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Beam Theory

Bakkalaureatsarbeit

zur Erlangung des akademischen Grades

Bakkalaureus der Technischen Wissenschaften

in der Studienrichtung

Technische Mathematik

Angefertigt am *Institut für Numerische Mathematik*

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Linz, November 2018

Abstract

Beam Theory deals with the question if it is possible to calculate the displacement of a beam when a load is applied to it. Since the 19th century it is an important part of engineering and even Galileo Galilei or Leonardo da Vinci tried to bring up a feasible model for beams, but failed to do so due to missing important material laws (Hooke's law). Only in the 18th century Leonhard Euler and Daniel Bernoulli succeeded to develop a useful theory for beam problems, although it should take the world another 100 years to bring this theory to practical uses. This model is the so called Euler-Bernoulli model, which was later further generalised in the Timoshenko model. In this thesis only the Euler-Bernoulli model will be presented and discussed.

Since the 19th century starting during the industrial revolution beam theory, as part of continuum mechanics, is used in many of the greater building projects of this time, for example the Eiffel Tower. It is also essential for building bridges and is an important part of engineering until today.

Zusammenfassung

In der Balkentheorie beschäftigt man sich mit der Frage, ob man modellieren kann wie sich ein Balken unter Belastung verbiegt. Seit dem 19. Jahrhundert ist es ein wichtiger Bestandteil des Ingenieurwesens und schon Galileo Galilei oder Leonardo da Vinci beschäftigten sich mit der Balkentheorie, jedoch führten diese Bemühungen zunächst zu nichts, da teils wichtige Materialgesetze fehlten (Hooke'sches Gesetz). Erst im 18. Jahrhundert gelang es Leonhard Euler und Daniel Bernoulli eine brauchbares Modell aufzustellen, obwohl es noch 100 Jahre länger dauern sollte bis diese Theorie praktische Anwendung fand. Dieses Modell ist das sogenannte Euler-Bernoulli Modell, das später im 19. Jahrhundert eine Verallgemeinerung im Timoshenko Modell fand. Weiters wird in dieser Arbeit nur das Euler-Bernoulli Modell behandelt.

Die Balkentheorie, als Teil der Kontinuumsmechanik, fand im Zuge der industriellen Revolution Anwendung in vielen größeren Bauprojekten und Konstruktionen des 19. Jahrhunderts wie zum Beispiel dem Eiffel Turm. Sehr wichtig ist diese Modellierung im Brückenbau und bis heutzutage ist sie ein wichtiger Bestandteil im Bauingenieurwesen.

Acknowledgments

I would like to thank my supervisor A. Univ. Prof. Dipl. Ing. Dr. Walter Zulehner for letting me write this thesis, the guidance through the theory and answering all my questions on the matter while working on this thesis.

Special thanks to O. Univ. Prof. Dipl. Ing. Dr. Ulrich Langer and Dipl.-Ing. Dr. Martin Neumüller for the lectures 'Mathematical Models in Engineering' and 'Numerical Methods for Partial Differential Equations', which helped me a lot in writing this thesis and understanding the theory.

Also special thanks to my parents and my brother for supporting me during my studies.

And i would like to thank my study colleagues during the last years and also for the years to come.

Stefan Tyoler
Linz, November 2018

Contents

1 Introduction

In this thesis the goal is to model a thin beam (length of beam » area of cross section), on which a transversal load is applied. For thin beams it suffices to use a 1D model. The function $w(x)$, which we want to compute is the displacement of the beam and describes the deflection of the beam in transversal direction (direction in which the load is applied) at some position x. Other than for a beam with longitudinal load, one will get not a differential equation of order 2 but a differential equation of order 4.

First the 1D model (also called Euler-Bernoulli model) will be derived from the 3D model. This method is also known as Reduction of Dimension. From this point on the weak form of the problem will be discussed more precisely and the existence and uniqueness of a solution will be shown.

Furthermore the Finite Element Method will be applied for this weak problem and also an algorithm will be presented, which computes an approximate solution to the function w. This algorithm was implemented in C++ and at last some numerical results will be shown based on the results of the algorithm.

1.1 Assumptions for the Euler-Bernoulli model

Further there are some assumptions for the beam, that will be modelled. The x_1-axis will be chosen as the middle line of the beam, the x_3-axis will point in the direction where the load is applied from.

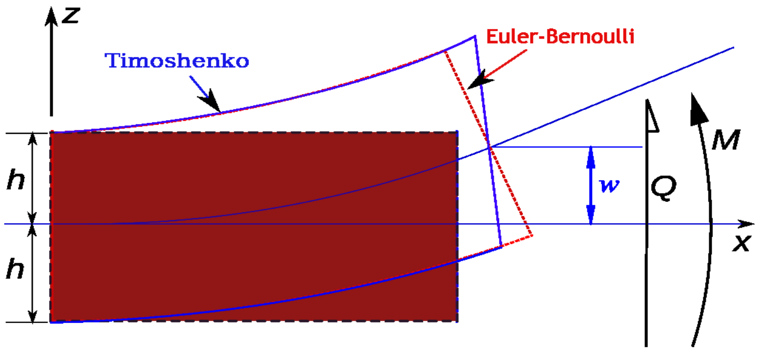

Figure 1.1: Beam model

Assumptions:

- Lines orthogonal to the middle line remain orthogonal after deformation.
- Only small deformations.
- No torsion around the x_1-Axis.
- No longitudinal load along the x_1-Axis.

One can see that in the Euler-Bernoulli model orthogonality w.r.t the middle line is preserved, whereas this is not the case in the Timoshenko model, as shown in Figure 1.1 For simplicity we will further assume the cross section of the beam is constant w.r.t x_1 so

$$Q(x_1) = Q$$

2 Deriving an Equation

To derive a differential equation for the beam problem one can use a direct approach by deriving an equilibrium with Hooke's law and the constitutive law in 1D, see e.g. [2], [3] and [6].

In this chapter a more physical approach will be chosen to derive a 1D-model. First of all we will look at the 3D-model.

2.1 3D-model

Let $\Omega \subset \mathbb{R}^3$ be the beam as a 3D object. On this beam there is an external force density vector $f(x)$ for $x \in \Omega$ applied which can be written as

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix}.$$

The function of displacement for $x \in \Omega$ will be denoted as

$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}.$$

For simplicity we will only look at fixed boundary conditions for the beam, i.e. the displacement of the beam at the sides is 0. (boundary conditions)

We now know from a physical point of view that the displacement function u minimizes the Ritz-energy functional

$$J(v) := \frac{1}{2} \int_\Omega \sigma(x) : \epsilon(x) \, dx - \int_\Omega f \cdot v \, dx,$$

and

$$J(u) = \min_v J(v),$$

where $\sigma \in \mathbb{R}^{3\times3}$ is the stress tensor and $\epsilon \in \mathbb{R}^{3\times3}$ is the strain tensor. The ':' - operator is defined as the euclidean scalar product between 2 matrices, if one considers 3x3 matrices as vectors of length 9.

$$\sigma(x) = \begin{pmatrix} \sigma_{11}(x) & \sigma_{12}(x) & \sigma_{13}(x) \\ \sigma_{21}(x) & \sigma_{22}(x) & \sigma_{23}(x) \\ \sigma_{31}(x) & \sigma_{32}(x) & \sigma_{33}(x) \end{pmatrix}, \epsilon(x) = \begin{pmatrix} \epsilon_{11}(x) & \epsilon_{12}(x) & \epsilon_{13}(x) \\ \epsilon_{21}(x) & \epsilon_{22}(x) & \epsilon_{23}(x) \\ \epsilon_{31}(x) & \epsilon_{32}(x) & \epsilon_{33}(x) \end{pmatrix}.$$

Furthermore there are the material law and the constitutive law for the 3D-model.

Material law:

$$\sigma = 2\mu\epsilon + \lambda tr(\epsilon)I \quad \text{in } \Omega,$$

where λ and μ are the so called Lamé-Coefficients, which are closely related to the elastic modulus. tr(ϵ) is the trace of the strain tensor and I is the identity matrix.

Constitutive law:

$$\epsilon = \frac{1}{2}(\nabla u + (\nabla u)^T) \quad \text{in } \Omega.$$

2.2 Reduction of Dimension

By inserting the data from the Euler-Bernoulli beam one gets

$$f(x) = \begin{pmatrix} 0 \\ 0 \\ -f_3(x) \end{pmatrix}$$

for the external force density (force is only applied from the x_3-direction), here f_3 is signed as negative because the force is applied from above.

For $u(x)$ we need to take some modelling errors into account. First of all we will denote

$$u_3(x_1, x_2, x_3) = w(x_1),$$

because w is our function for the displacement in x_3 direction in the 1D model. But by making u_3 only depend on x_1 we are basically assuming every point in a cross section has the same displacement in x_3 direction, which is of course not entirely true.

For u_2 we just get that

$$u_2(x_1, x_2, x_3) = 0,$$

because if there is only a load applied from x_3 then there is no displacement in x_2

And finally for u_1:

When looking at Figure 2.1 one can see that

$$w'(x_1) = tan(\phi(x_1)) \approx \phi(x_1) \quad \text{for small deformations, i.e. for small } \phi,$$

$$u_1(x_1, x_2, x_3) = -x_3 sin(\phi(x_1)) \approx -x_3\phi(x_1) \quad \text{for small deformations, i.e. for small } \phi,$$

and we get

$$u_1(x_1, x_2, x_3) \approx -x_3\phi(x_1) \approx -x_3 w'(x_1).$$

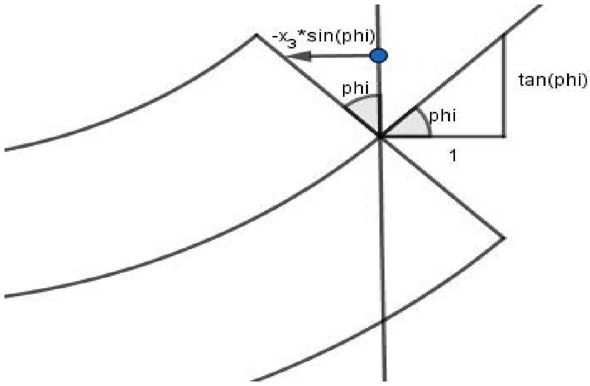

Figure 2.1: sketch

Then the displacement vector is

$$u(x_1, x_2, x_3) = \begin{pmatrix} -x_3 w'(x_1) \\ 0 \\ w(x_1) \end{pmatrix}.$$

The gradient of u is

$$\nabla u(x) = \begin{pmatrix} -x_3 w''(x_1) & 0 & -w'(x_1) \\ 0 & 0 & 0 \\ w'(x_1) & 0 & 0 \end{pmatrix}.$$

And by further inserting into the stress- and strain we get

$$\epsilon(x) = \begin{pmatrix} -x_3 w''(x_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} 2\mu\epsilon_{11}(x) + \lambda\epsilon_{11}(x) & 0 & 0 \\ 0 & \lambda\epsilon_{11}(x) & 0 \\ 0 & 0 & \lambda\epsilon_{11}(x) \end{pmatrix}.$$

The problem now is that this model is not entirely true for a 1D-model, so it will be modified a bit by either changing the material law and committing to the constitutive law or vice versa.
In this case we will omit the constitutive law but keep the material law.
we modify the 3D-model by enforcing a linear stress state

$$\epsilon(x) = \begin{pmatrix} \epsilon_{11} & 0 & 0 \\ 0 & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix}, \quad \sigma(x) = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \underline{0} & 0 \\ 0 & 0 & \underline{0} \end{pmatrix}.$$

With the material law

$$\sigma = 2\mu\epsilon + \lambda tr(\epsilon)I$$

and by enforcing

$$\sigma_{22} = 0 \ , \ \ \sigma_{33} = 0$$

we will now solve for ϵ_{22} and ϵ_{33}:

$$\begin{aligned}
&\sigma_{22} = 2\mu\epsilon_{22} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0 \\
&\sigma_{33} = 2\mu\epsilon_{33} + \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) = 0 \\
\Rightarrow \quad &\epsilon_{22} = \epsilon_{33} \\
&2\mu\epsilon_{22} + \lambda(\epsilon_{11} + 2\epsilon_{22}) = 0 \\
\Rightarrow \quad &2(\mu + \lambda)\epsilon_{22} + \lambda\epsilon_{11} = 0 \\
\Rightarrow \quad &\epsilon_{22} = \epsilon_{33} = -\frac{\lambda}{2(\lambda + \mu)}\epsilon_{11},
\end{aligned}$$

and compute

$$\begin{aligned}
\sigma_{11} = 2\mu\epsilon_{11} + \lambda tr(\epsilon) \quad &= 2\mu\epsilon_{11} + \lambda(\epsilon_{11} - \frac{2\lambda}{2(\lambda + \mu)}\epsilon_{11}) \\
&= (2\mu + \lambda(1 - \frac{\lambda}{\lambda + \mu}))\epsilon_{11} \\
&= \frac{2\mu(\lambda + \mu) + \lambda\mu}{\lambda + \mu}\epsilon_{11} = \underbrace{\frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}}_{\text{E}}\epsilon_{11}.
\end{aligned}$$

$$\Rightarrow \quad \sigma_{11} = E\epsilon_{11}$$

and obtain a linear stress state with factor E (the elastic modulus).
If we now insert u, σ and ϵ in the energy functional J we get

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_\Omega \begin{pmatrix} \sigma_{11}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \epsilon_{11}(x) & 0 & 0 \\ 0 & \epsilon_{22}(x) & 0 \\ 0 & 0 & \epsilon_{33}(x) \end{pmatrix} dx - \int_\Omega \begin{pmatrix} 0 \\ 0 \\ -f_3(x) \end{pmatrix} \cdot \begin{pmatrix} -x_3 w'(x_1) \\ 0 \\ w(x_1) \end{pmatrix} dx \\
&= \frac{1}{2}\int_\Omega \sigma_{11}\epsilon_{11}\, dx - \int_\Omega -f_3(x)w(x_1)\, dx \\
&= \frac{1}{2}\int_0^L \int_Q E\epsilon_{11}^2\, dx_1\, dQ - \int_0^L \int_Q -f_3(x)w(x_1)\, dx_1\, dQ \\
&= \frac{1}{2}E\int_0^L \underbrace{\int_Q x_3^2\, dQ}_{\text{I}}(w''(x_1))^2\, dx_1 - \int_0^L -\underbrace{\int_Q f_3(x)\, dQ}_{\text{q}(x_1)}\, w(x_1)\, dx_1 \\
&= \frac{1}{2}EI\int_0^L (w''(x_1))^2\, dx_1 - \int_0^L -q(x_1)w(x_1)\, dx_1.
\end{aligned}$$

Remark 2.1. *The function $q(x_1)$ can be interpreted as the load applied to a whole cross section of the beam and depends on x_1.*

Remark 2.2. *I is also called the second moment of area (ger. 'Flächenträgheitsmoment'). It depends on the cross section Q. Since Q is constant here, I is also a constant.*

We now define a new energy functional:

$$\bar{J}(v) := \frac{1}{2} EI \int_0^L (v''(x_1))^2 \, dx_1 - \int_0^L -q(x_1)v(x_1) \, dx_1$$

Since u minimizes J, it follows that w is a minimizer for $\bar{J}$.
For deriving a weak form we also need the following lemma.

Lemma 2.3. *Let V be a Hilbert space, V_0 a closed subspace of V, $V_g := g + V_0$ with $g \in V$, $a : V \times V \to \mathbb{R}$ a bounded symmetric, non-negative bilinear form, Q a bounded linear functional. Let the energy functional be defined as*

$$J(v) := \frac{1}{2} a(v, v) - \langle Q, v \rangle \quad \forall v \in V,$$

then the following equivalence holds for $w \in V_g$

$$J(w) = \min_{v \in V_g} J(v) \iff a(w, v) = \langle Q, v \rangle \quad \forall v \in V_0.$$

Remark 2.4. *A suitable Hilbert space and subspaces for the solutions of the beam model will be introduced in chapter 3.*

Remark 2.5. *A bounded, symmetric, non negative bilinear form $a : V \times V \to \mathbb{R}$ fulfills the following requirements:*

- *Linearity and homogeneity in both arguments:*

$$a(v_1 + v_2, v_3) = a(v_1, v_3) + a(v_2, v_3)$$

$$a(v_1, v_2 + v_3) = a(v_1, v_2) + a(v_1, v_3)$$

$$a(\lambda v_1, v_2) = \lambda a(v_1, v_2) = a(v_1, \lambda v_2)$$

- *it is symmetric:*

$$a(v_1, v_2) = a(v_2, v_1) \quad \forall v_1, v_2 \in V$$

- *it is non negative:*

$$a(v_1, v_2) \geq 0 \quad \forall v_1, v_2 \in V$$

- *it is bounded:*

$$\exists C > 0 : a(v_1, v_2) \leq C \|v_1\| \|v_2\| \quad \forall v_1, v_2 \in V.$$

If we define now

$$a(u,v) := EI \int_0^L u''(x_1)v''(x_1)\,dx_1 \quad \text{and} \quad \langle Q, v\rangle := \int_0^L -q(x_1)v(x_1)\,dx_1,$$

then we can rewrite the energy functional

$$\bar{J}(v) := \frac{1}{2}EI \int_0^L (v''(x_1))^2\,dx_1 - \int_0^L -q(x_1)v(x_1)\,dx_1 = \frac{1}{2}a(v,v) - \langle Q, v\rangle,$$

and since w minimizes $\bar{J}$ it follows with Lemma 2.3 that w is solution of

$$EI \int_0^L w''(x_1)v''(x_1)\,dx_1 = -\int_0^L q(x_1)v(x_1)\,dx_1 \quad \forall v \in V_0, \tag{2.1}$$

or more abstract: Find $w \in V_g$ such that

$$a(w,v) = \langle Q, v\rangle \quad \forall v \in V_0, \tag{2.2}$$

and we have finally obtain a weak form for the beam-model. As mentioned above, the fitting function space will be introduced in the next chapter.

Remark 2.6. *The homogeneous boundary conditions of the 3D-model also enforce homogeneous essential boundary conditions on the 1D- function w, i.e.*

$$w(0) = w(L) = w'(0) = w'(L) = 0.$$

Remark 2.7. *The method of reduction of dimension can also be done for 2D-models, so called plate-models. The Kirchhoff-Love plate model is the equivalent to the Euler-Bernoulli model in 1D and leads to a partial differential equation of order 4.*

2.3 Strong Form

Under certain requirements for w and q one can derive the strong form of the equation. But first we need an important lemma.

Lemma 2.8 (Fundamental Lemma of the Calculus of Variations)**.**
Let $I = [a,b] \subset \mathbb{R}$ be a compact real interval and $g : I \to \mathbb{R}$ a continuous function. If

$$\int_a^b g(x)v(x)\,dx = 0 \quad \forall v \in C_0^\infty(a,b),$$

then g is the null-function, i.e.

$$g(x) = 0 \quad \forall x \in I.$$

Now we will have a look at the weak problem (2.1) and demand that $w \in C^4(0, L)$ and $q \in C(0, L)$

With partial Integration we can rewrite the weak problem.

$$
\begin{aligned}
& EI \int_0^L w''(x)v''(x)\,dx = -\int_0^L q(x)v(x)\,dx \quad \forall v \in V_0 \\
\Rightarrow \quad & -EI \int_0^L w'''(x)v'(x)\,dx + EIw''(x)v'(x)\Big|_0^L = -\int_0^L q(x)v(x)\,dx \\
\Rightarrow \quad & EI \int_0^L w''''(x)v(x)\,dx + EIw''(x)v'(x)\Big|_0^L - EIw'''(x)v(x)\Big|_0^L = -\int_0^L q(x)v(x)\,dx
\end{aligned}
$$

By choosing $v \in C_0^\infty(0, L) \subset V_0$ we get that

$$
\begin{aligned}
& EI \int_0^L w''''(x)v(x)\,dx + \underbrace{EIw''(x)v'(x)\Big|_0^L}_{=0} - \underbrace{EIw'''(x)v(x)\Big|_0^L}_{=0} = -\int_0^L q(x)v(x)\,dx \\
\Rightarrow \quad & \int_0^L (\underbrace{EIw''''(x) + q(x)}_{\in C(0,L)})v(x)\,dx = 0 \quad \forall v \in C_0^\infty(0, L) \\
\Rightarrow \quad & EIw''''(x) + q(x) = 0 \\
\Rightarrow \quad & -EIw''''(x) = q(x).
\end{aligned}
$$

Together with the essential boundary conditions for w we get an ordinary differential equation of order 4 with

$$-EIw''''(x) = q(x) \quad \forall x \in (0, L),$$

$$w(0) = w(L) = w'(0) = w'(L) = 0.$$

This is the classical differential equation for the Euler-Bernoulli beam model.

Remark 2.9. *One can show that every solution of the classical differential equation is also a solution of the weak problem by multiplying with a test function and partially integrating the equation.*

As shown above a solution of the weak problem is only a classical solution if it fulfills a certain degree of smoothness, here being 4 times continuously differentiable.

3 Weak Form Analysis

In this chapter existence and uniqueness of a solution w for (2.1) will be proven, using the theorem of Lax-Milgram. But first we need a suitable Hilbert space for the functions in (2.1). This chapter is mainly based on [4].

3.1 Sobolev Spaces H^1 and H^2

Definition 3.1 (Weak derivative).
The function g is called the weak derivative of f if

$$\int_\Omega f\varphi' = -\int_\Omega g\varphi \quad \forall \varphi \in C_0^\infty(\Omega),$$

Furthermore the second weak derivative g of f is defined as

$$\int_\Omega f\varphi'' = \int_\Omega g\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

In this thesis, the weak derivatives will be written in the usual notation f' respectively f''.

Definition 3.2 (Sobolev space).
The Sobolev Space H^1 over Ω is defined as

$$H^1(\Omega) := \left\{ v \in L^2(\Omega) \mid v' \in L^2(\Omega) \right\}.$$

The Sobolev Space H^2 over Ω is defined as

$$H^2(\Omega) := \left\{ v \in L^2(\Omega) \mid v' \in L^2(\Omega), v'' \in L^2(\Omega) \right\},$$

which can be written as

$$H^2(\Omega) = \left\{ v \in L^2(\Omega) \mid v' \in H^1(\Omega) \right\}$$

with

$$\|v\|_{H^1} := \sqrt{(v,v)_{H^1(\Omega)}} = \sqrt{\|v\|^2_{L^2(\Omega)} + \|v'\|^2_{L^2(\Omega)}} = \sqrt{\int_\Omega |v(x)|^2 \, dx + \int_\Omega |v'(x)|^2 \, dx},$$

$$\|v\|_{H^2} := \sqrt{(v,v)_{H^2(\Omega)}} = \sqrt{\|v\|^2_{L^2(\Omega)} + \|v'\|^2_{L^2(\Omega)} + \|v''\|^2_{L^2(\Omega)}} = \sqrt{\|v\|^2_{L^2(\Omega)} + \|v'\|^2_{H^1(\Omega)}}.$$

Additionally we can define semi norms on the Sobolev spaces

$$|v|_{H^1} := \|v'\|_{L^2},$$
$$|v|_{H^2} := \|v''\|_{L^2}.$$

Remark 3.3. *The Sobolev spaces H^k form Hilbert spaces.*

Remark 3.4. *Another useful property of Sobolev spaces is that the function space $C^k(\Omega)$ is dense in $H^k(\Omega)$*

Remark 3.5. *Since we need for our weak problem (2.1) that*

$$\int_0^L w''v'' < \infty \quad \Leftarrow \quad w'', v'' \in L^2(0, L).$$

So $H^2(0, L)$ would be a fitting function space for our solution and test functions.

3.1.1 Trace Operator

Since functions in Sobolev spaces are L^2-functions we cannot evaluate them at some point x, we have to introduce an operator to do this. Furthermore the trace operator is linear and bounded.

Lemma 3.6 (Trace operator)**.**
Let $\gamma_y : H^2 \Rightarrow \mathbb{R}$ and $\gamma_y' : H^2 \Rightarrow \mathbb{R}$ with

$$\gamma_y(v) = v(y) \quad , \quad \gamma_y'(v) = v'(y).$$

Then there exists c_{tr} such that

$$|v(y)| \leq c_{tr}\|v\|_{H^2} \quad , \quad |v'(y)| \leq c_{tr}\|v\|_{H^2}$$

with

$$c_{tr} = \sqrt{2} max(\frac{1}{\sqrt{L}}, \sqrt{L}).$$

Proof.
Let $v \in C^2$

$$
\begin{aligned}
&\Rightarrow \quad v(y) = v(x) - \int_y^x v'(s)\,ds \quad x, y \in [0, L] \\
&\Rightarrow \quad |v(y)| \leq |v(x)| + \int_y^x |v'(s)|\,ds \overset{\mathsf{CS}}{\leq} |v(x)| + \underbrace{\|1\|_{L^2}}_{\sqrt{L}} \|v'\|_{L^2} \\
&\Rightarrow \quad \int_0^L |v(y)|\,dx \leq \int_0^L |v(x)|\,dx + \sqrt{L}\int_0^L \|v'\|_{L^2}\,dx \overset{\mathsf{CS}}{\leq} \sqrt{L}\|v\|_{L^2} + L^{\frac{3}{2}}\|v'\|_{L^2} \\
&\Rightarrow \quad L|v(y)| \leq \sqrt{L}\|v\|_{L^2} + L^{\frac{3}{2}}\|v'\|_{L^2} \\
&\Rightarrow \quad |v(y)| \leq \frac{1}{\sqrt{L}}\|v\|_{L^2} + \sqrt{L}\|v'\|_{L^2} \leq \max(\frac{1}{\sqrt{L}}, \sqrt{L})(\|v\|_{L^2} + \|v'\|_{L^2}) \\
&\qquad \leq \sqrt{2}\max(\frac{1}{\sqrt{L}}, \sqrt{L})\underbrace{(\|v\|_{L^2}^2 + \|v'\|_{L^2}^2)^{\frac{1}{2}}}_{\|v\|_{H^1}} \leq \underbrace{\sqrt{2}\max(\frac{1}{\sqrt{L}}, \sqrt{L})}_{c_{tr}}\underbrace{(\|v\|_{L^2}^2 + \|v'\|_{L^2}^2 + \|v''\|_{L^2}^2)^{\frac{1}{2}}}_{\|v\|_{H^2}}
\end{aligned}
$$

Since C^2 is a dense subset of H^2 this inequality also holds for all $v \in H^2$.
Analogously one can show the same inequality for v'.

$$
\begin{aligned}
&\quad\; v'(y) = v'(x) - \int_y^x v''(s)\,ds \quad x, y \in [0, L] \\
&\Rightarrow \quad |v'(y)| \leq |v'(x)| + \int_y^x |v''(s)|\,ds \overset{\mathsf{CS}}{\leq} |v'(x)| + \underbrace{\|1\|_{L^2}}_{\sqrt{L}} \|v''\|_{L^2} \\
&\Rightarrow \quad \int_0^L |v'(y)|\,dx \leq \int_0^L |v'(x)|\,dx + \sqrt{L}\int_0^L \|v''\|_{L^2}\,dx \overset{\mathsf{CS}}{\leq} \sqrt{L}\|v'\|_{L^2} + L^{\frac{3}{2}}\|v''\|_{L^2} \\
&\Rightarrow \quad L|v'(y)| \leq \sqrt{L}\|v'\|_{L^2} + L^{\frac{3}{2}}\|v''\|_{L^2} \\
&\Rightarrow \quad |v'(y)| \leq \frac{1}{\sqrt{L}}\|v'\|_{L^2} + \sqrt{L}\|v''\|_{L^2} \leq \max(\frac{1}{\sqrt{L}}, \sqrt{L})(\|v'\|_{L^2} + \|v''\|_{L^2}) \\
&\qquad \leq \sqrt{2}\max(\frac{1}{\sqrt{L}}, \sqrt{L})(\|v'\|_{L^2}^2 + \|v''\|_{L^2}^2)^{\frac{1}{2}} \leq \underbrace{\sqrt{2}\max(\frac{1}{\sqrt{L}}, \sqrt{L})}_{c_{tr}}\underbrace{(\|v\|_{L^2}^2 + \|v'\|_{L^2}^2 + \|v''\|_{L^2}^2)^{\frac{1}{2}}}_{\|v\|_{H^2}}
\end{aligned}
$$

□

Remark 3.7. *In the first part of the proof we didn't use the fact that v is 2-times differentiable but only that it is differentiable and one can see that we got*

$$|v(y)| \leq c_{tr}\|v\|_{H^1}$$

in the last line of the proof, so we have also shown this inequality for $v \in H^1$.

3.1.2 Friedrichs Inequality

Another important inequality that is needed to use the theorem of Lax-Milgram on (2.1) is Friedrichs inequality.

Lemma 3.8 (Friedrichs inequality)**.**
Let

$$V_0 := \left\{ v \in H^2(0,L) \;\middle|\; v(0) = v(L) = v'(0) = v'(L) = 0 \right\},$$

then there exist $c_{F_1} > 0$ *and* $c_{F_2} > 0$ *such that*

$$\|v\|_{L^2} \leq c_{F_1}|v|_{H^1} \quad \textit{and} \quad \|v\|_{L^2} \leq c_{F_2}|v|_{H^2}.$$

Proof.
Let

$$C_0^2 := \left\{ v \in C^2 \;\middle|\; v(0) = v(L) = v'(0) = v'(L) = 0 \right\} \subset V_0,$$

which is dense in V_0 and let $v \in C_0^2$:

$$\begin{aligned}
&\Rightarrow \quad v(x) = \int_0^x v'(s)\,ds \quad x \in [0,L] \\
&\Rightarrow \quad |v(x)| \leq \int_0^x |v'(s)|\,ds \overset{\text{CS}}{\leq} (\int_0^x 1\,ds)^{\frac{1}{2}}|v|_{H^1} = \sqrt{x}|v|_{H^1} \\
&\Rightarrow \quad \int_0^L |v(x)|^2\,dx \leq \int_0^L x|v|_{H^1}^2\,dx = \frac{1}{2}L^2|v|_{H^1}^2 \\
&\Rightarrow \quad \|v\|_{L^2} \leq \underbrace{\frac{1}{\sqrt{2}}L}_{c_{F_1}}|v|_{H^1}
\end{aligned}$$

If we substitute v with v' we get an inequality which we later need to use the theorem of Lax-Milgram:

$$|v|_{H^1} \leq \frac{1}{\sqrt{2}}L|v|_{H^2}$$

Furthermore if $v \in C_0^2$:

$$
\begin{aligned}
&\Rightarrow \quad v(x) = \int_0^x v'(s)\, ds = \int_0^x \int_0^s v''(u)\, du\, ds \\
&\Rightarrow \quad |v(x)| \leq \int_0^x \int_0^s |v''(u)|\, du\, ds \overset{\text{CS}}{\leq} \int_0^x \Big(\int_0^s 1\, du\Big)^{\frac{1}{2}} |v|_{H^2}\, ds = \int_0^x s^{\frac{1}{2}} |v|_{H^2} \\
&\qquad = \frac{s^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^x |v|_{H^2} = \frac{2}{3}\sqrt{x^3}|v|_{H^2} \\
&\Rightarrow \quad \int_0^L |v(x)|^2\, dx \leq \int_0^L \frac{4}{9} x^3 |v|_{H^2}^2\, dx = \frac{1}{9} L^4 |v|_{H^2}^2 \\
&\Rightarrow \quad \|v\|_{L^2} \leq \underbrace{\frac{1}{3}}_{c_{F_2}} L^2\, |v|_{H^2}
\end{aligned}
$$

And by using the density of C_0^2 in V_0 it also holds for $v \in V_0$.

$\square$

3.2 Homogenisation

In this section we will see that we can transform the abstract problem (2.2) such that the space for the solution and the space for the test functions will be the same. This would also be a requirement for the theorem of Lax-Milgram in the next section.

Let V be a Hilbert space and $w \in V_g \subset V$ such that

$$a(w, v) = \langle Q, v \rangle \quad \forall v \in V_0 \subset V.$$

If there exists a function $g \in V_g$ such that $V_g = g + V_0$,

$$\Rightarrow w \in V_g \Rightarrow w = w_0 + g \quad \text{for a} \quad w_0 \in V_0$$

$$\Rightarrow a(w, v) = a(w_0 + g, v) = \langle Q, v \rangle$$

$$\Rightarrow a(w_0, v) = \underbrace{\langle Q, v \rangle - a(g, v)}_{\langle \tilde{\mathrm{Q}}, v \rangle}.$$

So we have transformed the problem:

Find $w_0 \in V_0$ such that

$$a(w_0, v) = \langle \tilde{\mathrm{Q}}, v \rangle \quad \forall v \in V_0.$$

In our case for

$$V_0 := \left\{ v \in H^2(0,L) \;\middle|\; v(0) = v(L) = v'(0) = v'(L) = 0 \right\},$$

it will always be possible to find a piecewise cubic function g such that

$$g(0) = g_0 \;\;,\;\; g(L) = g_L \;\;,\;\; g'(0) = d_0 \;\;,\;\; g'(L) = d_L,$$

where g_0,g_L,d_0,d_L are the essential boundary conditions for the wanted function w.

3.3 Theorem of Lax-Milgram

In this chapter we will finally be able to give an answer about the existence and uniqueness of a solution for (2.1). Essential is the following theorem.

Theorem 3.9 (Lax-Milgram).
Let H be a Hilbert space, $V \subset H$ a closed subspace of H. Let $a : H \times H \Rightarrow \mathbb{R}$ be a bilinear form. Let $Q : H \Rightarrow \mathbb{R}$ be a linear functional in H.

If there are constants c_1^a,c_2^a,$c > 0$ such that

- $|a(u,u)| \geq c_1^a \|u\|^2 \quad \forall u \in V$ *(elliptic bilinear form).*
- $|a(u,v)| \leq c_2^a \|u\| \|v\| \quad \forall u, v \in V$ *(bounded bilinear form).*
- $|\langle Q, v \rangle| \leq c\|v\| \quad \forall v \in V$ *(bounded functional).*

Then the equation

$$a(u,v) = \langle Q, v \rangle \quad \forall v \in V$$

has a unique solution $u \in V$.

Claim 3.10. *The weak problem (2.1) fulfills the requirements of the theorem of Lax-Milgram.*

Proof.

- Closed subspace:
  We start by checking if V_0 is a closed subspace of H^2. Trivially it is a subspace of H^2.

  For the trace operators we have the upper bounds

$$|v(y)| \leq c_{tr} \|v\|_{H^2},$$

$$|v'(y)| \leq c_{tr} \|v\|_{H^2}.$$

Since the trace operators are bounded, it follows that they are continuous and therefore the kernels for the respective trace operators at the boundary are closed.

$$
\begin{aligned}
ker_1 &= \left\{ v \in H^2(0,L) \;\middle|\; v(0) = 0 \right\} \\
ker_2 &= \left\{ v \in H^2(0,L) \;\middle|\; v(L) = 0 \right\} \\
ker_3 &= \left\{ v \in H^2(0,L) \;\middle|\; v'(0) = 0 \right\} \\
ker_4 &= \left\{ v \in H^2(0,L) \;\middle|\; v'(L) = 0 \right\}
\end{aligned}
$$

Finally we can see that

$$V_0 = ker_1 \cap ker_2 \cap ker_3 \cap ker_4$$

and thus V_0 is a closed subspace of H^2.

- Elliptic bilinear form:

  For the H^2-Norm the following inequality holds for all $u \in V_0$:

  $\|u\|_{H^2}^2 = \|u\|_{L^2}^2 + |u|_{H^1}^2 + |u|_{H^2}^2 \le c_{F_2}^2 |u|_{H^2}^2 + c_{F_1}^2 |u|_{H^2}^2 + |u|_{H^2} = (c_{F_2}^2 + c_{F_1}^2 + 1)|u|_{H^2}^2$

  and for the bilinear form we have

$$|a(u,u)| = EI \int_0^L |u''(x)|^2 \, dx = EI|u|_{H^2}^2 \ge \underbrace{EI \frac{1}{c_{F_2}^2 + c_{F_1}^2 + 1}}_{c_1^a} \|u\|_{H^2}^2.$$

- Bounded bilinear form:

$$|a(u,v)| = EI \int_0^L u''v'' \, dx \overset{\mathsf{CS}}{\le} EI|u|_{H^2}|v|_{H^2} \le EI\|u\|_{H^2}\|v\|_{H^2}$$

- Bounded functional:

$$
\begin{aligned}
|\langle Q, v\rangle| &= |\langle Q, v\rangle - a(g,v)| \\
&\le \int_0^L |qv| \, dx + |a(g,v)| \\
&\overset{\mathsf{CS}}{\le} \|q\|_{L^2}\|v\|_{L^2} + EI\|g\|_{H^2}\|v\|_{H^2} \le (\|q\|_{L^2} + EI\|g\|_{H^2})\|v\|_{H^2}
\end{aligned}
$$

So the weak problem (2.1) satisfies the requirements for the theorem of Lax-Milgram and has a unique solution w.

□

4 Finite Element Method

In this chapter the Finite Element Method will be applied to the weak problem (2.1). The main goal is to define a subspace $V_h \subset H^2$ with a finite basis and solve the problem over this subspace instead of the infinite dimensional space H^2. We will see that this will lead to a linear equation system, see e.g. [4] and [1].

4.1 Basis Functions

The subspace will be created by discretizing the interval $[0, L]$, choosing nodes and defining polynomial basis functions on these nodes. This will lead to piecewise polynomial functions which will later interpolate our solution w.

Of course one has to make sure that this piecewise polynomial functions are H^2-functions. The following lemma will help to decide which basis functions we need.

Remark 4.1. *The set of nodes used to discretize the beam is called a mesh, the intervals in between those nodes are called elements.*

Lemma 4.2.
For a piecewise differentiable function f it holds that

$$f \textit{ continuous} \iff f \in H^1.$$

It follows for a piecewise 2-times differentiable function f that

$$f\textit{' continuous} \iff f' \in H^1 \iff f \in H^2.$$

This means we need piecewise polynomial basis functions which are continuously differentiable. We will create those basis functions in such a way that one type of basis function (φ) will be used to interpolate the function values of w and the other type of basis functions (ψ) will be used to interpolate the function values of the derivative.

In other words

$$\varphi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} \quad , \quad \varphi_i'(x_j) = 0 \quad \forall j,$$

$$\psi_i(x_j) = 0 \quad \forall j \quad , \quad \psi_i'(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} .$$

We now need to check how to compute those basis functions and how they will be defined in between 2 nodes. For this we will look at the so called shape functions.

4.1.1 Shape Functions

Shape functions are basically the basis functions defined in between 2 nodes, i.e. on one element of the discretization. One could say a basis function is composed of 2 shape functions. For one interval there are 4 different shape functions.

For simplicity we will compute the shape functions only on the interval $[0, 1]$, the reference interval. The corresponding shape functions for arbitrary elements can be computed by using a transformation from the reference element.

The definition of the basis functions above will yield for the 4 shape functions $\hat{\varphi}_0$, $\hat{\varphi}_1$, $\hat{\psi}_0$, $\hat{\psi}_1 \in \mathbb{P}_3(0,1)$. That means a shape function has 4 degrees of freedom.

- $\hat{\varphi}_0(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned} \hat{\varphi}_0(0) &= 1 \quad & \hat{\varphi}_0(1) &= 0 \\ \hat{\varphi}_0'(0) &= 0 \quad & \hat{\varphi}_0'(1) &= 0 \end{aligned}$$

$$\begin{aligned} &\hat{\varphi}_0(0) = \underline{d = 1} \\ &\hat{\varphi}_0'(0) = \underline{c = 0} \\ &\left.\begin{aligned} \hat{\varphi}_0(1) = a + b + c + d = 0 \quad &\Rightarrow \quad a + b + 1 = 0 \\ \hat{\varphi}_0'(1) = 3a + 2b + c = 0 \quad &\Rightarrow \quad 3a + 2b = 0 \end{aligned}\right\} \Rightarrow \underline{b = -3} \ , \ \underline{a = 2}. \\ &\qquad\qquad \Rightarrow \quad \underline{\hat{\varphi}_0(x) = 2x^3 - 3x^2 + 1} \end{aligned}$$

- $\hat{\varphi}_1(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned} \hat{\varphi}_0(0) &= 0 \quad & \hat{\varphi}_0(1) &= 1 \\ \hat{\varphi}_0'(0) &= 0 \quad & \hat{\varphi}_0'(1) &= 0 \end{aligned}$$

$$\begin{aligned} &\hat{\varphi}_0(0) = \underline{d = 0} \\ &\hat{\varphi}_0'(0) = \underline{c = 0} \\ &\left.\begin{aligned} \hat{\varphi}_0(1) = a + b + c + d = 1 \quad &\Rightarrow \quad a + b = 0 \\ \hat{\varphi}_0'(1) = 3a + 2b + c = 0 \quad &\Rightarrow \quad 3a + 2b = 0 \end{aligned}\right\} \Rightarrow \underline{b = 3} \ , \ \underline{a = -2}. \\ &\qquad\qquad \Rightarrow \quad \underline{\hat{\varphi}_1(x) = -2x^3 + 3x^2} \end{aligned}$$

- $\hat{\psi}_0(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned} \hat{\psi}_0(0) &= 0 \quad \hat{\psi}_0(1) = 0 \\ \hat{\psi}_0'(0) &= 1 \quad \hat{\psi}_0'(1) = 0 \end{aligned}$$

$$\begin{aligned} &\hat{\psi}_0(0) = \underline{d = 0} \\ &\hat{\psi}_0'(0) = \underline{c = 1} \\ &\left.\begin{aligned} \hat{\psi}_0(1) = a + b + c + d = 0 \quad &\Rightarrow \quad a + b + 1 = 0 \\ \hat{\psi}_0'(1) = 3a + 2b + c = 0 \quad &\Rightarrow \quad 3a + 2b + 1 = 0 \end{aligned}\right\} \Rightarrow \underline{b = -2} \ , \ \underline{a = 1}. \\ &\qquad \Rightarrow \quad \underline{\hat{\psi}_0(x) = x^3 - 2x^2 + x} \end{aligned}$$

- $\hat{\psi}_0(x) = ax^3 + bx^2 + cx + d$

$$\begin{aligned} \hat{\psi}_1(0) &= 0 \quad \hat{\psi}_1(1) = 0 \\ \hat{\psi}_1'(0) &= 0 \quad \hat{\psi}_1'(1) = 1 \end{aligned}$$

$$\begin{aligned} &\hat{\psi}_1(0) = \underline{d = 0} \\ &\hat{\psi}_1'(0) = \underline{c = 0} \\ &\left.\begin{aligned} \hat{\psi}_1(1) = a + b + c + d = 0 \quad &\Rightarrow \quad a + b = 0 \\ \hat{\psi}_1'(1) = 3a + 2b + c = 1 \quad &\Rightarrow \quad 3a + 2b = 1 \end{aligned}\right\} \Rightarrow \underline{b = -1} \ , \ \underline{a = 1}. \\ &\qquad \Rightarrow \quad \underline{\hat{\psi}_1(x) = x^3 - x^2} \end{aligned}$$

These 4 polynomials are also called the Hermite basis functions:

$$\begin{aligned} \hat{\varphi}_0(x) &= 2x^3 - 3x^2 + 1 \\ \hat{\varphi}_1(x) &= -2x^3 + 3x^2 \\ \hat{\psi}_0(x) &= x^3 - 2x^2 + x \\ \hat{\psi}_1(x) &= x^3 - x^2 \end{aligned}$$

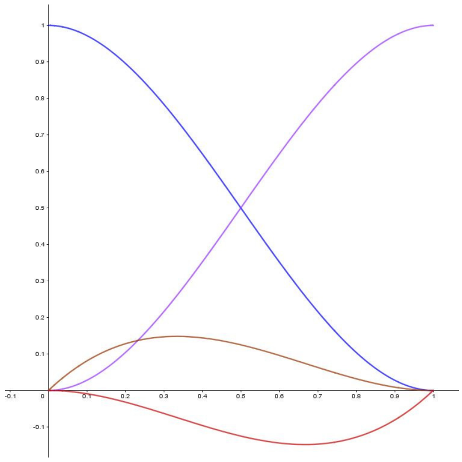

Figure 4.1: Hermite basis functions

Remark 4.3.
The function

$$F_k : [0,1] \to [x_{k-1}, x_k]$$

with

$$F_k(x) := x_{k-1} + (\underbrace{x_k - x_{k-1}}_{h_k})x$$

will be the transformation from the reference element and will be used to define the shape functions on all elements in the discretization. One can define the inverse transformation as well:

$$F_k^{-1}(x) := \frac{x - x_{k-1}}{x_k - x_{k-1}}$$

Also important to note is that

$$F_k'(x) = h_k \ \ , \ \ (F_k^{-1})'(x) = \frac{1}{h_k}.$$

We will now define the shape functions of an arbitrary element $T_k = [x_{k-1}, x_k]$ in the mesh of our beam:

$$\left.\begin{aligned}
\varphi_{k-1}(x) &:= \hat{\varphi}_0(F_k^{-1}(x)) \\
\varphi_k(x) &:= \hat{\varphi}_1(F_k^{-1}(x)) \\
\psi_{k-1}(x) &:= h_k\hat{\psi}_0(F_k^{-1}(x)) \\
\psi_k(x) &:= h_k\hat{\psi}_1(F_k^{-1}(x))
\end{aligned}\right\} \forall x \in T_k.$$

Respectively the whole basis functions can be defined as

$$\varphi_{k-1}(x) := \begin{cases} \hat{\varphi}_0(F_k^{-1}(x)) & x \in [x_{k-1}, x_k] \\ \hat{\varphi}_1(F_{k-1}^{-1}(x)) & x \in [x_{k-2}, x_{k-1}) \\ 0 & \text{else} \end{cases},$$

$$\psi_{k-1}(x) := \begin{cases} h_k\hat{\psi}_0(F_k^{-1}(x)) & x \in [x_{k-1}, x_k] \\ h_{k-1}\hat{\psi}_1(F_{k-1}^{-1}(x)) & x \in [x_{k-2}, x_{k-1}) \\ 0 & \text{else} \end{cases}.$$

It is easily checked that φ satisfies the requirements we proposed for a nodal basis for the function values namely

$$\varphi_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}, \ \varphi_i'(x_j) = 0 \quad \forall j.$$

The additional factor h_k in the definition of ψ is needed to guarantee the derivative value of 1 when 'stretching' the interval. Since we stretch the interval by the factor h_k we need to multiply the function with h_k.

$$\psi_{k-1}'(x_{k-1}) = \cancel{h_k}\hat{\psi}_0{}'(F_k^{-1}(x_{k-1}))\underbrace{(F_k^{-1})'(x_{k-1})}_{\frac{1}{\cancel{h_k}}} = \hat{\psi}_0{}'(0) = 1$$

Therefore ψ also satisfies

$$\psi_i(x_j) = 0 \quad \forall j \ , \ \psi_i'(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}.$$

Remark 4.4. *We will denote the basis functions exactly as the corresponding shape functions.*

With this basis we can now define the subspace $V_h \subset H^2$ and for every $v_h \in V_h$ we can write

$$v_h = \sum_{i=0}^{n} v_i\varphi_i + \sum_{i=0}^{n} v_i'\psi_i \ , \ v_i, v_i' \in \mathbb{R} \quad \forall i.$$

Remark 4.5. *The major advantage when choosing a nodal basis is that*

$$v_h(x_j) = v_j \ , \ v_h'(x_j) = v_j'.$$

so the coordinates in this basis are exactly the function values/derivative values.

With this subspace we can now define the approximate solution for the weak problem (2.1)

4.2 Approximate Solution

Definition 4.6. *For the weak problem (2.1)*

$$a(w,v) = \langle Q, v\rangle \quad \forall v \in V_0$$

the approximate solution $w_h \in V_{g,h}$ to the solution w is defined as the solution to the problem

$$a(w_h, v_h) = \langle Q, v_h\rangle \quad \forall v_h \in V_{0,h}. \tag{4.1}$$

To solve this problem we will make use of the representation of w_h and v_h as linear combination of the basis functions. As boundary conditions we will choose only essential boundary conditions (g_0,g_L,d_0,d_L).

Let

$$w_h = \underbrace{\sum_{i=1}^{n-1} w_i\varphi_i + \sum_{i=1}^{n-1} w_i'\psi_i}_{w_{0,h}\in V_{0,h}} + \underbrace{g_0\varphi_0 + d_0\psi_0 + g_L\varphi_n + d_L\psi_n}_{g_h} \quad , \quad v_h = \sum_{i=1}^{n-1} v_i\varphi_i + \sum_{i=1}^{n-1} v_i'\psi_i.$$

$$\begin{aligned}
& a(w_h, v_h) = \langle Q, v_h\rangle \quad \forall v_h \in V_{0,h} \\
\iff & \sum_{i=1}^{n-1} v_i a(w_h,\varphi_i) + \sum_{i=1}^{n-1} v_i' a(w_h,\psi_i) = \sum_{i=1}^{n-1} v_i\langle Q,\varphi\rangle + \sum_{i=1}^{n-1} v_i'\langle Q,\psi\rangle \quad \forall v_i, v_i' \in \mathbb{R} \\
& \text{Now we can choose the } v_i, v_i' \text{ such that} \\
\iff & a(w_h,\varphi_i) = \langle Q,\varphi_i\rangle \quad \forall i = 1, ..., n-1 \\
& a(w_h,\psi_i) = \langle Q,\psi_i\rangle \quad \forall i = 1, ..., n-1 \\
\iff & \sum_{j=1}^{n-1} (w_j a(\varphi_j,\varphi_i) + w_j' a(\psi_j,\varphi_i)) = \langle Q,\varphi_i\rangle - a(g_h,\varphi_i) \quad \forall i = 1, ..., n-1 \\
& \sum_{j=1}^{n-1} (w_j a(\varphi_j,\psi_i) + w_j' a(\psi_j,\psi_i)) = \langle Q,\psi_i\rangle - a(g_h,\psi_i) \quad \forall i = 1, ..., n-1
\end{aligned}$$

$$\iff$$

$$\underbrace{\begin{pmatrix} a(\varphi_1,\varphi_1) & a(\psi_1,\varphi_1) & a(\varphi_2,\varphi_1) & a(\psi_2,\varphi_1) & \cdots \\ a(\varphi_1,\psi_1) & a(\psi_1,\psi_1) & a(\varphi_2,\psi_1) & a(\psi_2,\psi_1) & \cdots \\ a(\varphi_1,\varphi_2) & a(\psi_1,\varphi_2) & a(\varphi_2,\varphi_2) & a(\psi_2,\varphi_2) & \cdots \\ a(\varphi_1,\psi_2) & a(\psi_1,\psi_2) & a(\varphi_2,\psi_2) & a(\psi_2,\psi_2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{K_h} \underbrace{\begin{pmatrix} w_1 \\ w_1' \\ w_2 \\ w_2' \\ \vdots \\ w_{n-1} \\ w_{n-1}' \end{pmatrix}}_{\underline{w}_h} = \underbrace{\begin{pmatrix} \langle Q,\varphi_1\rangle - a(g_h,\varphi_1) \\ \langle Q,\psi_1\rangle - a(g_h,\psi_1) \\ \langle Q,\varphi_2\rangle \\ \langle Q,\psi_2\rangle \\ \vdots \\ \langle Q,\varphi_{n-2}\rangle \\ \langle Q,\psi_{n-2}\rangle \\ \langle Q,\varphi_{n-1}\rangle - a(g_h,\psi_{n-1}) \\ \langle Q,\psi_{n-1}\rangle - a(g_h,\varphi_{n-1}) \end{pmatrix}}_{\underline{q}_h}$$

$$\iff K_h\underline{w}_h = \underline{q}_h$$

Remark 4.7. *The matrix K_h is called the stiffness matrix and the vector $\underline{q}_h$ is called the load vector*

Remark 4.8. *Since g_h is a linear combination of basis functions on the first and last element it follows that*

$$a(g_h,\varphi_i) = 0 \quad \forall i, 1 < i < n-1,$$

$$a(g_h,\psi_i) = 0 \quad \forall i, 1 < i < n-1.$$

So the additional terms in the load vector disappear, except for the first 2 and last 2 elements of the vector.

4.2.1 Stiffness Matrix

In this section we will compute the stiffness matrix which is needed for the linear equation system. One may notice that

$$a(f_i,g_j) = EI\int_0^L f_i''(x)g_j''(x)\,dx = 0 \qquad \text{for } |i-j| > 1 \text{ and } f,g \in \{\varphi,\psi\}$$

because the basis functions are defined on 2 neighbouring intervals, therefore only neighbouring basis functions have intersecting supports.

It follows that K_h is a band matrix with bandwidth 3. The computation of such a matrix will be done by first computing the element matrices, which will then be assembled into the global stiffness matrix.

As the name suggests the element matrices describe the values of the global stiffness matrix on one particular element.

The kth element matrix has the form

$$K_h^{(k)} = EI \int_{T_k} \begin{pmatrix} (\varphi''_{k-1})^2 & \psi''_{k-1}\varphi''_{k-1} & \varphi''_k\varphi''_{k-1} & \psi''_k\varphi''_{k-1} \\ \psi''_{k-1}\varphi''_{k-1} & (\psi''_{k-1})^2 & \varphi''_k\psi''_{k-1} & \psi''_k\psi''_{k-1} \\ \varphi''_k\varphi''_{k-1} & \varphi''_k\psi''_{k-1} & (\varphi''_k)^2 & \psi''_k\varphi''_k \\ \psi''_k\varphi''_{k-1} & \psi''_k\psi''_{k-1} & \psi''_k\varphi''_k & (\psi''_k)^2 \end{pmatrix} dx.$$

Since we have a symmetric bilinear form, we only need to compute the upper triangle of the matrix. Before integrating we need the following relation:

$$\hat{\varphi}_0''(x) = (\varphi_{k-1}(F_k(x)))'' = (\varphi'_{k-1}(F_k(x))\underbrace{F'_k(x)}_{h_k})' = h_k(\varphi'_{k-1}(F_k(x))' = h_k^2\varphi''_{k-1}(F_k(x))$$

$$\Rightarrow \quad \varphi''_{k-1}(F_k(x)) = \frac{1}{h_k^2}\hat{\varphi}_0''(x)$$

analogous

$$\Rightarrow \quad \varphi''_k(F_k(x)) = \frac{1}{h_k^2}\hat{\varphi}_1''(x)$$

$$\hat{\psi}_0''(x) = (\frac{1}{h_k}\psi_{k-1}(F_k(x)))'' = (\frac{1}{\cancel{h_k}}\psi'_{k-1}(F_k(x))\underbrace{F'_k(x)}_{\cancel{h_k}})' = (\psi'_{k-1}(F_k(x))' = h_k\psi''_{k-1}(F_k(x))$$

$$\Rightarrow \quad \psi''_{k-1}(F_k(x)) = \frac{1}{h_k}\hat{\psi}_0''(x)$$

analogous

$$\Rightarrow \quad \psi''_k(F_k(x)) = \frac{1}{h_k}\hat{\psi}_1''(x)$$

The second derivatives of the shape functions on the reference element:

$$\begin{aligned} \hat{\varphi}_0''(x) &= (2x^3 - 3x^2 + 1)'' = (6x^2 - 6x)' = \underline{12x - 6} \\ \hat{\varphi}_1''(x) &= (-2x^3 + 3x^2)'' = (-6x^2 + 6x)' = \underline{-12x + 6} \\ \hat{\psi}_0''(x) &= (x^3 - 2x^2 + x)'' = (3x^2 - 4x + 1)' = \underline{6x - 4} \\ \hat{\psi}_1''(x) &= (x^3 - x^2)'' = (3x^2 - 2x)' = \underline{6x - 2} \end{aligned}$$

$$k_{11} = \int_{T_k} (\varphi''_{k-1}(x))^2\,dx = \int_0^1 (\varphi''_{k-1}(F_k(x)))^2 \underbrace{F'_k(x)}_{h_k}\,dx = \frac{1}{h_k^4} h_k \int_0^1 (\hat{\varphi}_0{}''(x))^2\,dx$$

$$= \frac{1}{h_k^3}\int_0^1 (12x-6)^2\,dx = \frac{1}{h_k^3}\int_0^1 144x^2 - 144x + 36\,dx = \frac{1}{h_k^3}\left(\frac{144x^3}{3} - \frac{144x^2}{2} + 36x\right)\Big|_0^1$$

$$= \underline{\frac{1}{h_k^3}12}$$

$$k_{22} = \int_{T_k} (\psi''_{k-1}(x))^2\,dx = \int_0^1 (\psi''_{k-1}(F_k(x)))^2 \underbrace{F'_k(x)}_{h_k}\,dx = \frac{1}{h_k^2} h_k \int_0^1 (\hat{\psi}_0{}''(x))^2\,dx$$

$$= \frac{1}{h_k}\int_0^1 (6x-4)^2\,dx = \frac{1}{h_k}\int_0^1 36x^2 - 48x + 16\,dx = \frac{1}{h_k}\left(\frac{36x^3}{3} - \frac{48x^2}{2} + 16x\right)\Big|_0^1$$

$$= \underline{\frac{1}{h_k}4}$$

$$k_{33} = \int_{T_k} (\varphi''_{k}(x))^2\,dx = \int_0^1 (\varphi''_{k}(F_k(x)))^2 \underbrace{F'_k(x)}_{h_k}\,dx = \frac{1}{h_k^4} h_k \int_0^1 (\hat{\varphi}_1{}''(x))^2\,dx$$

$$= \frac{1}{h_k^3}\int_0^1 (-12x+6)^2\,dx = \frac{1}{h_k^3}\int_0^1 (12x-6)^2\,dx = \underline{\frac{1}{h_k^3}12}$$

$$k_{44} = \int_{T_k} (\psi''_{k}(x))^2\,dx = \int_0^1 (\psi''_{k}(F_k(x)))^2 \underbrace{F'_k(x)}_{h_k}\,dx = \frac{1}{h_k^2} h_k \int_0^1 (\hat{\psi}_1{}''(x))^2\,dx$$

$$= \frac{1}{h_k}\int_0^1 (6x-2)^2\,dx = \frac{1}{h_k}\int_0^1 36x^2 - 24x + 4\,dx = \frac{1}{h_k}\left(\frac{36x^3}{3} - \frac{24x^2}{2} + 4x\right)\Big|_0^1$$

$$= \underline{\frac{1}{h_k}4}$$

$$k_{12} = \int_{T_k} \psi''_{k-1}(x)\varphi''_{k-1}(x) = \int_0^1 \psi''_{k-1}(F_k(x))\varphi''_{k-1}(F_k(x)) \underbrace{F'_k(x)}_{h_k}\,dx = \frac{1}{h_k^3} h_k \int_0^1 \hat{\psi}_0{}''(x)\hat{\varphi}_0{}''(x)\,dx$$

$$= \frac{1}{h_k^2}\int_0^1 (6x-4)(12x-6)\,dx = \frac{1}{h_k^2}\int_0^1 (6x-4)(12x-6)\,dx = \frac{1}{h_k^2}\int_0^1 72x^2 - 84x + 24\,dx$$

$$= \frac{1}{h_k^2}\left(\frac{72x^3}{3} - \frac{84x^2}{2} + 24x\right)\Big|_0^1 = \underline{\frac{1}{h_k^2}6}$$

$$k_{13} = \int_{T_k} \varphi_k''(x)\varphi_{k-1}''(x) = \int_0^1 \varphi_k''(F_k(x))\varphi_{k-1}''(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = \frac{1}{h_k^4}h_k\int_0^1 \hat{\varphi}_1''(x)\hat{\varphi}_0''(x)\,dx$$

$$= \frac{1}{h_k^3}\int_0^1 (-12x+6)(12x-6)\,dx = -\frac{1}{h_k^3}\int_0^1 (12x-6)^2 = \underline{-\frac{1}{h_k^3}12}$$

$$k_{14} = \int_{T_k} \psi_k''(x)\varphi_{k-1}''(x) = \int_0^1 \psi_k''(F_k(x))\varphi_{k-1}''(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = \frac{1}{h_k^2}h_k\int_0^1 \hat{\psi}_1''(x)\hat{\varphi}_0''(x)\,dx$$

$$= \frac{1}{h_k^2}\int_0^1 (6x-2)(12x-6)\,dx = \frac{1}{h_k^2}\int_0^1 72x^2-60x+12\,dx$$

$$= \frac{1}{h_k^2}\left(\frac{72x^3}{3}-\frac{60x^2}{2}+12x)\right|_0^1 = \underline{\frac{1}{h_k^2}6}$$

$$k_{23} = \int_{T_k} \varphi_k''(x)\psi_{k-1}''(x) = \int_0^1 \varphi_k''(F_k(x))\psi_{k-1}''(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = \frac{1}{h_k^3}h_k\int_0^1 \hat{\varphi}_1''(x)\hat{\psi}_0''(x)\,dx$$

$$= \frac{1}{h_k^2}\int_0^1 (-12x+6)(6x-4)\,dx = \frac{1}{h_k^2}\int_0^1 -72x^2+84x-24\,dx$$

$$= \frac{1}{h_k^2}\left(-\frac{72x^3}{3}+\frac{84x^2}{2}-24x)\right|_0^1 = \underline{-\frac{1}{h_k^2}6}$$

$$k_{24} = \int_{T_k} \psi_k''(x)\psi_{k-1}''(x) = \int_0^1 \psi_k''(F_k(x))\psi_{k-1}''(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = \frac{1}{h_k^2}h_k\int_0^1 \hat{\psi}_1''(x)\hat{\psi}_0''(x)\,dx$$

$$= \frac{1}{h_k}\int_0^1 (6x-2)(6x-4)\,dx = \frac{1}{h_k}\int_0^1 36x^2-36x+8\,dx$$

$$= \frac{1}{h_k}\left(\frac{36x^3}{3}-\frac{36x^2}{2}+8x)\right|_0^1 = \underline{\frac{1}{h_k}2}$$

$$k_{34} = \int_{T_k} \psi_k''(x)\varphi_k''(x) = \int_0^1 \psi_k''(F_k(x))\varphi_k''(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = \frac{1}{h_k^2}h_k\int_0^1 \hat{\psi}_1''(x)\hat{\varphi}_1''(x)\,dx$$

$$= \frac{1}{h_k}\int_0^1 (6x-2)(-12x+6)\,dx = \frac{1}{h_k}\int_0^1 -72x^2+60x-12\,dx$$

$$= \frac{1}{h_k}\left(-\frac{72x^3}{3}+\frac{60x^2}{2}-12x)\right|_0^1 = \underline{-\frac{1}{h_k}6}$$

So the k-th element matrix is

$$K_h^{(k)} = \frac{EI}{h_k^3}\begin{pmatrix} 12 & 6h_k & -12 & 6h_k \\ 6h_k & 4h_k^2 & -6h_k & 2h_k^2 \\ -12 & -6h_k & 12 & -6h_k \\ 6h_k & 2h_k^2 & -6h_k & 4h_k^2 \end{pmatrix}.$$

The global stiffness matrix will now be assembled by taking every element matrix and adding them into the global stiffness matrix, but the k-th element matrix will be diagonally shifted (k-1)*2 slots. This way the integrals of the basis functions with same index will be added to the integrals from the last element matrix (which is what we want because 2 basis functions with same index share 2 elements) and the integrals of the basis functions with mixed index will only have values from one element.

For the boundary conditions one has to edit the stiffness matrix. In our case for only essential boundary conditions we need to cut out the first 2 rows/columns and 2 last rows/columns of the matrix, because we already know w_0,w_0',w_n,w_n'. Depending on the boundary conditions the stiffness matrix has to be configured individually.

Remark 4.9.
The stiffness matrix K_h is positive definite since one can show that

$$\langle K_h \underline{u}_h, \underline{u}_h \rangle_{\mathbb{R}^{2n-2}} = a(u_h, u_h) \geq c_1^a \|u_h\|_V^2 > 0 \quad \forall u_h \in V_h \longleftrightarrow \forall \underline{u}_h \in \mathbb{R}^{2n-2},$$

whereas $\underline{u}_h$ is the coordinate vector of u_h in the nodal basis functions. Therefore all eigenvalues of K_h are greater than 0 *and K_h is positive definite.*

4.2.2 Element Load Vector

In the same manner the global load vector can be assembled by computing the element vectors and adding them shiftwise to the global load vector.
But to integrate the arbitrary function q one must approximate the integral numerically. We will do this by using the $\frac{3}{8}$-formula of the Newton Cotes formulas.

Remark 4.10. *$\frac{3}{8}$-formula*

$$h = \frac{(b-a)}{3}$$

$$\int_a^b f(x)\,dx \approx (b-a)(\frac{1}{8}f(a) + \frac{3}{8}f(a+h) + \frac{3}{8}f(a+2h) + \frac{1}{8}f(b))$$

A useful property of the $\frac{3}{8}$-formula is that polynomials of degree 3 or less are integrated exactly. So if the load q on the beam is constant we will only integrate the basis functions, which are of degree 3 and thus we integrate exactly.

The k-th element vector is of the form

$$\underline{q}_h^{(k)} = -\int_{T_k} q(x) \begin{pmatrix} \varphi_{k-1}(x) \\ \psi_{k-1}(x) \\ \varphi_k(x) \\ \psi_k(x) \end{pmatrix} dx.$$

$$\begin{aligned}
\underline{q}_{h1}^{(k)} &= -\int_{T_k} q(x)\varphi_{k-1}(x)\,dx = -\int_0^1 q(F_k(x))\varphi_{k-1}(F_k(x))F_k'(x)\,dx = -h_k\int_0^1 q(F_k(x))\hat{\varphi}_0(x)\,dx \\
&\approx -h_k(\frac{1}{8}q(F_k(0))\underbrace{\hat{\varphi}_0(0)}_{=1}+\frac{3}{8}q(F_k(\frac{1}{3}))\underbrace{\hat{\varphi}_0(\frac{1}{3})}_{=\frac{20}{27}}+\frac{3}{8}q(F_k(\frac{2}{3}))\underbrace{\hat{\varphi}_0(\frac{2}{3})}_{=\frac{7}{27}}+\frac{1}{8}q(F_k(1))\underbrace{\hat{\varphi}_0(1)}_{=0}) \\
&= -h_k(\frac{1}{8}q(x_{k-1})+\frac{5}{18}q(x_{k-1}+\frac{h_k}{3})+\frac{7}{72}q(x_{k-1}+\frac{2h_k}{3}))
\end{aligned}$$

$$\begin{aligned}
\underline{q}_{h2}^{(k)} &= -\int_{T_k} q(x)\psi_{k-1}(x)\,dx = -\int_0^1 q(F_k(x))\psi_{k-1}(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = -h_k^2\int_0^1 q(F_k(x))\hat{\psi}_0(x)\,dx \\
&\approx -h_k^2(\frac{1}{8}q(F_k(0))\underbrace{\hat{\psi}_0(0)}_{=0}+\frac{3}{8}q(F_k(\frac{1}{3}))\underbrace{\hat{\psi}_0(\frac{1}{3})}_{=\frac{4}{27}}+\frac{3}{8}q(F_k(\frac{2}{3}))\underbrace{\hat{\psi}_0(\frac{2}{3})}_{=\frac{2}{27}}+\frac{1}{8}q(F_k(1))\underbrace{\hat{\psi}_0(1)}_{=0}) \\
&= -h_k^2(\frac{1}{18}q(x_{k-1}+\frac{h_k}{3})+\frac{1}{36}q(x_{k-1}+\frac{2h_k}{3})) \\
\underline{q}_{h3}^{(k)} &= -\int_{T_k} q(x)\varphi_{k}(x)\,dx = -\int_0^1 q(F_k(x))\varphi_{k}(F_k(x))F_k'(x)\,dx = -h_k\int_0^1 q(F_k(x))\hat{\varphi}_1(x)\,dx \\
&\approx -h_k(\frac{1}{8}q(F_k(0))\underbrace{\hat{\varphi}_1(0)}_{=1}+\frac{3}{8}q(F_k(\frac{1}{3}))\underbrace{\hat{\varphi}_1(\frac{1}{3})}_{=\frac{7}{27}}+\frac{3}{8}q(F_k(\frac{2}{3}))\underbrace{\hat{\varphi}_1(\frac{2}{3})}_{=\frac{20}{27}}+\frac{1}{8}q(F_k(1))\underbrace{\hat{\varphi}_1(1)}_{=1}) \\
&= -h_k(\frac{7}{72}q(x_{k-1}+\frac{h_k}{3})+\frac{5}{18}q(x_{k-1}+\frac{2h_k}{3})+\frac{1}{8}q(x_k)) \\
\underline{q}_{h4}^{(k)} &= -\int_{T_k} q(x)\psi_{k}(x)\,dx = -\int_0^1 q(F_k(x))\psi_{k}(F_k(x))\underbrace{F_k'(x)}_{h_k}\,dx = -h_k^2\int_0^1 q(F_k(x))\hat{\psi}_1(x)\,dx \\
&\approx -h_k^2(\frac{1}{8}q(F_k(0))\underbrace{\hat{\psi}_1(0)}_{=0}+\frac{3}{8}q(F_k(\frac{1}{3}))\underbrace{\hat{\psi}_1(\frac{1}{3})}_{=-\frac{2}{27}}+\frac{3}{8}q(F_k(\frac{2}{3}))\underbrace{\hat{\psi}_1(\frac{2}{3})}_{=-\frac{4}{27}}+\frac{1}{8}q(F_k(1))\underbrace{\hat{\psi}_1(1)}_{=0}) \\
&= h_k^2(\frac{1}{36}q(x_{k-1}+\frac{h_k}{3})+\frac{1}{18}q(x_{k-1}+\frac{2h_k}{3}))
\end{aligned}$$

So the element load vectors are

$$\underline{q}_h^{(k)} = \begin{pmatrix} -h_k(\frac{1}{8}q(x_{k-1})+\frac{5}{18}q(x_{k-1}+\frac{h_k}{3})+\frac{7}{72}q(x_{k-1}+\frac{2h_k}{3})) \\ -h_k^2(\frac{1}{18}q(x_{k-1}+\frac{h_k}{3})+\frac{1}{36}q(x_{k-1}+\frac{2h_k}{3})) \\ -h_k(\frac{7}{72}q(x_{k-1}+\frac{h_k}{3})+\frac{5}{18}q(x_{k-1}+\frac{2h_k}{3})+\frac{1}{8}q(x_k)) \\ h_k^2(\frac{1}{36}q(x_{k-1}+\frac{h_k}{3})+\frac{1}{18}q(x_{k-1}+\frac{2h_k}{3})) \end{pmatrix}.$$

The global load vector can then be assembled by again adding the element vectors shiftwise together. If one considers essential boundary conditions (i.e. boundary conditions for the function values and derivative values) the first 2 and the last 2 elements need to be cut. The additional terms are computed by

$$
\begin{aligned}
a(g_h,\varphi_1) &= a(g_0\varphi_0 + d_0\psi_0 + g_L\varphi_n + d_L\psi_n,\varphi_1) = g_0\underbrace{a(\varphi_0,\varphi_1)}_{-\frac{EI}{h_1^3}12} + d_0\underbrace{a(\psi_0,\varphi_1)}_{-\frac{EI}{h_1^2}6},\\
a(g_h,\psi_1) &= g_0\underbrace{a(\varphi_0,\psi_1)}_{\frac{EI}{h_1^2}6} + d_0\underbrace{a(\psi_0,\psi_1)}_{\frac{EI}{h_1}2},\\
a(g_h,\varphi_{n-1}) &= g_L\underbrace{a(\varphi_n,\varphi_{n-1})}_{-\frac{EI}{h_n^3}12} + d_L\underbrace{a(\psi_n,\varphi_{n-1})}_{\frac{EI}{h_n^2}6},\\
a(g_h,\psi_{n-1}) &= g_L\underbrace{a(\varphi_n,\psi_{n-1})}_{-\frac{EI}{h_n^2}6} + d_L\underbrace{a(\psi_n,\psi_{n-1})}_{\frac{EI}{h_n}2}.
\end{aligned}
$$

So for essential boundary conditions the global load vector needs to be configured even further.

$$
\underline{q}_h = \begin{pmatrix}
\underline{q}_{h1} - EI(-g_0\frac{12}{h_1^3} - d_0\frac{6}{h_1^2})\\
\underline{q}_{h2} - EI(g_0\frac{6}{h_1^2} + d_0\frac{2}{h_1})\\
\underline{q}_{h3}\\
\vdots\\
\underline{q}_{h2n}\\
\underline{q}_{h2n-1} - EI(-g_0\frac{12}{h_n^3} + d_0\frac{6}{h_n^2})\\
\underline{q}_{h2n-2} - EI(-g_0\frac{6}{h_n^2} + d_0\frac{2}{h_n})
\end{pmatrix}
$$

Remark 4.11. *For natural boundary conditions the load vector must be configured differently. For example, if we would have only natural boundary conditions ($d_0^{(2)}$,$d_0^{(3)}$,$d_L^{(2)}$,$d_L^{(3)}$) then the linear functional Q would change to*

$$\langle Q, v\rangle = -\int_0^L q(x)v(x)\,dx - EI\,\underbrace{w'''(L)}_{d_L^{(3)}}\,v(L) + EI\,\underbrace{w'''(0)}_{d_0^{(3)}}\,v(0) + EI\,\underbrace{w''(L)}_{d_L^{(2)}}\,v'(L) - EI\,\underbrace{w''(L)}_{d_L^{(2)}}\,v'(0).$$

And by deriving to a linear equation system as shown above the load vector would change to

$$\underset{\sim}{q}_h = \begin{pmatrix} \langle Q, \varphi_0\rangle \\ \langle Q, \psi_0\rangle \\ \langle Q, \varphi_1\rangle \\ \langle Q, \psi_1\rangle \\ \vdots \\ \langle Q, \varphi_{n-1}\rangle \\ \langle Q, \psi_{n-1}\rangle \\ \langle Q, \varphi_n\rangle \\ \langle Q, \psi_n\rangle \end{pmatrix} = \begin{pmatrix} -\int_0^L q\varphi_0\,dx + EId_0^{(3)} \\ -\int_0^L q\psi_0\,dx - EId_0^{(2)} \\ -\int_0^L q\varphi_1\,dx \\ -\int_0^L q\psi_1\,dx \\ \vdots \\ -\int_0^L q\varphi_{n-1}\,dx \\ -\int_0^L q\psi_{n-1}\,dx \\ -\int_0^L q\varphi_n\,dx - EId_0^{(3)} \\ -\int_0^L q\psi_n\,dx + EId_0^{(2)} \end{pmatrix}.$$

One can also mix essential and natural boundary conditions and configure the stiffness matrix and load vector accordingly.

5 Implementation

For this thesis an algorithm to compute an approximate solution to the weak problem (2.1) was implemented in C++.

It first assembles the stiffness matrix and the load vector by a user defined mesh of nodes for the interval $(0, L)$. The stiffness matrix and the load vector are further edited for different boundary conditions, which the user is able to configure.

The resulting linear equation system will either be solved directly via Gauß-algorithm (LU-decomposition) or iteratively via CG-method.

5.1 Numerical Results

5.1.1 Discretization Error

Since we now have an algorithm to compute an approximate solution for the weak problem, we can check how much closer we get to the real solution if we choose a finer mesh and estimate the discretization error.

To be able to do this we need to choose a load q such that we can solve the differential equation analytically and compare the exact solution w to our approximate solution w_h.

It is also important to note in which norm we will compare the 2 functions, in our case we will choose the L^2-norm. In the program the integral in the L^2-norm is approximated by the middle point formula.

Let $q \equiv 1$ then the differential equation can be solved analytically.

$$\begin{aligned}
& -EIw''''(x) = 1 \ , \ w(0) = w(L) = w'(0) = w'(L) = 0 \\
\Rightarrow \quad & EIw'''(x) = -x + c_1 \\
\Rightarrow \quad & EIw''(x) = -\frac{x^2}{2} + c_1 x + c_2 \\
\Rightarrow \quad & EIw'(x) = -\frac{x^3}{6} + c_1 \frac{x^2}{2} + c_2 x + c_3 \\
\Rightarrow \quad & EIw(x) = -\frac{x^4}{24} + c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4
\end{aligned}$$

By inserting the boundary conditions one gets a linear equation system for the constants c_1,c_2,c_3,c_4 and the final exact solution w is

$$w(x) = \frac{1}{EI}(-\frac{x^4}{24} + \frac{x^3}{12} - \frac{x^2}{24}).$$

We can now compare the computed approximate solution to the exact solution and watch how the error evolves when we choose a finer mesh. We will choose a uniform mesh with 2^n elements. For $E = 2.15 * 10^{11}$ (structural steel) , $I = \frac{1}{120000}$ (10x10cm^2 rectangle cross section) and $L = 1$. The linear equation system will be solved by Gauß Algorithm.
Output:

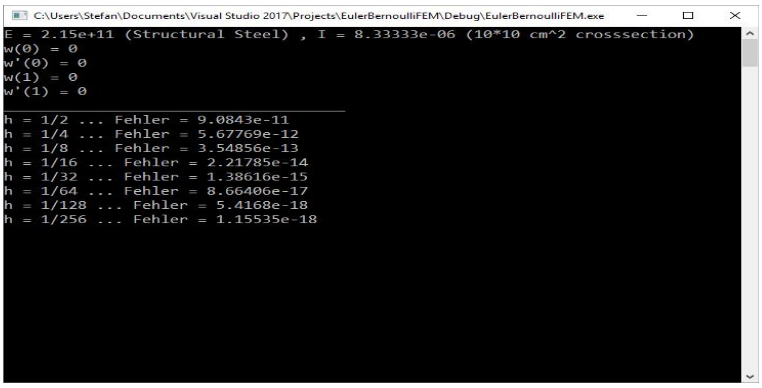

Figure 5.1: Discretization Error

As we can see if h is is divided by 2 (twice the number of nodes) the error is reduced by a factor ~ 16.
So we can assume that for the L^2-norm holds

$$\|w - w_h\|_{L^2} = \mathcal{O}(h^4).$$

5.1.2 Condition of $\mathbf{K}_h$

One can rigorously show what the bounds for the condition of the stiffness matrix are, but in this thesis we will only look at the numerical results when we compute the solution of the linear equation system by CG-method, inspect the number of iterations needed and based on this estimate the condition $\kappa(K_h)$ of the stiffness matrix.

We will look at the same configuration as shown above, but this time the linear equation system will be solved by CG-method. For the CG-method we have that

$$\#\text{CG-Iterations} \approx \sqrt{\kappa(K_h)}.$$

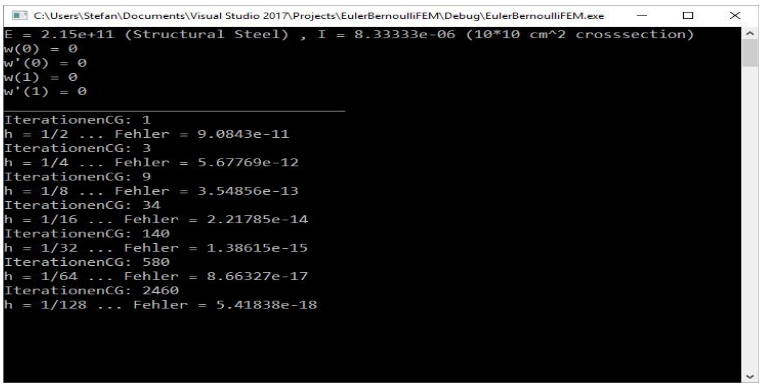

Figure 5.2: CG-method results

As seen in Figure 5.2 if h is divided by 2, the number of Iterations needed is multiplied by a factor ~ 4. From this we can assume that

$$\#\text{CG-Iterations} \approx \frac{1}{h^2}.$$

So we can estimate the condition of K_h by

$$\kappa(K_h) \approx \frac{1}{h^4}.$$

Obviously the condition of the matrix increases with the 4th power the smaller we choose h. This is not preferable, so most of the time preconditioning to reduce the condition of K_h is used.

Remark 5.1.
Preconditioning is an important part when solving linear equation systems numerically. In the case of numerically solving differential equations, special methods were developed for preconditioning these stiffness matrices, so called multigrid methods.

5.2 Examples

5.2.1 Example 1

The first example is not really a practical example but more a test if the boundary conditions are built into the algorithm correctly. We will again choose $q(x) = 1$ and $L = 1$ with the following boundary conditions.

$$-EIw''''(x) = 1$$

$$w(0) = 0 \quad w'(1) = 1 \quad w''(0) = 5 \quad w'''(1) = 3$$

The exact solution is

$$w(x) = \frac{1}{EI}(-\frac{x^4}{24} + \frac{(3EI+1)}{6}x^3 + (-\frac{11}{2}EI - \frac{1}{3})x) + \frac{5}{2}x^2$$

For simplicity we will just choose

$$E = 1 \quad I = 1$$

When computing the approximate solution and comparing it to the exact solution with respect to the L^2-norm one can see that the approximate solution converges to the exact solution, so the boundary conditions were built in correctly.

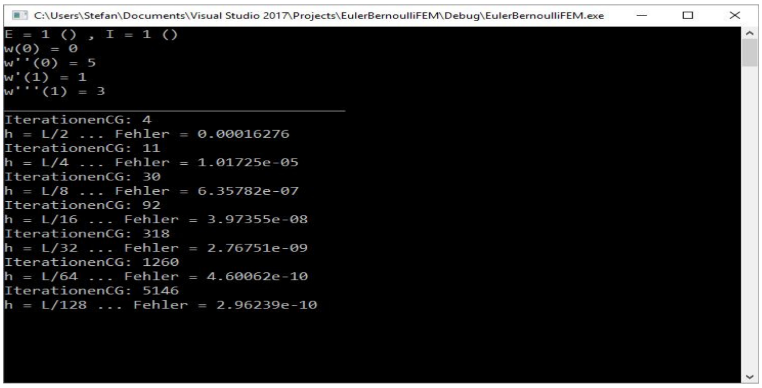

Figure 5.3: Discretization error for example 1

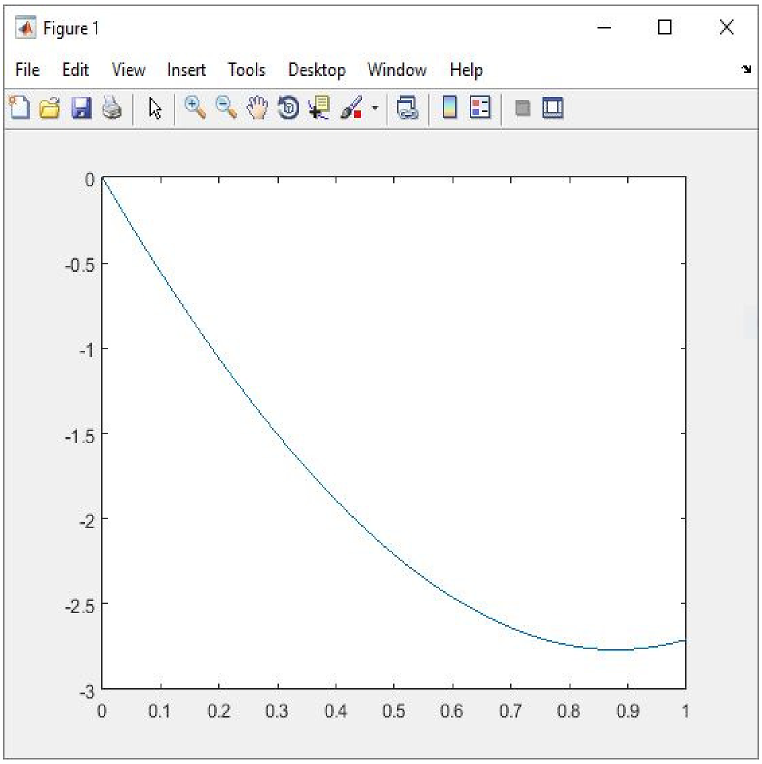

Figure 5.4: solution of example 1

5.2.2 Example 2

For the second example we will again take a steel beam with cross section $10 * 10\ cm^2$. The beam will be 15 meters long. The most basic load on a beam would be its own weight. So first we have to compute the load distribution q.

For the volume of the beam we get

$$V = 3 * 0,1 * 0,1m^3 = \frac{15}{100}\ m^3$$

For the density of steel we have

$$\rho(\text{structural steel}) = 7856\ \ kg/m^3$$

and thus

$$M = 7856 * \frac{15}{100} = 15 * 78.56\ \ kg$$

and for the weight

$$F_G = M * g = 78.56 * 15 * 9.81 = 15 * 770.6736\ \ N$$

where g is the gravitational acceleration on Earth. For the distributed load q that yields

$$q(x) = \frac{F_G}{L} = \frac{15 * 770.6736}{15} = 770.6736\ \ N$$

For the constants E and I we get

$$E = 215 \;\; GPa = 2.15 * 10^{11} \;\; Pa$$

$$I = \int_Q z^2 dQ = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \, dz \, dy = b \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 \, dz = b \frac{z^3}{3} \bigg|_{-\frac{h}{2}}^{\frac{h}{2}} = b \frac{\frac{h^3}{8} + \frac{h^3}{8}}{3} = b \frac{h^3}{12}$$

and for our case $b = h = \frac{1}{10}$

$$I = \frac{1}{120000}$$

with these values the approximate solution can be computed via FEM and be visualized with MATLAB.

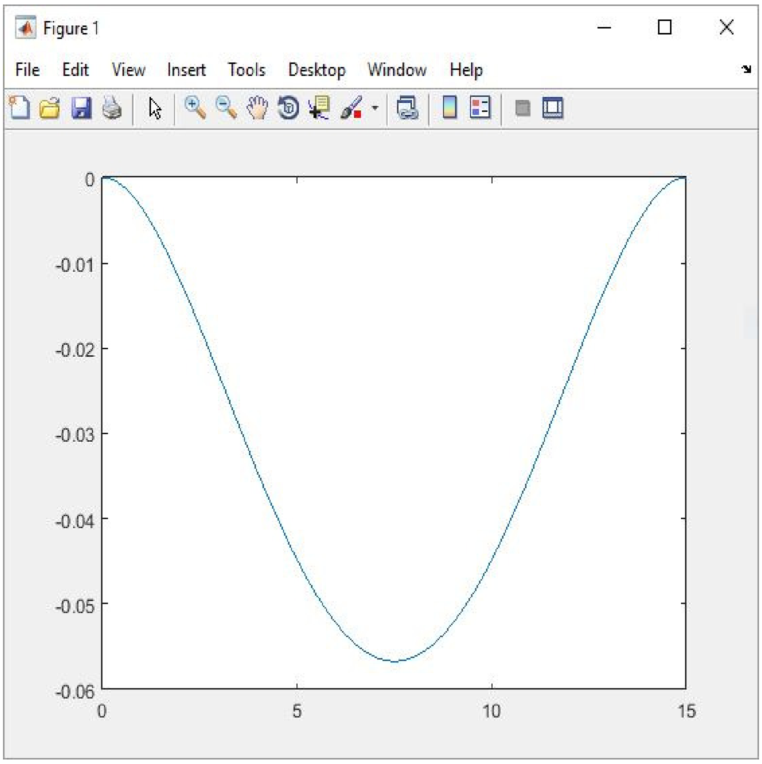

Figure 5.5: Deflection of example 2

5.2.3 Example 3

For example 3 we will look at a more practical beam. The most common form of a steel beam are I-beams. As the name suggests the cross section of these beams looks like an 'I'. We will look at an I-beam with length 15 m. The measurements for the cross section can be looked up in Figure 5.6

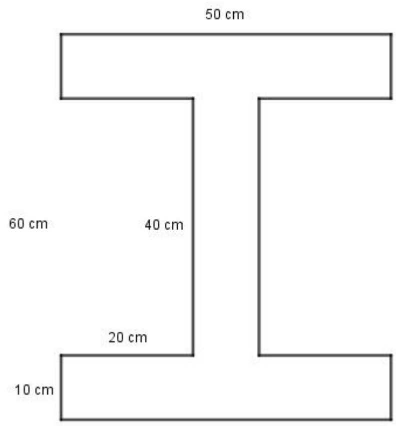

Figure 5.6: Measurements of the I-beam

For E we have again

$$E = 215 \ GPa = 2.15 * 10^{11} \ Pa$$

To compute I we can use the formula for rectangles we derived in the last example. Therefore we have the integral over the big rectangle minus the integral over the parts we don't need (since the integrand does not depend on y, integrals over these rectangles can be shifted to the left and right and stay the same so we can again use the rectangle formula for the smaller parts)

$$I = \frac{1}{12}(BH^3 - bh^3)$$

whereas $B = 0.5 \ m$, $H = 0.6 \ m$, $h = 0.4 \ m$, $b = 0.4 \ m$.

$$I = \frac{103}{15000}$$

To compute the weight force density we have

$$\begin{aligned} V &= 15 * (0.05 + 0.05 + 0.04) = 15 * 0.14 = 2.1 m^3 \\ M &= 7856 * 2.1 = 16497.6 kg \\ F_G &= M * 9.81 = 161841.456 N \\ f_G &= \frac{F_G}{15} = 10789.4304 N \end{aligned}$$

and for the load q we will choose the weight force density and some additional load.

$$q(x) = \underbrace{10789.4304}_{f_G} + \underbrace{5000000 * sin(\frac{\pi x}{15})}_{\text{additional load}}$$

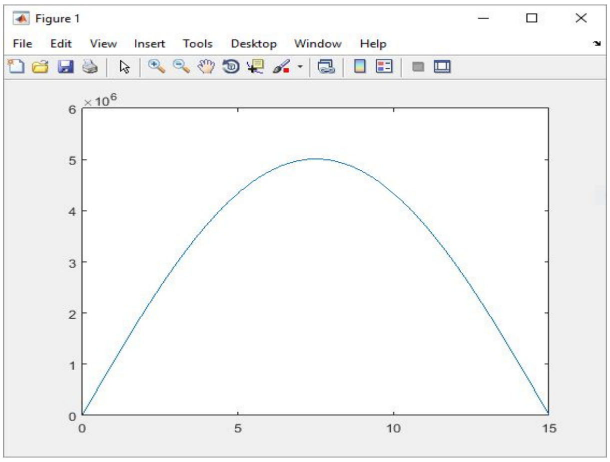

Figure 5.7: Load on I-beam

As seen in Figure 5.7 in the middle of the beam ($x = 7.5$) the additional load is about 500-times heavier than the weight force density of the beam.
The deflection of the beam can be seen in Figure 5.8.

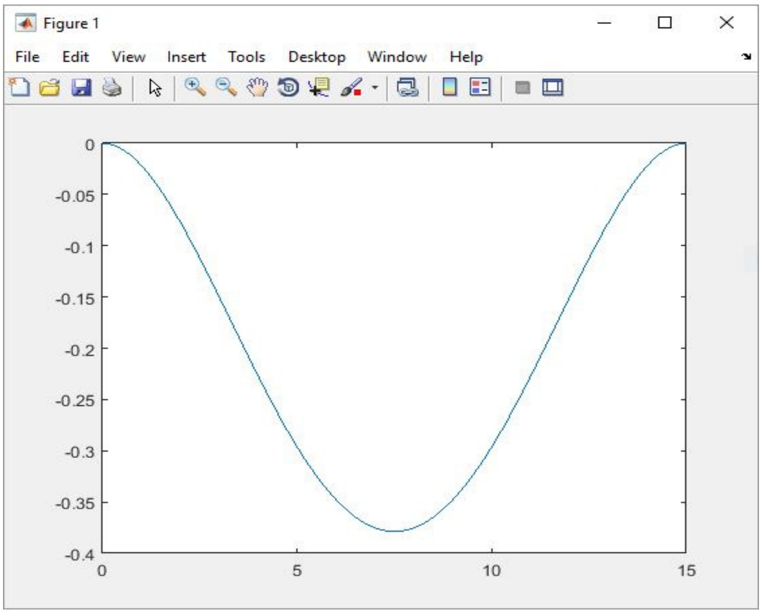

Figure 5.8: Deflection of example 3

5.2.4 Example 4

It is also possible to model a beam which has a fixed end on one side and a free end on the other side. A free end is characterized by $w''(L) = w'''(L) = 0$ for the deflection w. This time we will look at a beam made out of oak wood with length 5m and again a cross section of $10 * 10cm^2$. This situation is similar to applying a load to the branch of a tree and seeing how much it bends. For this configuration we get

$$E = 1.3 * 10^{10} \ Pa \text{ (Oak wood)} \quad , \quad I = \frac{1}{120000}$$

and for the weight force density

$$\begin{aligned} V &= \frac{5}{100} m^3 \\ \rho(oak) &= 670 kg/m^3 \\ M &= 670 * \frac{5}{100} = 5 * 6.7 kg \\ F_G &= 5 * 6.7 * 9.81 = 5 * 65.727 N \\ f_G &= \frac{5 * 65.727}{5} = 65.727 N \end{aligned}$$

and for the load density we choose

$$q(x) = 65.727 + 200x$$

so the additional load increases linearly until it reaches 1000N at the end. The deflection of the beam can be seen in Figure 5.10

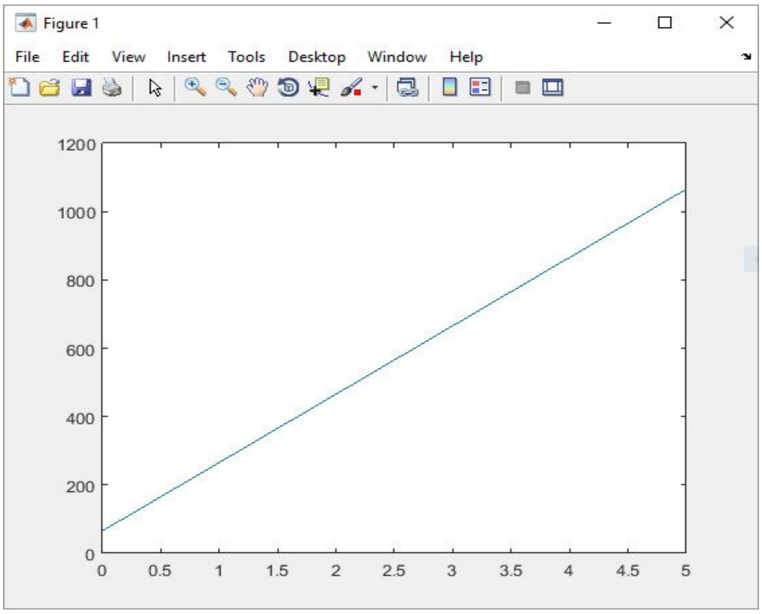

Figure 5.9: Load of example 4

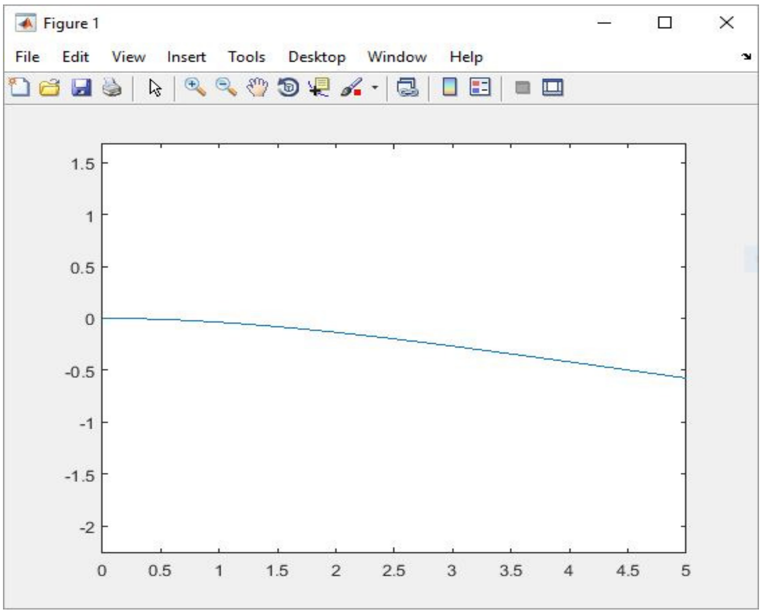

Figure 5.10: Deflection of example 4

6 Conclusion

We started to derive a model for our beam (which we wanted to be 1D) with only a transversal load by looking at the 3D-model of our beam, inserting these special values into the 3D-model and slightly modifying the known material law and constitutive law for 3D to derive a weak equation for the deflection of the beam in 1D. We further discussed how to derive the strong form of this equation by enforcing some requirements on the deflection w.

Additionally we showed the existence and uniqueness of a solution of the weak problem by using special function spaces for the solution space and the testfunction space, which were subspaces of the Sobolev space H^2. We showed that for this problem all requirements for the theorem of Lax-Milgram were satisfied and therefore existence and uniqueness of a solution followed.

The next step was to attempt to solve the weak problem by numerically approximate the solution, using a discretization of the the beam and defining basis functions on these nodes along the beam. By reducing the solution space and test space to subspaces with a finite basis (these nodal basis functions) we transformed the weak problem to a linear equation system where we introduced and computed the stiffness matrix and the load vector. Furthermore we discussed how to implement boundary conditions in this linear equation system.

As part of this thesis, an algorithm based on the theory that was presented was implemented in C++ and the resulting approximate solution visualized with MATLAB. At the end some basic examples were presented.

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Eidesstattliche Erklärung

Ich, Stefan Tyoler, erkläre an Eides statt, dass ich die vorliegende Bachelorarbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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