\mathbf{h},\sigma_\varepsilon}{}^{-1}\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}\|_{\mathbf{P}_{\mathbf{h},\sigma_\varepsilon}} \leq \sqrt{2}. \end{aligned}$$

In these cases, the condition number estimates are even independent of the small regularization parameter ε.

Remark 4. *It is very common to discretize the time-harmonic eddy current problems in terms of symmetrically coupled finite and boundary element method (e.g.* [22]*), taking care of the different physical behavior in the conducting and non-conducting subdomains, respectively. Also in this case the resulting symmetric system of linear equations can be preconditioned by block-diagonal preconditioner, that involves the evaluation of standard* $\mathbf{H_0}(\mathbf{curl})$ *inner products, see* [28]. *Beside the fact, that we do not have to deal with a regularization error, the main advantage of the finite element - boundary element approach is the treatability of possible unbounded domains.*

6.2. **A block-diagonal preconditioner for a non-symmetric system.** Throughout this work, we use a symmetric reformulation of the original frequency domain equations. However, in the non-linear case, the Newton linearization yields to a non-symmetric Jacobi system, that cannot be reformulated as symmetric system as in the linear case. Therefore, it is important to investigate the non-symmetric system

$$\underbrace{\begin{pmatrix} \mathbf{K_h} & \mathbf{M}_{\omega\sigma,\mathbf{h}} \\ -\mathbf{M}_{\omega\sigma,\mathbf{h}} & \mathbf{K_h} \end{pmatrix}}_{=:\bar{\mathbf{A}}_\mathbf{h}} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ \mathbf{f_h^s} \end{pmatrix}, \tag{30}$$

that reflects the structure of the Jacobi system in the non-linear case. Due to the non-symmetry the MinRes method is no longer applicable, but we can use, for instance, the GMRes method [47] or QMR method [18]. Indeed, such a kind of system has already be considered in [5], wherein a multigrid-preconditioned QMR solver is proposed. The main ingredient of this solver is a specific block preconditioner $\hat{\mathbf{C}}_\mathbf{h}$, that only involves the inversion of $\mathbf{H}(\mathbf{curl})$ standard problems

$$\hat{\mathbf{C}}_\mathbf{h}^{-1} := \frac{1}{2}\begin{pmatrix} (\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}.$$

This preconditioner leads to a parameter-robust bound for the condition number of the preconditioned system, i.e.

$$\kappa_{\mathbf{P_{h,1}}}(\hat{\mathbf{C}}_{\mathbf{h}}^{-1}\bar{\mathbf{A}}_{\mathbf{h}}) := \|\hat{\mathbf{C}}_{\mathbf{h}}^{-1}\bar{\mathbf{A}}_{\mathbf{h}}\|_{\mathbf{P_{h,1}}}\|\bar{\mathbf{A}}_{\mathbf{h}}^{-1}\hat{\mathbf{C}}_{\mathbf{h}}\|_{\mathbf{P_{h,1}}} \leq 2.$$

Following the approach of Section 2, we can even do better. Inspired by the parameter-robust preconditioner (17) we propose the following preconditioner for the non-symmetric case

$$\bar{\mathbf{C}}_{\mathbf{h}} = \begin{pmatrix} \mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}} & \mathbf{0} \\ \mathbf{0} & -(\mathbf{K_h} + \mathbf{M}_{\omega\sigma,\mathbf{h}}) \end{pmatrix}.$$

Again we can verify an inf-sup condition and a sup-sup condition in a non-standard norm, and analogous to the symmetric case we obtain the condition number estimate

$$\kappa_{\mathbf{P_{h,1}}}(\bar{\mathbf{C}}_{\mathbf{h}}^{-1}\bar{\mathbf{A}}_{\mathbf{h}}) := \|\bar{\mathbf{C}}_{\mathbf{h}}^{-1}\bar{\mathbf{A}}_{\mathbf{h}}\|_{\mathbf{P_{h,1}}}\|\bar{\mathbf{A}}_{\mathbf{h}}^{-1}\bar{\mathbf{C}}_{\mathbf{h}}\|_{\mathbf{P_{h,1}}} \leq \sqrt{2}. \tag{31}$$

Anyhow, a condition number estimate is not enough for GMRes convergence (we need a field of value estimate, see e.g. [37]). Nevertheless, the condition number estimate indicates, that the preconditioner $\bar{\mathbf{C}}_{\mathbf{h}}$ is the right one to be used in the GMRes or QMR method applied to (30).

7. Conclusion

The method developed in this work shows great potential for solving time-harmonic and time-periodic eddy current problems in an efficient and optimal way. The key ingredients of our method are the usage of a non-standard time discretization technique in terms of a truncated Fourier series, and the construction of parameter-independent solvers for the resulting system of equations in the frequency domain by a special operator interpolation technique. The theory developed in this paper allows us to establish a theoretical estimate of the convergence rate of MinRes as a solver when our proposed preconditioners are applied. Numerical experiments confirm these convergence rate estimates. Due to the natural decoupling of the frequency domain equations an efficient parallel implementation of the solution procedure is straight-forward.

In the non-linear case, i.e. $\nu = \nu(|\mathbf{curl}\,\mathbf{u}|)$, it turns out, that even for a harmonic excitation of the right-hand side, we have to take all other frequencies $k\omega$ into account, see e.g. [3, 45, 16]. Additionally, due to the nonlinearity, we lose the advantageous block-diagonal structure, and, therefore, we have to deal with a fully-coupled system of non-linear equations in the Fourier coefficients. Since the Fréchet derivative of the non-linear frequency domain equations is explicitly computable, the nonlinearity can easily be overcome by applying Newton's method. Anyhow, at each step of Newton's iteration, a huge and fully block-coupled Jacobi system with sparse blocks has to be solved. It turns out, that the Jacobi system is not symmetric, and therefore the preconditioned MinRes method is not suitable any more. Anyhow, due to § 6.2 the preconditioners developed in this work are very promising to be usable also in the non-linear case.

References

[1] D. N. Arnold, R. S. Falk, and R. Winther. Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$. *Numer. Math.*, 85(2):197–217, 2000.

[2] I. Babuška. Error-bounds for finite element method. *Numer. Math.*, 16(4):322–333, 1971.

[3] F. Bachinger, M. Kaltenbacher, and S. Reitzinger. An Efficient Solution Strategy for the HBFE Method. In *Proceedings of the IGTE '02 Symposium Graz, Austria*, pages 385–389, 2002.

[4] F. Bachinger, U. Langer, and J. Schöberl. Numerical analysis of nonlinear multiharmonic eddy current problems. *Numer. Math.*, 100(4):593–616, 2005.

[5] F. Bachinger, U. Langer, and J. Schöberl. Efficient solvers for nonlinear time-periodic eddy current problems. *Comput. Vis. Sci.*, 9(4):197–207, 2006.
[6] J. Bergh and J. Löfström. *Interpolation spaces: An introduction*. Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[7] J. H. Bramble, J. E. Pasciak, and P. S. Vassilevski. Computational scales of Sobolev norms with application to preconditioning. *Math. Comp.*, 69(230):463–480, 2000.
[8] F. Brezzi. On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge*, 8(R-2):129–151, 1974.
[9] J. Chen, Z. Chen, T. Cui, and L.-B. Zhang. An adaptive finite element method for the eddy current model with circuit/field couplings. *SIAM J. Sci. Comput.*, 32(2):1020–1042, 2010.
[10] D. Copeland, M. Kolmbauer, and U. Langer. Domain decomposition solvers for frequency-domain finite element equation. In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *LNCSE*, pages 301–308, Heidelberg, 2011. Springer.
[11] D. Copeland and U. Langer. Domain Decomposition Solvers for Nonlinear Multiharmonic Finite Element Equations. *J. Numer. Math.*, 18(3):157–175, 2010.
[12] T. Cui. Efficient parallel adaptive computation of 3d time-harmonic maxwell's equations using the toolbox phg. In *High Performance Computing and Communications, 2008. HPCC '08. 10th IEEE International Conference on*, pages 876 –881, sept. 2008.
[13] T. A. Davis. Algorithm 832: Umfpack v4.3—an unsymmetric-pattern multifrontal method. *ACM Trans. Math. Softw.*, 30:196–199, June 2004.
[14] T. A. Davis. A column pre-ordering strategy for the unsymmetric-pattern multifrontal method. *ACM Trans. Math. Softw.*, 30:165–195, June 2004.
[15] T. A. Davis and I. S. Duff. A combined unifrontal/multifrontal method for unsymmetric sparse matrices. *ACM Trans. Math. Softw.*, 25:1–20, March 1999.
[16] H. de Gersem, H. V. Sande, and K. Hameyer. Strong coupled multiharmonic finite element simulation package. *COMPEL*, 20:pp. 535–546, 2001.
[17] R. D. Falgout and P. S. Vassilevski. On generalizing the algebraic multigrid framework. *SIAM J. Numer. Anal.*, 42(4):1669–1693 (electronic), 2004.
[18] R. W. Freund and N. M. Nachtigal. QMR: a quasi-minimal residual method for non-Hermitian linear systems. *Numer. Math.*, 60(3):315–339, 1991.
[19] A. Greenbaum. *Iterative methods for solving linear systems*, volume 17 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1997.
[20] J. Gyselinck, P. Dular, C. Geuzaine, and W. Legros. Harmonic-balance finite-element modeling of electromagnetic devices: a novel approach. *Magnetics, IEEE Transactions on*, 38(2):521 –524, mar. 2002.
[21] R. Hiptmair. Multigrid method for Maxwell's equations. *SIAM J. Numer. Anal.*, 36(1):204–225 (electronic), 1999.
[22] R. Hiptmair. Symmetric coupling for eddy current problems. *SIAM J. Numer. Anal.*, 40(1):41–65 (electronic), 2002.
[23] R. Hiptmair. Operator preconditioning. *Comput. Math. Appl.*, 52(5):699–706, 2006.
[24] R. Hiptmair and J. Xu. Nodal auxiliary space preconditioning in $\mathbf{H}(\mathbf{curl})$ and $\mathbf{H}(\mathrm{div})$ spaces. *SIAM J. Numer. Anal.*, 45(6):2483–2509 (electronic), 2007.
[25] Q. Hu and J. Zou. A nonoverlapping domain decomposition method for Maxwell's equations in three dimensions. *SIAM J. Numer. Anal.*, 41(5):1682–1708 (electronic), 2003.
[26] M. Kollmann and M. Kolmbauer. A preconditioned minres solver for time-periodic optimal control problems. NuMa-Report 2011-06, Institute of Computational Mathematics, JKU Linz, Linz, August 2011.
[27] M. Kollmann, M. Kolmbauer, U. Langer, M. Wolfmayr, and W. Zulehner. A finite element solver for a multiharmonic parabolic optimal control problem. NuMa-Report 2011-10, Institute of Computational Mathematics, JKU Linz, Linz, December 2011.
[28] M. Kolmbauer and U. Langer. A robust fem-bem solver for time-harmonic eddy current problems. NuMa-Report 2011-05, Institute of Computational Mathematics, JKU Linz, Linz, May 2011.
[29] M. Kolmbauer and U. Langer. A robust preconditioned-minres-solver for distributed time-periodic eddy current optimal control problems. NuMa-Report 2011-04, Institute of Computational Mathematics, JKU Linz, Linz, May 2011.
[30] M. Kolmbauer and U. Langer. A frequency-robust solver for the time-harmonic eddy current problem. In *Scientific Computing in Electrical Engineering SCEE 2010*, volume 16 of *Mathematics in Industry*, pages 97–105. Springer Berlin Heidelberg, 2012.

[31] F. Kramer, M. Bebendorf, J. Ostrowski, and R. Hiptmair. Hierarchical matrix preconditioners for eddy current problems. In *Electromagnetics in Advanced Applications, 2009. ICEAA '09. International Conference on*, pages 724 –727, sept. 2009.

[32] J. Kraus. Additive schur complement approximation and application to multilevel preconditioning. RICAM Report 2011-22, Johann Radon Institute for Computational and Applied Mathematics (RICAM) of the Austrian Academy of Sciences, Linz, August 2011.

[33] J. Kraus and S. Margenov. *Robust algebraic multilevel methods and algorithms*. Walter de Gruyter, 2009.

[34] J. K. Kraus. Algebraic multilevel preconditioning of finite element matrices using local Schur complements. *Numer. Linear Algebra Appl.*, 13(1):49–70, 2006.

[35] M. Kuhn. *Efficient Parallel Numerical Simulation of Magnetic Field Problems*. PhD thesis, Johannes Kepler University, Linz, 1998.

[36] Y. A. Kuznetsov. Efficient iterative solvers for elliptic finite element problems on nonmatching grids. *Russ. J. Numer. Anal. Math. Model.*, 10:187–211, 1995.

[37] D. Loghin and A. J. Wathen. Analysis of preconditioners for saddle-point problems. *SIAM J. Sci. Comput.*, 25(6):2029–2049 (electronic), 2004.

[38] K. Mardal and R. Winther. Preconditioning discretizations of systems of partial differential equations. *Numer. Linear Algebra Appl.*, 18(1):1–40, 2011.

[39] P. Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[40] M. F. Murphy, G. H. Golub, and A. J. Wathen. A note on preconditioning for indefinite linear systems. *SIAM J. Sci. Comput.*, 21:1969–1972, 2000.

[41] J.-C. Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[42] J.-C. Nédélec. A new family of mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 50(1):57–81, 1986.

[43] M. A. Olshanskii and A. Reusken. On the convergence of a multigrid method for linear reaction-diffusion problems. *Computing*, 65(3):193–202, 2000.

[44] C. C. Paige and M. A. Saunders. Solutions of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 12(4):617–629, 1975.

[45] G. Paoli, O. Biro, and G. Buchgraber. Complex representation in nonlinear time harmonic eddy current problems. *Magnetics, IEEE Transactions on*, 34(5):2625 –2628, Sep. 1998.

[46] S. Reitzinger and J. Schöberl. An algebraic multigrid method for finite element discretizations with edge elements. *Numer. Linear Algebra Appl.*, 9(3):223–238, 2002.

[47] Y. Saad and M. H. Schultz. GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Statist. Comput.*, 7(3):856–869, 1986.

[48] J. Schöberl and W. Zulehner. Symmetric indefinite preconditioners for saddle point problems with applications to PDE-constrained optimization problems. *SIAM J. Matrix Anal. Appl.*, 29(3):752–773 (electronic), 2007.

[49] A. Toselli. Overlapping Schwarz methods for Maxwell's equations in three dimensions. *Numer. Math.*, 86(4):733–752, 2000.

[50] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

[51] J. Xu and Y. Zhu. Robust preconditioner for H(curl) interface problems. In *Domain Decomposition Methods in Science and Engineering XIX*, volume 78 of *LNCSE*, pages 173–180, Heidelberg, 2011. Springer.

[52] S. Yamada and K. Bessho. Harmonic field calculation by the combination of finite element analysis and harmonic balance method. *Magnetics, IEEE Transactions on*, 24(6):2588 –2590, nov 1988.

[53] W. Zulehner. Nonstandard norms and robust estimates for saddle point problems. *SIAM J. Matrix Anal. Appl.*, 32:536 – 560, 2011.

(M. Kolmbauer) Institute of Computational Mathematics, JKU Linz
E-mail address: kolmbauer@numa.uni-linz.ac.at

(U. Langer) RICAM, Austrian Academy of Sciences, Linz
E-mail address: ulanger@numa.uni-linz.ac.at

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