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On Directional Metric Subregularity and Second-Order Optimality Conditions for a Class of Nonsmooth Mathematical Programs

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On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs

Helmut Gfrerer*

Abstract

We study infinite dimensional optimization problems where the constraint mapping is given as the sum of a smooth function and a generalized polyhedral multifunction, e.g. the normal cone mapping of a convex polyhedral set. By using advanced techniques of variational analysis we obtain first-order and second-order characterizations, both necessary and sufficient, for directional metric subregularity of the constraint mapping. These results are used to obtain second-order optimality conditions for the optimization problem.
Key words. Metric subregularity, second-order optimality conditions
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1 Introduction

We study in this paper optimization problems of the form

$$\text{minimize } f(x) \qquad \text{subject to} \qquad 0 \in M(x) := F(x) + S(x) \tag{1}$$

where the objective $f : X \to \mathbb{R}$ and $F : X \to Y$ are smooth mappings taking values in the real numbers respectively another Banach space Y and $S : X \rightrightarrows Y$ is a generalized polyhedral multifunction.

Definition 1. *1. Let Z be a Banach space. A convex set $P \subset Z$ is called* polyhedral*, if there exist continuous linear functionals $z_j^* \in Z^*$ and real numbers $\zeta_j \in \mathbb{R}$, $j = 1,\ldots,m$ such that*

$$P = \{z \in Z \,|\, \langle z_j^*, z\rangle \le \zeta_j,\ j = 1,\ldots,m\}$$

A convex set $P \subset Z$ is called generalized polyhedral*, if there exist a closed linear subspace $L \subset Z$, a point $z' \in Z$ and $z_j^* \in Z^*$, $\zeta_j \in \mathbb{R}$, $j = 1,\ldots,m$ such that*

$$P = \{z \in z' + L \,|\, \langle z_j^*, z\rangle \le \zeta_j,\ j = 1,\ldots,m\}$$

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2. *Let X,Y be Banach spaces. A multifunction $S : X \rightrightarrows Y$ is called* (generalized) polyhedral*, if its graph* $\operatorname{gph} S := \{(x,s) \,|\, s \in S(x)\}$ *is the union of finitely many convex (generalized) polyhedral sets, called components, in $X \times Y$.*

It has been well recognized that problem (1) is a convenient model to describe optimization problems where, among the constraints, so called equilibrium constraints occur. This equilibrium is often described by a variational inequality, a complementarity constraint or a lower-level optimization problem. Let us consider some examples.

Example 1 (Complementarity constraints)**.** *Let X, W denote Banach spaces and let $f : X \to \mathbb{R}$, $h : X \to W$, $g : X \to \mathbb{R}^m$, $G : X \to \mathbb{R}^q$, $H : X \to R^q$ be mappings. Consider the mathematical program with complementarity constraints*

$$
\begin{array}{lll}
(MPCC') & \textit{minimize} & f(x) \\
 & \textit{subject to} & h(x) = 0 \\
 & & g(x) \le 0 \\
 & & G(x) \ge 0,\; H(x) \ge 0,\; G(x)^T H(x) = 0.
\end{array}
$$

Here we have written down the complementarity constraints in form of smooth constraints. However, it is well known that all the classical constraint qualification conditions are violated for this formulation. Hence it is convenient to study the following equivalent nonsmooth program. Let $\mathscr{Q} := \{(a,b) \in \mathbb{R}^2_- \,|\, ab = 0\}$. Then (MPCC') is equivalent with the problem

$$
\begin{array}{lll}
(MPCC) & \textit{minimize} & f(x) \\
 & \textit{subject to} & h(x) = 0 \\
 & & g(x) \le 0 \\
 & & 0 \in (G_i(x), H_i(x)) + \mathscr{Q},\; i = 1,\dots,q.
\end{array}
$$

which is also of the form (1) *with $F(x) = (h(x), g(x), (G_i(x), H_i(x))_{i=1,\dots,q})$ and $S(x) \equiv \{0_W\} \times \mathbb{R}^m_+ \times \prod_{i=1}^q \mathscr{Q}$. $\mathscr{Q}$ is the union of the 2 convex polyhedral sets $\mathscr{Q}_1 := \{(a,0) \,|\, a \le 0\}$ and $\mathscr{Q}_2 := \{(0,b) \,|\, b \le 0\}$ and therefore* $\operatorname{gph} S$ *is the union of 2^q convex polyhedral sets of the form*

$$
X \times \{0_W\} \times \mathbb{R}^m_+ \times \prod_{i=1}^q \mathscr{Q}_{j_i},
$$

where $j_i \in \{1,2\}$, $i = 1,\dots,q$.

Example 2. *Given a generalized convex polyhedral subset $C = \{y \in y' + L \,|\, \langle y_j^*, y\rangle \le \xi_j, j = 1,\dots,m\}$ of the Banach space Y, consider the problem*

$$
\begin{array}{lll}
(MPVI) & \textit{minimize}_{(x,y)} & f(x,y) \\
 & \textit{subject to} & 0 \in g(x,y) + N(y;C)
\end{array}
$$

where $f : X \times Y \to \mathbb{R}$, $g : X \times Y \to Y^*$ are mappings and $N(y;C)$ denotes the normal cone in the sense of convex analysis to C at y. Defining the active index set mapping $\mathscr{A}(y) := \{j \in$

$\{1,\dots,m\}\,|\,\langle y_j^*,y\rangle=\xi_j\}$, by the well known Generalized Farkas lemma (see, e.g., Theorem 2 below) we have

$$N(y;C)=\begin{cases}L^\perp+\{\sum_{j\in\mathscr{A}(y)}\lambda_j y_j^*\,|\,\lambda_j\geq 0, j\in\mathscr{A}(y)\} & \text{if } y\in C\\ \emptyset & \text{else}\end{cases}$$

Now we show that the normal cone mapping $N(\cdot;C):Y\rightrightarrows Y^*$ is a generalized polyhedral multifunction. This is due to the observation that for every $y^*\in N(y;C)$ there exist an index set $J\subset\mathscr{A}(y)$, a linear functional $l^*\in L^\perp$ and positive numbers $\lambda_j>0,\ j\in J$, such that $y^*=l^*+\sum_{j\in J}\lambda_j y_j^*$ and the functionals y_j^* are linearly independent on L. In fact, every $y^*\in N(y;C)$ can be written in the form $y^*=l^*+\sum_{j\in\mathscr{A}(y)}\lambda_j y_j^*$ with $l^*\in L^\perp$ and $\lambda_j\geq 0,\ j\in\mathscr{A}^(y)$. Then we can take $J=\{j\in\mathscr{A}(y)\,|\,\lambda_j>0\}$ but possibly the property of linear independence of the functional y_j^*, $j\in J$ on L_i is not fulfilled.. If the linear functionals y_j^*, $j\in J$ were linearly dependent on L, then we could find coefficients μ_j, $j\in J$, not all zero, such that $\sum_{j\in J}\mu_j y_j^*$ vanishes on L, i.e. $\sum_{j\in J}\mu_j y_j^*\in L^\perp$. Then we can choose some real α such that $y^*=\tilde{l}^*+\sum_{j\in J}\tilde{\lambda}_j y_j^*$ with $\tilde{l}^*:=l^*-\alpha\sum_{j\in J}\mu_j y_j^*\in L^\perp$, $\tilde{\lambda}_j:=\lambda_j+\alpha\mu_j\geq 0$, $j\in J$ and such that $|\tilde{J}|<|J|$, where $\tilde{J}:=\{j\in J\,|\,\tilde{\lambda}_j>0\}$. By eventually repeating this procedure we obtain the desired representation for y^*.

Now let $\mathscr{J}$ denote the collection of all index sets $J\subset\{1,\dots,m\}$ such that y_j^*, $j\in J$ are linearly independent on L. Then we can find for every index set $J\in\mathscr{J}$ some points $y_k^J\in L$, $k\in J$, such that

$$\langle y_j^*,y_k^J\rangle=\begin{cases}1 & \text{if } k=j\\ 0 & \text{if } k\neq j\end{cases}$$

and a point $\bar{y}^J\in y'+L$ with $\langle y_j^*,\bar{y}^J\rangle=\xi_j$, $j\in J$. Denoting by $\operatorname{lin}(y_j^*)_{j\in J}$ the linear hull of the functionals y_j^*, $j\in J$ and $\ker(y_j^*)_{j\in J}:=\{y\,|\,\langle y_j^*,y\rangle=0,j\in J\}$, it follows that $\operatorname{gph}N(\cdot;C)=\bigcup_{J\in\mathscr{J}}P_J$, where each of the sets

$$P_J:=\{(y,y^*)\in(\bar{y}^J,0)+(L\cap\ker(y_j^*)_{j\in J})\times(L^\perp+\operatorname{lin}(y_j^*)_{j\in J})\,|\,\begin{array}{l}\langle y_j^*,y\rangle\leq\xi_j, j\in\{1,\dots,m\}\setminus J;\\ \langle y^*,y_j^J\rangle\geq 0, j\in J\end{array}\}$$

is a convex generalized polyhedral set. Hence $N(\cdot;C)$ is a generalized polyhedral multifunction and the problem $(MPVI)$ is of the form (1).

In connection with optimality conditions for the problem (1) the so-called *M-stationarity* concept plays an important role, see e.g. [4, 21, 22, 26, 27]. The M-stationarity concept is associated with the generalized differentiation calculus of Mordukhovich and uses so-called *coderivatives*. For a comprehensive theory on this subject and several applications we refer the reader to the monographs [20]. For the calculation of coderivatives and stability analysis for the normal cone mapping of convex (generalized) polyhedral sets as presented in Example 2 we refer to the recent papers [1, 11].

Optimality conditions are ultimately related with some constraint qualification condition. Such constraint qualification conditions are, for instance, the properties of metric regularity and subregularity of the constraint mapping. The property of metric regularity for general multifunctions can be equivalently characterized by the so-called *Mordukhovich criterion* (see, e.g., [20, Theorem 4.18]), which is stated as a condition on the coderivative of M and a *partial sequential normal compactness* (PSNC) property.

Characterizations of metric subregularity based on objects of generalized differentiation can be found for instance in [6, 9, 10, 12, 13, 18]. It is well-known that generalized polyhedral multifunctions in finite dimensions are always metrically subregular. This result is due to Robinson [24], some extensions to the infinite dimensional case are given in [2, Section 2.5.7] and in Section 3 of this paper. Due to this property we will pay special attention to the case that the constraint mapping $M(x)$ can be partitioned into a nonlinear part $M_1(x) = F_1(x) + S_1(x)$ and a polyhedral part $M_2(x) = Ax + S_2(x)$, the latter being known in advance to be metrically subregular.

In the recent paper [7] the M-stationarity conditions have been extended, even for more general problems than (1), by showing that at a locally optimal solution for every critical direction a M-stationarity condition, with possibly different multipliers, is fulfilled. The underlying constraint qualification condition for these extended M-stationarity conditions is given by the property of directional metric (sub)regularity as introduced in [7] and its characterization by directional objects of generalized differentiation. However, although these results are applicable for problems with a quite general structure, there are some restrictions on the class of spaces involved: either all spaces have to be Asplund spaces or at least one space has to be Fréchet smooth.

The aim of this paper is to carry over and even extend these results concerning regularity properties and optimality conditions to the problem (1) in general Banach spaces. We will see that this is possible by replacing the (PSNC) property by the requirement that certain subspaces are closed. As a byproduct we obtain that the Mordukhovich criterion for characterizing metric regularity of M is valid in general Banach spaces. Moreover, the linear structures inherited by the polyhedral multifunction S allow us to formulate second-order conditions, both sufficient and necessary, for metric subregularity which are the base for second-order optimality conditions for the problem (1). Let us note that these optimality conditions, when applied to the nonlinear programming problem with equality and inequality constraints, even slightly extend the well known second-order conditions due to Ioffe [15].

The rest of the paper is organized as follows: In Section 2 we recall the basic definitions of metric regularity and subregularity together with their directional versions as well as the basic definitions from generalized differentiation. In section 3 we collect some results on generalized polyhedral sets and polyhedral multifunctions. First-order and second-order characterizations of directional metric (sub)regularity are presented in Section 4. Just by observing that at a locally optimal solution to problem (1) a certain multifunction associated with the problem cannot behave metrically (sub)regular, we obtain the first-order and second-order optimality conditions as presented in Section 5.

Our notation is fairly standard. Throughout this paper let X, Y and Z be Banach spaces equipped with norm $\|\cdot\|$. By X^* we denote the topological dual of X with the canonical pairing $\langle\cdot,\cdot\rangle$ between X and X^*. $\mathscr{B}_X := \{x \in X \mid \|x\| \leq 1\}$ denotes the closed unit ball and $\mathscr{S}_X := \{x \in X \mid \|x\| = 1\}$ denotes the unit sphere. Unless otherwise stated, we assume that the product space

$X \times Y$ of two spaces X and Y is equipped with a norm satisfying $\max\{\|x\|,\|y\|\} \le \|(x,y)\| \le \|x\|+\|y\|$. Given an element of a product space, e.g. $z \in X \times Y$ respectively $w \in W_1 \times \ldots \times W_m$, we denote by appropriate indices the corresponding components, e.g. $z = (z_x, z_y) \in X \times Y$ respectively $w = (w_1,\ldots,w_m) \in \prod_{i=1}^m W_i$.

Given a subspace $L \subset X$, we denote by $L^\perp := \{x^* \in X^* \mid \langle x^*, l\rangle = 0 \forall l \in L\}$ the annihilator of L. Given a functional $x^* \in X^*$, we denote by $x^*_{|L}$ the restriction of x^* to L. As a consequence of the Hahn-Banach Theorem we know that $L^* = \{x^*_{|L} \mid x^* \in X^*\}$. Consider the equivalence relation on X defined by $x_1 \sim x_2$ iff $x_1 - x_2 \in L$. The quotient space X/L of equivalence classes, equipped with the algebraic operations induced by X and the norm $\|[x]_L\| := \inf_{x' \in [x]_L} \|x'\|$, where $[x]_L$ denotes the equivalence class of x, is again a Banach space. Note that $(X/L)^*$ can be identified with $L^\perp$, see e.g. [23]. Any closed affine subspace $L' = x' + L$ of X can be represented in the form $L' = \{x \in X \mid Ax = b\}$, where $A : X \to W$ is a continuous linear operator from X onto another Banach space W and $b \in W$. We can use the surjective mapping $A : X \to X/L$ which assigns to a point x its equivalence class $[x]_L$ and $b = Ax'$ to obtain the desired representation. Moreover we have $L = \ker A$.

2 Directional metric (sub)regularity and preliminaries from generalized differentiation

Recall that a multifunction $M : X \rightrightarrows Y$ between Banach spaces is called *metrically regular* with modulus $\kappa > 0$ near the point $(\bar{x},\bar{y}) \in \operatorname{gph} M := \{(x,y) \in X \times Y \mid y \in M(x)\}$ from its graph, provided there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathrm{d}(x, M^{-1}(y)) \le \kappa \mathrm{d}(y, M(x)) \quad \forall (x,y) \in U \times V. \tag{2}$$

Here $\mathrm{d}(x,\Omega)$ denotes the usual distance between a point x and a set Ω. When fixing $y = \bar{y}$ in (2) we obtain the weaker property of *metric subregularity* of M at $(\bar{x},\bar{y})$, i.e. we require the estimate

$$\mathrm{d}(x, M^{-1}(\bar{y})) \le \kappa \mathrm{d}(\bar{y}, M(x)) \quad \forall x \in U \tag{3}$$

with some neighborhood $U$of $\bar{x}$ and a positive real $\kappa > 0$.

It is well known that a multifunction $M : X \rightrightarrows Y$ is metrically regular near $(\bar{x},\bar{y}) \in \operatorname{gph} M$ if the inverse multifunction $F = M^{-1}$ has the *Aubin property (local Lipschitz–like property, pseudo–Lipschitzian property)* near $(\bar{y},\bar{x})$, i.e.

$$F(y') \cap U \subset F(y) + L\|y - y'\|\mathscr{B}_X \quad \forall y', y \in V,$$

with $L \ge 0$ und neighborhoods U of $\bar{x}$ and V of $\bar{y}$. Further, the property of metric subregularity is equivalent to *calmness* of the inverse multifunction, see [3]. For a survey on the theory of metric regularity and the Aubin property we refer the reader to [17] and to the monographs [19, 20, 25] and the references therein.

To study the directional behavior of multifunctions, it is convenient to introduce the following neighborhoods of directions: Given a Banach space Z, a direction $d \in Z$ and positive numbers

$\varepsilon,\delta>0$, the set $V_{\varepsilon,\delta}(d)$, is given by

$$V_{\varepsilon,\delta}(d):=\{z\in\varepsilon\mathscr{B}_Z\,|\,\big\|\|d\|z-\|z\|d\big\|\le\delta\|z\|\,\|d\|\}. \tag{4}$$

This can be written also in the form

$$V_{\varepsilon,\delta}(d)=\begin{cases}\{0\}\cup\big\{z\in\varepsilon\mathscr{B}_Z\setminus\{0\}\,\big|\,\big|\frac{z}{\|z\|}-\frac{d}{\|d\|}\big|\le\delta\big\} & \text{if } d\neq 0,\\ \varepsilon\mathscr{B}_Z & \text{if } d=0.\end{cases}$$

Note that $V_{\varepsilon,\delta}(d)=V_{\varepsilon,\delta}(\alpha d)$, $\forall\alpha>0$ and that, given $\bar z\in Z$ and a sequence $(z_k)\to\bar z$, there exist sequences $(t_k)\downarrow 0$, $(d_k)\to d$ with $z_k=\bar z+t_kd_k$ if and only if for every $\varepsilon>0$, $\delta>0$ there is some index $k_{\varepsilon,\delta}$ such that $z_k\in\bar z+\operatorname{int}V_{\varepsilon,\delta}(d)$, $\forall k\ge k_{\varepsilon,\delta}$.

The following definition was given in [7].

Definition 2. *Let $M:X\rightrightarrows Y$ be a multifunction and let $(\bar x,\bar y)\in\operatorname{gph}M$.*

1. *Given $d:=(u,v)\in X\times Y$, M is called* metrically regular in direction (u,v) *at $(\bar x,\bar y)$, provided there exist positive reals $\rho>0,\delta>0$ and $\kappa>0$ such that*

   $$\mathrm{d}(x,M^{-1}(y))\le\kappa\mathrm{d}(y,M(x)) \tag{5}$$

   holds for all $(x,y)\in(\bar x,\bar y)+V_{\rho,\delta}(d)$ with $\|d\|\mathrm{d}((x,y),\operatorname{gph}M)\le\delta\|d\|\|(x,y)-(\bar x,\bar y)\|$.

2. *For given $u\in X$, M is said to be* metrically subregular in direction u *at $(\bar x,\bar y)$, if there are positive reals $\rho>0,\delta>0$ and $\kappa'>0$ such that*

   $$\mathrm{d}(x,M^{-1}(\bar y))\le\kappa'\mathrm{d}(\bar y,M(x)) \tag{6}$$

   holds for all $x\in\bar x+V_{\rho,\delta}(u)$.

It is well known [7, Lemma 1] that metric regularity in direction $(u,0)$ implies subregularity in direction u. By the definition, M is metrically regular in direction $(0,0)$ if and only if M is metrically regular. Similarly, metric subregularity in direction 0 is equivalent to the property of metric subregularity.

We consider now the case that our multifunction M is composed by two multifunctions $M_i:X\rightrightarrows Y_i$, $i=1,2$, i.e. M has the form

$$M=(M_1,M_2):X\rightrightarrows Y:=Y_1\times Y_2. \tag{7}$$

In what follows we denote the components of $y\in Y=Y_1\times Y_2$ by y_i, i.e. $y=(y_1,y_2)$ and we set $\tilde Y_1:=Y_1\times\{0_{Y_2}\}$.

Definition 3. *Let M be given by* (7)*, $(\bar x,\bar y)\in\operatorname{gph}M$ and let $(u,v_1)\in X\times Y_1$. We say that M is* mixed regular/subregular in direction (u,v_1) *at $(\bar x,\bar y)$ if there are numbers $\rho>0,\delta>0$ and $\kappa>0$ such that*

$$\mathrm{d}(x,M^{-1}(y_1,\bar y_2))\le\kappa\mathrm{d}((y_1,\bar y_2),M(x))$$

holds for all $(x,y_1)\in(\bar x,\bar y_1)+V_{\rho,\delta}(u,v_1)$ satisfying

$$\|(u,v_1)\|\mathrm{d}((x,(y_1,\bar y_2)),\operatorname{gph}M)\le\delta\|(u,v_1)\|\|(x,y_1)-(\bar x,\bar y_1)\|.$$

We call M mixed regular/subregular *at $(\bar x,\bar y)$ if it is mixed regular/subregular in direction $(0,0)$ at $(\bar x,\bar y)$.*

Again one can show that the multifunction M given by (7) is metrically subregular in direction u if it is mixed regular/subregular in direction $(u,0)$.

We will use the concept of mixed regularity/subregularity in situations when the multifunction M_2 is known to be metrically subregular in advance. More exactly, we will use the following definition.

Definition 4. *Let M be given by* (7)*, $(\bar{x},\bar{y}) \in \operatorname{gph} M$ and let $(u,v_1) \in X \times Y_1$. We say that M_2 is* proper subregular in direction u relative to M_1 and v_1 *at $(\bar{x},\bar{y})$, if there are positive constants $\kappa',\rho',\delta',L > 0$ such that for all $(x,y_1) \in ((\bar{x},\bar{y}_1)+V_{\rho',\delta'}(u,v_1)) \cap \operatorname{gph} M_1$ there is some $\check{x} \in M_2^{-1}(\bar{y}_2)$ satisfying $\|x-\check{x}\| \le \kappa' \mathrm{d}(\bar{y}_2, M_2(x))$ and $\mathrm{d}(y_1, M_1(\check{x})) \le L\|x-\check{x}\|$.*

A sufficient condition for proper subregularity is provided by the following lemma:

Lemma 1 ([7, Lemma 3])**.** *Let M be given by* (7)*, $(\bar{x},\bar{y}) \in \operatorname{gph} M$ and let $u \in X$. Assume that M_2 is metrically subregular in direction u at $(\bar{x},\bar{y}_2)$ and that M_1 has the Aubin property near $(\bar{x},\bar{y}_1)$. Then, M_2 is proper subregular in direction u relative to M_1 and every $v_1 \in Y_1$ at $(\bar{x},\bar{y})$.*

Next we recall some generalized differential constructions from variational analysis as can be found in [20].

Let Ω be a nonempty subset of a Banach space X and let $x \in \Omega$. The *contingent cone* to Ω at x, denoted by $T(x;\Omega)$, is given by

$$T(x;\Omega) := \{d \in X \,|\, \exists (x_k) \in \Omega, (t_k) \downarrow 0 : \frac{x_k - x}{t_k} \to d\}.$$

Given $\varepsilon \ge 0$ we denote by

$$\hat{N}_\varepsilon(x;\Omega) = \{x^* \in X^* \,|\, \limsup_{x' \overset{\Omega}{\to} x} \frac{\langle x^*, x'-x\rangle}{\|x'-x\|} \le \varepsilon\} \tag{8}$$

the *set of ε-normals* to Ω. When $\varepsilon = 0$, elements of (8) are called *Fréchet normals* or *normals in a regular sense* and their collection is denoted by $\hat{N}(x;\Omega)$. Finally, the *limiting normal cone* to Ω at x is defined by

$$N(x;\Omega) := \{x^* \,|\, \exists (\varepsilon_k) \downarrow 0,\ (x_k) \overset{\Omega}{\to} x,\ (x_k^*) \overset{w^*}{\to} x^* : x_k^* \in \hat{N}_{\varepsilon_k}(x_k;\Omega) \forall k\}.$$

If $x \notin \Omega$ we put $T(x;\Omega) = \emptyset$, $N(x;\Omega) = \emptyset$ and $\hat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \ge 0$.

The limiting normal cone sometimes is also called basic normal cone or Mordukhovich normal cone. It is generally nonconvex whereas the Fréchet normal cone is always convex. In the case of a convex set Ω, both the Fréchet normal cone and the limiting normal cone coincide with the standard normal cone from convex analysis and we have the relation

$$\hat{N}_\varepsilon(x;\Omega) = \{x^* \in X^* \,|\, \mathrm{d}(x^*, N(x,\Omega)) \le \varepsilon\}.$$

Moreover, the contingent cone is equal to the tangent cone in the sense of convex analysis.

Given a multifunction $M : X \rightrightarrows Y$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} M$, the *contingent derivative* of M at $(\bar{x},\bar{y})$ is defined as the set-valued mapping $CM(\bar{x},\bar{y}) : X \rightrightarrows Y$ with the values $CM(\bar{x},\bar{y})(u) := \{v \in Y \mid (u,v) \in T((\bar{x},\bar{y}); \operatorname{gph} M)\}$, i.e. $CM(\bar{x},\bar{y})(u)$ is the collection of all $v \in Y$ such that there are sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ with $(\bar{x}+t_k u_k, \bar{y}+t_k v_k) \in \operatorname{gph} M$.

The *normal coderivative* of M at $(\bar{x},\bar{y})$ is a multifunction $D^*_N M(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$, where $D^*_N M(\bar{x},\bar{y})(y^*)$ is the collection of all $x^* \in X^*$ for which there are sequences $(\varepsilon_k) \downarrow 0$, $(x_k,y_k) \to (\bar{x},\bar{y})$ and $(x^*_k,y^*_k) \xrightarrow{w^*} (x^*,y^*)$ with $(x^*_k,-y^*_k) \in \hat{N}_{\varepsilon_k}((x_k,y_k); \operatorname{gph} M)$.

The following directional versions of these differentiation constructions were introduced in [7].

Definition 5. *1. Let $\Omega \subset X$, $x \in \Omega$ and $u \in X$ be given. The* limiting normal cone to Ω in direction *u at x is defined by*

$$N(x;\Omega;u) := \{x^* \mid \exists (\varepsilon_k) \downarrow 0,\ (t_k) \downarrow 0,\ (u_k) \to u,\ (x^*_k) \xrightarrow{w^*} x^* : x^*_k \in \hat{N}_{\varepsilon_k}(x+t_k u_k;\Omega) \forall k\}.$$

2. *Let $M : X \rightrightarrows Y$ and let $(\bar{x},\bar{y}) \in \operatorname{gph} M$, $(u,v) \in X \times Y$.*

The normal coderivative of M in direction (u,v) *at $(\bar{x},\bar{y})$ is defined as the set-valued mapping $D^*_N M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$, where $D^*_N M((\bar{x},\bar{y});(u,v))(y^*)$ is the collection of all $x^* \in X^*$ for which there exist sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ and $(x^*_k,y^*_k) \xrightarrow{w^*} (x^*,y^*)$ with $(x^*_k,-y^*_k) \in \hat{N}_{\varepsilon_k}((\bar{x}+t_k u_k, \bar{y}+t_k v_k); \operatorname{gph} M)$.*

For $u \neq 0$ our definition of the limiting normal cone in direction u coincides with the definition of the *basic normal cone in direction u* as presented in [5].

Note that $N(x;\Omega;0) = N(x,\Omega)$, $D^*_N M((\bar{x},\bar{y});(0,0)) = D^*_N M(\bar{x},\bar{y})$. Further we have the relation

$$D^*_N M((\bar{x},\bar{y});(u,v))(y^*) = \{x^* \in X^* \mid (x^*,-y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} M;(u,v))\}.$$

In the context of this paper it also makes sense to replace in the definitions above ε-normals respectively weak* convergence by Fréchet normals respectively strong convergence.

Definition 6. *1. Let $\Omega \subset X$, $x \in \Omega$ and $u \in X$ be given. The* strongly limiting normal cone to Ω in direction *u at x is defined by*

$$\check{N}(x;\Omega;u) := \{x^* \mid \exists (t_k) \downarrow 0,\ (u_k) \to u,\ (x^*_k) \to x^* : x^*_k \in \hat{N}(x+t_k u_k;\Omega) \forall k\}.$$

2. *Let $M : X \rightrightarrows Y$ and let $(\bar{x},\bar{y}) \in \operatorname{gph} M$, $(u,v) \in X \times Y$.*

The strong coderivative of M in direction (u,v) *at $(\bar{x},\bar{y})$ is defined as the set-valued mapping $D^*_S M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$, where $D^*_S M((\bar{x},\bar{y});(u,v))(y^*)$ is the collection of all $x^* \in X^*$ for which there exist sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ and $(x^*_k,y^*_k) \to (x^*,y^*)$ with $(x^*_k,-y^*_k) \in \hat{N}((\bar{x}+t_k u_k, \bar{y}+t_k v_k); \operatorname{gph} M)$.*

The multifunction M is called strongly dually norm stable in direction (u,v) *at $(\bar{x},\bar{y})$ if*

$$D^*_S M((\bar{x},\bar{y});(u,v)) = D^*_N M((\bar{x},\bar{y});(u,v)) =: D^* M((\bar{x},\bar{y});(u,v)).$$

Further we set $\check{N}(x;\Omega) := \check{N}(x;\Omega;0)$, $D^*_S M(\bar{x},\bar{y}) := D^*_S M((\bar{x},\bar{y});(0,0))$.

It follows immediately from the definition that strong dual-norm stability in direction $(0,0)$ of a set-valued mapping M implies *strong coderivative normality* [20, Definition 4.8].

3 Generalized polyhedral multifunctions

We start this section with the following generalization of the well-known Hoffman's lemma [14] to the infinite dimensional case [16, Theorem 3]:

Theorem 1 (Hoffman's lemma)**.** *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear continuous operator with closed range. Given $x_j^* \in X^*$, $j = 1,\dots,m$, consider the multifunction $S : Y \times \mathbb{R}^m \rightrightarrows X$ with values*

$$S(y,\xi) := \{x \in X \,|\, Ax = y,\ \langle x_j^*, x\rangle \le \xi_j,\ j = 1,\dots,m\}$$

Then there exists a constant $\gamma > 0$ such that for any $x \in X$ and any $(y,\xi) \in \operatorname{dom} S$,

$$\mathrm{d}(x, S(y,\xi)) \le \gamma(\|Ax-y\| + \sum_{j=1}^{m} \max\{\langle x_j^*, x\rangle - \xi_j, 0\})$$

In what follows we will also use the following infinite dimensional version of the Farkas lemma which can be found e.g. in [2, Proposition 2.201]

Theorem 2 (Generalized Farkas lemma)**.** *Let X and Y be Banach spaces, $A : X \to Y$ a continuous linear operator with closed range and $x_j^* \in X^*$, $j = 1,\dots,m$. Then the following holds:*

1. *The polar cone of*

$$K := \{x \in X \,|\, Ax = 0,\ \langle x_j^*, x\rangle \le 0,\ j = 1,\dots,m\}$$

can be written in the form

$$K^* = A^*Y^* + \{\sum_{j=1}^{m} \lambda_j x_j^* \,|\, \lambda_j \ge 0,\ j = 1,\dots,m\}$$

2. *There exists $\gamma > 0$ such that for every $x^* \in K^*$ there exist $y^* \in Y^*$ and $\lambda \in \mathbb{R}^m_+$ satisfying*

$$\|y^*\| + \|\lambda\| \le \gamma\|x^*\|,\ x^* = A^*y^* + \sum_{j=1}^{m} \lambda_j x_j^*.$$

Now let L be a closed linear subspace of the Banach space X, let $x_j^* \in X^*$, $j = 1,\dots,m$ and define the multifunction $S : X \times \mathbb{R}^m \rightrightarrows X$ by

$$S(x',\xi) := \{x \in X \,|\, x \in x' + L,\ \langle x_j^*, x\rangle \le \xi_j,\ j = 1,\dots,m\}.$$

Taking $A : X \to X/L$, $Ax := [x]_L$, Hoffman's lemma states, that there exist $\gamma > 0$ such that for any $x \in X$ and any $(x',\xi) \in \operatorname{dom} S$

$$\mathrm{d}(x, S(x',\xi)) \le \gamma(\mathrm{d}(x-x', L) + \sum_{j=1}^{m} \max\{\langle x_j^*, x\rangle - \xi_j, 0\}).$$

Defining $K := L \cap \{x \in X \,|\, \langle x_j^*, x\rangle \le 0\}$ and taking into account the relation $A^*W^* = (\ker A)^\perp$ valid for operators $A : X \to W$ with closed range between Banach spaces, we obtain

$$K^* = L^\perp + \{\sum_{j=1}^{m} \lambda_j x_j^* \,|\, \lambda_j \ge 0, j = 1,\dots,m\}$$

Further, for any $(x', \xi) \in \operatorname{dom} S$ and any $\bar{x} \in P := S(x', \xi)$ we have

$$T(\bar{x}; P) = \{x \in L \,|\, \langle x_j^*, x\rangle \le 0,\ j \in \mathscr{A}(\bar{z})\}$$

and

$$N(\bar{x}; P) = L^\perp + \{\sum_{j \in \mathscr{A}(\bar{x})} \lambda_j x_j^* \,|\, \lambda_j \ge 0,\ j \in \mathscr{A}(\bar{x})\},$$

where $\mathscr{A}(\bar{x}) = \{j \in \{1,\dots,m\} \,|\, \langle x_j^*, \bar{x}\rangle = \xi_j\}$ denotes the index set of active inequality constraints.

Next let us consider the intersection of two convex generalized polyhedral sets.

Lemma 2. *Let Z be a Banach space and let $P_1, P_2 \subset Z$ denote two convex generalized polyhedral sets with representation*

$$P_l = \{z \in z_l' + L_l \,|\, \langle z_{lj}^*, z\rangle \le \zeta_{lj}, j = 1,\dots,m_l\},\ l = 1,2. \tag{9}$$

Then $P := P_1 \cap P_2$ is again a convex generalized polyhedral set and if $L_1^\perp + L_2^\perp$ is closed, then

$$N(\bar{z}; P_1 \cap P_2) = N(\bar{z}; P_1) + N(\bar{z}; P_2),\ \forall \bar{z} \in P_1 \cap P_2.$$

Proof. If $z' \in (z_1' + L_1) \cap (z_2' + L_2) \ne \emptyset$, then we have $z' + L_l = z_l' + L_l$, $l = 1,2$ and therefore $P = \{z \in z' + L_1 \cap L_2 \,|\, \langle z_{lj}^*, z\rangle \le \zeta_{lj}, j = 1,\dots,m_l,\ l = 1,2\}$ is a convex generalized polyhedral set. On the other hand, if $(z_1' + L_1) \cap (z_2' + L_2) = \emptyset$ then $P = \emptyset$ is also a convex generalized polyhedral set. Now consider the operator $A : Z \to Z/L_1 \times Z/L_2$ defined by $Az := ([z]_{L_1}, [z]_{L_2})$. Identifying $(Z/L_l)^*$, $l = 1,2$ with $L_l^\perp$, we have $A^* : L_1^\perp \times L_2^\perp \to Z^*$, $A^*(l_1^*, l_2^*) = l_1^* + l_2^*$. Hence the range of A^* is $L_1^\perp + L_2^\perp$ and therefore closed. Utilizing the Closed Range Theorem we obtain $(L_1 \cap L_2)^\perp = (\ker A)^\perp = L_1^\perp + L_2^\perp$ and the assertion about the normal cones follows. □

Now we consider the union of finitely many convex generalized polyhedral sets

$$P = \bigcup_{i=1}^{p} P_i,\ P_i = \{z \in z' + L_i \,|\, \langle z_{ij}^*, z\rangle \le \zeta_{ij},\ j = 1,\dots,m_i\}, i = 1,\dots,p, \tag{10}$$

where for each $i = 1,\dots,p$, $L_i \subset Z$ is a closed linear subspace of the Banach space Z, $z_i' \in Z$ and $z_{ij}^* \in Z^*$, $\zeta_{ij} \in \mathbb{R}$ for $j = 1,\dots,m_i$. We denote by $L_i' = z_i' + L_i$ the closed affine subspace parallel to L_i. For arbitrary $z \in Z$ we denote by $\mathscr{P}(z) := \{i \in \{1,\dots,p\} \,|\, z \in P_i\}$ the index set of the generalized polyhedral sets containing z and for every $i \in \mathscr{P}(z)$ we denote the index set of active inequality constraints by $\mathscr{A}_i(z) := \{j \in \{1,\dots,m_i\} \,|\, \langle z_{ij}^*, z\rangle = \zeta_{ij}\}$.

Given $\bar{z} \in P$ and a direction $w \in Z$ we denote by $\mathscr{I}(\bar{z};w)$ the collection of index sets $\mathscr{P} \subset \{1,\dots,p\}$ such that there exist sequences $(t_k) \downarrow 0$, $(w_k) \to w$ with $\bar{z}+t_k w_k \in P$ and $\mathscr{P} = \mathscr{P}(\bar{z}+t_k w_k)$ for all k. Note that for every $\mathscr{P} \in \mathscr{I}(\bar{z};w)$ we have $\mathscr{P} \subset \mathscr{P}(\bar{z};w) := \{i \in \mathscr{P}(\bar{z}) \,|\, w \in T(\bar{z};P_i)\}$.

Further, for every $\mathscr{P} \in \mathscr{I}(\bar{z};w)$ we denote by $\mathscr{J}_{\mathscr{P}}(\bar{z};w)$ the collection of all sets $(\mathscr{A}_i)_{i\in\mathscr{P}}$ such that there exist sequences $(t_k) \downarrow 0$, $(w_k) \to w$ with $\bar{z}+t_k w_k \in P$, $\mathscr{P} = \mathscr{P}(\bar{z}+t_k w_k)$ and $\mathscr{A}_i = \mathscr{A}_i(\bar{z}+t_k w_k)$, $i \in \mathscr{P}$ for all k.

Since there are only finitely many subsets of $\{1,\dots,p\}$ respectively $\{1,\dots,m_i\}$, $i = 1,\dots,p$, there exist $\bar{\rho} > 0$ and $\bar{\delta} > 0$ such that

$$\begin{aligned}&\{(\mathscr{P},(\mathscr{A}_i)_{i\in\mathscr{P}}) \,|\, \mathscr{P} \in \mathscr{I}(\bar{z};w),\ (\mathscr{A}_i)_{i\in\mathscr{P}} \in \mathscr{J}_{\mathscr{P}}(\bar{z};w)\}\\ &\quad= \bigcup_{z\in\bar{z}+\operatorname{int}V_{\rho,\delta}(w)} \{(\mathscr{P}(z),(\mathscr{A}_i(z))_{i\in\mathscr{P}(z)})\},\ \forall 0<\rho<\bar{\rho},\ 0<\delta<\bar{\delta}.\end{aligned} \tag{11}$$

For every $z \in P$ we have

$$T(z;P) = \bigcup_{i\in\mathscr{P}(z)} T(z;P_i) = \bigcup_{i\in\mathscr{P}(z)} \{w \in L_i \,|\, \langle z^*_{ij}, w\rangle \le 0, j \in \mathscr{A}_i(z)\}$$

and

$$\hat{N}(z;P) = \bigcap_{i\in\mathscr{P}(z)} \hat{N}(z;P_i) = \bigcap_{i\in\mathscr{P}(z)} (L_i^{\perp} + \{\sum_{j\in\mathscr{A}_i(z)} \lambda_{ij} z^*_{ij} \,|\, \lambda_{ij} \ge 0, j \in \mathscr{A}_i(z)\}).$$

Since the sets P_i are convex generalized polyhedral sets, for every $z \in P$ there exists some neighborhood U of z such that $P \cap U = (z + T(z;P)) \cap U$. Taking into account that $T(z;P)$ is a cone we obtain $N(z;P;w) = N(0;T(z;P);w) = N(w;T(z;P))$ for every direction w.

Lemma 3. *Let the subset P of the Banach space Z be given by* (10)*, let $\bar{z} \in P$ and $w \in Z$. Then there exist $\bar{\rho} > 0$ and $\bar{\delta} > 0$ such that for every $0 < \rho < \bar{\rho}$, $0 < \delta < \bar{\delta}$ we have*

$$\begin{aligned}N(\bar{z};P;w) &= \check{N}(\bar{z};P;w) = \bigcup_{z\in\bar{z}+\operatorname{int}V_{\rho,\delta}(w)} \hat{N}(z;P)\}\\ &= \bigcup_{\substack{\mathscr{P}\in\mathscr{I}(\bar{z};w)\\ (\mathscr{A}_i)_{i\in\mathscr{P}}\in\mathscr{J}_{\mathscr{P}}(\bar{z};w)}} \bigcap_{i\in\mathscr{P}} (L_i^{\perp} + \{\sum_{j\in\mathscr{A}_i} \lambda_{ij} z^*_{ij} \,|\, \lambda_{ij} \ge 0, j \in \mathscr{A}_i\})\Big\}.\end{aligned}$$

Proof. Let $z^* \in N(\bar{z};P;w)$ and consider sequences $(t_k) \downarrow 0$, $(\varepsilon_k) \downarrow 0$, $(w_k) \to w$, $(z^*_k) \xrightarrow{w^*} z^*$ with $z^*_k \in \hat{N}_{\varepsilon_k}(z_k;P)$, where $z_k := \bar{z}+t_k w$. Since there are only finitely many subsets $\mathscr{P} \subset \{1,\dots,p\}$, $\mathscr{A}_i \subset \{1,\dots,m_i\}$, $i \in \{1,\dots,p\}$, by passing to a subsequence if necessary, we can assume that there are index sets $\mathscr{P} \in \mathscr{I}(\bar{z};w)$, $(\mathscr{A}_i)_{i\in\mathscr{P}} \in \mathscr{J}_{\mathscr{P}}(\bar{z};w)$ satisfying $\mathscr{P}(z_k) = \mathscr{P}$ and $\mathscr{A}_i(z_k) = \mathscr{A}_i$, $i \in \mathscr{P}$ for all k. Then $z^*_k \in \hat{N}_{\varepsilon_k}(z_k;P) = \hat{N}_{\varepsilon_k}(z_k;\bigcup_{i\in\mathscr{P}} P_i) = \bigcap_{i\in\mathscr{P}} \hat{N}_{\varepsilon_k}(z_k;P_i)$. Hence, for every $i \in \mathscr{P}$ and every k there exist some functional $z^*_{ik} \in \hat{N}(z_k;P_i)$ with $\|z^*_k - z^*_{ik}\| \le \varepsilon_k$. Since the Fréchet normal cones of convex sets are always weak* closed and $\hat{N}(z_k;P_i) = \hat{N}(z_{k'};P_i)$ for all $i \in \mathscr{P}$, k, k',we obtain $(z^*_{ik}) \xrightarrow{w^*} z^* \in \hat{N}(z_k;P_i)$, $i \in \mathscr{P}$ and therefore

$$z^* \in \bigcap_{i\in\mathscr{P}} \hat{N}(z_k;P_i) = \bigcap_{i\in\mathscr{P}} (L_i^{\perp} + \{\sum_{j\in\mathscr{A}_i} \lambda_{ij} z^*_{ij} \,|\, \lambda_{ij} \ge 0, j \in \mathscr{A}_i\}).$$

Since $\bigcap_{i\in\mathscr{P}}\hat{N}(z_k;P_i)=\hat{N}(z_k;P)$ we can also conclude $z^*\in\bigcup_{(x,y)\in\bar{z}+\mathrm{int}\,V_{\rho,\delta}(w)}\hat{N}(z;P)$ for all positive $\rho>0$, $\delta>0$.

Now choose $\bar{\rho}>0$, $\bar{\delta}>0$ such that (11) holds and let $\mathscr{P}\in\mathscr{I}(\bar{z};w)$, $(\mathscr{A}_i)_{i\in\mathscr{P}}\in\mathscr{J}_{\mathscr{P}}(\bar{z};w)$,

$$z^*\in\bigcap_{i\in\mathscr{P}}(L_i^{\perp}+\{\sum_{j\in\mathscr{A}_i}\lambda_{ij}z_{ij}^*\,|\,\lambda_{ij}\geq 0, j\in\mathscr{A}_i\})$$

and consider sequences $(t_k)\downarrow 0$, $w_k\to w$ with $\bar{z}+t_kw_k\in P$, $\mathscr{P}=\mathscr{P}(\bar{z}+t_kw_k)$ and $\mathscr{A}_i=\mathscr{A}_i(\bar{z}+t_kw_k)$, $i\in\mathscr{P}$ for all k. Then we have

$$z^*\in\bigcap_{i\in\mathscr{P}}\hat{N}(\bar{z}+t_kw_k;P_i)=\hat{N}(\bar{z}+t_kw_k;P)$$

and $z^*\in\check{N}(\bar{z};P;w)$ follows. Since $\check{N}(\bar{z};P;w)\subset N(\bar{z};P;w)$, the assertion follows. □

Next we consider a generalized polyhedral multifunction $S:X\rightrightarrows Y$ between Banach spaces. By the definition, its graph can be written in the form

$$\mathrm{gph}\,S=\bigcup_{i=1}^{p}P_i,\ P_i=\{(x,y)\in(x_i',y_i')+L_i\,|\,\langle x_{ij}^*,x\rangle+\langle y_{ij}^*,y\rangle\leq\zeta_{ij},\ j=1,\ldots,m_i\},i=1,\ldots,p \quad (12)$$

where for each $i=1,\ldots,p$, $L_i\subset X\times Y$ is a closed linear subspace, $(x_i',y_i')\in X\times Y$ and $x_{ij}^*\in X^*$, $y_{ij}^*\in Y^*$, $\zeta_{ij}\in\mathbb{R}$ for $j=1,\ldots,m_i$, i.e. (12) is a special case of (10) with $Z=X\times Y$, $z_i'=(x_i',y_i')$ and $z_{ij}^*=(x_{ij}^*,y_{ij}*$. Hence, given $(x,y),(\bar{x},\bar{y})\in\mathrm{gph}\,S$ and $(u,v)\in X\times Y$, the sets $\mathscr{P}(x,y)$, $\mathscr{P}((x,y);(u,v))$, $\mathscr{A}_i(x,y)$, $\mathscr{I}((\bar{x},\bar{y});(u,v))$ and $\mathscr{J}_{\mathscr{P}}((\bar{x},\bar{y});(u,v))$ are already defined. Later we will also use the set $\mathscr{I}(\bar{x},\bar{y}):=\mathscr{I}((\bar{x},\bar{y});(0,0))$ respectively the index set $\bar{\mathscr{I}}(\bar{x},\bar{y})$ which is defined as the collection of all index sets $\bar{\mathscr{P}}\subset\{1,\ldots,p\}$ such that there exist a sequence $(t_k)\downarrow 0$ and a bounded sequence $(u_k,v_k)\in\mathscr{S}_X\times Y$ with $(\bar{x}+t_ku_k,\bar{y}+t_kv_k)\in\mathrm{gph}\,S$ for all k and

$$\lim_{k\to\infty}t_k^{-1}\mathrm{d}((\bar{x}+t_ku_k,\bar{y}+t_kv_k),P_i)=0,\ i\in\bar{\mathscr{P}},\quad \liminf_{k\to\infty}t_k^{-1}\mathrm{d}((\bar{x}+t_ku_k,\bar{y}+t_kv_k),P_i)>0,\ i\notin\bar{\mathscr{P}}.$$

Further we define the multifunctions $S_i:X\rightrightarrows Y$ by $S_i(x):=\{y\in Y\,|\,(x,y)\in P_i\}$ and we denote by $\pi_X(L_i):=\{x\in X\,|\,\exists y\in Y:(x,y)\in L_i\}$ respectively $\pi_Y(L_i):=\{y\in Y\,|\,\exists x\in X:(x,y)\in L_i\}$ the projection of the subspace L_i on X respectively Y.

Lemma 4. *Let $S:X\rightrightarrows Y$ be a generalized polyhedral multifunction with its graph given in the form* (12)*. Assume that for some $i\in\{1,\ldots,p\}$ the projection $\pi_X(L_i)$ is closed in X. Then the domain of S_i is a generalized convex polyhedral set and there is a constant $\gamma_i>0$ such that for every $\bar{x}\in\mathrm{dom}\,S_i$ we have*

$$S_i(x)\subset S_i(\bar{x})+\gamma_i\|x-\bar{x}\|\mathscr{B}_Y,\ \forall x\in X.$$

Proof. Let $(A_{ix},A_{iy}):X\times Y\to W_i$ be a surjective continuous linear mapping onto another Banach space W_i, such that $L_i=\ker(A_{ix},A_{iy})$. Let $(y_k)\in Y$ denote a sequence such that $A_{iy}y_k$ converges to some element $w\in W_i$. By the Open Mapping Theorem we can find a sequence $(x_k',y_k')\in X\times Y$ converging to $(0,0)$ with $A_{ix}x_k'+A_{iy}y_k'=w-A_{iy}y_k$. Let $(\tilde{x},\tilde{y})$ be chosen such that $w=A_{ix}\tilde{x}+A_{iy}\tilde{y}$. Then $(x_k'-\tilde{x},y_k'+y_k-\tilde{y})\in L_i$, $\lim_{k\to\infty}x_k'-\tilde{x}=-\tilde{x}$ and since $\pi_X(L_i)$ is closed, there is some $\hat{y}$ with $(-\tilde{x},\hat{y})\in L_i$, i.e. $-A_{ix}\tilde{x}+A_{iy}\hat{y}=0$ showing $w=A_{iy}(\hat{y}+\tilde{y})$. Hence the range of A_{iy} is closed and the assertion of the lemma follows from [2, Theorem 2.207]. □

The following theorem is a generalization of a result of Robinson [24] to the infinite dimensional case.

Theorem 3. *Let $S : X \rightrightarrows Y$ be a generalized polyhedral multifunction with its graph given in the form* (12)*.*

1. *Assume that for each $i = 1,\dots,p$ the projection $\pi_X(L_i)$ is closed in X. Then there exists a constant $\gamma > 0$ such that for every $\bar{x} \in X$ there exists a neighborhood U of $\bar{x}$ with*

$$S(x) \subset S(\bar{x}) + \gamma\|x-\bar{x}\|\mathscr{B}_Y,\ \forall x \in U.$$

2. *Assume that for each $i = 1,\dots,p$ the projection $\pi_Y(L_i)$ is closed in Y. Then there is a constant $\kappa > 0$ such that S is metrically subregular at every $(\bar{x},\bar{y}) \in \operatorname{gph} S$ with modulus κ.*

Proof. Let $\bar{x} \in X$ be arbitrarily fixed and consider the index set $I := \{i \in \{1,\dots,p\} \,|\, \bar{x} \in \operatorname{dom} S_i\}$. Since $\operatorname{dom} S_i$ is closed for every i we can find a neighborhood U of $\bar{x}$ such that $x \notin \operatorname{dom} S_i$, $\forall x \in U$, $i \in \{1,\dots p\} \setminus I$. If $I \neq \emptyset$, by Lemma 4 we obtain $S(x) = \bigcup_{i\in I} S_i(x) \subset \bigcup_{i\in I} S_i(\bar{x}) + \gamma_i\|x-\bar{x}\|\mathscr{B}_Y \subset S(\bar{x}) + \gamma\|x-\bar{x}\|\mathscr{B}_Y$, where $\gamma := \max\{\gamma_i \,|\, i = 1,\dots,p\}$. If $I = \emptyset$ then $S(x) = \emptyset$, $\forall x \in U$ and the assertion is automatically true. Hence the first part of the theorem is proved. To show the second part note that $\operatorname{gph} S^{-1} = \{(y,x) \,|\, (x,y) \in \operatorname{gph} S\}$ and hence S^{-1} is a generalized polyhedral multifunction. Using the first part of the theorem, there exists some constant $\kappa > 0$ such that for every $\bar{y} \in Y$ there exists a neighborhood V of $\bar{y}$ with $S^{-1}(y) \subset S^{-1}(\bar{y}) + \kappa\|y-\bar{y}\|\mathscr{B}_X$, $\forall y \in V$. Now let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ be arbitrarily fixed and choose some radius $\rho > 0$ with $\bar{y} + \rho\mathscr{B}_Y \subset V$. Set $\delta := \rho\kappa$ and consider $x \in \bar{x} + \delta\mathscr{B}_X$. If $\mathrm{d}(\bar{y}, S(x)) < \rho$ then for every $\varepsilon > 0$ we can find some $y \in S(x) \cap V$ with $\|y-\bar{y}\| \le \mathrm{d}(\bar{y}, S(x)) + \varepsilon$ implying $\mathrm{d}(x, S^{-1}(\bar{y})) \le \kappa\|y-\bar{y}\|$ and consequently $\mathrm{d}(x, S^{-1}(\bar{y})) \le \kappa \mathrm{d}(\bar{y}, S(x))$. On the other hand, if $\mathrm{d}(\bar{y}, S(x)) \ge \rho$ then $\mathrm{d}(x, S^{-1}(\bar{y})) \le \delta = \kappa\rho \le \kappa \mathrm{d}(\bar{y}, S(x))$ also holds. □

The following lemma states that the sum of a generalized polyhedral multifunction and a continuous linear operator is again a generalized polyhedral multifunction.

Lemma 5. *Let $S : X \rightrightarrows Y$ be a generalized polyhedral multifunction and let $A : X \to Y$ be a continuous linear operator. Then $\tilde{S} : X \rightrightarrows Y$, $\tilde{S}(x) := Ax + S(x)$ is also a generalized polyhedral multifunction.*

Proof. Assuming that $\operatorname{gph} S$ has the representation (12) we obtain $\operatorname{gph} \tilde{S} = \bigcup_{i=1}^p \tilde{P}_i$ with

$$\tilde{P}_i := \{(x,y) \in (x_i', y_i' + Ax_i') + \tilde{L}_i \,|\, \langle x_{ij}^* - A^* y_{ij}^*, x\rangle + \langle y_{ij}^*, y\rangle \le \xi_{ij},\ j = 1,\dots,m_i\},\ i = 1,\dots,p,$$

where $\tilde{L}_i := \{(x,y) \in X \times Y \,|\, (x, y - Ax) \in L_i\}$ is a closed subspace of $X \times Y$ and therefore $\tilde{S}$ is a generalized polyhedral multifunction. □

The following lemma follows immediately from Lemma 3.

Lemma 6. *Let the generalized polyhedral multifunction $S : X \rightrightarrows Y$ between Banach spaces be given by* (12) *and let $(\bar{x},\bar{y}) \in \operatorname{gph} S$ and $(u,v) \in X \times Y$. Then S is strongly dually norm stable in direction (u,v) at $(\bar{x},\bar{y})$ and there exist $\bar{\rho} > 0$ and $\bar{\delta} > 0$ such that for every $0 < \rho < \bar{\rho}$, $0 < \delta < \bar{\delta}$ and every $y^* \in Y^*$ we have*

$$D^*S((\bar{x},\bar{y});(u,v))(y^*) = \{x^* \mid (x^*,-y^*) \in \bigcup_{(x,y)\in(\bar{x},\bar{y})+\operatorname{int} V_{\rho,\delta}(u,v)} \hat{N}((x,y);\operatorname{gph} S)\}$$

$$= \Big\{x^* \in X^* \mid (x^*,-y^*) \in \bigcup_{\substack{\mathscr{P}\in\mathscr{I}((\bar{x},\bar{y});(u,v))\\ (\mathscr{A}_i)_{i\in\mathscr{P}}\in\mathscr{J}_{\mathscr{P}}((\bar{x},\bar{y});(u,v))}} \bigcap_{i\in\mathscr{P}} (L_i^\perp + \{\sum_{j\in\mathscr{A}_i} \lambda_{ij}(x^*_{ij},y^*_{ij}) \mid \lambda_{ij} \geq 0, j \in \mathscr{A}_i\})\Big\}.$$

Next we consider two generalized polyhedral multifunctions $S_1 : X \rightrightarrows Y_1$ respectively $S_2 : X \rightrightarrows Y_1$ with representations

$$\operatorname{gph} S_l = \bigcup_{i=1,\ldots,p_l} P_{li} \subset X \times Y_l,\ l = 1,2 \tag{13}$$

$$P_{li} = \{(x,y_l) \in (x'_{li},y'_{li}) + L_{li} \mid \langle x^*_{lij},x\rangle + \langle y^*_{lij},y_l\rangle \leq \zeta_{lij},\ j = 1,\ldots,m_{li}\},\ i = 1,\ldots,p_l,\ l = 1,2, \tag{14}$$

and study the composite multifunction $S : X \rightrightarrows Y_1 \times Y_2$, $S(x) = S_1(x) \times S_2(x)$.

In what follows we denote for $(x,y_l) \in \operatorname{gph} S_l$, $l = 1,2$ by $\mathscr{P}_l(x,y_l) := \{i \in \{1,\ldots,p_l\} \mid (x,y_l) \in P_{li}\}$ the index set of the generalized polyhedral sets containing (x,y_l) and by $\mathscr{A}_{li}(x,y_l) = \{i \in \{1,\ldots,m_{li}\} \mid \langle x^*_{lij},x\rangle + \langle y^*_{lij},y_l\rangle = \zeta_{lij}\}$, $i =\in \mathscr{P}_l(x,y_l)$ the index sets of active inequality constraints.

Given $(\bar{x},\bar{y}_l) \in \operatorname{gph} S_l$ and a direction $(u,v_l) \in X \times Y_l$ we denote by $\mathscr{I}_l((\bar{x},\bar{y}_l);(u,v_l))$ the collection of index sets $\mathscr{P} \subset \{1,\ldots,p_l\}$ such that there exist sequences $(t_k) \downarrow 0$, $(u_k,v_{lk}) \to (u,v_l)$ with $(\bar{x}+t_ku_k,\bar{y}+t_kv_{lk}) \in \operatorname{gph} S_l$ and $\mathscr{P} = \mathscr{P}_l(\bar{x}+t_ku_k,\bar{y}_l+t_kv_{lk})$.

Lemma 7. *Let $S_l : X \rightrightarrows Y_l$, $l = 1,2$ be two multifunctions between Banach spaces with representation* (13), (14). *Then the multifunction $S : X \rightrightarrows Y_1 \times Y_2$, $S(x) := S_1(x) \times S_2(x)$ is again a generalized polyhedral multifunction. Given $(\bar{x},(\bar{y}_1,\bar{y}_2)) \in \operatorname{gph} S$ and a direction $(u,(v_1,v_2)) \in X \times Y_1 \times Y_2$, assume that for every pair of index sets $(\mathscr{P}_1,\mathscr{P}_2) \in \mathscr{I}_1((\bar{x},\bar{y}_1);(u,v_1)) \times \mathscr{I}_2((\bar{x},\bar{y}_2);(u,v_2))$ the subspace*

$$\sum_{i_1\in\mathscr{P}_1} \pi_X(L_{1i_1}) - \sum_{i_2\in\mathscr{P}_2} \pi_X(L_{2i_2})$$

is closed. Then

$$D^*S((\bar{x},(\bar{y}_1,\bar{y}_2)),(u,(v_1,v_2)))(y_1^*,y_2^*) \subset D^*S_1((\bar{x},\bar{y}_1),(u,v_1))(y_1^*) + D^*S_2((\bar{x},\bar{y}_2),(u,v_2))(y_2^*). \tag{15}$$

Proof. Let $\tilde{P}_{li} := \{(x,y_1,y_2) \in X \times Y_1 \times Y_2 \mid (x,y_l) \in P_{li}\}$, $i = 1,\ldots p_l$, $l = 1,2$. Then

$$\operatorname{gph} S = \bigcup_{\substack{i_1\in\{1,\ldots,p_1\}\\ i_2\in\{1,\ldots,p_2\}}} (\tilde{P}_{1i_1} \cap \tilde{P}_{2i_2})$$

and each of the sets $\hat{P}_{i_1 i_2} := \tilde{P}_{1i_1} \cap \tilde{P}_{2i_2}$ is a convex generalized polyhedral set as the intersection of two convex generalized polyhedral sets as stated by Lemma 2. Hence S is a generalized polyhedral multifunction. Now fix $x^* \in D^*S((\bar{x},(\bar{y}_1,\bar{y}_2)),(u,(v_1,v_2)))(y_1^*,y_2^*)$. Then, by Lemma 6 we know that there exist an index set $\mathscr{P} \subset \{1,\dots,p_1\} \times \{1,\dots,p_2\}$ and sequences $(t_k) \downarrow 0$, $(u_k,(v_{k1},v_{k2})) \to (u,(v_1,v_2))$ such that $\mathscr{P} = \{(i_1,i_2) \,|\, (x_k,y_k) \in \hat{P}_{i_1 i_2}\}$ and $(x^*,-(y_1^*,y_2^*)) \in \hat{N}((x_k,y_k);\bigcup_{(i_1,i_2)\in\mathscr{P}} \hat{P}_{i_1 i_2})$ for all k, where $x_k := \bar{x} + t_k u_k$, $y_k := (\bar{y}_1 + t_k v_{k1}, \bar{y}_2 + t_k v_{k2})$. Taking $\mathscr{P}_1 := \{i_1 \,|\, \exists i_2 : (i_1,i_2) \in \mathscr{P}\}$, $\mathscr{P}_2 := \{i_2 \,|\, \exists i_1 : (i_1,i_2) \in \mathscr{P}\}$ we have $\mathscr{P} = \mathscr{P}_1 \times \mathscr{P}_2$, $\mathscr{P}_l = \mathscr{P}_l(\bar{x}+t_k u_k, \bar{y}_l + t_k v_{lk}) \in \mathscr{I}_l((\bar{x},\bar{y}_l);(u,v_l))$, $l=1,2$ and $\bigcup_{(i_1,i_2)\in\mathscr{P}} \hat{P}_{i_1 i_2} = (\bigcup_{i_1\in\mathscr{P}_1} \tilde{P}_{1i_1}) \cap (\bigcup_{i_2\in\mathscr{P}_2} \tilde{P}_{2i_2})$. Hence, the functional $(x^*,-(y_1^*,y_2^*))$ belongs to the weak* closure of the set

$$
\begin{aligned}
N_k &:= \bigcap_{i_1\in\mathscr{P}_1} \hat{N}((x_k,y_k);\tilde{P}_{1i_1}) + \bigcap_{i_2\in\mathscr{P}_2} \hat{N}((x_k,y_k);\tilde{P}_{2i_2}) \\
&= \bigcap_{i_1\in\mathscr{P}_1} \{(x^*,y_1^*,0) \,|\, (x^*,y_1^*) \in \hat{N}((x_k,y_{k1});P_{1i_1})\} + \bigcap_{i_2\in\mathscr{P}_2} \{(x^*,0,y_2^*) \,|\, (x^*,y_2^*) \in \hat{N}((x_k,y_{k2});P_{2i_2})\}
\end{aligned}
$$

for all k, see e.g [8, Lemma 5.8], and there remains to show that N_k is weak* closed. Indeed, if N_k is weak* closed, then there exist functionals $x_1^*,x_2^* \in X^*$ with $(x_1^*,-y_1^*,0) \in \bigcap_{i_1\in\mathscr{P}_1} \hat{N}((x_k,y_k);\tilde{P}_{1i_1})$, $(x_2^*,0,-y_2^*) \in \bigcap_{i_2\in\mathscr{P}_2} \hat{N}((x_k,y_k);\tilde{P}_{2i_2})$ and $x_1^*+x_2^* = x^*$, implying $x_l^* \in D^*S_l((\bar{x},\bar{y}_1),(u,v_l))(y_l^*)$, $l=1,2$.

To show that N_k is weak* closed, let $Z_l := X \times Y_l$, $l=1,2$ and consider the operator

$$
T : X \times Y_1 \times Y_2 \times \prod_{i_1\in\mathscr{P}_1} Z_1 \times \prod_{i_2\in\mathscr{P}_2} Z_2 \to Z_1 \times Z_2 \times \prod_{i_1\in\mathscr{P}_1} (Z_1/L_{1i_1}) \times \prod_{i_2\in\mathscr{P}_2} (Z_2/L_{2i_2}),
$$

$$
\begin{aligned}
&T((\hat{x},\hat{y}_1,\hat{y}_2),(z_{1i_1})_{i_1\in\mathscr{P}_1},(z_{1i_2})_{i_2\in\mathscr{P}_2}) \\
&\quad := \left((\hat{x},\hat{y}_1) - \sum_{i_1\in\mathscr{P}_1} z_{1i_1}, (\hat{x},\hat{y}_2) - \sum_{i_2\in\mathscr{P}_2} z_{2i_2}, ([z_{1i_1}]_{L_{1i_1}})_{i_1\in\mathscr{P}_1}, ([z_{2i_2}]_{L_{2i_2}})_{i_2\in\mathscr{P}_2} \right).
\end{aligned}
$$

We will now show that T has closed range. Let $(\hat{x}_k),(\hat{y}_{1k}),(\hat{y}_{2k}),(z_{1i_1k})_{i_1\in\mathscr{P}_1},(z_{1i_2}k)_{i_2\in\mathscr{P}_2}$ denote sequences such that $(\hat{x}_k,\hat{y}_{lk}) - \sum_{i_l\in\mathscr{P}_l} z_{li_lk} \to (\xi_l,\eta_l)$, $[z_{li_lk}]_{L_{li_l}} \to [\bar{z}_{li_l}]_{L_{li_l}}$, $i_l \in \mathscr{P}_l$, $l=1,2$. Then we can find elements $f_{li_lk} \in L_{li_l}$ with $z_{li_lk} + f_{li_lk} \to \bar{z}_{li_l}$ and therefore $\sum_{i_1\in\mathscr{P}_1} f_{1i_1kx} - \sum_{i_2\in\mathscr{P}_2} f_{2i_2kx} \to \xi_1 + \sum_{i_1\in\mathscr{P}_1} \bar{z}_{1i_1x} - \xi_2 - \sum_{i_2\in\mathscr{P}_2} \bar{z}_{2i_2x}$. The assumptions of the lemma guarantee that there exist elements $\bar{f}_{li_l} \in L_{li_l}$, $i_l \in \mathscr{P}_l$, $l=1,2$ with $\sum_{i_1\in\mathscr{P}_1} \bar{f}_{1i_1x} - \sum_{i_2\in\mathscr{P}_2} \bar{f}_{2i_2x} = \xi_1 + \sum_{i_1\in\mathscr{P}_1} \bar{z}_{1i_1x} - \xi_2 - \sum_{i_2\in\mathscr{P}_2} \bar{z}_{2i_2x}$ and we conclude that

$$
\begin{aligned}
&T(\xi_1 + \sum_{i_1\in\mathscr{P}_1} (\bar{z}_{1i_1x} - \bar{f}_{1i_1x}), \eta_1 + \sum_{i_1\in\mathscr{P}_1} (\bar{z}_{1i_1y_1} - \bar{f}_{1i_1y_1}), \eta_2 + \sum_{i_2\in\mathscr{P}_2} (\bar{z}_{2i_2y_2} - \bar{f}_{2i_2y_2}), \\
&\quad (\bar{z}_{1i_1} - \bar{f}_{1i_1})_{i_1\in\mathscr{P}_1}, (\bar{z}_{2i_2} - \bar{f}_{2i_2})_{i_2\in\mathscr{P}_2}) = ((\xi_1,\eta_1),(\xi_2,\eta_2),([\bar{z}_{1i_1}]_{L_{1i_1}})_{i_1\in\mathscr{P}_1},([\bar{z}_{2i_2}]_{L_{2i_2}})_{i_2\in\mathscr{P}_2}),
\end{aligned}
$$

showing that T has closed range. Then, by the Closed Range Theorem, Range(T^*) equals to $(\ker T)^\perp$ and is therefore weak* closed. We have

$$
T^* : Z_1^* \times Z_2^* \times \prod_{i_1\in\mathscr{P}_1} L_{1i_1}^\perp \times \prod_{i_2\in\mathscr{P}_2} L_{2i_2}^\perp \to X^* \times Y_1^* \times Y_2^* \times \prod_{i_1\in\mathscr{P}_1} Z_1^* \times \prod_{i_2\in\mathscr{P}_2} Z_2^*,
$$

$$T^*(z_1^*,z_2^*,(l_{1i_1}^*)_{i_1\in\mathscr{P}_1},(\tilde{l}_{2i_2}^*)_{i_2\in\mathscr{P}_2}) = (z_{1x^*}^*+z_{2x^*}^*,z_{1y_1^*}^*,z_{2y_2^*}^*,(-z_1^*+l_{1i_1}^*)_{i_1\in\mathscr{P}_1},(-z_2^*+l_{2i_2}^*)_{i_2\in\mathscr{P}_2})$$

Next define the cones

$$K_{li_lk} := \{\sum_{j\in\mathscr{A}_{li_l}(x_k,y_{kl})} \lambda_{li_lj}(x_{li_lj}^*,y_{li_lj}^*)\,|\,\lambda_{li_lj}\geq 0,\ j\in\mathscr{A}_{li_l}(x_k,y_{kl})\}\subset Z_l^*,\ i_l\in\mathscr{P}_l,\ l=1,2$$

and $K_k := \{0_{X^*}\}\times\{0_{Y_1^*}\}\times\{0_{Y_2^*}\}\times\prod_{i_1\in\mathscr{P}_1}K_{1i_1k}\times\prod_{i_2\in\mathscr{P}_2}K_{2i_2k}$. Then $\mathrm{Range}(T^*)+K_k$ is weak* closed as the sum of the weak* closed subspace $\mathrm{Range}(T^*)$ and the finitely generated cone K_k. It follows that

$$\begin{aligned} &N_k\times\prod_{i_1\in\mathscr{P}_1}\{0_{Z_1^*}\}\times\prod_{i_2\in\mathscr{P}_2}\{0_{Z_2^*}\}\\ &=(X^*\times Y_1^*\times Y_2^*\times\prod_{i_1\in\mathscr{P}_1}\{0_{Z_1^*}\}\times\prod_{i_2\in\mathscr{P}_2}\{0_{Z_2^*}\})\cap(\mathrm{Range}(T^*)+K_k)\end{aligned}$$

is weak* closed and consequently also N_k. □

At the end of this section we state a sufficient condition for proper subregularity of generalized polyhedral multifunctions:

Lemma 8. *Let $M_l : X\rightrightarrows Y_l$, $M_l(x)=A_l(x)+S_l(x)$, $l=1,2$ denote two generalized polyhedral multifunctions, where $A_l : X\to Y_l$ are continuous linear operators and the generalized polyhedral multifunctions $S_l : X\rightrightarrows Y_l$, $l=1,2$ have the representation* (13). *Further let $(\bar{x},(\bar{y}_1,\bar{y}_2))\in$ $\mathrm{gph}\,(M_1,M_2)$ and assume that for each $i_1\in\mathscr{P}_1(\bar{x},\bar{y}_1-A_1\bar{x})$ and each $i_2\in\mathscr{P}_2(\bar{x},\bar{y}_2-A_2\bar{x})$ the projection $\pi_X(L_{1i_1})$ and the subspace*

$$\{(x_1+x_2,y_2+A_2x_2)\,|\,(x_l,y_l)\in L_{li_l},\ l=1,2\}$$

are closed. Then for every $(u,v_1)\in X\times Y_1$ the multifunction M_2 is proper subregular in direction u relative to M_1 and v_1.

Proof. Choose $\rho>0$ small enough that $((\bar{x},\bar{y}_l-A_l\bar{x})+2\rho(1+\|A_l\|)\mathscr{B}_{X\times Y_l})\cap P_{li_l}=\emptyset$ holds for all $i_l\notin\mathscr{P}_l(\bar{x},\bar{y}_l-A_l\bar{x})$, $l=1,2$ and fix $(\tilde{x},\tilde{y}_1)\in((\bar{x},\bar{y}_1)+\rho\mathscr{B}_{X\times Y_1})\cap\mathrm{gph}\,M_1$ and $i_1\in\mathscr{P}_1(\tilde{x},\tilde{y}_1-A_1\tilde{x})$. Since $\|(\tilde{x},\tilde{y}_1-A_1\tilde{x})-(\bar{x},\bar{y}_1-A_1\bar{x})\|\leq(1+\|A_1\|)\|(\tilde{x},\tilde{y}_1)-(\bar{x},\bar{y}_1)\|\leq\rho(1+\|A_1\|)$ we conclude $i_1\in\mathscr{P}_1(\bar{x},\bar{y}_1-A_1\bar{x})$. Next take some $(x^+,y_2^+)\in\mathrm{gph}\,M_2$ with $\|(\tilde{x},\bar{y}_2)-(x^+,y_2^+)\|\leq 2\mathrm{d}((\tilde{x},\bar{y}_2),\mathrm{gph}\,M_2)$ and select any $i_2\in\mathscr{P}_2(x^+,y_2^+-A_2x^+)$. Since $\mathrm{d}((\tilde{x},\bar{y}_2),\mathrm{gph}\,M_2)\leq\|\tilde{x}-\bar{x}\|\leq\rho$ we obtain $\|(x^+,y_2^+-A_2x^+)-(\bar{x},\bar{y}_2-A_2\bar{x})\|\leq\|(x^+,y_2^+)-(\bar{x},\bar{y}_2)\|(1+\|A_2\|)\leq 2\rho(1+\|A_2\|)$ and consequently $i_2\in\mathscr{P}_2(\bar{x},\bar{y}_2-A_2\bar{x})$. Now our assumptions imply that the operator $T_{i_1i_2}$: $X\times Y_1\times L_{1i_1}\times L_{2i_2}\to(X\times Y_1)\times(X\times Y_2)$,

$$T_{i_1i_2}(x,y_1,l_{1i_1},l_{2i_2}) := ((x,y_1-A_1x)-l_{1i_1},(x,-A_2x)-l_{2i_2})$$

has closed range. In fact, for any sequence $(x_k,y_{1k},l_{1i_1k},l_{2i_2k}$ such that $T_{i_1i_2}(x_k,y_{1k},l_{1i_1k},l_{2i_2k})$ converges to some element $(\xi_1,\eta_1,\xi_2,\eta_2)\in(X\times Y_1)\times(X\times Y_2)$ we conclude $l_{2i_2kx}-l_{1i_1kx}\to$ $\xi_1-\xi_2$ and $A_2l_{2i_2kx}+l_{2i_2ky}\to-\eta_2-A_2\xi_2$, where $l_{li_lk}=(l_{li_lkx},l_{li_lky})\in X\times Y_l$, $l=1,2$. By our assumption there exist $l_{li_l} = (l_{li_lx}, l_{li_ly}) \in L_{li_l}$, $l = 1,2$ such that $l_{2i_2x} - l_{1i_1x} = \xi_1 - \xi_2$ and $A_2 l_{2i_2x} + l_{2i_2y} = -\eta_2 - A_2\xi_2$. Taking $x = \xi_1 + l_{1i_1x} (= \xi_2 + l_{2i_2x})$, $y_1 = \eta_1 + l_{1i_1y} + A_1\xi$ we obtain

$$T_{i_1i_2}(x, y_1, l_{1i_1}, l_{2i_2}) = (x - l_{1i_1x}, y_1 - A_1x - l_{1i_1y}, x - l_{2i_2x}, -Ax - l_{2i_2y}) = (\xi_1, \eta_1, \xi_2, \eta_2)$$

and therefore $\mathrm{Range}\, T_{i_1i_2}$ is closed. Consider the set

$$\hat{P}_{i_1i_2} = \left\{ (x, y_1, l_{1i_1}, l_{2i_2}) \in X \times Y_1 \times L_{1i_1} \times L_{2i_2} \;\middle|\; \begin{array}{l} T_{i_1i_2}(x, y_1, l_{1i_1}, l_{2i_2}) = (x'_{1i_1}, y'_{1i_1}, x'_{2i_2}, y'_{2i_2} - \bar{y}_2) \\ \langle x^*_{1i_1j}, x\rangle + \langle y^*_{1i_1j}, y_1 - A_1x\rangle \le \zeta_{1i_1j},\ j = 1, \ldots m_{1i_1} \\ \langle x^*_{2i_2j}, x\rangle + \langle y^*_{2i_2j}, \bar{y}_2 - A_2x\rangle \le \zeta_{2i_2j},\ j = 1, \ldots m_{2i_2} \end{array} \right\}$$

whose projection on $X \times Y_1$ is exactly the set $\bar{P}_{i_1i_2} := \{(x, y_1) \in X \times Y_1 \mid (x, y_1 - A_1x) \in P_{1i_1},\ (x, \bar{y}_2 - A_2x) \in P_{2i_2}\}$. Setting $\bar{l}_{li_l} := (\bar{x} - x'_{li_l}, \bar{y}_l - A_l\bar{x} - y'_{li_l}) \in L_{li_l}$, $l = 1,2$ we have $(\bar{x}, \bar{y}_1, \bar{l}_{1i_1}, \bar{l}_{2i_2}) \in \hat{P}_{i_1i_2} \neq \emptyset$ and therefore we can use Hoffman's lemma to find a constant $\gamma_{i_1i_2}$ such that

$$\mathrm{d}((x, y_1, l_{1i_1}, l_{2i_2}), \hat{P}_{i_1i_2}) \le \gamma_{i_1i_2}(\|T_{i_1i_2}(x, y_1, l_{1i_1}, l_{2i_2}) - (x'_{1i_1}, y'_{1i_1}, x'_{2i_2}, y'_{2i_2} - \bar{y}_2)\| + r_{i_1i_2}(x, y_1))$$

holds for all $(x, y_1, l_{1i_1}, l_{2i_2}) \in X \times Y_1 \times L_{1i_1} \times L_{2i_2}$, where

$$\begin{aligned} r_{i_1i_2}(x, y_1) \;:=\; & \sum_{j=1}^{m_{1i_1}} \max\{\langle x^*_{1i_1j}, x\rangle + \langle y^*_{1i_1j}, y_1 - A_1x\rangle - \zeta_{1i_1j}, 0\} \\ & + \sum_{j=1}^{m_{2i_2}} \max\{\langle x^*_{2i_2j}, x\rangle + \langle y^*_{2i_2j}, \bar{y}_2 - A_2x\rangle - \zeta_{2i_2j}, 0\} \end{aligned}$$

Putting $\tilde{l}_{i_1} = (\tilde{x} - x'_{1i_1}, \tilde{y}_1 - A_1\tilde{x} - y'_{1i_1}) \in L_{1i_1}$, $l^+_{i_2} = (x^+ - x'_{2i_2}, y^+_2 - A_2x^+ - y'_{2i_2}) \in L_{2i_2}$ we obtain

$$\begin{aligned} \mathrm{d}((\tilde{x}, \tilde{y}_1, \tilde{l}_{i_1}, l^+_{i_2}), \hat{P}_{i_1i_2}) &\le \gamma_{i_1i_2}(\|(0, 0, \tilde{x} - x^+, -A_2(\tilde{x} - x^+) + \bar{y}_2 - y^+_2)\| + r(\tilde{x}, \tilde{y}_1)) \\ &\le \gamma_{i_1i_2}((1 + \|A_2\|)\|(\tilde{x} - x^+, \bar{y}_2 - y^+_2)\| + r(\tilde{x}, \tilde{y}_1)) \end{aligned}$$

and

$$\begin{aligned} r(\tilde{x}, \tilde{y}_1)) &= \sum_{j=1}^{m_{2i_2}} \max\{\langle x^*_{2i_2j}, \tilde{x}\rangle + \langle y^*_{2i_2j}, \bar{y}_2 - A_2\tilde{x}\rangle - \zeta_{2i_2j}, 0\} \\ &\le \sum_{j=1}^{m_{2i_2}} \max\{\langle x^*_{2i_2j}, \tilde{x}\rangle + \langle y^*_{2i_2j}, \bar{y}_2 - A_2\tilde{x}\rangle - (\langle x^*_{2i_2j}, x^+\rangle + \langle y^*_{2i_2j}, y^+_2 - A_2x^+\rangle), 0\} \\ &\le \sum_{j=1}^{m_{2i_2}} (\|x^*_{2i_2j}\| + \|y^*_{2i_2j}\|(1 + \|A_2\|))\|(\tilde{x} - x^+, \bar{y}_2 - y^+_2)\|. \end{aligned}$$

Hence we can find $(\check{x}, \check{y}_1) \in \bar{P}_{i_1i_2}$ such that

$$\|(\check{x}, \check{y}_1) - (\tilde{x}, \tilde{y}_1)\| \le \tilde{\gamma}_{i_1i_2}\|(\tilde{x} - x^+, \bar{y}_2 - y^+_2)\| \le 2\tilde{\gamma}_{i_1i_2}\mathrm{d}((\tilde{x}, \bar{y}_2), \mathrm{gph}\, M_2) \le 2\tilde{\gamma}_{i_1i_2}\mathrm{d}(\bar{y}_2, M_2(\tilde{x})),$$

where $\tilde\gamma_{i_1i_2} := \gamma_{i_1i_2}((1+\|A_2\|)+\sum_{j=1}^{m_{2i_2}}(\|x^*_{2i_2j}\|+\|y^*_{2i_2j}\|(1+\|A_2\|)))+1$. Finally let $M_{1i_1}(x) := A_1x+\{y_1 \in Y_1 \,|\, (x,y_1) \in P_{1i_1}\}$. By Lemma 4 there exists a constant $\gamma_{1i_1} > 0$ such that

$$M_{1i_1}(\tilde x) \subset M_{1i_1}(x) + \gamma_{1i_1}\|\tilde x - x\|\mathscr{B}_{Y_1}$$

holds for every $x \in \operatorname{dom} M_{1i_1}$. Because of $\check y_1 \in M_{1i_1}(\check x)$ we have $\check x \in \operatorname{dom} M_{1i_1}$ and therefore $\mathrm{d}(\tilde y_1, M_1(\check x)) \le \mathrm{d}(\tilde y_1, M_{1i1}(\check x)) \le \gamma_{1i_1}\|\tilde x - \check x\|$. This shows that the property of proper subregularity holds with constants $\kappa' := 2\max\{\tilde\gamma_{i_1i_2} \,|\, i_l \in \mathscr{P}_l(\bar x, \bar y_l), l = 1,2\}$ and $L := \max\{\gamma_{1i_1} \,|\, i_1 \in \mathscr{P}_1(\bar x, \bar y_1)\}$. □

4 First-order and second-order characterizations of directional regularity properties

In this section we study different directional regularity properties such as mixed directional regularity/subregularity or directional subregularity of multifunctions of the form

$$M = (M_1, M_2) : X \rightrightarrows Y_1 \times Y_2 =: Y,\ M_l(x) = F_l(x) + S_l(x),\ l = 1,2 \tag{16}$$

where X, Y_1,Y_2 are Banach spaces, $F_l : X \to Y_l$, $l = 1,2$ are single-valued mappings and the multifunctions $S_l : X \rightrightarrows Y_l$, $l = 1,2$ are generalized polyhedral. We further assume that we are given a point $\bar x \in X$ such that $0 \in M(\bar x)$ and $F := (F_1, F_2)$ is strictly differentiable at $\bar x$.

A common assumption of this section will be that the multifunction M_2 is proper subregular (eventually in some direction) with respect to M_1 at some point $(\bar x, 0)$. This is certainly the case if $F_2(x) = A_2x$ is a linear mapping and $A_2x + S_2(x)$ is proper subregular relative to $F_1(\bar x) + \nabla F_1(\bar x)(x - \bar x) + S_1(x)$, the latter being true under the weak assumptions of Lemma 8. On the other hand, given an arbitrary multifunction $M_1 : X \rightrightarrows Y_1$ with $0 \in M_1(\bar x)$ the multifunction $M_2 : X \rightrightarrows \{0\}$, $M_2(x) := \{0\}$ is always proper subregular relative to M_1 at $(\bar x, 0)$.

By Lemma 7 we know that the multifunction $S : X \rightrightarrows Y$, $S(x) := (S_1(x), S_2(x))$ is also generalized polyhedral. In what follows we assume that the graph of S has the representation (12).

We will also use some kind of second-order differentiability for the mapping F. Given a direction $0 \ne u \in X$, we say that F is *second-order directional differentiable in direction u* at $\bar x$ if the limit

$$F''(\bar x; u) := \lim_{\substack{t \downarrow 0 \\ u' \to u}} \frac{F(\bar x + tu') - F(\bar x) - t\nabla F(\bar x)u'}{t^2/2} \tag{17}$$

exists.

The results of this section are valid under the assumption that certain linear operators have closed range. For each nonempty index set $\emptyset \ne \mathscr{P} \subset \{1, \ldots, p\}$ we define the linear operators

$$A_{\mathscr{P}} : \prod_{i \in \mathscr{P}} L_i \to Y, \quad A_{\mathscr{P}}((l_i)_{i \in \mathscr{P}}) := \sum_{i \in \mathscr{P}} (\nabla F(\bar x) l_{ix} + l_{iy}),$$

$$\tilde A^*_{\mathscr{P}} : Y^* \times \prod_{i \in \mathscr{P}} L_i^\perp \to \prod_{i \in \mathscr{P}} (X \times Y)^*, \quad \tilde A^*_{\mathscr{P}}(y^*, (l_i^*)_{i \in \mathscr{P}}) := ((\nabla F(\bar x)^* y^*, y^*) + l_i^*)_{i \in \mathscr{P}}$$

and

$$B_{\mathscr{P}} : X \times \prod_{i\in\mathscr{P}} L_i \to \prod_{i\in\mathscr{P}} (X\times Y), \quad B_{\mathscr{P}}(x,(l_i)_{i\in\mathscr{P}}) := ((x, -\nabla F(\bar{x})x) - l_i)_{i\in\mathscr{P}}.$$

Lemma 9. *Let $\emptyset \neq \mathscr{P} \subset \{1,\ldots,p\}$ be some index set. Then $A_{\mathscr{P}}$ has closed range if and only if $\tilde{A}^*_{\mathscr{P}}$ has closed range.*

Proof. Since all spaces involved are Banach spaces, by the Closed Range Theorem, $A_{\mathscr{P}}$ has closed range if and only if this holds for the adjoint operator $A^*_{\mathscr{P}} : Y^* \to \prod_{i\in\mathscr{P}} L_i^*$, $A^*_{\mathscr{P}}(y^*) = ((\nabla F(\bar{x})y^*, y^*)_{|L_i})_{i\in\mathscr{P}}$. Therefore it remains to show the equivalence of the closed range property of the operators $A^*_{\mathscr{P}}$ and $\tilde{A}^*_{\mathscr{P}}$. Assume that $A^*_{\mathscr{P}}$ has closed range and let $(y^*_k, (l^*_{ik})_{i\in\mathscr{P}}) \subset Y^* \times \prod_{i\in\mathscr{P}} L_i^\perp$ be a sequence such that $\tilde{A}^*_{\mathscr{P}}(y^*_k, (l^*_{ik})_{i\in\mathscr{P}}) =: (x^*_{ik}, y^*_{ik})_{i\in\mathscr{P}}$ converges to some element $(\bar{x}^*_i, \bar{y}^*_i)_{i\in\mathscr{P}} \in \prod_{i\in\mathscr{P}}(X\times Y)^*$. Since for every $i \in \mathscr{P}$ we have $((\nabla F(\bar{x})^* y^*_k, y^*_k) + l^*_{ik})_{|L_i} = (\nabla F(\bar{x})^* y^*_k, y^*_k)_{|L_i} = (x^*_{ik}, y^*_{ik})_{|L_i} \to (\bar{x}^*_i, \bar{y}^*_i)_{|L_i}$, there is some $\bar{y}^* \in Y^*$ such that $(\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*)_{|L_i} = (\bar{x}^*_i, \bar{y}^*_i)_{|L_i}$ and by defining $\bar{l}^*_i := (\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*) - (\bar{x}^*_i, \bar{y}^*_i) \in L_i^\perp$ we obtain $\tilde{A}^*_{\mathscr{P}}(\bar{y}^*, (\bar{l}^*_i)_{i\in\mathscr{P}}) = (\bar{x}^*_i, \bar{y}^*_i)_{i\in\mathscr{P}}$, showing that $\tilde{A}^*_{\mathscr{P}}$ has closed range. On the other hand, if the range of $\tilde{A}^*_{\mathscr{P}}$ is closed and the sequence $((\nabla F(\bar{x})^* y^*_k, y^*_k)_{|L_i})_{i\in\mathscr{P}} \subset \prod_{i\in\mathscr{P}} L_i^*$ converges to some element $(\bar{l}^*_i)_{i\in\mathscr{P}} \in \prod_{i\in\mathscr{P}} L_i^*$, for each $i \in \mathscr{P}$ consider an extension $\bar{z}^*_i \in (X\times Y)^*$ of $\bar{l}^*_i$ due to the Hahn-Banach Theorem. Let for each k define the numbers

$$\delta_{ik} := \sup\{((\nabla F(\bar{x})^* y^*_k, y^*_k) - \bar{z}^*_i)(l) \,|\, l \in \mathscr{B}_{L_i}\} = \|(\nabla F(\bar{x})^* y^*_k, y^*_k)_{|L_i} - \bar{l}^*_i\|, \; i \in \mathscr{P}.$$

By invoking the Hahn-Banach Theorem once more we can find functionals $z^*_{ik} \in (X\times Y)^*$ with $\|z^*_{ik}\| \le \delta_{ik}$ which coincide with $(\nabla F(\bar{x})^* y^*_k, y^*_k) - \bar{z}^*_i$ on L_i. Defining $l^*_{ik} := z^*_{ik} - (\nabla F(\bar{x})^* y^*_k, y^*_k) + \bar{z}^*_i$ we have $l^*_{ik} \in L_i^\perp$ and $\|(\nabla F(\bar{x})^* y^*_k, y^*_k) + l^*_{ik} - \bar{z}^*_i\| \le \delta_{ik}$. Since $\lim_{k\to\infty} \delta_{ik} = 0$, $i \in \mathscr{P}$ and the range of $\tilde{A}^*_{\mathscr{P}}$ is closed, there are elements $\bar{y}^* \in Y^*$, $\tilde{l}^*_i \in L_i^\perp$, $i \in \mathscr{P}$ with $(\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*) + \tilde{l}^*_i = \bar{z}^*_i$, $i \in \mathscr{P}$. Restricting both sides of the last equation to L_i we obtain $(\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*)_{|L_i} = \bar{z}^*_{i\,|L_i} = \bar{l}^*_i$, showing that the range of $A^*_{\mathscr{P}}$ is closed. □

It follows that the operators $A_{\mathscr{P}}$, $\tilde{A}^*_{\mathscr{P}}$ and $B_{\mathscr{P}}$ have closed range for each nonempty index set $\mathscr{P} \subset \{1,\ldots,p\}$, if S is polyhedral. In fact, if $L_i = X\times Y$, $i = 1,\ldots,p$, then $L_i^\perp = \{0\}$ and consequently $\mathrm{Range}(\tilde{A}^*_{\mathscr{P}}) = \prod_{i\in\mathscr{P}} \mathrm{gph}\, \nabla F(\bar{x})^*$ is closed by the Closed Graph Theorem, showing that $\mathrm{Range}(A_{\mathscr{P}})$ is closed by Lemma 9. Further it is easily seen, that the operator $B_{\mathscr{P}}$ has closed range $\prod_{i\in\mathscr{P}}(X\times Y)$.

The following lemma is crucial for our analysis. It allows us to work in general Banach spaces and hence to avoid the usage of tools from variational analysis which are only valid in Asplund spaces.

Lemma 10. *Let Z be a Banach space, let $\varphi : Z \to \mathbb{R}$ be a convex continuous function, let $P \subset Z$ be the union of finitely man convex generalized polyhedral sets given by* (10) *and assume that $\bar{z}$ is a local minimizer of the problem*

$$\min \varphi(z) \quad \textit{subject to} \quad z \in P.$$

Then

$$0 \in \partial\varphi(\bar{z}) + N(\bar{z}; P).$$

Proof. For each $i \in \mathscr{P}(\bar{z})$ the point $\bar{z}$ is also a local minimizer of the convex problem $\min_{z\in P_i} \varphi(z)$ and hence there exists some $g_i^* \in \partial\varphi(\bar{z})) \cap (-\hat{N}(\bar{z};P_i))$, implying $\langle g_i^*, z\rangle \geq 0, \forall z \in T(\bar{z};P_i) = \{z \in L_i \,|\, \langle z_{ij}^*, z\rangle \leq 0, j \in \mathscr{A}_i(\bar{z})\}$. Hence 0 is a solution of the problem

$$\min_{z\in Z} \bar{\varphi}(z) := \max_{i\in\mathscr{P}(\bar{z})} \langle g_i^*, z\rangle \quad \text{subject to} \quad z \in \bigcup_{i\in\mathscr{P}(\bar{z})} T(\bar{z};P_i).$$

Now define for each $i \in \mathscr{P}(\bar{z})$ the linear operator

$$H_i : L_i \to \prod_{l\in\mathscr{P}(\bar{z})} \mathbb{R} \times \prod_{j\in\mathscr{A}_i(\bar{z})} \mathbb{R},\; H_i(z) := ((\langle g_l^*, z\rangle)_{l\in\mathscr{P}(\bar{z})}, (\langle z_{ij}^*, z\rangle)_{j\in\mathscr{A}_i(\bar{z})}).$$

Range(H_i) is a finite dimensional space and therefore we can find some finite dimensional subspace $E_i \subseteq L_i \subseteq Z$ with $H_i(E_i) = \text{Range}(H_i)$. Next put $\bar{Z} := \sum_{i\in\mathscr{P}(\bar{z})} E_i$, $\bar{L}_i := L_i \cap \bar{Z}$ and $\bar{T}_i := \{z \in \bar{L}_i \,|\, \langle z_{ij}^*, z\rangle \leq 0,\; j \in \mathscr{A}_i(\bar{z})\}$ for each $i \in \mathscr{P}(\bar{z})$. Obviously we have $\bar{T}_i \subset T(\bar{z};P_i)$ and consequently 0 is a solution of the problem $\min_{z\in\bar{Z}} \bar{\varphi}(z),\; z \in \bigcup_{i\in\mathscr{P}(\bar{z})} \bar{T}_i$. Since $\bar{Z}$ is finite dimensional, it follows from [20, Proposition 5.3] that $0 \in \partial\bar{\phi}(0) + N(0; \cup_{i\in\mathscr{P}(\bar{z})} \bar{T}_i)$ and therefore by Lemma 3 there exist index sets $\emptyset \neq \bar{\mathscr{P}} \subseteq \mathscr{P}(\bar{z})$, $\bar{\mathscr{A}_i} \subseteq \mathscr{A}_i(\bar{z})$, $i \in \bar{\mathscr{P}}$, a direction $h \in \bar{Z}$ and nonnegative numbers $\mu_i \geq 0$, $i \in \mathscr{P}(\bar{z})$, $\lambda_{ij} \geq 0$, $j \in \bar{\mathscr{A}_i}$, $i \in \bar{\mathscr{P}}$ and functionals $l_i^* \in \bar{L}_i^\perp$ such that $\sum_{i\in\mathscr{P}} \mu_i = 1$

$$h \in \bar{T}_i, \langle z_{ij}^*, h\rangle \begin{cases} = 0 & j \in \bar{\mathscr{A}_i}, \\ < 0 & j \in \mathscr{A}_i \setminus \bar{\mathscr{A}_i}, \end{cases} \; i \in \bar{\mathscr{P}}; \quad h \notin \bar{T}_i, i \notin \bar{\mathscr{P}}$$

and for every $i \in \bar{\mathscr{P}}$

$$\sum_{l\in\mathscr{P}(\bar{z})} \mu_l g_l^*{}_{|\bar{Z}} + l_i^*{}_{|\bar{Z}} + \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} z_{ij}^*{}_{|\bar{Z}} = 0.$$

Now we show by contradiction that $\sum_{l\in\mathscr{P}(\bar{z})} \mu_l g_l^* + \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} z_{ij}^* \in L_i^\perp$ holds for every $i \in \bar{\mathscr{P}}$. Assuming that there were some $i \in \bar{\mathscr{P}}$ and some $z \in L_i$ with $\sum_{l\in\mathscr{P}(\bar{z})} \mu_l \langle g_l^*, z\rangle + \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} \langle z_{ij}^*, z\rangle \neq 0$, we could find some $e \in E_i \subset \bar{L}_i$ with $H_i e = H_i z$ and therefore

$$0 \neq \sum_{l\in\mathscr{P}(\bar{z})} \mu_l \langle g_l^*, z\rangle + \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} \langle z_{ij}^*, z\rangle = \sum_{l\in\mathscr{P}(\bar{z})} \mu_l \langle g_l^*, e\rangle + \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} \langle z_{ij}^*, e\rangle = -\langle l_i^*, e\rangle = 0,$$

a contradiction. Hence $-\sum_{l\in\mathscr{P}(\bar{z})} \mu_l g_l^* \in \sum_{j\in\bar{\mathscr{A}_i}} \lambda_{ij} z_{ij}^* + L_i^\perp \subseteq \hat{N}(h; T(\bar{z};P_i))$, $i \in \bar{\mathscr{P}}$. Next we will show that $h \notin T(\bar{z};P_i)$ for every $i \in \mathscr{P}(\bar{z}) \setminus \bar{\mathscr{P}}$. If on the contrary $h \in T(\bar{z};P_i)$ for some $i \in \mathscr{P}(\bar{z}) \setminus \bar{\mathscr{P}}$, then $h \in L_i \cap \bar{Z} = \bar{L}_i$ and $\langle z_{ij}^*, h\rangle \leq 0$, $j \in \mathscr{A}_i(\bar{z})$, yielding the contradiction $h \in \bar{T}_i$. Putting all together we see that $\bar{\mathscr{P}} = \{i \,|\, h \in T(\bar{z};P_i)\}$ and since $\sum_{l\in\mathscr{P}(\bar{z})} \mu_l g_l^* \in \partial\phi(\bar{z})$ as a convex combination of subgradients, we have $0 \in \partial\varphi(\bar{z}) + \bigcap_{i\in\bar{\mathscr{P}}} \hat{N}(h; T(\bar{z};P_i)) \subseteq \partial\varphi(\bar{z}) + N(\bar{z};P)$. □

Theorem 4. *Let X,Y be Banach spaces, let M be given by* (16)*, let $\bar{x} \in X$ with $0 \in M(\bar{x})$ and assume that F is strictly differentiable at $\bar{x}$. Let $(u, v_1) \in X \times Y_1$ and assume that M_2 is proper subregular in direction u relative to M_1 and v_1 at $(\bar{x}, 0)$. Further assume that for every $\mathscr{P} \in \mathscr{I}((\bar{x}, -F(\bar{x})); (u, (v_1, 0) - \nabla F(\bar{x})u)$ the operator $A_{\mathscr{P}}$ has closed range.*

1. *If for every* $y^* \in Y^*$ *satisfying* $0 \in \nabla F(\bar{x})^* y^* + D^* S((\bar{x}, -F(\bar{x})); (u, (v_1, 0) - \nabla F(\bar{x})u)(y^*)$ *we have* $y^*_{|_{\tilde{Y}_1}} = 0$*, then* M *is mixed regular/subregular in direction* (u, v_1) *at* $(\bar{x}, 0)$.

2. *If* $u \neq 0$*,* $v_1 = 0$*,* F *is second-order differentiable in direction* u *at* $\bar{x}$ *and if*

$$\langle y^*, F''(\bar{x}; u) \rangle < 0$$

holds for every $y^* \in Y^*$ *satisfying* $0 \in \nabla F(\bar{x})^* y^* + D^* S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(y^*)$ *and* $\|y^*_{|_{\tilde{Y}_1}}\| > 0$*, then* M *is metrically subregular in direction* u *at* $(\bar{x}, 0)$.

Proof. Let κ', L be given in accordance with Definition 4 and assume that Y is equipped with the norm

$$|||(y_1, y_2)||| := \sqrt{\|y_1\|^2 + (L\kappa' + 2)^2 \|y_2\|^2}.$$

We proof both parts by contradiction. Let $\bar{z} := (\bar{x}, 0)$, $v := (v_1, 0)$, $w := (u, v)$, $\bar{s} := -F(\bar{x}) \in S(\bar{x})$. Assuming that M is not mixed regular/subregular in direction (u, v_1) there exists a sequence $z_k := (x_k, y_k) \in \bar{z} + V_{\frac{1}{k}, \frac{1}{k}}(w)$, $y_{k2} = 0$ with $\|w\| \mathrm{d}(z_k, \mathrm{gph} M) \leq \frac{\|w\|}{k} \|z_k - \bar{z}\|$ such that $\mathrm{d}(x_k, M^{-1}(y_k)) > 4k \mathrm{d}(y_k, M(x_k))$. Then we can exactly proceed as in the proof of [7, Theorem 4] to find sequences $(\tilde{z}_k) \to \bar{z}$, $(t_k) \downarrow 0$, $(\tilde{u}_k) \to u$ and $(\tilde{v}_k) \to v$ such that $\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) = \bar{z} + t_k(\tilde{u}_k, \tilde{v}_k) \in \mathrm{gph} M$, $\tilde{y}_k \neq y_k$ and

$$|||\tilde{y}_k - y_k||| \leq |||y - y_k||| + \frac{1}{\sqrt{k}} \|(x, y) - (\tilde{x}_k, \tilde{y}_k)\|, \ \forall (x, y) \in \mathrm{gph} M. \tag{18}$$

Using the same arguments as in the proof of [7, Theorem 6] we obtain $\|\tilde{y}_{k2} - y_{k2}\| \leq \|\tilde{y}_{k1} - y_{k1}\|$, where we have eventually passed to a subsequence. Now let

$$\eta_k := \sup\{ \frac{|||F(x) - F(\tilde{x}_k) - \nabla F(\bar{x})(x - \tilde{x}_k)|||}{\|x - \tilde{x}_k\|} \,|\, \tilde{x}_k \neq x \in \tilde{x}_k + \frac{1}{k} \mathscr{B}_X \}$$

and put $\tilde{s}_k := \tilde{y}_k - F(\tilde{x}_k) \in S(\tilde{x}_k)$. Setting $y = F(x) + s$ we obtain from (18)

$$\begin{aligned}
&|||F(\tilde{x}_k) + \tilde{s}_k - y_k||| \\
&\leq |||F(\tilde{x}_k) + \nabla F(\bar{x})(x - \tilde{x}_k) + s - y_k||| + \frac{1}{\sqrt{k}} \|(x - \tilde{x}_k, F(x) + s - (F(\tilde{x}_k) + \tilde{s}_k))\| + \eta_k \|x - \tilde{x}_k\| \\
&\leq |||F(\tilde{x}_k) + \nabla F(\bar{x})(x - \tilde{x}_k) + s - y_k||| + \varepsilon_k \|(x - \tilde{x}_k, s - \tilde{s}_k)\|, \ \forall (x, s) \in \mathrm{gph} S : \|x - \tilde{x}_k\| \leq \frac{1}{k},
\end{aligned}$$

where $\varepsilon_k := \frac{1}{\sqrt{k}}(1 + |||\nabla F(\bar{x})||| + \eta_k) + \eta_k$. Obviously we have $\varepsilon_k \to 0$. Putting $\varphi_k(x, s) := |||F(\tilde{x}_k) + \nabla F(\bar{x})(x - \tilde{x}_k) + s - y_k||| + \varepsilon_k \|(x - \tilde{x}_k, s - \tilde{s}_k)\|$ and assuming that S has the representation (12), it follows that $(\tilde{x}_k, \tilde{s}_k)$ is a local minimizer of the problem $\min \varphi_k(x, s)$, $(x, s) \in \bigcup_{i=1}^p P_i$ and by Lemma 10 we have $0 \in \partial \varphi_k(\tilde{x}_k, \tilde{s}_k) + N((\tilde{x}_k, \tilde{s}_k); \cup_{i=1}^p P_i)$. By convex analysis there exist a functional $y_k^* \in \partial |||\tilde{y}_k - y_k|||$, functionals $(\hat{x}_k^*, \hat{y}_k^*) \in \mathscr{B}_{X^* \times Y^*}$ and, as a consequence of Lemma 3, $(x_k', s_k') \in (\tilde{x}_k, \tilde{s}_k) + \frac{t_k^2}{2k} \mathscr{B}_{X \times Y}$ with

$$0 \in (\nabla F(\bar{x})^* y_k^*, y_k^*) + \varepsilon_k (\hat{x}_k^*, \hat{y}_k^*) + \hat{N}((x_k', s_k'); \cup_{i=1}^p P_i)$$

Since $\tilde{y}_k \neq y_k$ and $\|\tilde{y}_{k2} - y_{k2}\| \leq \|\tilde{y}_{k1} - y_{k1}\|$ we obtain $\tilde{y}_{k1} \neq y_{k1}$ and therefore there exist functionals $(y^*_{k1}, y^*_{k2}) \in \mathscr{S}_{Y_1^*} \times \mathscr{B}_{Y_2^*}$ with $\langle y^*_{k1}, \tilde{y}_{k1} - y_{k1}\rangle = \|\tilde{y}_{k1} - y_{k1}\|$ such that

$$\langle y^*_k, (y_1, y_2)\rangle = \frac{\|\tilde{y}_{k1} - y_{k1}\|\langle y^*_{k1}, y_1\rangle + (L\kappa' + 2)^2\|\tilde{y}_{k2} - y_{k2}\|\langle y^*_{k2}, y_2\rangle}{|||\tilde{y}_k - y_k|||},\ \forall (y_1, y_2) \in Y$$

Thus

$$\langle y^*_k, (\tilde{y}_{k1} - y_{k1}, 0)\rangle = \frac{\|\tilde{y}_{k1} - y_{k1}\|^2}{|||\tilde{y}_k - y_k|||} \geq \frac{\|\tilde{y}_{k1} - y_{k1}\|}{\sqrt{1 + (L\kappa' + 2)^2}}$$

showing $\|y^*_{k|_{\tilde{Y}_1^*}}\| \geq 1/\sqrt{1 + (L\kappa' + 2)^2} =: 2\beta$. Since $t_k^{-1}(x'_k - \bar{x}) = t_k^{-1}(x'_k - \tilde{x}_k) + \tilde{u}_k \to u$, $t_k^{-1}(s'_k - \bar{s}) = t_k^{-1}(s'_k - \tilde{s}_k) - t_k^{-1}(F(\tilde{x}_k) - F(\bar{x})) + \tilde{v}_k \to (v_1, 0) - \nabla F(\bar{x})u$, by passing to a subsequence we can assume that there are index sets

$$\mathscr{P} \in \mathscr{I}((\bar{x}, -F(\bar{x})); (u, (v_1, 0) - \nabla F(\bar{x})u), (\mathscr{A}_i)_{i \in \mathscr{P}} \in \mathscr{J}_{\mathscr{P}}((\bar{x}, -F(\bar{x})); (u, (v_1, 0) - \nabla F(\bar{x})u),$$

such that for every k we have $\mathscr{P}(x'_k, s'_k) = \mathscr{P}$, $\mathscr{A}_i(x'_k, s'_k) = \mathscr{A}_i$, $i \in \mathscr{P}$. Then for each k there exists an element $p^*_k \in \bigcap_{i \in \mathscr{P}} \hat{N}((x'_k, s'_k); P_i)$ such that $\|(\nabla F(\bar{x})^* y^*_k, y^*_k) + p^*_k\| \leq \varepsilon_k$, and therefore there are multipliers $l^*_{ki} \in L_i^\perp$, $\lambda_{kij} \geq 0$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ with

$$\|(\nabla F(\bar{x})^* y^*_k, y^*_k) + l^*_{ki} + \sum_{j \in \mathscr{A}_i} \lambda_{kij}(x^*_{ij}, y^*_{ij})\| \leq \varepsilon_k,\ i \in \mathscr{P}$$

Next consider the operator, which assign to every $y^* \in Y^*$, $(l^*_i)_{i \in \mathscr{P}}$ and $(\lambda_{ij})_{j \in \mathscr{A}_i, i \in \mathscr{P}}$ the value

$$((\nabla F(\bar{x})^* y^*, y^*) + l^*_i + \sum_{j \in \mathscr{A}_i} \lambda_{ij}(x^*_{ij}, y^*_{ij}))_{i \in \mathscr{P}}.$$

The range of this operator is the sum of the range of $\tilde{A}^*_{\mathscr{P}}$ and a finite dimensional subspace and therefore closed by Lemma 9 and our assumption. Hence we can invoke Hoffman's lemma to find for every k the elements $\tilde{y}^*_k \in Y^*$, $\tilde{l}^*_{ki} \in L_i^\perp$, $\tilde{\lambda}_{kij} \geq 0$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ such that $(\nabla F(\bar{x})^* \tilde{y}^*_k, \tilde{y}^*_k) + \tilde{l}^*_{ki} + \sum_{j \in \mathscr{A}_i} \tilde{\lambda}_{kij}(x^*_{ij}, y^*_{ij}) = 0$, $i \in \mathscr{P}$ and $\|y^*_k - \tilde{y}^*_k\| + \sum_{i \in \mathscr{P}}(\|l^*_{ki} - \tilde{l}^*_{ki}\| + \sum_{j \in \mathscr{A}_i} |\lambda_{kij} - \tilde{\lambda}_{kij}|) \leq \gamma\varepsilon_k$ for some constant $\gamma > 0$ independent of k. Hence for all $\bar{k}$ sufficiently large we have $\|\tilde{y}^*_{\bar{k}|_{\tilde{Y}_1}}\| > \beta$ and $-(\nabla F(\bar{x})^* \tilde{y}^*_{\bar{k}}, \tilde{y}^*_{\bar{k}}) \in \bigcap_{i \in \mathscr{P}} \hat{N}((x'_k, s'_k); P_i) = \hat{N}((x'_k, s'_k); \bigcup_{i \in \mathscr{P}} P_i)$, $\forall k$, implying

$$-\nabla F(\bar{x})^* \tilde{y}^*_{\bar{k}} \in D^* S((\bar{x}, -F(\bar{x})); (u, (v_1, 0) - \nabla F(\bar{x})u))(\tilde{y}^*_{\bar{k}}).$$

However this contradicts the assumption of the first assertion and thus the first part of the theorem is proved.

To prove the second part let us assume that M is not subregular in direction u. Then we can find a sequence $x_k \in \bar{x} + V_{\frac{1}{k}, \frac{1}{k}}(u)$ with $\mathrm{d}(x_k, M^{-1}(0)) > 4k\mathrm{d}(0, M(x_k))$. Putting $y_k = 0$, $z_k := (x_k, y_k)$ we have again a sequence fulfilling $z_k \in \bar{z} + V_{\frac{1}{k}, \frac{1}{k}}(w)$, $y_{k2} = 0$, $\|w\|\mathrm{d}(z_k, \mathrm{gph}\,M) \leq \frac{\|w\|}{k}\|z_k - \bar{z}\|$ and $\mathrm{d}(x_k, M^{-1}(y_k)) > 4k\mathrm{d}(y_k, M(x_k))$, since

$$\mathrm{d}(z_k, \mathrm{gph}\,M) = \mathrm{d}((x_k, 0), \mathrm{gph}\,M) \leq \mathrm{d}(0, M(x_k)) < \frac{1}{4k}\mathrm{d}(x_k, M^{-1}(0)) \leq \frac{1}{4k}\|x_k - \bar{x}\| = \frac{1}{4k}\|z_k - \bar{z}\|$$

and we can proceed exactly as in the proof of the first part of the theorem to find the sequences (x'_k), (s'_k), $(\tilde{x}_k)$, $(\tilde{s}_k)$, (y^*_k), $(\tilde{y}^*_k)$, $(\tilde{u}_k)$, (ε_k), (t_k) and the constants β and γ. Then $\langle \tilde{y}^*_k, F''(\bar{x};u)\rangle < 0$ $\forall k$ and $\limsup_k \langle \tilde{y}^*_k, F''(\bar{x};u)\rangle := -2\sigma < 0$ follows, since otherwise we could construct $\tilde{y}^*_k$ such that $\langle \tilde{y}^*_k, F''(\bar{x};u)\rangle = 0$ because adding the single equation $\langle y^*, F''(\bar{x};u)\rangle = 0$ to the system defining $\tilde{y}^*_k$ does not affect the applicability of Hoffman's lemma.

Since $\mathscr{P} \subset \mathscr{P}(\bar{x}, -F(\bar{x}))$, $\mathscr{A}_i \subset \mathscr{A}_i(\bar{x}, -F(\bar{x}))$, $i \in \mathscr{P}$ and for each $i \in \mathscr{P}$ the functional $(\nabla F(\bar{x})^* \tilde{y}^*_k, \tilde{y}^*_k))$ can be written in the form $(\nabla F(\bar{x})^* \tilde{y}^*_k, \tilde{y}^*_k) = -\tilde{l}_{ki} - \sum_{j\in\mathscr{A}_i} \tilde{\lambda}_{kij}(x^*_{ij}, y^*_{ij})$ with $\tilde{l}_{ki} \in L_i^{\perp}$ and $\tilde{\lambda}_{kij} \geq 0$, $j \in \mathscr{A}_i$, we obtain

$$\langle (\nabla F(\bar{x})^* \tilde{y}^*_k, \tilde{y}^*_k), (x'_k - \bar{x}, s'_k - \bar{s})\rangle = \langle \tilde{y}^*_k, \nabla F(\bar{x})(x'_k - \bar{x}) + s'_k + F(\bar{x})\rangle = 0. \tag{19}$$

Then, for all k sufficiently large we obtain the contradiction

$$\begin{aligned}
0 &\leq (1-\gamma\varepsilon_k)|||F(\tilde{x}_k)+\tilde{s}_k||| = \langle y^*_k, F(\tilde{x}_k)+\tilde{s}_k\rangle - \gamma\varepsilon_k|||F(\tilde{x}_k)+\tilde{s}_k||| \leq \langle \tilde{y}^*_k, F(\tilde{x}_k)+\tilde{s}_k\rangle \\
&= \langle \tilde{y}^*_k, F(\tilde{x}_k) - F(\bar{x}) - \nabla F(\bar{x})(\tilde{x}_k - \bar{x}) - \nabla F(\bar{x})(x'_k - \tilde{x}_k) - (s'_k - \tilde{s}_k)\rangle \\
&= \frac{t_k^2}{2}\Bigg(\langle \tilde{y}^*_k, F''(\bar{x};u)\rangle + \langle \tilde{y}^*_k, \frac{F(\bar{x}+t_k\tilde{u}_k) - F(\bar{x}) - t_k\nabla F(\bar{x})\tilde{u}_k}{t_k^2/2} - F''(\bar{x};u)\rangle \\
&\qquad - \langle \tilde{y}^*_k, \frac{(s'_k - \tilde{s}_k) + \nabla F(\bar{x})(x'_k - \tilde{x}_k)}{t_k^2/2}\rangle\Bigg) \\
&\leq \frac{t_k^2}{2}\Bigg(-\sigma + (1+\gamma\varepsilon_k)(|||\frac{F(\bar{x}+t_k\tilde{u}_k) - F(\bar{x}) - t_k\nabla F(\bar{x})\tilde{u}_k}{t_k^2/2} - F''(\bar{x};u)||| \\
&\qquad + \frac{|||\nabla F(\bar{x})|||+1}{k})\Bigg) \\
&< 0
\end{aligned}$$

and hence M is metrically subregular in direction u. $\square$

The second-order sufficient conditions for directional metric subregularity as stated in Theorem 4 are also close to be necessary.

Proposition 1. *Let $M : X \rightrightarrows Y$ be a multifunction between Banach spaces of the form $M(x) = F(x) + S(x)$, where the mapping $F : X \to Y$ is strictly differentiable at the point $\bar{x}$ with $0 \in M(\bar{x})$ and $S : X \rightrightarrows Y$ is a generalized polyhedral multifunction. Assume that M is metrically subregular in direction $u \neq 0$ at $(\bar{x}, 0)$ and that F is second-order directionally differentiable in direction u. Then*

$$\langle y^*, F''(\bar{x};u)\rangle \leq 0, \forall y^* \in Y^* : -(\nabla F(\bar{x})^* y^*, y^*) \in \hat{N}((u, -\nabla F(\bar{x})u); T((\bar{x}, -F(\bar{x})); \operatorname{gph} S))$$

Proof. If $(u, -\nabla F(\bar{x})u) \notin T((\bar{x}, -F(\bar{x})); \operatorname{gph} S)$ then $\hat{N}((u, -\nabla F(\bar{x})u); T((\bar{x}, -F(\bar{x})); \operatorname{gph} S)) = \emptyset$ and there is nothing to prove. Hence let $(u, -\nabla F(\bar{x})u) \in T((\bar{x}, -F(\bar{x})); \operatorname{gph} S)$. Since $\operatorname{gph} S$ is the union of finitely many convex generalized polyhedral sets, we have $(\bar{x}+tu, -F(\bar{x}) - t\nabla F(\bar{x})u) \in \operatorname{gph} S$ for all $t \geq 0$ sufficiently small and, since F is second-order differentiable in direction u, we have $\mathrm{d}(0, M(\bar{x}+tu)) = \mathrm{d}(-F(\bar{x}+tu), S(\bar{x}+tu)) \leq \gamma t^2$ for some $\gamma \geq 0$. Since M is metrically subregular in direction u we conclude that there is some radius $r > 0$ such that for each $t \geq 0$ sufficiently small there is some $w_t \in r\mathscr{B}_X$ with $-F(\bar{x}+tu+t^2w_t) \in S(\bar{x}+tu+t^2w_t)$. We can also argue that $(t^2w_t, -F(\bar{x}+tu+t^2w_t)+F(\bar{x})+t\nabla F(\bar{x})u) \in T((u,-\nabla F(\bar{x})u); T((\bar{x},-F(\bar{x})); \operatorname{gph} S))$, since for every convex generalized polyhedral set $P_i \subset \operatorname{gph} S$ with $(u,-\nabla F(\bar{x})u) \notin T((\bar{x},-F(\bar{x})); P_i)$ we have $\mathrm{d}((\bar{x}+tu, -F(\bar{x})-t\nabla F(\bar{x})u)), P_i) \geq c_i t$ for some constant $c_i > 0$ and therefore $(\bar{x} + tu + t^2w_t, -F(\bar{x}+tu+t^2w_t)) \notin P_i$. Hence, for every $y^* \in Y^*$ satisfying $-(\nabla F(\bar{x})^*y^*, y^*) \in \hat{N}((u,-\nabla F(\bar{x})u); T((\bar{x},-F(\bar{x})); \operatorname{gph} S))$ we obtain

$$
\begin{aligned}
0 &\geq \langle -\nabla F(\bar{x})^*y^*, t^2w_t\rangle + \langle -y^*, -F(\bar{x}+tu+t^2w_t)+F(\bar{x})+t\nabla F(\bar{x})u\rangle \\
&= \frac{t^2}{2}(\langle y^*, F''(\bar{x};u)\rangle + \langle y^*, \frac{F(\bar{x}+t(u+tw_t))-F(\bar{x})-t\nabla F(\bar{x})(u+tw_t)}{t^2/2} - F''(\bar{x}:u)\rangle).
\end{aligned}
$$

Dividing by $t^2/2$ and passing to the limit $t \downarrow 0$ yields the claimed estimate $\langle y^*, F''(\bar{x};u)\rangle \leq 0$. $\square$

Note that we always have

$$\hat{N}((u,-\nabla F(\bar{x})u); T((\bar{x},-F(\bar{x})); \operatorname{gph} S)) \subset N((\bar{x},-F(\bar{x})); \operatorname{gph} S; (u,-\nabla F(\bar{x})u))$$

and equality holds if $\mathscr{I}((\bar{x},-F(\bar{x})); (u,-\nabla F(\bar{x})u))$ contains only one index set $\{\bar{i}\}$ being a singleton, i.e. the points $(\bar{x}+tu, -F(\bar{x})-t\nabla F(\bar{x})u)$ belong to exactly one convex generalized polyhedral set $P_{\bar{i}}$ for all $t > 0$ sufficiently small. Hence, $0 \in \nabla F(\bar{x})^*y^* + D^*S((\bar{x},-F(\bar{x})); (u,-\nabla F(\bar{x})u))(y^*)$ holds for all multipliers y^* with $-(\nabla F(\bar{x})^*y^*, y^*) \in \hat{N}((u,-\nabla F(\bar{x})u); T((\bar{x},-F(\bar{x})); \operatorname{gph} S))$.

For the sake of completeness we formulate the following result concerning (directional) metric regularity:

Theorem 5. *Let $M : X \rightrightarrows Y$ be a multifunction between Banach spaces the form $M(x) = F(x) + S(x)$, where the mapping $F : X \to Y$ is strictly differentiable at a point $\bar{x}$ with $0 \in M(\bar{x})$ and $S : X \rightrightarrows Y$ is a generalized polyhedral multifunction whose graph has the representation* (12) *and let $(u,v) \in X \times Y$. Then the following statements are equivalent:*

(1) M is metrically regular in direction (u,v) at $(\bar{x},0)$.

(2) For every $\mathscr{P} \in \mathscr{I}((\bar{x},-F(\bar{x})); (u,v-\nabla F(\bar{x})u))$ the operator $A_{\mathscr{P}}$ has closed range and $\ker D^*M((\bar{x},0);(u,v)) = \ker(\nabla F(\bar{x})^* + D^*S(\bar{x},-F(\bar{x});(u,v-\nabla F(\bar{x})u))) = \{0\}$.

Proof. The implication (2)⇒(1) follows immediately from the first part of Theorem 4 applied to the multifunction $\tilde{M} = (\tilde{M}_1, \tilde{M}_2) : X \rightrightarrows Y_1 \times Y_2$ with $Y_1 := Y$, $\tilde{M}_1 := M$, $Y_2 := \{0\}$, $\tilde{M}_2(x) := \{0\}$. We prove the other implication by contradiction. If there is some $0 \neq y^* \in Y^*$ with $0 \in D^*M((\bar{x},0);(u,v))(y^*) = \nabla F(\bar{x})^*y^* + D^*S((\bar{x},-F(\bar{x}));(u,v-\nabla F(\bar{x})u))(y^*)$, then there are sequences $(t_k) \downarrow 0$, $(u_k,w_k) \to (u,v-\nabla F(\bar{x})u)$ and some index set $\mathscr{P} \in \mathscr{I}((\bar{x},-F(\bar{x});(u,v))$ such that $-(\nabla F(\bar{x})^*y_k^*, y_k^*) \in \bigcap_{i\in\mathscr{P}} \hat{N}_{\varepsilon_k}((\bar{x}+t_ku_k, -F(\bar{x})+t_kw_k); P_i) = \hat{N}_{\varepsilon_k}((\bar{x}+t_ku_k, -F(\bar{x})+t_kw_k); \operatorname{gph} S)$, where $y_k^* := y^*$, $\varepsilon_k := 0$. Without loss of generality we can assume $\|y^*\| = 1$.

On the other hand, if for some $\mathscr{P} \in \mathscr{I}((\bar{x},-F(\bar{x}));(u,v-\nabla F(\bar{x})u))$ the range of the operator $A_{\mathscr{P}}$ is not closed, then by Lemma 9 Range$(\tilde{A}^*_{\mathscr{P}})$ is also not closed. Hence there is a sequence $(\tilde{y}_k^*, (\tilde{l}_{ik}^*)_{i\in\mathscr{P}}) \in Y^* \times \prod_{i\in\mathscr{P}} L_i^\perp$ with $\tilde{A}^*_{\mathscr{P}}(\tilde{y}_k^*, (\tilde{l}_{ik}^*)_{i\in\mathscr{P}}) = ((\nabla F(\bar{x})^*\tilde{y}_k^*, \tilde{y}_k^*) + \tilde{l}_{ik}^*)_{i\in\mathscr{P}} \to$

$z^* \notin \mathrm{Range}(\tilde{A}^*_{\mathscr{P}})$. Then the sequence $(\tilde{y}^*_k, (\tilde{l}^*_{ik})_{i\in\mathscr{P}})$ must be unbounded, since if the sequence were bounded, by the Alaoglu-Bourbaki Theorem, together with the weak-* closedness of $L_i^\perp$, $i \in \mathscr{P}$, the sequence would have a weak-* accumulation point $(\bar{y}^*, (\bar{l}^*_i)_{i\in\mathscr{P}}) \in Y^* \times \prod_{i\in\mathscr{P}} L_i^\perp$ with $\tilde{A}^*_{\mathscr{P}}(\bar{y}^*, (\bar{l}^*_i)_{i\in\mathscr{P}}) = z^*$. But then it follows that the sequence $\tilde{y}^*_k$ must be unbounded and therefore $\varepsilon_k := \max_{i\in\mathscr{P}} \mathrm{d}(-(\nabla F(\bar{x})^* y^*_k, y^*_k), L_i^\perp) \to 0$, where $y^*_k := \tilde{y}^*_k/\|\tilde{y}^*_k\|$ and from the definition of the set $\mathscr{I}((\bar{x}, -F(\bar{x})); (u, v - \nabla F(\bar{x})u))$ it follows again, that there exist sequences $(t_k) \downarrow 0$, $(u_k, w_k) \to (u, v - \nabla F(\bar{x})u)$ such that $-(\nabla F(\bar{x})^* y^*_k, y^*_k) \in \bigcap_{i\in\mathscr{P}} \hat{N}_{\varepsilon_k}((\bar{x}+t_k u_k, -F(\bar{x})+t_k w_k); P_i) = \hat{N}_{\varepsilon_k}((\bar{x}+t_k u_k, -F(\bar{x})+t_k w_k); \mathrm{gph}\, S)$.

In both cases, for each k we can find a positive radius $\rho_k < \frac{1}{k}$ such that

$$-\langle \nabla F(\bar{x})^* y^*_k, x - x_k\rangle - \langle y^*_k, s - s_k)\rangle \le \varepsilon'_k \|(x - x_k, s - s_k)\|$$

holds for all $(x,s) \in \mathrm{gph}\, S \cap ((x_k, s_k) + \rho_k \mathscr{B}_{X\times Y})$, where $x_k := \bar{x} + t_k u_k$, $s_k := -F(\bar{x}) + t_k w_k \in S(x_k)$ and $\varepsilon_{k'} := \varepsilon_k + \frac{1}{k}$. Defining

$$\eta_k := \sup_{x \in x_k + \rho_k \mathscr{B}_X} \frac{\|F(x) - F(x_k) - \nabla F(\bar{x})(x - x_k)\|}{\|x - x_k\|}$$

we obtain

$$-\langle y^*_k, F(x) + s - (F(x_k) + s_k)\rangle \le \varepsilon''_k \|(x - x_k, s - s_k)\|$$

for all $(x,s) \in \mathrm{gph}\, S \cap ((x_k, s_k) + \rho_k \mathscr{B}_{X\times Y})$, where $\varepsilon''_k = \varepsilon'_k + \eta_k$, and we conclude $(0, -y^*_k) \in \hat{N}_{\varepsilon''_k}((x_k, F(x_k) + s_k), \mathrm{gph}\, M)$. Since

$$\lim_{k\to\infty} \frac{F(x_k) + s_k}{t_k} = \lim_{k\to\infty} \frac{F(\bar{x} + t_k u_k) - F(\bar{x})}{t_k} + w_k = \nabla F(\bar{x})u - \nabla F(\bar{x})u + v = v$$

it follows from [7] that $(v, 0) \in \mathrm{Cr}M((\bar{x}, 0); u)$ and therefore M is not metrically regular in direction (u, v) at $(\bar{x}, 0)$ by [7, Theorem 4]. □

Note that the case of metric regularity of M near $(\bar{x}, 0)$ is covered by the case $(u, v) = (0, 0)$

Let us compare Theorem 5 with the results of [7, Theorem 5]: Firstly, in Theorem 5 the spaces X and Y can be arbitrary Banach spaces, whereas in [7] either the space Y must be Fréchet smooth or both X and Y must be Asplund spaces. Secondly, in [7] the characterization for directional metric regularity was stated in terms of the limit set $\mathrm{Cr}M((\bar{x}, 0); u)$ involving sequences $(y^*_k) \in \mathscr{S}_{Y^*}$, whereas in Theorem 5 we can pass to the limit and formulate the condition in terms of single elements y^*. To perform the limiting process, the so-called (directional) PSNC property of M, which is usually required for the characterization of (directional) metric regularity in terms of coderivatives, is now replaced by the requirement that the operators $A_{\mathscr{P}}$, $\mathscr{P} \in \mathscr{I}((\bar{x}, -F(\bar{x})); (u, v - \nabla F(\bar{x})u)$ have closed range. Recall that the latter is for instance automatically fulfilled if the multifunction S is polyhedral.

We now state a second-order sufficient condition for the property of metric subregularity. In what follows we define for given $\bar{x} \in M^{-1}(0)$ and $u \in X$ the sets of multipliers

$$\Lambda(\bar{x}; u) := \{y^* \in Y^* \,|\, 0 \in \nabla F(\bar{x})^* y^* + D^* S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(y^*)\}.$$

and

$$\tilde{\Lambda}(\bar{x};u) := \{y^* \in \Lambda(x;u) \mid \|y^*_{|\bar{Y}_1^*}\| > 0\}.$$

Further we define $\mathscr{D}(\bar{x}) := \{0 \neq u \in X \mid \tilde{\Lambda}(\bar{x};u) \neq \emptyset\}$.

Theorem 6. *Let X,Y be Banach spaces, let M be given by* (16)*, let $\bar{x} \in M^{-1}(0)$ and assume that F is strictly differentiable at $\bar{x}$. Further assume that M_2 is proper subregular relative to M_1 at $(\bar{x},0)$ and assume that for every $\mathscr{P} \in \mathscr{I}(\bar{x},-F(\bar{x}))$ the operator $A_{\mathscr{P}}$ has closed range and that for every $\bar{\mathscr{P}} \in \bar{\mathscr{I}}(\bar{x},-F(\bar{x}))$ the operator $B_{\bar{\mathscr{P}}}$ has closed range. If either*

1. *$\mathscr{D}(\bar{x}) = \emptyset$, or*
2. *for every $u \in \mathscr{D}(\bar{x})$ the mapping F is second-order directional differentiable in direction u, the limit* (17) *is uniform for $u \in \mathscr{D}(\bar{x}) \cap \mathscr{S}_X$ and there exists a real $\sigma > 0$ such that*

$$\langle y^*, F''(\bar{x};u)\rangle \leq -\sigma \|y^*_{|\bar{Y}_1}\| \|u\|^2$$

holds for all $u \in \mathscr{D}(\bar{x})$ and all $y^* \in \tilde{\Lambda}(\bar{x};u)$,

then M is metrically subregular at $(\bar{x},0)$.

Proof. The theorem is proved by contradiction. If M is not metrically subregular at $(\bar{x},0)$ then there exists a sequence (x_k) with $x_k \in \bar{x} + \frac{1}{k}\mathscr{B}_X$ and $\mathrm{d}(x_k, M^{-1}(0)) > 4k\mathrm{d}(0, M(x_k))$. Putting $y_k = 0$, $u = 0$ and $v = 0$ we can exactly proceed as in the proof of Theorem 4 to find the sequences (x'_k), (s'_k), $(\tilde{x}_k)$, $(\tilde{s}_k)$, (y^*_k), $(\tilde{y}^*_k)$, $(\tilde{u}_k)$, (ε_k) together with the constants $\beta > 0$ and γ and the index sets $\mathscr{P} \in \mathscr{I}(\bar{x},-F(\bar{x}))$, $(\mathscr{A}_i)_{i\in\mathscr{P}} \in \mathscr{J}_{\mathscr{P}}(\bar{x},-F(\bar{x}))$ such that for every k we have $\mathscr{P}(x'_k,s'_k) = \mathscr{P}, \mathscr{A}_i(x'_k,s'_k) = \mathscr{A}_i$, $i \in \mathscr{P}$ and for all $\bar{k}$ sufficiently large $\|\tilde{y}^*_{\bar{k}|\bar{Y}_1}\| \geq \beta > 0$ and $-(\nabla F(\bar{x})^*\tilde{y}^*_{\bar{k}}, y^*_{\bar{k}}) \in \hat{N}((x'_k,s'_k), \bigcup_{i\in\mathscr{P}} P_i)$, $\forall k$. We obtain from (18) with $(x,y) = (\bar{x},0)$

$$|||\tilde{y}_k||| \leq |||0||| + \frac{1}{\sqrt{k}}\|(\bar{x},0) - (\tilde{x}_k,\tilde{y}_k)\| \leq \frac{1}{\sqrt{k}}\|\bar{x} - \tilde{x}_k\| + \frac{1}{\sqrt{k}}|||\tilde{y}_k|||,$$

showing, together with $\tilde{y}_k \neq y_k = 0$ and $\tilde{x}_k \to \bar{x}$, $\|\tilde{x}_k - \bar{x}\| \downarrow 0$ and $\tilde{y}_k/\|\tilde{x}_k - \bar{x}\| \to 0$. We may also assume that according to Lemma 3 we have chosen the points (x'_k,s'_k) so close to $(\tilde{x}_k,\tilde{s}_k)$ such that $\|(x'_k,s'_k) - (\tilde{x}_k,\tilde{s}_k)\|/\|\tilde{x}_k - \bar{x}\|^2 \leq \frac{1}{k}$. Let us denote $\tau_k := \|x'_k - \bar{x}\|$, $u_k := (x'_k - \bar{x})/\tau_k$. Note that $\|\tilde{x}_k - \bar{x}\|/\tau_k \to 1$ and $(F(x'_k) + s'_k)/\tau_k = \tilde{y}_k/\tau_k + (F(x'_k) - F(\tilde{x}_k))/\tau_k + (s'_k - \tilde{s}_k)/\tau_k \to 0$.

By passing to subsequences if necessary we can find an index set $\bar{\mathscr{P}} \in \bar{\mathscr{I}}(\bar{x},-F(\bar{x}))$ such that $\mathscr{P} \subset \bar{\mathscr{P}}$ and

$$\lim_{k\to\infty} \tau_k^{-1}\mathrm{d}((x'_k,s'_k),P_i) = 0,\ i \in \bar{\mathscr{P}},\ \liminf_{k\to\infty} \tau_k^{-1}\mathrm{d}((x'_k,s'_k),P_i) > 0,\ i \notin \bar{\mathscr{P}}$$

and subsequently index sets $\bar{\mathscr{A}_i}$, $i \in \bar{\mathscr{P}}$ and numbers η_{ij}, $j \in \bar{\mathscr{A}_i}$ such that

$$\lim_{k\to\infty} \frac{\langle x^*_{ij}, x'_k\rangle + \langle y^*_{ij}, s'_k\rangle - \zeta_{ij}}{\tau_k} = 0,\ j \in \bar{\mathscr{A}_i},\ \limsup_{k\to\infty} \frac{\langle x^*_{ij}, x'_k\rangle + \langle y^*_{ij}, s'_k\rangle - \zeta_{ij}}{\tau_k} \leq \eta_{ij} < 0,\ j \notin \bar{\mathscr{A}_i}.$$

Let

$$\delta_k := \sum_{i\in\bar{\mathscr{P}}}\left(\mathrm{d}((u_k,-\nabla F(\bar{x})u_k),L_i)+\sum_{j\in\bar{\mathscr{A}_i}}|\langle x^*_{ij},u_k\rangle-\langle y^*_{ij},\nabla F(\bar{x})u_k\rangle|\right).$$

Since $\mathscr{P}(\bar{x},-F(\bar{x}))\supset\bar{\mathscr{P}}$, $\mathscr{A}_i(\bar{x},-F(\bar{x}))\supset\bar{\mathscr{A}_i}$, $i\in\bar{\mathscr{P}}$ and $\nabla F(\bar{x})u_k+(s'_k-\bar{s})/\tau_k=\nabla F(\bar{x})u_k-(F(x'_k)-F(\bar{x}))/\tau_k+(s'_k+F(x'_k))/\tau_k\to 0$ we have

$$\lim_{k\to\infty}\mathrm{d}((u_k,-\nabla F(\bar{x})u_k),L_i) \;=\; \lim_{k\to\infty}\tau_k^{-1}\mathrm{d}((x'_k-\bar{x},s'_k-\bar{s}),L_i)=\lim_{k\to\infty}\tau_k^{-1}\mathrm{d}((x'_k,s'_k),L'_i)=0,\ i\in\bar{\mathscr{P}}$$

and

$$\begin{aligned}0 &= \lim_{k\to\infty}\frac{\langle x^*_{ij},x'_k\rangle+\langle y^*_{ij},s'_k\rangle-\zeta_{ij}}{\tau_k}=\lim_{k\to\infty}\frac{\langle x^*_{ij},x_k-\bar{x}\rangle+\langle y^*_{ij},s_k-\bar{s}\rangle}{\tau_k}\\ &= \lim_{k\to\infty}\langle x^*_{ij},u_k\rangle-\langle y^*_{ij},\nabla F(\bar{x})u_k\rangle,\ j\in\bar{\mathscr{A}_i},\ i\in\bar{\mathscr{P}},\end{aligned}$$

implying $\delta_k\to 0$.

Since the range of $B_{\bar{\mathscr{P}}}$ is assumed to be closed and therefore also the operator

$$(u,(l_i)_{i\in\bar{\mathscr{P}}})\to((u,-\nabla F(\bar{x})u)-l_i,(\langle x^*_{ij},u\rangle-\langle y^*_{ij},\nabla F(\bar{x})u\rangle)_{j\in\bar{\mathscr{A}_i}})_{i\in\bar{\mathscr{P}}}$$

has closed range, there exist some $\gamma'>0$ such that for each k we can find some $\bar{u}_k$ with

$$(\bar{u}_k,-\nabla F(\bar{x})\bar{u}_k)\in L_i,\quad \langle x^*_{ij},\bar{u}_k\rangle-\langle y^*_{ij},\nabla F(\bar{x})\bar{u}_k\rangle=0,\ j\in\bar{\mathscr{A}_i},\ i\in\bar{\mathscr{P}}$$

and $\|u_k-\bar{u}_k\|\le\gamma'\delta_k$. By setting $\bar{x}_k:=\bar{x}+\tau_k\bar{u}_k$, $\bar{s}_k:=-F(\bar{x})-\tau_k\nabla F(\bar{x})\bar{u}_k$ we have

$$(\bar{x}_k,\bar{s}_k)\in P_i,\ i\in\bar{\mathscr{P}},(\bar{x}_k,\bar{s}_k)\notin P_i,\ i\notin\bar{\mathscr{P}}$$

and

$$\langle x^*_{ij},\bar{x}_k\rangle+\langle y^*_{ij},\bar{s}_k\rangle-\zeta_{ij}=0,\ j\in\bar{\mathscr{A}_i},\ \langle x^*_{ij},\bar{x}_k\rangle+\langle y^*_{ij},\bar{s}_k\rangle-\zeta_{ij}\le\frac{\tau_k\eta_{ij}}{2}<0,\ j\notin\bar{\mathscr{A}_i},\ i\in\bar{\mathscr{P}}$$

for all k sufficiently large. Next we show that

$$\mathscr{P}((\bar{x}_k,\bar{s}_k)+\tau(x'_k-\bar{x}_k,s'_k-\bar{s}_k))=\mathscr{P}\tag{20}$$

holds for all k sufficiently large and for all $\tau\in(0,1]$. Since $\mathscr{P}(\bar{x}_k,\bar{s}_k)=\bar{\mathscr{P}}\supset\mathscr{P}$ we have $\mathscr{P}((\bar{x}_k,\bar{s}_k)+\tau(x'_k-\bar{x}_k,s'_k-\bar{s}_k))\supset\mathscr{P}$, $\forall\tau\in(0,1]$ by convexity arguments. Now assume on the contrary to (20) that, after eventually passing to a subsequence, there exist a sequence $(\rho_k)\in(0,1]$ and an index $\bar{i}\in\mathscr{P}(\hat{x}_k,\hat{s}_k)\setminus\mathscr{P}$, $\forall k$, where we have set $(\hat{x}_k,\hat{s}_k):=(\bar{x}_k,\bar{s}_k)+\rho_k(x'_k-\bar{x}_k,s'_k-\bar{s}_k)$. Since

$$\begin{aligned}&\tau_k^{-1}(x'_k-\bar{x}_k,s'_k-\bar{s}_k)\\ &= (u_k-\bar{u}_k,\nabla F(\bar{x})u_k-\frac{F(\bar{x}+\tau_ku_k)-F(\bar{x})}{\tau_k}+\nabla F(\bar{x})(\bar{u}_k-u_k)+\frac{F(x'_k)+s'_k}{\tau_k})\to(0,0)\end{aligned}$$

we conclude $\bar{i} \in \bar{\mathscr{P}} \setminus \mathscr{P}$ by the construction of the set $\bar{\mathscr{P}}$. Furthermore, since $(\hat{x}_k, \hat{s}_k), (\bar{x}_k, \bar{s}_k) \in P_{\bar{i}}$, $\mathscr{A}_{\bar{i}}(\bar{x}_k, \bar{s}_k) = \bar{\mathscr{A}_{\bar{i}}}$ and $(x'_k, s'_k) = (\bar{x}_k, \bar{s}_k) + \rho_k^{-1}((\hat{x}_k, \hat{s}_k) - (\bar{x}_k, \bar{s}_k))$ we conclude $(x'_k, s'_k) \in L'_{\bar{i}}$ and $\langle x^*_{\bar{i}j}, x'_k \rangle + \langle y^*_{\bar{i}j}, s'_k \rangle \leq \zeta_{\bar{i}j}$, $j \in \bar{\mathscr{A}_{\bar{i}}}$. Finally we have $\langle x^*_{\bar{i}j}, x'_k \rangle + \langle y^*_{\bar{i}j}, s'_k \rangle - \zeta_{\bar{i}j} \leq \frac{\tau_k \eta_{\bar{i}j}}{2} < 0$, $j \notin \bar{\mathscr{A}_{\bar{i}}}$ and thus $(x'_k, s'_k) \in P_{\bar{i}}$, implying the contradiction $\bar{i} \in \mathscr{P}(x'_k, s'_k) = \mathscr{P}$.

Hence (20) holds true for all k sufficiently large and all $\tau \in (0,1]$, and it follows that

$$\mathscr{P}((\bar{x}, -F(\bar{x})) + \eta((\bar{x}_k, \bar{s}_k) + \tau(x'_k - \bar{x}_k, s'_k - \bar{s}_k) - (\bar{x}, -F(\bar{x})))) = \mathscr{P} \tag{21}$$

is valid for all k sufficiently large and all $\tau \in (0,1]$, $\eta \in (0,1]$. In fact, if there would be some $\bar{i} \in \mathscr{P}((\bar{x}, -F(\bar{x})) + \eta((\bar{x}_k, \bar{s}_k) + \tau(x'_k - \bar{x}_k, s'_k - \bar{s}_k) - (\bar{x}, -F(\bar{x})))) \setminus \mathscr{P}$ for some $\tau \in (0,1]$, $\eta \in (0,1]$, then $(\bar{x}_k, \bar{s}_k) + \tau(x'_k - \bar{x}_k, s'_k - \bar{s}_k) - (\bar{x}, -F(\bar{x})) \in T((\bar{x}, -F(\bar{x})); P_{\bar{i}})$ and $\bar{i} \in \mathscr{P}((\bar{x}_k, \bar{s}_k) + \tau(x'_k - \bar{x}_k, s'_k - \bar{s}_k))$ would follow for all k sufficiently large.

Since we also have $\mathscr{A}_i \subset \bar{\mathscr{A}_i} \subset \mathscr{A}_i(\bar{x}, -F(\bar{x}))$, $i \in \mathscr{P}$, it follows that

$$-(\nabla F(\bar{x})^* \tilde{y}^*_k, \tilde{y}^*_k) \in \hat{N}((\bar{x}, -F(\bar{x})) + \eta((\tilde{x}_k, \tilde{s}_k) + \tau(x_k - \tilde{x}_k, s_k - \tilde{s}_k) - (\bar{x}, -F(\bar{x})))); P_i), i \in \mathscr{P}$$

for all $\tau \in (0,1]$, $\eta \in (0,1]$ showing

$$-\nabla F(\bar{x})^* \tilde{y}^*_k \in D^* S((\bar{x}, -F(\bar{x})); (\bar{u}_k, -\nabla F(\bar{x})\bar{u}_k))(\tilde{y}^*_k).$$

Since also $\lim_k \|\bar{u}_k\| = \lim_k \|u_k\| = 1$, we conclude $\bar{u}_k \in \mathscr{D}(\bar{x})$ for all k sufficiently large and we obtain a contradiction in the first case $\mathscr{D}(\bar{x}) = \emptyset$.

To obtain a contradiction also in the second case, note that by our assumption on the uniformness of the limit (17) we have

$$\lim_{k\to\infty} \frac{F(\bar{x} + \tau_k u_k) - F(\bar{x}) - \tau_k \nabla F(\bar{x}) u_k}{\tau_k^2/2} - F''(\bar{x}; \bar{u}_k) = 0.$$

Utilizing (19) and taking into account that $((s'_k - \tilde{s}_k) + F(x'_k) - F(\tilde{x}_k))/(\tau_k^2/2) \to 0$ due to the Lipschitz continuity of F near $\bar{x}$, we obtain the contradiction

$$\begin{aligned}
0 &\leq (1 - \gamma\varepsilon_k)|||F(\tilde{x}_k) + \tilde{s}_k||| = \langle y^*_k, F(\tilde{x}_k) + \tilde{s}_k \rangle - \gamma\varepsilon_k |||F(\tilde{x}_k) + \tilde{s}_k||| \leq \langle \tilde{y}^*_k, F(\tilde{x}_k) + \tilde{s}_k \rangle \\
&= \langle \tilde{y}^*_k, F(\tilde{x}_k) - F(\bar{x}) - \nabla F(\bar{x})(x'_k - \bar{x}) - (s'_k - \tilde{s}_k) \rangle \\
&= \frac{\tau_k^2}{2} \Bigg(\langle \tilde{y}^*_k, F''(\bar{x}; \bar{u}_k) \rangle + \langle \tilde{y}^*_k, \frac{F(\bar{x} + \tau_k u_k) - F(\bar{x}) - \tau_k \nabla F(\bar{x}) u_k}{\tau_k^2/2} - F''(\bar{x}; \bar{u}_k) \rangle \\
&\qquad - \langle \tilde{y}^*_k, \frac{(s'_k - \tilde{s}_k) + F(x'_k) - F(\tilde{x}_k)}{\tau_k^2/2} \rangle \Bigg) \\
&\leq \frac{\tau_k^2}{2} \Bigg(-\sigma\beta \|\bar{u}_k\|^2 + (1 + \gamma\varepsilon_k)(|||\frac{F(\bar{x} + \tau_k u_k) - F(\bar{x}) - \tau_k \nabla F(\bar{x}) u_k}{\tau_k^2/2} - F''(\bar{x}; \bar{u}_k)||| \\
&\qquad + |||\frac{(s'_k - \tilde{s}_k) + F(x'_k) - F(\tilde{x}_k)}{\tau_k^2/2}|||) \Bigg) \\
&< 0
\end{aligned}$$

□

In case of $(u,v_1)=(0,0)$ the condition of Theorem 4(1) sufficient for mixed regularity/subregularity of M can be formulated as $\tilde{\Lambda}(\bar{x};0)=\emptyset$ which readily implies $\tilde{\Lambda}(\bar{x};u)=\emptyset\ \forall u\neq 0$ and consequently $\mathscr{D}(\bar{x})=\emptyset$. An example where $\tilde{\Lambda}(\bar{x};0)\neq\emptyset$ but $\mathscr{D}(\bar{x})=\emptyset$ and $M^{-1}(0)$ is not a singleton, can be found in [7, Example 3].

To the end of this section we present some result which is important for showing sufficient second-order optimality conditions, but which can also be interpreted as a sufficient condition for some kind of metric Hölder subregularity.

Proposition 2. *Let $M:X\rightrightarrows Y$ be a multifunction between Banach spaces of the form $M(x)=F(x)+S(x)$, where the mapping $F:X\to Y$ is strictly differentiable at the point $\bar{x}$ with $0\in M(\bar{x})$ and $S:X\rightrightarrows Y$ is a generalized polyhedral multifunction with representation* (12). *Assume that for every $\bar{\mathscr{P}}\in\bar{\mathscr{I}}(\bar{x},-F(\bar{x}))$ the operator $B_{\bar{\mathscr{P}}}$ has closed range. Further suppose that there is some constant $\sigma>0$ such that for every $u\neq 0$ with $(u,-\nabla F(\bar{x})u)\in T((\bar{x},-F(\bar{x}));\operatorname{gph}S)$ the mapping F is second-order directionally differentiable in direction u and there is some multiplier y^* with $-(\nabla F(\bar{x})^*y^*,y^*)\in\hat{N}((u,-\nabla F(\bar{x})u);T((\bar{x},-F(\bar{x}));\operatorname{gph}S))$ and*

$$\langle y^*,F''(\bar{x};u)\rangle>\sigma\|u\|^2\|y^*\|.$$

If in addition the limit (17) *is uniform for $u\in\{u\,|\,(u,-\nabla F(\bar{x})u)\in T((\bar{x},-F(\bar{x}));\operatorname{gph}S)\}\cap\mathscr{S}_X$, then there are a neighborhood U of $\bar{x}$ and a constant $\beta>0$ such that*

$$\mathrm{d}(0,M(x))\geq\beta\|x-\bar{x}\|^2\ \forall x\in U. \tag{22}$$

Proof. Assume on the contrary that there is some sequence $(x_k,s_k)\in\operatorname{gph}S$ with $(x_k)\to\bar{x}$ such that

$$\|F(x_k)+s_k\|\leq 2\mathrm{d}(0,M(x_k))<\frac{1}{k}\|x_k-\bar{x}\|^2$$

Defining $t_k:=\|x_k-\bar{x}\|$, $u_k:=(x_k-\bar{x})/t_k$ and passing to subsequences if necessary, we can assume that there are index sets $\mathscr{P}\in\mathscr{I}(\bar{x},-F(\bar{x}))$, $(\mathscr{A}_i)_{i\in\mathscr{P}}\in\mathscr{J}_{\mathscr{P}}(\bar{x},-F(\bar{x}))$ such that for every k we have $\mathscr{P}(x_k,s_k)=\mathscr{P}$, $\mathscr{A}_i(x_k,s_k)=\mathscr{A}_i$, $i\in\mathscr{P}$. Further we can assume that there is an index set $\bar{\mathscr{P}}\in\bar{\mathscr{I}}(\bar{x},-F(\bar{x}))$ such that $\mathscr{P}\subset\bar{\mathscr{P}}$ and

$$\lim_{k\to\infty}t_k^{-1}\mathrm{d}((x_k,s_k),P_i)=0,\ i\in\bar{\mathscr{P}},\ \liminf_{k\to\infty}t_k^{-1}\mathrm{d}((x_k,s_k),P_i)>0,\ i\notin\bar{\mathscr{P}}$$

and that for every $i\in\bar{\mathscr{P}}$ there are an index set $\bar{\mathscr{A}_i}$ and numbers η_{ij}, $j\in\bar{\mathscr{A}_i}$ such that

$$\lim_{k\to\infty}\frac{\langle x_{ij}^*,x_k\rangle+\langle y_{ij}^*,s_k\rangle-\zeta_{ij}}{t_k}=0,\ j\in\bar{\mathscr{A}_i},\ \limsup_{k\to\infty}\frac{\langle x_{ij}^*,x_k\rangle+\langle y_{ij}^*,s_k\rangle-\zeta_{ij}}{t_k}\leq\eta_{ij}<0,\ j\notin\bar{\mathscr{A}_i}.$$

Using the same arguments as in the proof of the preceding theorem there exist sequences $(\delta_k)\to 0$, $(\bar{u}_k)\in X$ and a real $\gamma'>0$ such that for each k we can find some $\bar{u}_k$ with

$$(\bar{u}_k,-\nabla F(\bar{x})\bar{u}_k)\in L_i,\quad\langle x_{ij}^*,\bar{u}_k\rangle-\langle y_{ij}^*,\nabla F(\bar{x})\bar{u}_k\rangle=0,\ j\in\bar{\mathscr{A}_i},\ i\in\bar{\mathscr{P}}$$

and $\|u_k-\bar{u}_k\|\leq\gamma'\delta_k$. Further we have $\mathscr{P}(\bar{x}_k,\bar{s}_k)+\tau(x_k-\bar{x}_k,s_k-\bar{s}_k))=\mathscr{P}\subset\mathscr{P}(\bar{x},-F(\bar{x}))$ $\forall\tau\in(0,1]$ for all k sufficiently large, where $(\bar{x}_k,\bar{s}_k):=(\bar{x},-F(\bar{x}))+t_k(\bar{u}_k,-\nabla F(\bar{x})\bar{u}_k)$. Hence

$t_k^{-1}((\bar{x}_k,\bar{s}_k)+\tau(x_k-\bar{x}_k,s_k-\bar{s}_k)-(\bar{x},-F(\bar{x})))=(1-\tau)(\bar{u}_k,-\nabla F(\bar{x})\bar{u}_k)+\tau(u_k,t_k^{-1}(s_k+F(\bar{x})))\in T((\bar{x},-F(\bar{x}));\mathrm{gph}\,S)$ $\forall\tau\in[0,1]$. By our assumption we can choose some multiplier $y_k^*\in\mathscr{S}_{Y^*}$ with $-(\nabla F(\bar{x})^*y_k^*,y_k^*)\in\hat{N}((\bar{u}_k,-\nabla F(\bar{x})\bar{u}_k);T((\bar{x},-F(\bar{x}));\mathrm{gph}\,S))$ and $\langle y_k^*,F''(\bar{x};\bar{u}_k)\rangle>\sigma\|\bar{u}_k\|^2$ and because of $\lim_{k\to\infty}\frac{F(\bar{x}+t_ku_k)-F(\bar{x})-t_k\nabla F(\bar{x})u_k}{t_k^2/2}-F''(\bar{x};\bar{u}_k)=0$ and $\lim_{k\to\infty}\|\bar{u}_k\|=1$ we obtain the contradiction

$$\begin{aligned}0 &\geq \limsup_{k\to\infty}2t_k^{-1}\langle-(\nabla F(\bar{x})^*y_k^*,y_k^*),(u_k-\bar{u}_k,t_k^{-1}(s_k+F(\bar{x}))+\nabla F(\bar{x})\bar{u}_k)\rangle\\ &= \limsup_{k\to\infty}\langle y_k^*,\frac{F(\bar{x}+t_ku_k)-F(\bar{x})-t_k\nabla F(\bar{x})u_k}{t_k^2/2}-F''(\bar{x};\bar{u}_k)+F''(\bar{x};\bar{u}_k)+\frac{F(x_k)+s_k}{t_k^2/2}\rangle\\ &= \limsup_{k\to\infty}\langle y_k^*,F''(\bar{x};\bar{u}_k)\rangle\geq\sigma>0.\end{aligned}$$

□

Note that from the relation (22) the Hölder–type estimate

$$\mathrm{d}(x,M^{-1}(0))\leq\frac{1}{\sqrt{\beta}}\mathrm{d}(0,M(x))^{1/2}$$

follows.

5 Necessary and sufficient optimality conditions

In this section we consider the following optimization problem

$$\begin{aligned}(P)\qquad &\min_{x\in X}f(x)\\ s.t.\qquad &0\in M(x):=(M_1(x),M_2(x)):=(F_1(x)+S_1(x),Ax+S_2(x))\end{aligned}$$

where the objective $f:X\to\mathbb{R}$ maps a Banach space X into the real numbers, $F_1:X\to Y_1$ takes values in another Banach space Y_1, $A:X\to Y_2$ is a continuous linear operator having values in the Banach space Y_2 and $S_l:X\rightrightarrows Y_l$, $l=1,2$ are generalized polyhedral multifunctions. We assume that S_l, $l=1,2$ have the representations (13), (14) and define the multifunction $S:X\rightrightarrows Y:=Y_1\times Y_2$ respectively the mapping $F:X\to Y$ by $S(x):=(S_1(x),S_2(x))$ respectively $F(x):=(F_1(x),Ax)$

Throughout this section we denote by $\bar{x}$ a point feasible for the problem (P) such that f and F_1 are strictly differentiable at $\bar{x}$. We define the cone of critical directions by

$$\mathscr{C}(\bar{x}):=\{u\in X\,|\,\nabla f(\bar{x})u\leq 0,-(u,\nabla F(\bar{x})u)\in T((\bar{x},-F(\bar{x}));\mathrm{gph}\,S)\}$$

Given a functional $x^*\in X^*$ we define the multifunction

$$M^{x^*}:X\rightrightarrows\mathbb{R}\times W,\ M^{x^*}(x)=:=(f(x)-f(\bar{x})+\langle x^*,x-\bar{x}\rangle^3+\mathbb{R}_+,F(x)+S(x)),\qquad(23)$$

which is again of the form (16) with components $M_1^{x^*}(x):=f(x)-f(\bar{x})+\langle x^*,x-\bar{x}\rangle^3+\mathbb{R}_+$ and $M_2^{x^*}(x):=F(x)+S(x)$. Our necessary optimality conditions are motivated by the following observation.

Proposition 3. *Let $\bar{x}$ be a local minimizer for the problem (P). Then M^0 is not mixed regular/subregular at $(\bar{x},0)$ and for every critical direction $0\neq u\in\mathscr{C}(\bar{x})$ there exists some $x^*\in X^*$ such that M^{x^*} is not metrically subregular in direction u.*

Proof. Assume first on the contrary that M^0 is mixed regular/subregular at $\bar{x}$. Then there are some constants $\bar{t}>0$, $\kappa>0$ such that

$$\mathrm{d}(\bar{x},M^{0^{-1}}(-t^2,0))\leq\kappa\mathrm{d}((-t^2,0),M^0(\bar{x}))\leq\kappa t^2\ \forall t\in(0,\bar{t}),$$

i.e., for every $t\in(0,\bar{t})$ we can find a point x_t with $\|x_t-\bar{x}\|\leq(\kappa+1)t^2$ such that $0\in F(x_t)+S(x_t)$ and $-t^2\geq f(x_t)-f(\bar{x})$, contradicting the optimality of $\bar{x}$.

Now let $0\neq u\in\mathscr{C}(\bar{x})$ be fixed, let $x^*\in S_{X^*}$ be chosen such that $\langle x^*,u\rangle=1$ and assume that M^{x^*} is metrically subregular in direction u. Then there exist some $\kappa>0$ such that for all $t>0$ sufficiently small we have

$$\begin{aligned}\mathrm{d}(\bar{x}+tu,M^{x^{*}-1}(0)) &\leq \kappa\mathrm{d}(0,M^{x^*}(\bar{x}+tu))\\ &\leq \kappa(\max\{f(\bar{x}+tu)-f(\bar{x})+t^3,0\}+\mathrm{d}(F(\bar{x}+tu),S(\bar{x}+tu)))\end{aligned}$$

Since S is a generalized polyhedral multifunction and $-(u,\nabla F(\bar{x})u)\in T((\bar{x},-F(\bar{x}));\mathrm{gph}\,S)$ we have $(\bar{x},-F(\bar{x}))+t(u,-\nabla F(\bar{x})u)\in\mathrm{gph}\,S$ for all $t>0$ sufficiently small and consequently $\lim_{t\downarrow 0}t^{-1}\mathrm{d}(-F(\bar{x}+tu),S(\bar{x}+tu))=0$. Since $\nabla f(\bar{x})u\leq 0$ we also have $\lim_{t\downarrow 0}t^{-1}\max\{f(\bar{x}+tu)-f(\bar{x})+t^3,0\}=0$ and therefore we can find for every $t>0$ sufficiently small a point x_t with $\lim_{t\downarrow 0}t^{-1}\|\bar{x}+tu-x_t\|=0$ such that $0\in F(x_t)+S(x_t)$ and $0\geq f(x_t)-f(\bar{x})+\langle x^*,x_t-\bar{x}\rangle^3\geq f(x_t)-f(\bar{x})+t^3/2$, contradicting again the optimality of $\bar{x}$. □

Associated with the multifunction M^{x^*} is the multifunction $\hat{S}:X\rightarrow\mathbb{R}\times Y$, $\hat{S}(x):=\mathbb{R}_+\times S(x)$ with representation $\mathrm{gph}\,\hat{S}=\bigcup_{i=1}^p\hat{P}_i$, where

$$\hat{P}_i=\{(x,\phi,y)\in(x_i',0,y_i')+\hat{L}_i\,|\,\phi\geq 0,\langle x_{ij}^*,x\rangle+\langle y_{ij}^*,y\rangle\leq\zeta_{ij},\ j=1,\ldots,m_i\}$$

and $\hat{L}_i:=\{(x,\phi,y)\,|\,(x,y,\phi)\in L_i\times\mathbb{R}\}$ for $i=1,\ldots,p$.

Now consider for arbitrary nonempty index sets $\mathscr{P}\subset\{1,\ldots,p\}$ the operators

$$\hat{A}_{\mathscr{P}}:\prod_{i\in\mathscr{P}}\hat{L}_i\rightarrow Y,\quad \hat{A}_{\mathscr{P}}((l_{ix},l_{i\phi},l_{iy})_{i\in\mathscr{P}}):=\sum_{i\in\mathscr{P}}(\nabla f(\bar{x})l_{ix}+l_{i\phi},\nabla F(\bar{x})l_{ix}+l_{iy}),$$

and

$$\hat{B}_{\mathscr{P}}:X\times\prod_{i\in\mathscr{P}}\hat{L}_i\rightarrow\prod_{i\in\mathscr{P}}(X\times\mathbb{R}\times Y),\quad \hat{B}_{\mathscr{P}}(x,(l_i)_{i\in\mathscr{P}}):=((x,-\nabla f(\bar{x})x,-\nabla F(\bar{x})x)-l_i)_{i\in\mathscr{P}}.$$

Thus $\mathrm{Range}(\hat{A}_{\mathscr{P}})=\mathbb{R}\times\mathrm{Range}(A_{\mathscr{P}})$ and $\mathrm{Range}(\hat{B}_{\mathscr{P}})=\{(x,\phi,y)\,|\,(x,y,\phi)\in\mathrm{Range}(B_{\mathscr{P}})\times\mathbb{R}\}$ and it follows that $\mathrm{Range}(\hat{A}_{\mathscr{P}})$ respectively $\mathrm{Range}(\hat{B}_{\mathscr{P}})$ is closed if and only if $\mathrm{Range}(A_{\mathscr{P}})$ respectively $\mathrm{Range}(B_{\mathscr{P}})$ is closed.

Now, all what we have to do in order to state optimality conditions is to apply the results of the preceding section. Consider the following assumptions:

(A1) For each pair of index sets $(\mathscr{P}_1,\mathscr{P}_2)\in\mathscr{I}_1(\bar{x},-F_1(\bar{x}))\times\mathscr{I}_2(\bar{x},-A\bar{x})$ the subspace

$$\sum_{i_1\in\mathscr{P}_1}\pi_X(L_{1i_1})-\sum_{i_2\in\mathscr{P}_2}\pi_X(L_{2i_2})$$

is closed.

(A2) For each $\mathscr{P}\in\mathscr{I}(\bar{x},-F(\bar{x}))$ the operator

$$A_{\mathscr{P}}:\prod_{i\in\mathscr{P}}L_i\to Y,\quad A_{\mathscr{P}}((l_{ix},l_{iy})_{i\in\mathscr{P}}):=\sum_{i\in\mathscr{P}}(\nabla F(\bar{x})l_{ix}+l_{iy})$$

has closed range.

(A3) For each $i_1\in\mathscr{P}_1(\bar{x},-F_1(\bar{x}))$ and each $i_2\in\mathscr{P}_2(\bar{x},-A\bar{x})$ the projection $\pi_X(L_{1i_1})$ and the subspace

$$\{(x_1+x_2,y_2+Ax_2)\,|\,(x_l,y_l)\in L_{li_l},\ l=1,2\}$$

are closed.

(A4) For each $\bar{\mathscr{P}}\in\bar{\mathscr{I}}(\bar{x},-F(\bar{x}))$ the operator

$$B_{\bar{\mathscr{P}}}:X\times\prod_{i\in\bar{\mathscr{P}}}L_i\to\prod_{i\in\bar{\mathscr{P}}}(X\times Y),\quad B_{\bar{\mathscr{P}}}(x,(l_i)_{i\in\bar{\mathscr{P}}}):=((x,-\nabla F(\bar{x})x)-l_i)_{i\in\bar{\mathscr{P}}}$$

has closed range.

Theorem 7. *Let $\bar{x}$ be a local minimizer for the problem (P), assume that f and F are strictly differentiable at $\bar{x}$ and that assumptions (A1)-(A3) are fulfilled. Then for all critical directions $u\in\mathscr{C}(\bar{x})$ there exist some multiplier $(\lambda_0,y_1^*,y_2^*)\in\mathbb{R}_+\times Y_1^*\times y_2^*$, $(\lambda_0,y_1^*)\neq 0$, $\lambda_0\langle\nabla f(\bar{x}),u\rangle=0$ such that*

$$\begin{aligned}0\ \in\ &\lambda_0\nabla f(\bar{x})+\nabla F_1(\bar{x})^*y_1^*+A^*y_2^*\\&+D^*S_1((\bar{x},-F_1(\bar{x}));(u,-\nabla F_1(\bar{x})u))(y_1^*)+D^*S_2((\bar{x},-A\bar{x});(u,-Au))(y_2^*)\end{aligned}\tag{24}$$

and, provided f and F_1 are twice Fréchet differentiable at $\bar{x}$,

$$\lambda_0\nabla^2 f(\bar{x})(u,u)+\langle y_1^*,\nabla^2F_1(\bar{x})(u,u)\rangle\geq 0.\tag{25}$$

If in addition M is metrically subregular in direction u at $(\bar{x},0)$, then these conditions also hold with $\lambda_0>0$.

Proof. Let x^* be chosen according to Proposition 3. Assumption (A3) together with Lemma 8 guarantee that M_2 is proper subregular in direction u with relative to $F(\bar{x})+\nabla F(\bar{x})(x-\bar{x})+S_1(x)$ and 0 at $(\bar{x},0)$. Then since $F(x)-F(\bar{x})-\nabla F(\bar{x})(x-\bar{x})$ is Lipschitzian near $\bar{x}$ and the multifunction $x\rightrightarrows f(x)-f(\bar{x})+\langle x^*,x-\bar{x}\rangle^3+\mathbb{R}_+$ has the Aubin property near $(\bar{x},0)$, it follows by the definition that M_2 is also proper subregular in direction u relative to $x \rightrightarrows (f(x) - f(\bar{x}) + \langle x^*, x - \bar{x}\rangle^3 + \mathbb{R}_+, M_1(x))$ and 0 at $(\bar{x}, 0)$. Straightforward application of Lemma 7 yields

$$D^*\hat{S}((\bar{x},(0,-F(\bar{x}))); (u,(-\langle\nabla f(\bar{x}),u\rangle, -\nabla F(\bar{x})u)))(\lambda_0, y^*) = \begin{cases} D^*S((\bar{x},-F(\bar{x})); (u,-\nabla F(\bar{x})u))(y^*) & \text{if } \lambda_0 \geq 0,\, \lambda_0\langle\nabla f(\bar{x}),u\rangle = 0 \\ \emptyset & \text{else.}\end{cases}$$

Then the assertions follow immediately from Theorem 4 together with Lemma 7, where in case that M is metrically subregular in direction u we also have to take into account that M is proper subregular in direction u relative to $x \rightrightarrows f(x) - f(\bar{x}) + \langle x^*, x - \bar{x}\rangle^3 + \mathbb{R}_+$ and 0 at $(\bar{x}, 0)$ by Lemma 1. □

Note that, if (24) respectively (25) cannot be fulfilled with $\lambda_0 = 0$, this is a sufficient condition for mixed regularity/subregularity respectively metric subregularity of M in direction u.

Theorem 7 contains as the special case $u = 0$ the usual M-stationarity conditions: If M is metrically subregular at $(\bar{x}, 0)$ then there exist multipliers y_1^*, y_2^* such that

$$0 \in \nabla f(\bar{x}) + \nabla F_1(\bar{x})^* y_1^* + A^* y_2^* + D^*S_1(\bar{x}, -F_1(\bar{x}))(y_1^*) + D^*S_2(\bar{x}, -A\bar{x}))(y_2^*),$$

where we have taken without loss of generality $\lambda_0 = 1$. But (24) holds not only for the critical direction $u = 0$ with $\lambda_0 = 1$, but for every critical direction u such that the constraint mapping M is metrically subregular in this direction. Hence we call the first-order optimality conditions (24) *extended M-stationarity conditions*.

We now state the second-order sufficient conditions which follow immediately from Proposition 2 applied to the multifunction M^0.

Theorem 8. *Let $\bar{x}$ be feasible for the problem (P), assume that f and F are twice Fréchet differentiable at $\bar{x}$ and that assumption (A4) is fulfilled. Further assume that there is some constant $\sigma > 0$ such that for every critical direction $0 \neq u \in \mathscr{C}(\bar{x})$ there are multipliers $(\lambda_0, y^*) \in \mathbb{R}_+ \times Y^*$ such that $\lambda_0\langle\nabla f(\bar{x}), u\rangle = 0$,*

$$-(\lambda_0\nabla f(\bar{x}) + \nabla F(\bar{x})^* y^*, y^*) \in \hat{N}((u, -\nabla F(\bar{x})u); T((\bar{x}, -F(\bar{x})); \operatorname{gph} S)) \tag{26}$$

satisfying

$$\lambda_0\nabla^2 f(\bar{x})(u,u) + \langle y_1^*, \nabla^2 F_1(\bar{x})(u,u)\rangle > \sigma(\lambda_0 + \|y\|)\|u\|^2,$$

where $y^* = (y_1^*, y_2^*) \in Y_1^* \times Y_2^*$. Then there exist some neighborhood U of $\bar{x}$ and some real $\beta > 0$ such that

$$\max\{f(x) - f(\bar{x}), 0\} + \mathrm{d}(0, F(x) + S(x)) \geq \beta\|x - \bar{x}\|^2 \; \forall x \in U.$$

In particular $\bar{x}$ is a strictly local minimizer fulfilling a quadratic growth condition.

Note that multipliers (λ_0, y^*) fulfilling (26) also fulfill

$$0 \in \lambda_0\nabla f(\bar{x}) + \nabla F(\bar{x})^* y^* + D^*S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(y^*)$$

and consequently (24), provided (A1) is fulfilled.

When applied to the nonlinear programming problem with equality and inequality constraints, the optimality conditions of Theorems 7, 8 imply the standard second-order optimality conditions [15]. Further note that the necessary second-order optimality conditions of Theorem 7 even extend the standard second-order optimality conditions in case when M_2, the polyhedral part of the constraints, is not metrically regular near $(\bar{x}, 0)$.

References

[1] L. BAN,B, B. S. MORDUKHOVICHC, W. SONG, *Lipschitzian stability of parametric variational inequalities over generalized polyhedra in Banach spaces*, Nonlinear Anal., 74 (2011), pp. 441–461.

[2] J. F. BONNANS, A. SHAPIRO, *Perturbation analysis of optimization problems*, Springer, New York, 2000.

[3] A. L. DONTCHEV, R. T. ROCKAFELLAR, *Regulartity and conditioning of solution mappings in variatonal anlysis*, Set-Valued Anal., 12 (2004), pp. 79–109.

[4] M. L. FLEGEL, C. KANZOW, J. V. OUTRATA, *Optimality conditions for disjunctive programs with application to mathematical programs with equilibrium constraints*, Set-Valued Anal., 15 (2007), pp. 139–162.

[5] I. GINCHEV, B. S. MORDUKHOVICH, *On directionally dependent subdifferentials*, C. R. Bulg. Acad. Sci., 64 (2011), 497-508.

[6] H. GFRERER, *First order and second order characterizations of metric subregularity and calmness of constraint set mappings*, SIAM J. Optim., 21 (2011), pp. 1439–1474.

[7] H. GFRERER,*On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs*, (2012), Set-Valued Var. Anal., DOI: 10.1007/s11228-012-0220-5

[8] I. V. GIRSANOV, *Lectures on mathematical theory of extremum problems*, in Lecture Notes in Economics and Mathematical Systems, Vol. 67, Springer, Berlin, 1972.

[9] R. HENRION, A. JOURANI, *Subdifferential conditions for calmness of convex constraints*, SIAM J. Optim., 13(2002), pp. 520–534.

[10] R. HENRION, A. JOURANI, J. V. OUTRATA, *On the calmness of a class of multifunctions*, SIAM J. Optim., 13 (2002), pp. 603–618.

[11] R. HENRION, B. S. MORDUKHOVICH, N. M. NAM, *Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities*, SIAM J. Optim., 20 (2010), pp. 2199–2237.

[12] R. HENRION, J. V. OUTRATA, *A subdifferential condition for calmness of multifunctions*, J. Math. Anal. Appl., 258 (2001), pp. 110–130.

[13] R. HENRION, J. V. OUTRATA, *Calmness of constraint systems with applications*, Math. Program., Ser. B, 104 (2005), pp. 437–464.

[14] A. HOFFMAN, *On approximate solutions of systems of linear inequalities*, Journal of Research of the National Bureau of Standards, 49 (1952), pp. 263–265.

[15] A. D. IOFFE, *Necessary and sufficient conditions for a local minimum 3: Second order conditions and augmented duality*, SIAM J. Control Optim., 17 (1979), pp. 266–288.

[16] A. D. IOFFE, *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc., 251 (1979), pp. 61–69.

[17] A. D. IOFFE, *Metric regularity and subdifferential calculus*, Russian Math. Surveys, 55 (2000), pp. 501–558.

[18] A. D. IOFFE, J. V. OUTRATA, *On metric and calmness qualification conditions in subdifferential calculus*, Set-Valued Anal., 16 (2008), pp. 199–227.

[19] D. KLATTE, B. KUMMER, *Nonsmooth equations in optimization. Regularity, calculus, methods and applications*, Nonconvex Optimization and its Applications 60, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[20] B. S. MORDUKHOVICH, *Variational analysis and generalized differentiation, I: Basic theory, II: Applications*, Springer, Berlin, Heidelberg, 2006.

[21] J. V. OUTRATA, *Optimality conditions for a class of mathematical programs with equilibrium constraints*, Math. Oper. Res., 24 (1999), pp. 627644.

[22] J. V. OUTRATA, *A generalized mathematical program with equilibrium constraints*, SIAM J. Control Optim., 38 (2000), pp. 16231638

[23] A. P. ROBERTSON, W. ROBERTSON, *Topological vector spaces*, Canbridge University Press, Cambridge, 1964,

[24] S. M. ROBINSON, *Some continuity properties of polyhedral multifunctions*, Math. Program. Studies, 14 (1981), pp. 206–214.

[25] R. T. ROCKAFELLAR, R. J-B. WETS, *Variational analysis*, Springer, Berlin, 1998.

[26] J. J. YE, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM J. Optim., 10 (2000), pp. 943–962.

[27] J. J. YE, X. Y. YE, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Math. Oper. Res., 22 (1997), pp. 977997.

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