Monotone operators in nonlinear PDEs

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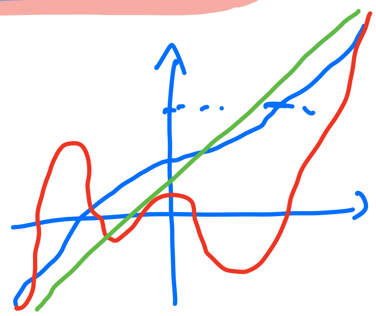

0. Motivation

We want to generalize a result like:

A function $F:\mathbb{R}\to\mathbb{R}$ fulfilling:
- F monotonically increasing
- F continuous
- F coercive, $F(u)\to\pm\infty$ if $x\to\pm\infty$

then the equation $F(u)=b$ has a solution $u\in\mathbb{R}$ $\forall b\in\mathbb{R}$ ie F is surjective

If F is strictly monotone, then the solution u is unique.

The theory on monotone operators wants to generalize this result to equations of the form $\boxed{Au=b}$ in a reflexive Banach space.

$X\cong X''$, $\langle Jx, x'\rangle_{X'} = \langle x', x\rangle_X$ $\forall x'\in X'$

Theorem 0.1 Let X be a separable & reflexive Banach space, and let the operator $A: X\to X'$ fulfill:
- monotone i.e. $\langle Au - Av, u-v\rangle_X \geq 0$ $\forall u,v\in X$ $\quad \in X^*$
- hemicontinuous ie. $t\mapsto \underbrace{\langle A(u+tv), w\rangle_X}_{\in\mathbb{R}}$ is continuous in $[0,1]$ for any $u,v,w\in X$, $\xrightarrow{t\to 0} \langle Au, w\rangle_X$
- coercive ie. $\lim_{\|u\|_X\to\infty} \frac{\langle Au, u\rangle_X}{\|u\|_X} = \infty$

$\Longrightarrow$ A is surjective ie. $\forall b\in X^*\ \exists u\in X$: $Au=b$

Sketch of a proof:

1) Galerkin approximation: Since X is separable, there is a basis $(w_i)_{i\in\mathbb{N}}$ of X s.t. for $X_n = \operatorname{span}\{w_1,\dots,w_n\}$ it holds

$$X = \overline{\bigcup_{n=1}^{\infty} X_n} = \overline{\operatorname{span}\{w_1,\dots\}}$$

We want to approximate $Au=b$ in the space X_n
$\rightsquigarrow u_n\in X_n$ (show this by Brouwer fixed pt. theorem)
every cont. mapping in a closed ball in $\mathbb{R}^d$ has a fixed pt.

2) A priori estimate We want to show that u_n is bounded.
Since $A: X\to X'$ is coercive, there exists a $R_0>0$ s.t. for any $\|u\|_X > R_0$ it holds:

$$\frac{\langle Au, u\rangle_X}{\|u\|_X} \geq 1 + \|b\|_{X^*}$$

$\langle b, u\rangle \leq \|b\|\cdot\|u\|$

$$\Rightarrow \langle Au, u\rangle_X \geq (1 + \|b\|_{X^*}) \cdot \|u\|_X$$

$$\Rightarrow \underbrace{\langle Au, u\rangle_X - \langle b, u\rangle_X}_{=0} \geq (1 + \|b\|_{X^*})\|u\|_X - \langle b, u\rangle_X$$

$$\geq (1 + \|b\|_{X^*})\, \|u\|_X - \|b\|_{X^*} \|u\|_X$$

$$\geq \|u\|_X > R_0$$

Now, if $u \in X$ with $\|u\|_X > R_0$ is a solution of $Au = b$, it would mean $0 \geq R_0 > 0$ ↯

$\Rightarrow$ $u \in X$ the solution of $Au = b$ fulfills $\|u\|_X \leq R_0$

3) Weak convergence Since X is reflexive, it follows by the Eberlein-Smulian theorem that there exists a weakly converging subsequence $(u_{n_k})_{k \in \mathbb{N}}$ & a limit pt. $\underline{u}$ s.t.

$$u_{n_k} \rightharpoonup \underline{u} \text{ in } X \qquad (f(u_n) \to f(u) \ \forall f \in X^*)$$

4) Existence We need to show that this $\underline{u}$ is a solution of $Au = b$ (Minty trick)

$\langle A u_{n_k}, w\rangle = \langle b, w\rangle \quad \forall w \in X_{n_k}$

we need to prove

$\langle A u, w\rangle = \langle b, w\rangle \quad \forall w \in X$

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Lemma 0.2 (Minty) Let X be a Banach space & $A: X \to X^*$ a hemicontinuous & monotone operator. Then:

(i) If A is maximal monotone ie. for any fixed $u \in X$, $b \in X^*$ it holds

$$\langle b - Av, u - v\rangle_X \geq 0 \quad \forall v \in X$$

$$\Rightarrow Au = b$$

$\langle Au - Av, u - v\rangle \geq 0$ take $b = Au$

(ii) If A type M ie.

$$u_n \rightharpoonup u \text{ in } X$$

$$Au_n \rightharpoonup b \ (= Au) \text{ in } X^*$$

$$\limsup_{n \to \infty} \langle Au_n, u_n\rangle_X \leq \langle b, u\rangle_X$$

$$\Rightarrow Au = b$$

(iii) If $u_n \to u$ in X & $Au_n \rightharpoonup b$ in X^*

or $u_n \rightharpoonup u$ in X & $Au_n \to b$ in X^*

$\Rightarrow Au = b$

Proof (i) Let $u \in X$, $b \in X^*$ be given s.t. A is max. monotone. $w \in X$ arbitrary but fixed. Then we set $v := u - tw$, $t > 0$, $w \in X$.
Then we have

$$\langle b - Av, u - v\rangle \geq 0 \Rightarrow \langle b - A(u + t\cdot(-w)), w\rangle_X \geq 0$$

A is hemicontinuous $\Rightarrow \langle b - Au, w\rangle_X \geq 0$

We can do the same for $v = u + tw \Rightarrow \langle b - Au, w\rangle_X \leq 0$

$\Rightarrow \langle b - Au, w\rangle_X = 0 \; \forall w \in X \Rightarrow b = Au$

(ii) $$0 \overset{A \text{ monotone}}{\leq} \langle Au_n - Av, u_n - v\rangle_X = \langle Au_n, u_n\rangle_X - \langle Av, u_n\rangle_X - \langle Au_n - Av, v\rangle_X$$

Take $\limsup\limits_{n\to\infty}$

$\langle Au_n, u_n\rangle \to \langle b, u\rangle$

$$0 \leq \underbrace{\limsup \langle Au_n, u_n\rangle_X}_{\leq \langle b, u\rangle_X} - \underbrace{\limsup \langle Av, u_n\rangle_X}_{= \langle Av, u\rangle_X \text{ since } u_n \rightharpoonup u} - \underbrace{\limsup \langle Au_n - Av, v\rangle_X}_{= \langle b - Av, v\rangle_X}$$

$$= \langle b - Av, u - v\rangle_X$$

$\Rightarrow A$ max. monotone $\overset{(i)}{\Rightarrow} Au = b$

(iii) follows from Lemma 0.3

Lemma 0.3 Let X be a Banach space.

(i) If $x_n \to x$ in X, then $\|x_n\|_X \leq C \; \forall n \in \mathbb{N}$

(ii) If $x_n \to x$ in X, $f_n \to f$ in X^* then $\langle f_n, x_n\rangle_X \to \langle f, x\rangle_X$

(iii) If $x_n \to x$ in X, $f_n \to f$ in X^* then $\langle f_n, x_n\rangle_X \to \langle f, x\rangle_X$

(iv) Let X be reflexive. Let $(x_n)_n$ be bounded. If all weakly convergent subsequences of (x_n) converge to the same limit point x, then $x_n \to x$ in X

Proof: Functional analysis & see Exercise class

1. Monotone operators

Definition 1.1 Let X be a Banach space, let $A: X \to X^*$. Then A is called:

(i) monotone iff. $\langle Au - Av, u-v \rangle_X \geq 0 \quad \forall u, v \in X$

(ii) strictly monotone iff $\langle Au - Av, u-v \rangle_X > 0 \quad \forall u, v \in X, u \neq v$

(iii) strongly monotone iff $\exists c > 0$ s.t. $\langle Au - Av, u-v \rangle_X \geq c \cdot \|u-v\|_X^2 \quad \forall u, v \in X$
> 0 if $u \neq v$

(iv) coercive iff $\displaystyle\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle_X}{\|u\|_X} = \infty$

It holds: (i) A strongly monotone $\Rightarrow$ A strictly mon. $\Rightarrow$ A mon.

(ii) A strongly monotone, then A is coercive since

$$\langle Au, u \rangle_X = \langle Au + A(0), u \rangle_X - \langle A(0), u \rangle_X$$
$$\geq c\,\|u - 0\|_X^2 - \|A(0)\|_{X^*} \|u\|_X$$

divide by $\|u\|_X$

$$\frac{\langle Au, u \rangle_X}{\|u\|_X} \geq \underbrace{c\,\|u\|_X}_{\to\infty} - \|A(0)\| \longrightarrow \infty \quad \text{for } \|u\|_X \to \infty$$

Example: 1) let $f: \mathbb{R} \to \mathbb{R}$, $X = \mathbb{R}$, $X^* = \mathbb{R}$,

$$\langle f(u) - f(v), u-v \rangle_X = (f(u) - f(v)) \cdot \overbrace{(u-v)}^{\geq 0} \geq 0$$

(i). f (strictly) monotone $\Leftrightarrow$ f (strictly) monotonically increasing

(ii) f coercive $\Leftrightarrow$ $f(u) \to \pm\infty$ if $u \to \pm\infty$

2) $g: \mathbb{R} \to \mathbb{R}$, $g(u) = \begin{cases} |u|^{p-2} u, & u \neq 0 \\ 0 & u = 0 \end{cases}$

Exercise: (i) g is strictly monotone for $p > 1$

(ii) $\langle g(u) - g(v), u - v \rangle \geq c \cdot |u-v|^p$ for $p \geq 2$

(iii) g strongly mon. for $p = 2$

Definition 1.2

Let X, Y be Banach spaces, $A: X \to Y$. Then A is called

(i) **completely continuous** iff

$$u_n \rightharpoonup u \text{ in } X \implies Au_n \to Au \text{ in } Y$$

(A is weak-strong continuous)

Ex: $Id: H^1(\Omega) \to L^2(\Omega)$, $u_n \rightharpoonup u$ in $H^1(\Omega)$ $\implies u_n \to u$ in $L^2(\Omega)$

(ii) **demicontinuous** iff

$$u_n \to u \text{ in } X \implies Au_n \rightharpoonup Au \text{ in } Y$$

(A is strong-weak convergent)

(iii) **hemicontinuous** if $Y = X^*$ and $t \mapsto \langle A(u+tv), w\rangle_X$ is continuous in $[0,1]$ for any $u, v, w \in X$

(A weakly continuous)

(iv) **bounded** iff A maps bounded sets in X to bounded sets in Y

(v) **locally bounded** iff $\forall u \in X \; \exists \varepsilon(u) > 0, K(u) : \|Av\|_Y \le K$ for any $v \in X$ with $\underline{\|u-v\|_X < \varepsilon}$

strong - strong,
weak - weak conv.

A is monotone $\Longleftarrow$

A compl. cont. $\implies$ A cont. $\implies$ A demicontinuous $\implies$ A hemicontinuous

A bounded $\implies$ A locally bounded.

Lemma 1.3

Let X be a reflexive Banach space, $A: X \to X^*$

(i) If A completely continuous, then A is compact.

(ii) If A demicontinuous, then A is locally bounded.

(iii) If A monotone, then A is locally bounded.

(iv) If A monotone & hemicontinuous, then A is demicontinuous.

Proof (i) compact $\triangleq$ relatively compact wrt sequence

We want to show that for every bdd subset $M \subseteq X$ the image $A(M)$ is relatively compact wrt sequences.

Let $(Au_n)_n \subseteq A(M)$. Since M is bounded, we know $(u_n)_n$ is bounded. By Eberlein-Smulian theorem (bc. X refl. BS) $\exists (u_{n_k})_k, u \in X$ s.t. $u_{n_k} \rightharpoonup u$ in $X \implies Au_{n_k} \to Au$ in $X \implies A(M)$ is relatively compact.

(ii) Proof by contradiction: Let A not be locally bounded i.e. $\exists u \in X$ & a sequence $(u_n) \subseteq X$ s.t. $u_n \to u$ but $\|Au_n\|_{X^*} \to \infty$.

However, A is demicont., it holds $Au_n \to Au$ in X^*. Thus, $\|Au_n\| \le C$ ↯

(iii) Proof by contradiction: Let A not be locally bounded, then there is a $u \in X$ & a sequence $(u_n)_n \subseteq X$ s.t. $u_n \to u$ in X but $\|Au_n\|_{X^*} \to \infty$

We define:
$$a_n = (1 + \|Au_n\|_{X^*} \cdot \|u_n - u\|_X)^{-1} > 0 \qquad \frac{1}{1+\dots}$$
$$\le 1$$

Since A is monotone, we know $\forall v \in X$
$$0 \le \langle Au_n - Av, u_n - v \rangle_X$$
$$= \langle Au_n - Av, (u_n - u) + (u - v) \rangle_X$$

$$\Rightarrow \; a_n \cdot \langle Au_n, \underbrace{v - u}_{u-v} \rangle \le a_n \left(\langle Au_n, u_n - u \rangle_X - \langle Av, u_n - v \rangle \right)$$
$$\le \underbrace{a_n \|Au_n\|_{X^*} \cdot \|u_n - u\|_X}_{\le 1} + \underbrace{a_n}_{\le 1} \underbrace{\|Av\|_{X^*}}_{=C_1} (\underbrace{\|u_n\|}_{\le C_2} + \underbrace{\|v\|}_{=C_3})$$
$$\le 1 + c(v, u)$$

Since this holds for any $v \in X$, we can select $v = 2u - v$
$$\Rightarrow \; - a_n \langle Au_n, v - u \rangle_X \le 1 + c(v,u) \qquad \forall n$$
$$\Rightarrow \; \sup_n | a_n \langle Au_n, w \rangle_X | \le \tilde{c}(w, u) < \infty$$

Uniform boundedness principle (linear & cont. operator $a_n Au_n : X \to \mathbb{R}$ are pointwise bounded) tells us that
$$\sup_n \|a_n Au_n\|_{X^*} \le c(u) < \infty$$

$$\Rightarrow \; \|Au_n\|_{X^*} \le \frac{c(u)}{a_n} = c(u) \cdot (1 + \|Au_n\|_{X^*} \cdot \underbrace{\|u_n - u\|_X}_{\to 0}) \le c(u) + \tfrac{1}{2} \|Au_n\|_{X^*}$$

Since $\|u_n - u\|_X \to 0$, there is a $n_0 \in \mathbb{N}$ s.t. $c(u) \cdot \|u_n - u\|_X < \frac{1}{2}$ $\forall n \ge n_0$

$$\Rightarrow \; \|Au_n\|_{X^*} \le 2c(u) \quad ↯$$

(iv) Let $(u_n)_n \subseteq X$ be a sequence with $u_n \to u \in X$.

A is monotone $\overset{(iii)}{\Longrightarrow}$ A locally bounded $\Rightarrow \|Au_n\|_{X^*} \le C$

$\underset{\text{Smulian}}{\overset{\text{Eberlein}}{\Longrightarrow}}$ $\exists (u_{n_k})$ s.t. $Au_{n_k} \rightharpoonup b$ in X^*

1) Minty's trick: $b = Au$
(Lemma 0.2 (iii))

2) Every subsequence of $(Au)_n$ is converging weakly to Au

since otherwise there would be a subsequence $(Au_{n_\ell})_\ell \subseteq X^*$ s.t.

$Au_{n_\ell} \longrightarrow c \neq b$ in X^*

$\overset{\text{Minty}}{\Longrightarrow}$ $c = Au$ ↯ $Au = b$

$\Rightarrow$ every subsequence is converging weakly to $b = Au$

$\langle A\underset{u-v}{u}, \underset{u-v}{u} \rangle \geq c\|\underset{u-v}{u}\|_X \geq 0$

$\langle \underbrace{Au - Av}_{A(u-v)}, u-v \rangle \geq 0$

2. Theorem of Browder – Minty

Theorem 2.1 (Browder-Minty) Let X be a separable & reflexive BS. Further, let $A: X \to X^*$ be monotone, coercive, hemicontinuous. Then for any $b \in X^*$ there is a solution $u \in X$ of

$$Au = b. \qquad \langle Au - b, w \rangle_X = 0 \quad \forall w \in X$$

The solution set is closed, bounded & convex. If A is strictly monotone, then the solution of $Au = b$ is unique.

Proof Since X is separable, there is a basis $(w_i)_{i\in\mathbb{N}}$ of X. We set $X_n = \text{span}(w_1, \dots, w_n)$ & we look for an approximate solution $u_n \in X_n$ of the form $u_n = \sum_{j=1}^n c_n^j w_j$ to the Galerkin system

$$\langle Au_n - b, w_k \rangle_X = 0 \quad \forall k = 1, \dots, n \tag{G}$$

$$Au_n - \Pi_n b = 0, \quad \Pi_n : X, X^* \to X_n \text{ orth. projection}$$

① (G) has a solution. Define $c_n = \begin{pmatrix} c_n^1 \\ \vdots \\ c_n^n \end{pmatrix} \in \mathbb{R}^n$, $|c_n| := \|\sum_{j=1}^n c_n^j w_j\|_X$

$$g_n^k(c_n) = \langle A\Big(\sum_{j=1}^n c_n^j w_j\Big) - b, w_k \rangle_X = 0 \quad \forall k = 1, \dots, n$$

$g_n^k : \mathbb{R}^n \to \mathbb{R}$

We look for a vector c_n s.t. $g_n^k(c_n) = 0 \quad \forall k = 1, \dots, n$

$$\longrightarrow \underbrace{\begin{pmatrix} g_n^1(c_n) \\ \vdots \\ g_n^n(c_n) \end{pmatrix}}_{= g_n(c_n)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \mathbf{0}$$

A is monotone & hemicontinuous $\Rightarrow$ A demicontinuous by Lemma 1.3(iv)

$\Rightarrow g_n : \mathbb{R}^n \to \mathbb{R}^n$ is continuous since if $c_\ell \to c$ as $\ell \to \infty$ w.r.t. $|\cdot|$ in $\mathbb{R}^n$ then $\sum_{j=1}^{n} c_\ell^j w_j \longrightarrow \sum_{j=1}^{n} c^j w_j$ in X

$\Rightarrow g_n(c_\ell) \longrightarrow g_n(c) \Rightarrow g_n$ is continuous

Brouwer: Every continuous mapping of a closed ball in $\mathbb{R}^n$ into itself has a fixed pt.

Corollary: Let $g = (g^1, \dots, g^n) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, that fulfills $\exists R > 0 : \sum_{i=1}^{d} g^i(x) x_i = (g(x), x) \geq 0 \;\; \forall x$ with $|x| = R$.

Then there exists a solution x_0 of $g(x_0) = 0$ with $|x_0| \leq R$.

Proof: Proof by contradiction. Let $g(x_0) = 0$ have no solution in $\overline{B_R(0)} \subseteq \mathbb{R}^n$. We define $f^i(x) = -R \cdot \frac{g^i(x)}{|g(x)|}$, $i = 1, \dots, n$

Since $g(x_0) = 0$ has no solution, it holds $|g(x)| > 0 \;\; \forall x \in \overline{B_R(0)}$

$\Rightarrow f = (f^1, \dots, f^n)^T$ is well-defined, continuous, maps the closed ball $\overline{B_R(0)}$ into itself.

$$\overset{\text{Brouwer}}{\Longrightarrow} \exists x^* \in \overline{B_R(0)} \text{ s.t. } x^* = f(x^*)$$

$$\& \;\; |x^*| = |f(x^*)| = \left| -R \frac{g(x^*)}{|g(x^*)|} \right| = R$$

$$\Rightarrow 0 \leq \sum_{i=1}^{n} g^i(x^*) \cdot x_i^* = -\sum_{i=1}^{n} \underbrace{f^i(x^*)}_{= x_i^*} \cdot x_i^* \frac{|g(x^*)|}{R}$$

$$= -\underbrace{|x^*|^2}_{= R^2} \cdot \frac{|g(x^*)|}{R} = -R|g(x^*)| < 0 \;\; ⚡$$

Let $c = (c^1, \dots, c^n)^T$, $v = \sum_{j=1}^{n} c^j w_j$

$$\sum_{k=1}^{n} g_n^k(c) \cdot c^k \underset{\text{Def. of } g_n^k}{=} \langle Av, v\rangle_X - \underline{\langle b, v\rangle_X}$$

$\langle b, v\rangle_X \leq \|b\|_{X^*} \cdot \|v\|_X$

$-\langle b, v\rangle_X \geq -\|b\|_{X^*} \cdot \|v\|_X$

Since A is coercive i.e. $\frac{\langle Aw, w\rangle_X}{\|w\|_X} \to \infty$ as $\|w\|_X \to \infty$

$\Rightarrow \exists R_0 > 0$ s.t. $\forall \|w\|_X \geq R_0$ it holds $\langle Aw, w\rangle_X \geq \|b\|_{X^*} \cdot \|w\|_X$

Hence, for any c with $|c| = \|v\|_X = R_0$ it holds

$$\langle Av, v\rangle_X \geq \|b\|_{X^*} \|v\|_X$$

$$\Rightarrow \sum_{k=1}^{n} g_n^k(c) \cdot c^k \geq \|b\|_{X^*} \|v\|_X - \|b\|_{X^*} \|v\|_X = 0$$

Corollary of Brouwer: $\exists u_n$ of (G) with $\|u_n\|_X \le R_0$ — a priori estimate, independent of n

$\rightsquigarrow u_n \rightharpoonup u$ in X

② Boundedness of $(Au_n)_n$

A is monotone. Thus, it is locally bounded by Lemma 1.3. Then, there are constants $r, M > 0$ s.t. it holds

$$\|w\|_X \le r \implies \|Aw\|_{X^*} \le M.$$

We proved that u_n is a solution to (G) ie. $\langle Au_n, u_n\rangle_X = \langle b, u_n\rangle_X$

$$\implies |\langle Au_n, u_n\rangle_X| \le \|b\|_{X^*} \cdot \|u_n\|_X \le R_0 \cdot \|b\|_{X^*}$$

Moreover, A is monotone so it holds:

$$\langle Au_n - Aw, u_n - w\rangle_X \ge 0$$

$$\implies \langle Au_n, w\rangle_X \le \langle Au_n, u_n\rangle_X - \langle Aw, u_n\rangle_X + \langle Aw, w\rangle_X$$

$$\|Au_n\|_{X^*} = \sup_{\|w\|_X \le r} \frac{1}{r} \langle Au_n, w\rangle_X$$

$$\le \sup_{\|w\|_X \le r} \frac{1}{r} \left(\langle Au_n, u_n\rangle_X - \langle Aw, u_n\rangle_X + \langle Aw, w\rangle_X\right)$$

$$\le \frac{1}{r}\left(R_0 \|b\|_{X^*} + MR_0 + Mr\right) < \infty$$

③ Convergence of the Galerkin method

X & X^* reflexive, so use the Eberlein–Šmulian theorem to infer a subsequence $(u_{n_k}) \subseteq X$ s.t. it holds

1. $u_{n_k} \rightharpoonup u$ in X
2. $Au_{n_k} \rightharpoonup c$ in X^*, $c \in X^*$

$$\boxed{Au = b}$$

We want to show $b = c$.

$$\langle Au_{n_k}, w\rangle_X = \langle b, w\rangle_X \quad \forall w \in X_{n_k} = \operatorname{span}\{w_1, \dots, w_{n_k}\}$$

Taking limit $k \to \infty$: $\quad \langle c, w\rangle_X = \langle b, w\rangle_X \quad \forall w \in \bigcup_{j=1}^{\infty} \operatorname{span}\{w_1, \dots, w_j\} = \bigcup_{j=1}^{\infty} X_j$

BLT theorem $\Rightarrow$ $\underset{X^*}{\langle c, w\rangle} = \underset{X^*}{\langle b, w\rangle_X}$ $\forall w \in \overline{\bigcup_{j=1}^{\infty} X_j} = \underset{X \text{ separable.}}{X}$

$\Rightarrow b = c$

$\Rightarrow Au_{nk} \longrightarrow b$ in X^*

Want to show that A is type M

3 $\limsup_{n\to\infty} \langle Au_{nn}, u_{nn}\rangle_X \underset{\text{a solution of (0)}}{\overset{u_{nn} \text{ is}}{=}} \limsup_{n\to\infty} \langle b, u_{nn}\rangle_X \overset{u_{kn}\to u \text{ in } X}{=} \langle b, u\rangle_X$

Minty's trick (Lemma 0.2): $\Rightarrow Au = b$

$\Rightarrow u$ is a solution

④ <u>Uniqueness</u> Let A be strictly monotone. If there are two solutions $u \neq v$, then we have

$Au = b = Av$ &

$$0 < \langle Au - Av, u - v\rangle_X = \langle b - b, u - v\rangle_X = 0 \ \lightning$$

⑤ <u>Solution set $S = \{u \in X : Au = b\}$ is closed, convex, bounded</u>

(a) S is non-empty; ✓

(b) S is convex. Let $u_1, u_2 \in S$ so that $Au_1 = b = Au_2$,

convex combination $w = tu_1 + (1-t)u_2$, $t \in [0,1]$,

then for any $v \in X$ it holds

$$0 \underset{\text{Minty}}{\overset{\text{! goal}}{\leq}} \langle b - Av, w - v\rangle_X = \langle b - Av, tu_1 + (1-t)u_2 \overbrace{- tv + (1-t)v}^{v}\rangle_X$$

$$= \langle b - Av, t\cdot(u_1 - v)\rangle_X + \langle b - Av, (1-t)(u_2 - v)\rangle_X$$

$$= t\cdot\langle Au_1 - Av, u_1 - v\rangle_X + (1-t)\langle Au_2 - Av, u_2 - v\rangle_X$$

≥ 0 ≥ 0

≥ 0

$\overset{\text{Minty}}{\underset{\text{Lemma 0.2}}{\Longrightarrow}}$ $Aw = b \implies w \in S \implies S$ convex

(c) S is bounded: If S is not bounded, then for any $R > 0$ $\exists u \in S : \|u\|_X \geq R > 0$

This implies:

$$0 = \langle Au, u\rangle_X - \underline{\langle b, u\rangle_X}$$

$$\geq (\|b\|_{X^*} + 1)\, \|u\|_X - \|b\|_{X^*} \|u\|_X = \|u\|_X$$

$\dfrac{\langle Au, u\rangle_X}{\|u\|_X} \geq \underline{\|b\|_{X^*} + 1}$

(d) S closed: Take $(u_n) \subseteq S$ so it holds $Au_n = b$ $\forall n \in \mathbb{N}$ & we know that $u_n \to u$ in X

We have to show that $u \in S$ (ie. $Au = b$)

$$\langle b - Av, u - v\rangle_X \overset{u_n \to u}{=} \lim_{n \to \infty} \langle b - Av, u_n - v\rangle_X$$

$$= \lim_{n\to\infty} \langle Au_n - Av, u_n - v\rangle_X \geq 0 \quad \forall v \in X$$

A is monotone

$\overset{\text{Minty's trick}}{\underset{\text{Lemma 0.2}}{\Longrightarrow}}$ $Au = b \implies u \in S$

$\square$

Remark: This holds also for non-separable spaces, see 3.2.

Corollary 2.2 X separable & reflexive BS, $A: X \to X^*$ strictly monotone, coercive, hemicontinuous. Then there exists the operator $A^{-1}: X^* \to X$ & it is strictly monotone & demicontinuous.

Proof: Exercise

Carathéodory: $x \in W^{1,1}(0,1) \subseteq C([0,1])$

$x'(t) = f(t, x(t))$
$= (Fx)(t)$

$F: \overset{\text{domain}}{L^p(\Omega)} \to L^p(\Omega)$

$u \in L^p(\Omega)$, $Fu \in L^p(\Omega)$

$(Fu)(x)$

3. Nemychii operators

Definition We call $(Fu)(x) := f(x,u(x))$ a Nemychii operator for $u = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} : G \subseteq \mathbb{R}^N \to \mathbb{R}^n$ with a domain $G \subseteq \mathbb{R}^n$ (bounded, connected) if $f: G \times \mathbb{R}^n \to \mathbb{R}$ fulfills:

(i) Caratheodory conditions:
- $f(\cdot,\eta) : x \mapsto f(x,\eta)$ is measurable on G $\forall \eta \in \mathbb{R}^n$
- $f(x,\cdot) : \eta \mapsto f(x,\eta)$ is continuous on $\mathbb{R}^n$ for a.e. x in G

(ii) growth condition

$$|f(x,\eta)| \le |a(x)| + b \sum_{i=1}^{n} |\eta^i|^{p_i/q}$$

?

where $b>0$, $a \in L^q(G)$, $1 \le q < \infty$, $p_i \in [1,\infty)$, $i = 1,\dots,n$

$L^\infty \subseteq L^4 \subseteq L^2 \subseteq L^1$

should hold still for $\le$

Lemma 3.1 With the assumptions on f, a, in the definition, the Nemychii operator $F : \prod_{i=1}^{n} L^{p_i}(G) \to L^q(G)$ is continuous & bounded. Moreover, it holds

$F(u^1,\dots,u^n)$, $u^i \in L^{p_i}(G)$

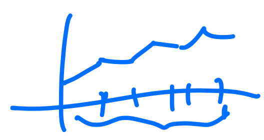
$$\underbrace{\|Fu\|_q}_{= \|Fu\|_{L^q(G)}} \le c\left(\|a\|_q + \sum_{i=1}^{n} \|u^i\|_{p_i}^{p_i/q}\right)$$

Proof We only consider $n = 1$, $u = u_1$, $p = p_1$

1) Measurability of Fu: Since $u \in L^p(G)$, we know that $x \mapsto u(x)$ is Lebesgue measurable on G. Then there is a sequence of step functions (u_n) s.t.

$$u_n \to u \quad \text{a.e. on } G$$

$$u_n = \sum_{j=0}^{M(n)} \underbrace{c_j^n}_{\text{coefficients}} \chi_{G_j^n}(\cdot)$$



G

Then it holds for a.e. $x \in G$

$$(Fu)(x) = f(x, u(x)) = \lim_{n\to\infty} f(x, u_n(x))$$

f is continuous in 2nd component

$$f(x, u_n(x)) = f\left(x, \sum_{j=0}^{M(n)} c_j^n \chi_{G_j^n}(x)\right) = \sum_{j=0}^{M(n)} f(x, c_j^n)\, \chi_{G_j^n}(x)$$

$\Rightarrow$ Fu is measurable.

2) Boundedness of F:

$$\|Fu\|_q^q = \int_G |f(x, u(x))|^q \, dx$$

$$\underset{\text{estimate}}{\overset{\text{growth}}{\leq}} \int_G \left(|a(x)| + b\,|u(x)|^{p/q}\right)^q dx$$

$$\leq C \int_G |a(x)|^q + b^q |u(x)|^p \, dx$$

$$\leq C\left(\|a\|_q^q + \|u\|_p^p\right)$$

$(a-b)^2 \geq 0$

$a^2 + b^2 \geq 2ab$

$\Rightarrow (a+b)^2 = a^2 + 2ab + b^2$

$\leq 2a^2 + 2b^2$

$\rightsquigarrow (a+b)^q \leq C(a^q + b^q)$

2^{q-1}

3) Continuity of F: $\quad F: L^p(G) \to L^q(G)$

Let $(u_n) \subset L^p(G)$ be s.t. $u_n \to u$ in $L^p(G)$

Show: $\quad Fu_n \to Fu$ in $L^q(G)$

Lebesgue dominated convergence theorem:

g_n measurable with $g_n \to g$ a.e.

& $|g_n| \leq h \in L^1$

$\Rightarrow \int g_n \to \int g$

or $\int |g_n - g| \to 0$

$$|(Fu_n)(x) - Fu(x)|^q$$

$$= |f(x, u_n(x)) - f(x, u(x))|^q$$

$$\leq C\left(|f(x, u_n(x))|^q + |f(x, u(x))|^q\right)$$

$(Fu)(x) \in L^q(G)$

growth condition

$$\leq \underbrace{C\left(|a(x)|^q + b^q |u_n(x)|^p + |f(x, u(x))|^q\right)}_{=: h_n(x)}$$

$$\int |h_n| \leq \underbrace{\int |a(x)|^q}_{\|a\|_q^q} + \underbrace{\int |u_n(x)|^p}_{\|u_n\|_p^p}$$

For h_n it holds:

- $(h_n) \subset L^1(G)$
- $h_n(x) \to h(x)$ a.e. in G
- $\int_G h_n(x)\,dx \to \int_G h(x)\,dx$

$g_n^{(x)} - g(x) = |Fu_n(x) - Fu(x)|^q$

However, we also know $|(Fu_n)(x) - F(u(x))|^q \to 0$ for a.e. $x \in G$ since f is continuous in the 2nd component

Generalized Lebesgue dominated convergence thm:
g_n measurable & $g_n \to g$ a.e. & $|g_n| \le h_n$ $\forall n$ for $h_n \in L^1(G)$ $\forall n$, $h_n(x) \to h(x)$ a.e. & $\int h_n(x) \to \int h(x)$

$$\Longrightarrow \int_G g_n(x) \to \int_G g(x)$$

$$\Longrightarrow 0 \leftarrow \int_G |g_n^{(x)} - g(x)| = \int |(Fu_n)(x) - (Fu)(x)|^q$$

$$\Longrightarrow Fu_n \to Fu \text{ in } L^q(G)$$

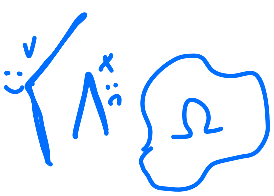
$$\Longrightarrow F \text{ is continuous} \qquad \square$$

$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|^{2-p}+\varepsilon}\right)$ $\boxed{\varepsilon \to 0}$

[BM $\Rightarrow$ LAx ?]

4 Quasilinear elliptic Equations

We study the following BVP (boundary value problem)



$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + su = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \qquad (Q)$$

Here, $p \in (1,\infty)$, Ω bdd Lipschitz domain, $s \geq 0$

If we formally multiply the PDE by u, integrate over Ω & use integration by parts, we get the apriori estimate

$$\int_\Omega \underbrace{|\nabla u|^{p-2}|\nabla u|^2}_{|\nabla u|^p} + s|u|^2 \leq c(f)$$

From here, we see that the canonical Sobolev space is $W_0^{1,p}(\Omega) = \{u \in \underbrace{W^{1,p}(\Omega)}_{\substack{u\in L^p(\Omega)\\ \nabla u \in L^p(\Omega)}} : u = 0 \text{ on } \partial\Omega\}$

$p \geq \frac{2d}{d+2} \overset{d=3}{=} \frac{6}{5}$

$\|u\|_{W_0^{1,p}\cap L^2} = \max\{\|u\|_{W_0^{1,p}}, \|u\|_{L^2}\}$

However, for $s>0$ one might consider $\underbrace{W_0^{1,p}(\Omega) \cap L^2(\Omega)}_{\neq W_0^{1,p}(\Omega)}$

$\ulcorner$ $\Omega = (0,1)$: $W_0^{1,1}(\Omega) \hookrightarrow L^\infty(\Omega) \subseteq L^2(\Omega)$ $\lrcorner$

Weak formulation: For a given $f \in L^p(\Omega)^* \cong L^{p'}(\Omega)$, $\frac{1}{p'}+\frac{1}{p}=1$, we look for a $u \in X = W_0^{1,p}(\Omega)$ s.t.

$$(Q_{weak}) \qquad \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi + s u \varphi \, dx = \int f\varphi \, dx$$

for any $\varphi \in X$.

We define an operator A by

$$\langle Au, \varphi\rangle_X := \int_\Omega \underbrace{|\nabla u|^{p-2}\nabla u}_{=g(\nabla u)} \cdot \nabla\varphi + s u \varphi \, dx$$

& b by

$$\langle b, \varphi\rangle_X := \int f\varphi \, dx \quad \forall \varphi \in X$$

Weak form: $\quad Au = b \text{ in } X^* \qquad (Q_{op})$

$W^{1,p} \subseteq L^2$ for $2 \leq \frac{np}{n-p} \Rightarrow 2n - 2p \leq np$

$\Rightarrow 2n \leq p\cdot(n+2)$

$\Rightarrow p \geq \frac{2n}{n+2} = \frac{6}{5}$

$W^{1,p} \subset L^q$ for $q \leq \frac{np}{n-p}$

then $W_0^{1,p} \cap L^2 = W_0^{1,p}$

Remark If $p \in (1,2)$, then $|\nabla u(x)|^{p-2}$ is not defined ($p > 1$)
for any $x \in \Omega$ with $\nabla u(x) = 0$. Therefore, we write

$$g = \begin{pmatrix} g^1 \\ \vdots \\ g^d \end{pmatrix} : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$$
$$\xi \mapsto |\xi|^{p-2}\xi$$

which can be extended to the whole $\mathbb{R}^d$ by setting $g(0) = 0$.

$$\frac{|\xi|}{|\xi|^{2-p}} = |\xi|^{1-2+p} = |\xi|^{\overbrace{p-1}^{>0}} \longrightarrow 0 \quad \text{if } |\xi| \to 0$$

$\langle G, \varphi \rangle = \int f \varphi \leq \|f\|_{W^{-1,p}} \|\varphi\|_{W_0^{1,p}}$

Lemma 4.1 Ω bdd domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$.
Further, let $f \in L^{p'}(\Omega)$ ($W^{-1,p'}(\Omega)$), $p' = \frac{p}{p-1}$, $p \in (1,\infty)$, $s \geq 0$. For $p \geq \frac{2d}{d+2}$ (Sobolev conjugate of 2, $W^{1,p} \hookrightarrow L^2$)
the operator $A: X \to X^*$ is bounded. & $b \in X^*$. Further, the weak form of (Q) is equivalent to (Qop) i.e. $Au = b$ in X^*.

Proof $X = W_0^{1,p}(\Omega)$, $\|u\|_X := \|\nabla u\|_p$ is equivalent to $\|u\|_{1,p}$ since

$$\|\nabla u\|_p^p \leq \|u\|_{1,p}^p = \|u\|_p^p + \|\nabla u\|_p^p \overset{\text{Poincaré}}{\leq} C\|\nabla u\|_p^p + \|\nabla u\|_p^p$$
$$\leq \max\{C,1\} \cdot \|\nabla u\|_p^p$$

1) $A: X \to X^*$. For any $u, \varphi \in X$ it holds

$$|\langle Au, \varphi \rangle_X| \leq \int |\nabla u|^{p-1} |\nabla \varphi| \, dx + s \int |u\varphi| \, dx$$
$$\overset{\text{Hölder}}{\leq} \left(\int_\Omega |\nabla u|^{\underbrace{(p-1)p'}_{=p}} dx \right)^{1/p'} \left(\int_\Omega |\nabla \varphi|^p dx \right)^{1/p} + s\|u\|_2 \|\varphi\|_2$$
$$= \|\nabla u\|_p^{p-1} \|\nabla \varphi\|_p + s\|u\|_2 \|\varphi\|_2$$

For $q \leq \frac{dp}{d-p}$ it holds $X = W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ & such that $X \hookrightarrow L^2(\Omega)$
because $p \geq \frac{2d}{d+2}$

If $p \geq d$, we use that $X \overset{p \geq d}{\hookrightarrow} W^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$

$$\Rightarrow \|\varphi\|_2 \leq c_1 \|\varphi\|_X = c_1 \|\nabla \varphi\|_p$$

$\Rightarrow \quad |\langle Au, \varphi \rangle_X| \leq c \left(\|Du\|_p^{p-1} + s\, \|Du\|_p \right) \|D\varphi\|_p$

$\Rightarrow \quad \|Au\|_{X^*} = \sup_{\|\varphi\|_X \leq 1} |\langle Au, \varphi \rangle| \leq c \left(\|Du\|_p^{p-1} + s \|Du\|_p \right)$

$\Rightarrow \quad Au \in X^*$ & A bounded

2) $\|b\|_{X^*} = \sup_{\|\varphi\|_X \leq 1} |\underbrace{\langle b, \varphi \rangle_X}_{= \int f \varphi}| \leq \sup_{\|\varphi\|_X \leq 1} \|f\|_{p'} \|\varphi\|_p \leq c \, \|f\|_{p'}$

3) The weak form of (Q) reads

$$\langle Au, \varphi \rangle_X = \langle b, \varphi \rangle_X \quad \forall \varphi \in X$$

$$\Leftrightarrow \quad Au = b \quad \text{in } X^* \qquad \square$$

Remark For $s = 0$ we do not need the assumption $p \geq \frac{2d}{d+2}$

& we also do not need it in the case when working with

$X = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ for $s > 0$

Lemma 4.2 $A : X \to X^*$ is strictly monotone, coercive & continuous

Proof

1) A is strictly monotone: $\quad g(\xi) = \begin{cases} |\xi|^{p-2} \xi, \\ 0, \quad \xi = 0 \end{cases}, \quad g \in C(\mathbb{R}^d)$

$g^i(\xi) = |\xi|^{p-2} \xi^i, \quad \partial_j g^i(\xi) = |\xi|^{p-2} \delta_{ij} + (p-2) |\xi|^{p-4} \xi^i \xi^j$

$\frac{1}{\sqrt{x^2 + y^2}} = (x^2 + y^2)^{-\frac{1}{2}}$

$\frac{-1/2 x}{(x^2 + y^2)^{3/2}} = \frac{-x}{|x|^3}$

$\langle Au - Av, u - v \rangle_X$

$= \int_\Omega \sum_{j=1}^d \left(g^i(Du) - g^i(Dv) \right) (\partial_i u - \partial_i v) \, dx + \underbrace{s \int_\Omega |u - v|^2 \, dX}_{\geq 0}$

$\vdots$

$> 0 \quad \forall u \neq v \in X$

$\eta \in \mathbb{R}^d$

$(*) \quad \sum_{i,j=1}^d \partial_j g^i(\xi) \eta^i \eta^j = |\xi|^{p-2} \left(|\eta|^2 + \underbrace{(p-2) \frac{(\xi \cdot \eta)^2}{|\xi|^2}}_{\geq 0 \text{ for } p \geq 2} \right)$ & $1 < p < 2$: $\frac{\xi_i^2 |\eta_i|^2}{|\xi|^2} \leq \eta_i^2$

$\geq \min(1, p-1) \, |\xi|^{p-2} |\eta|^2$

Goal: $g(\xi) - g(\eta) \overset{\text{FTC}}{=} \int_0^1 Dg(\eta + \tau \cdot (\xi - \eta)) \cdot (\xi - \eta) \, d\tau$

Since g is only in $C(\mathbb{R}^d) \cap C^1(\mathbb{R}^d \setminus \{0\})$, we approximate g by

$g_\varepsilon(\xi) = (\varepsilon^2 + |\xi|^2)^{\frac{p-2}{2}} \xi, \quad g_\varepsilon \in C^1(\mathbb{R}^d)$

it holds $g_\varepsilon(\xi) \to g(\xi)$ $\forall \xi \in \mathbb{R}^d$, $\varepsilon \to 0$,

$Dg_\varepsilon(\xi) \to Dg(\xi)$ $\forall \xi \in \mathbb{R}^d \setminus \{0\}$

$|Dg_\varepsilon(\xi)| \le C(p,d)\, |\xi|^{p-2}$ $\forall \xi \in \mathbb{R}^d$ $\quad \rightarrow z$

Exercise:

$$c(|\xi|+|\eta|)^{p-2} \le \int_0^1 |\xi + \tau(\eta-\xi)|^{p-2}\, d\tau \le C(|\xi|+|\eta|)^{p-2}$$

$\overset{\text{FTC}}{\Longrightarrow}$
$$g_\varepsilon(\xi) - g_\varepsilon(\eta) = \int_0^1 \frac{d}{d\tau} g_\varepsilon(\eta + \tau(\xi-\eta))\, d\tau = \int_0^1 Dg_\varepsilon(\eta+\tau(\xi-\eta))\cdot(\xi-\eta)\, d\tau$$

$\overset{\varepsilon\to 0}{\Longrightarrow}$ (Lebesgue dominated convergence theorem)
$$g(\xi) - g(\eta) = \int_0^1 Dg(\eta+\tau(\xi-\eta))\cdot(\xi-\eta)\, d\tau$$

$$\boxed{f(x) - f(y) = \int_y^x f'(z)\, dz}$$
weak FTC (exercise 2) $\quad f \in C^1$, $f \in W^{1,1}$

Continue: $\forall u \neq v \in X$ it holds (Dirichlet!)

$$\langle Au - Av, u-v\rangle_X = \int_\Omega \sum_{i=1}^d \big(g^i(\underbrace{Du}_{=\xi}) - g^i(\underbrace{Dv}_{=\eta})\big)(\partial_i u - \partial_i v)\, dx + \underbrace{s\int_\Omega |u-v|^2\, dx}_{\ge 0}$$

$$\ge \int_\Omega \int_0^1 \sum_{i,j=1}^d \partial_j g^i(Dv + \tau(Du-Dv))\cdot(\partial_j u - \partial_j v)\cdot(\partial_i u - \partial_i v)\, d\tau\, dx$$

$$\overset{(*)}{\ge} c\int_\Omega |Du-Dv|^2 \int_0^1 |\underbrace{Dv}_{=\xi} + \tau(\underbrace{Du}_{=\eta} - Dv)|^{p-2}\, d\tau\, dx$$

$$\overset{\text{Ex.}}{\ge} c\int_\Omega |Du-Dv|^2 (|Du| + |Dv|)^{p-2}\, dx > 0$$

$\Longrightarrow$ A strictly monotone $\qquad \left| |Du|^{-1} Du \right| = 1$, $u \in L^2$

2) **A coercive**

$$\langle Au, u\rangle_X = \|Du\|_p^p + \overbrace{s\|u\|_2^2}^{\ge 0} \ge \|Du\|_p^p$$

$$\Longrightarrow \frac{\langle Au, u\rangle_X}{\|u\|_X} \ge \|Du\|_p^{p-1} \to \infty \quad \text{for } \|Du\|_p = \|u\|_X \to \infty \ \& \ p > 1$$

3) A continuous: $u_n \subset X$ s.t. $u_n \to u$ in X i.e. $\nabla u_n \to \nabla u$ in L^p

show: $Au_n \to Au$ in X^*

$$F(\nabla u)(x) := g(\nabla u(x))$$

since it holds $|g^1(\xi)| = |\xi|^{p-1} = |\xi|^{p/p'}$ for $p' = \frac{p}{p-1}$,

we know that F is a Nemychii operator

Lem. 1.20 $\Rightarrow$ $F: L^p(\Omega)^d \to L^{p'}(\Omega)^d$ is continuous

$\Rightarrow$ $F(\nabla u_n) \to F(\nabla u)$ in $L^{p'}(\Omega)^d$

$$\Rightarrow \langle Au_n - Au, \varphi \rangle_X = \int_\Omega (F(\nabla u_n) - F(\nabla u)) \cdot \nabla \varphi \, dx + s \int_\Omega (u_n - u) \varphi \, dx$$

$$\leq \| F(\nabla u_n) - F(\nabla u) \|_{p'} \cdot \underbrace{\| \nabla \varphi \|_p}_{= \|\varphi\|_X} + s \| u_n - u \|_2 \underbrace{\|\varphi\|_2}_{\leq c \|\varphi\|_X}$$

$$\leq c \Big(\underbrace{\| F(\nabla u_n) - F(\nabla u) \|_{p'}}_{\to 0} + \underbrace{\| u_n - u \|_X}_{\to 0} \Big) \cdot \| \varphi \|_X$$

$\Rightarrow$ $Au_n \to Au$ in X^* $\square$

Theorem 4.3 Ω bdd domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, $s \geq 0$, $p \in (1, \infty)$. For any $f \in L^{p'}(\Omega)$, $p' = p/_{p-1}$, there exist a unique weak solution to (Q).

Proof: $X = W_0^{1,p}(\Omega)$ reflexiv & separable, so Browder-Minty is applicable.

compactly embedded

$$X = W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2 \quad \text{for } p > \frac{2d}{d-2}$$

$\pm su \hookrightarrow L^{2-\varepsilon}$

?

$$X = W_0^{1,p}(\Omega) \cap L^2(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$$

5. Pseudomonotone operators

<u>Example</u>: $$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + g(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

e.g. $g(u) = su$, $s \in \mathbb{R}$

Typical pseudomonotone operators are of the form
$$A = \underbrace{A_1}_{A_1: X \to X^* \text{ monotone \& hemicontinuous}} + \underbrace{A_2}_{A_2: X \to X^* \text{ completely continuous operator}}$$

<u>Definition 5.1</u> X Banach space, $A: X \to X^*$ is of type (M) if from
$$u_n \rightharpoonup u \text{ in } X$$
$$Au_n \rightharpoonup b \text{ in } X^*$$
$$\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \leq \langle b, u\rangle_X$$
it follows $Au = b$.

<u>Lemma 5.2</u> Let X be a reflexive Banach space, $A: X \to X^*$, $B: X \to X^*$

(i) A monotone & hemicont. $\Longrightarrow$ A of type M

(ii) A type (M), B completely cont $\Rightarrow$ $A+B$ type M

<u>Proof</u> (i) Minty's trick, see Lemma 0.2(ii)

(ii) Consider $u_n \in X$ s.th
$$u_n \rightharpoonup u \text{ in } X$$
$$Au_n + Bu_n \rightharpoonup b \text{ in } X^*$$
$$\limsup_{n\to\infty} \langle Au_n + Bu_n, u_n\rangle_X \leq \langle b, u\rangle_X$$

<u>Show</u> $(A+B)u = b$

Since B is completely cont., it holds $Bu_n \to Bu$ in X^*

$\Longrightarrow$ $$Au_n \rightharpoonup b - Bu \text{ in } X^*$$
$$\limsup \langle Au_n, u_n\rangle_X \leq \langle b - Bu, u\rangle \quad \text{(by Lemma 0.3(iii))}$$

$\Longrightarrow$ A of type M & thus: $Au = b - Bu \Rightarrow Au + Bu = b$

Remark Typically, the sum of two type M operators is not of type M.

Definition 5.3 $A : X \to X^*$ is called pseudomonotone if from

$$u_n \rightharpoonup u \text{ in } X$$
$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \le 0$$

it follows

$$\langle Au, u - w\rangle_X \le \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \quad \forall w \in X.$$

Lemma 5.4 X reflexive BS, $A, B : X \to X^*$

(i) A monotone & hemicont. $\Rightarrow$ A pseudomonotone
(ii) A completely continuous $\Rightarrow$ A pseudomonotone
(iii) A, B pseudomonotone $\Rightarrow$ $A+B$ — " —
(iv) A pseudomonotone $\Rightarrow$ A type M
(v) A — " — & locally bdd $\Rightarrow$ A demicontinuous (strong-weak)

Proof (i) Let $u_n \in X$ with $u_n \rightharpoonup u$ in X &

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \le 0$$

Show:

$$\langle Au, u - w\rangle_X \le \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \quad \forall w \in X.$$

Since A is monotone, we have

$$\langle Au_n - Au, u_n - u\rangle_X \ge 0$$

$$\Rightarrow \langle Au_n, u_n - u\rangle_X \ge \langle Au, u_n - u\rangle_X$$

$$\Rightarrow \liminf \langle Au_n, u_n - u\rangle_X \ge \liminf \langle Au, u_n - u\rangle_X \overset{u_n \rightharpoonup u}{=} 0$$

$$\Longrightarrow \langle Au_n, u_n - u\rangle_X \to 0 \quad (n \to \infty)$$

For any $w \in X$ we set $z_t = (1-t)u + tw \in X$, $t > 0$

A is monotone, so that

$$\langle Au_n - Az_t, u_n - z_t \rangle_X \geq 0$$

$$\Rightarrow t\langle Au_n, u_n - w\rangle_X \geq -(1-t)\underbrace{\langle Au_n, u_n - u\rangle_X}_{\to 0} + (1-t)\underbrace{\langle Az_t, u_n - u\rangle_X}_{\to 0} + t\langle Az_t, u_n - w\rangle_X$$

$$\overset{\text{liminf}}{\Longrightarrow} \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \geq \langle Az_t, u - w\rangle_X$$

$$z_t = u + t\cdot(w-u) \overset{\text{Hemi.}}{\Longrightarrow} Az_t \rightharpoonup Au \quad \text{for } t \to 0$$

$$\overset{t\to 0}{\Longrightarrow} \liminf \langle Au_n, u_n - w\rangle_X \geq \langle Au, u-w\rangle_X$$

$\Rightarrow$ A pseudomonotone

(ii) $u_n \rightharpoonup u$ in X, $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \leq 0$

Show: $\langle Au, u-w\rangle_X \leq \liminf_{n\to\infty} \langle Au_n, u_n - w\rangle_X \quad \forall w \in X.$

A is completely continuous $\Rightarrow Au_n \to Au$ in X^*

$$\Rightarrow \langle Au, u-w\rangle = \lim_{n\to\infty} \langle Au_n, u_n - w\rangle_X \quad \forall w \in X$$

Lem 0.3 (ii)

$\Rightarrow$ A pseudo.

(iii) Take $u_n \in X$ with $u_n \rightharpoonup u$ in X &

$$\limsup_{n\to\infty} \langle Au_n + Bu_n, u_n - u\rangle_X \leq 0$$

Show: $\langle Au + Bu, u - w\rangle_X \leq \liminf_{n\to\infty} \langle Au_n + Bu_n, u_n - w\rangle_X$

Claim: It holds $\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X \leq 0$

& $\limsup \langle Bu_n, u_n - u\rangle_X \leq 0$

Proof by $\lightning$: Assume that it holds

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle_X = a > 0$$

Then

$$\limsup_{n\to\infty} \langle Bu_n, u_n-u\rangle_X$$

$$= \limsup \langle (A+B)u_n, u_n-u\rangle - \langle Au_n, u_n-u\rangle$$

$$\leq 0 - \limsup_{n\to\infty} \langle Au_n, u_n-u\rangle$$

$$\overset{\cdot}{=} -a < 0$$

But B is pseudomonotone Δ then it holds

$$\langle Bu, u-w\rangle_X \leq \liminf_{n\to\infty} \langle Bu_n, u_n-w\rangle \quad \forall w\in X$$

$\overset{w=u}{\Longrightarrow}$ $$0 \leq \liminf_{n\to\infty} \langle Bu_n, u_n-u\rangle$$

$$\leq \limsup \langle Bu_n, u_n-u\rangle < 0 \ \lightning$$

$\liminf (a) + \liminf (b) \leq \liminf (a+b)$

A & B are pseudo

$$\Rightarrow \langle Au, u-w\rangle_X \leq \liminf \langle Au_n, u_n-w\rangle_X \quad \forall w\in X$$

$$\langle Bu, u-w\rangle_X \leq \liminf \langle Bu_n, u_n-w\rangle_X \quad \forall w\in X$$

$$\Rightarrow \langle Au+Bu, u-w\rangle_X \leq \liminf \langle Au_n+Bu_n, u_n-w\rangle_X \quad \forall v\in X$$

$\Rightarrow$ $A+B$ pseudo

(iv) $u_n\in X$ fulfilling

$u_n \to u$ in X

$Au_n \to b$ in X^*

$$\limsup_{n\to\infty} \langle Au_n, u_n\rangle_X \leq \langle b, u\rangle_X$$

<u>Show</u>: $Au=b$

The (ii) condition gives: $\limsup \langle Au_n, u_n\rangle_X - \underbrace{\langle b, u\rangle_X}_{= \lim \langle Au_n, u\rangle} \leq 0$

$$\Longrightarrow \limsup \langle Au_n, u_n-u\rangle_X \leq 0$$

$$\xRightarrow{A\text{ pseudo}} \langle Au, u-w\rangle_X \leq \liminf \langle Au_n, u_n-w\rangle_X$$

$$\leq \langle b, u\rangle_X - \langle b, w\rangle_X \quad \forall w\in X$$

$$= \langle b, u-w\rangle_X$$

Choose $w \hat{=} 2u-w$ to get.

$$\langle Au, u-w\rangle_X = \langle b, u-w\rangle_X \quad \forall w\in X$$

Choose $w \hat{=} u-w$: $\langle Au-b, w\rangle_X = 0 \Rightarrow Au=b$ in X^*

(v) $u_n \in X$ s.t. $u_n \rightharpoonup u$ in X

A locally bdd. $\Rightarrow$ $(Au_n)_n$ bdd. $\Rightarrow$ $Au_{n_k} \rightharpoonup b$ in X^*

$\Rightarrow$ $\langle \underbrace{Au_{n_k}}_{\text{bdd.}}, \underbrace{u_{n_k} - u}_{\to 0} \rangle_X \longrightarrow 0$

$\overset{A \text{ pseudo}}{\Rightarrow}$ $\langle Au, u-w \rangle_X \leq \liminf \langle Au_{n_k}, u_{n_k} - w \rangle_X$
$= \langle b, u-w \rangle_X \quad \forall w \in X$

$\Rightarrow$ $Au = b$

$\Rightarrow$ $Au_{n_k} \rightharpoonup Au$ in X^*

$\Rightarrow$ A demi. $\square$

Theorem 5.5 (Brezis 1968) $A : X \to X^*$ pseudomonotone, (locally) bounded, coercive operator, X separable & reflexive BS

$\Rightarrow$ $\forall b \in X^* \; \exists u \in X : \; Au = b$

Proof By Lemma 5.4, we know that A is demicontinuous ($u_n \to u \Rightarrow Au_n \rightharpoonup Au$) & it is of type M.

We follow the structure of the proof to the Browder-Minty thm.

Choose basis $(w_i)_i$ of X & consider $u_n = \sum_{k=1}^n c_n^k w_k$ where c_n^k are coefficients such that

$$g_n^k(\vec{c}_n) = g_n^k(u_n) := \langle Au_n - b, w_k \rangle_X = 0 \quad \forall k = 1, \dots, n$$

$$\Leftrightarrow \langle Au_n - b, v \rangle_X = 0 \quad \forall v \in X_n = \operatorname{span}(w_1, \dots, w_n)$$

A is demicontinuous & coercive, so as in the Browder-Minty theorem, we obtain the existence of c_n^k such that

$$\|u_n\|_X \leq R_0 \quad \forall n \in \mathbb{N}$$

$\Rightarrow$ $\exists (u_{n_k})_k$ s.t. $u_{n_k} \rightharpoonup u$ in X

$\lim_{k \to \infty} \langle Au_{n_k}, v \rangle = \langle b, v \rangle \quad \forall v \in \bigcup_{n \in \mathbb{N}} \operatorname{span}(w_1, \dots, w_{n_k})$ this is dense in X

Further, A is bounded ie. $\|Au_{n_k}\|_{X^*} \leq C$

$\Rightarrow$ $\exists (Au_{n_{k_j}})_j$ s.t. $Au_{n_{k_j}} \rightharpoonup c$ in X^*

from here we have $Au_{n_{k_j}} \rightharpoonup b$ in X^*
$\Rightarrow c = b$

$$\langle Au_{n_{k_j}}, u_{n_{k_j}} \rangle_X = \langle b, u_{n_{k_j}} \rangle \longrightarrow \langle b, u \rangle_X, \ j \to \infty$$

$\Rightarrow Au = b$ since A is type M. $\square$

6. Quasilinear elliptic PDE II

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + g(u) = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$(g(u_1) - g(u_2), u_1 - u_2)$
$\leq L \|u_1 - u_2\|_{L^2}^2$
$\leq K \|\nabla(u_1 - u_2)\|_{L^2}^2$

We had $g(u) = su$, $s \geq 0$, before.
We want to use the theory of pseudomonotone operators and write

$$A_1 u + A_2 u = b \quad \text{in } X^*$$

for

$$\langle A_1 u, \varphi \rangle_X = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \quad \forall \varphi \in X = W_0^{1,p}(\Omega)$$
$$\langle A_2 u, \varphi \rangle_X = \int_\Omega g(u) \varphi \, dx \quad \text{-- " --}$$
$$\langle b, \varphi \rangle_X = \int_\Omega f \varphi \, dx \quad \text{-- " --}$$

Lemma 6.1 Let Ω be a bdd domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. Let $g \in C^0(\mathbb{R})$ fulfill the growth condition

$$|g(t)| \leq c(1 + |t|^{r-1}), \quad r \in [1, \infty).$$

If $p \in [1, d)$ & $r \in [1, \frac{dp}{d-p}]$, then $A_2 : X \to X^*$ is bounded.
If $r < \frac{dp}{d-p} = q$, then A_2 is completely continuous.

$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$
$W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, $r < q$

$d = 3$: $2 < \frac{3p}{3-p} \Leftrightarrow 6 - 2p < 3p \Leftrightarrow p > 6/5$

$g(u) = su$, this means $r = 2$: $2 < \frac{dp}{d-p}$
$g(u) = su^2$
$g(u) = su^p$, $r = p+1$ $\quad p + 1 < \frac{3p}{3-p} \Rightarrow \cancel{(3-p)}$

Proof: $|\langle A_2 u, \varphi\rangle_X| \leq \int_\Omega c(1+|u|^{r-1})|\varphi|\,dx$ (Hölder conj of q)

$$= c\underbrace{\int_\Omega |\varphi|\,dx}_{\|\varphi\|_1 \leq c\|\varphi\|_X} + c\underbrace{\left(\int_\Omega |u|^{(r-1)q'}dx\right)^{1/q'}}_{\|u\|^{r-1}_{(r-1)q'}}\underbrace{\left(\int_\Omega |\varphi|^q dx\right)^{1/q}}_{\|\varphi\|_q \leq c_q\|\varphi\|_X = \|D\varphi\|_p}$$

$\|u\|^{r-1}_{(r-1)q'} \leq C\|u\|^{r-1}_q \leq C\|u\|^{r-1}_X$

$L^{(r-1)q'} \supseteq L^q$ because $(r-1)q' \leq q$ since $r-1 \leq \frac{q}{q'} = q(1-\frac{1}{q}) = q-1$

$$\Rightarrow \|A_2 u\|_{X^*} = \sup_{\|\varphi\|_X \leq 1} |\langle A_2 u, \varphi\rangle_X| \leq c(1+\|u\|_X^{r-1}) \Rightarrow A_2 u \in X^* \ \forall u \in X$$

$\Rightarrow A_2 : X \to X^*$ bdd.

Next, consider $(u_n) \subset X$ s.t. $u_n \to u$ in X weakly

Show: $A_2 u_n \to A_2 u$ in $X^* = W^{-1,p}(\Omega)$

By assumption we have $X = W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, $r < \frac{dp}{d-p}$

by Rellich-Kondrachov's theorem. & thus

$u_n \to u$ in $L^r(\Omega)$ strongly!

$\frac{1}{r} + \frac{1}{r'} = 1 \Rightarrow 1 + \frac{r}{r'} = r$, $\frac{r}{r'} = r-1$

Define $(Fv)(x) = g(v(x))$ & it holds $|(Fv)(x)| \leq c(1+|v(x)|^{r-1})$

& thus $F: L^r(\Omega) \to L^{r'}(\Omega)$ is continuous by the theory of Nemytskii operators

$$\Rightarrow \|A_2 u_n - A_2 u\|_{X^*} = \sup_{\|\varphi\|_X \leq 1} |\langle A_2 u_n - A_2 u, \varphi\rangle_X|$$

$$\leq \sup_{\|\varphi\|_X \leq 1} \int_\Omega |g(u_n) - g(u)|\,|\varphi|\,dx$$

$$\overset{\text{Hölder}}{\leq} \sup_{\|\varphi\|_X \leq 1} \|F(u_n) - F(u)\|_{r'} \underbrace{\|\varphi\|_r}_{\leq C\|\varphi\|_X} \quad X \hookrightarrow L^r$$

$$\leq C\,\|F(u_n) - F(u)\|_{r'} \to 0 \qquad \square$$

<u>Lemma 6.3</u> Additionally to Lemma 6.2, let g fulfill

$$\inf_{t \in \mathbb{R}} g(t)t > -\infty$$

$g(t) = st$
$st^2 \geq 0$, $s > 0$
Fails for $s < 0$

Then $A_1 + A_2 : X \to X^*$ coercive

<u>Proof</u>: We have $\langle A_1 u, u \rangle_X = \|\nabla u\|_p^p$ &

$$\langle A_2 u, u \rangle_X = \int_\Omega g(u) u \, dx \geq -c_0$$

Then

$$\frac{\langle A_1 u + A_2 u, u \rangle_X}{\|u\|_X} \geq \frac{\langle A_1 u, u \rangle}{\|\nabla u\|_p} - \frac{c_0}{\|\nabla u\|_p} = \underbrace{\|\nabla u\|_p^{p-1}}_{\overset{p>1}{\to}\infty} - \underbrace{\frac{c_0}{\|\nabla u\|_p}}_{\to 0} \to \infty$$

for $\|\nabla u\|_p \to \infty$

<u>Theorem 6.4</u> Let Ω be a bdd domain with $\partial\Omega \in C^{0,1}$. Let $p \in (1, d)$ & g fulfill the assumptions of Lemma 6.2 & 6.3. Then for all $f \in L^{p'}(\Omega)$ $\exists$ weak solution $u \in X$ to (Q) i.e.

$$(A_1 + A_2) u = b \text{ in } X^*.$$

<u>Proof</u>: We have A_1 is strictly monotone & cont. & bdd.

$\Rightarrow$ A_1 is pseudomonotone

A_2 is completely cont. & bdd by Lemma 6.2 & 6.3

$\Rightarrow$ $A_1 + A_2$ is pseudomonotone & bdd. & coercive

$\Rightarrow$ Brezis' theorem $\square$

<u>Remark</u>: (i) $p \geq d$ can be done in the same way but is more technical. However, $r \in [1, \infty)$ is possible.

(ii) $g(t) = -st$, $s>0$, does not fulfill $\inf g(t)t > -\infty$. But A_1 and A_2 is still coercive in this case since

$$\frac{\langle A_1 u + A_2 u, u\rangle_X}{\|u\|_X} = \|\nabla u\|_p^{p-1} - s\frac{\|u\|_2^2}{\|\nabla u\|_p}$$

$$\overset{W_0^{1,p} \hookrightarrow L^2}{\geq} \|\nabla u\|_p^{p-1} - sC\|\nabla u\|_p \to \infty \text{ if } p>2$$

$p \geq 2d/(d+2)$

or $p=2$ & $sC<1$

7. Stationary Navier-Stokes equation

$$-\Delta u + [\nabla u]u + \nabla p = f \quad \text{in } \Omega$$
$$\operatorname{div} u = 0 \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$-\operatorname{div}(\nu(x)\nabla u)$

Ω bounded domain in $\mathbb{R}^3$ with Lipschitz bdy $\partial\Omega$

$u \in L^8(0,T;L^8(\Omega))$
$\Downarrow$
uniqueness
$u \in L^{8/3}(0,T;L^{8/3}(\Omega))$

- weak sol. ex., but unique? 3D
- strong sol. unique but exist.?

where $[\nabla u]u = \sum_{j=1}^{3} u^j(\partial_j u^i) = (u\cdot\nabla)u$

- $u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}: \Omega \to \mathbb{R}^3$, velocity; $p: \Omega \to \mathbb{R}$ pressure
- Condition $\int_\Omega p\,dx = 0$ bc. p only relevant up to a constant

$$X = \{\varphi \in H_0^1(\Omega;\mathbb{R}^3) : \operatorname{div}\varphi = 0\} \subseteq H_0^1(\Omega;\mathbb{R}^3)$$

$H_0^1(\Omega;\mathbb{R}^3) = W_0^{1,2}(\Omega,\mathbb{R}^3)$

$$\|u\|_X = \|\nabla u\|_{L^2(\Omega;\mathbb{R}^{3\times3})}$$

For all $u, \varphi \in X$ and $p \in L^2(\Omega)$ with $\int_\Omega p\,dx = 0$ we define the operators

$$\langle A_1 u, \varphi\rangle_X = \int_\Omega \nabla u \cdot \nabla\varphi\,dx$$
$$\langle A_2 u, \varphi\rangle_X = \int_\Omega [\nabla u]u\cdot\varphi\,dx$$
$$\langle P, \varphi\rangle = \langle\nabla p, \varphi\rangle_X := -\int p \operatorname{div}\varphi\,dx \overset{!}{=} 0 \quad \varphi \in X$$
$$\langle b, \varphi\rangle = \int_\Omega f\cdot\varphi\,dx$$

$$A_1 u + A_2 u = b$$

$$\iff \int_\Omega \nabla u\cdot\nabla\varphi\,dx + \int_\Omega [\nabla u]u\cdot\varphi\,dx = \int_\Omega f\cdot\varphi\,dx \qquad \forall\varphi\in X$$

- $A_1 : X\to X^*$ is obv. linear, continuous, coercive, strictly monotone & bounded

<u>Lemma 7.2</u> $A_2 : X\to X^*$ is completely cont. & bdd

<u>Proof</u> 1) $\forall u,\varphi\in X$ we have

$$|\langle A_2 u,\varphi\rangle_X| \le \int_\Omega |u||\nabla u||\varphi|\,dx$$

$$X\subseteq H_0^1 \overset{3D}{\subseteq} L^4$$

$$\overset{\text{Hölder}}{\le} \underbrace{\|u\|_{L^4}}_{\le c\|u\|_X}\ \underbrace{\|\varphi\|_{L^4}}_{\le c\|\varphi\|_X}\ \underbrace{\|\nabla u\|_{L^2}}_{\|u\|_X}$$

$$\le C\,\|u\|_X\,\|\varphi\|_X$$

$$\Rightarrow A_2 : X\to X^* \ \&\ A_2 \text{ is bounded.}$$

2) Take $u_n\in X\ \forall n$ s.t. $u_n\rightharpoonup u$ in X weakly $\Rightarrow u_n\to u$ in L^4 in 3D

To prove: $A_2 u_n \to A_2 u$ in X^* strongly

$$|\langle A_2 u_n - A_2 u,\varphi\rangle_X|$$

$$=\left|\int_\Omega [\nabla u_n]u_n\cdot\varphi - [\nabla u]u\cdot\varphi\,dx\right|$$

$$=\left|\int_\Omega [\nabla u_n](u_n-u)\cdot\varphi + [\nabla(u_n-u)]u\cdot\varphi\,dx\right|$$

$$=\left|\int_\Omega [\nabla u_n](u_n-u)\cdot\varphi - [\nabla\varphi]u\cdot(u_n-u)\,dx\right|$$

$$\int[\nabla(u_n-u)]u\cdot\varphi$$

$$= \int \partial_j (u_n^i - u^i)\, u_j\, \varphi_i$$
$$= -\int_\Omega (u_n^i - u^i)\, \underbrace{\partial_j u_j}_{=0}\, \varphi_i - \int_\Omega (u_n^i - u^i)\, \partial_j \varphi_i\, u_j$$

$$\Rightarrow \|A_2 u_n - A_2 u\|_{X^*} = \sup_{\|\varphi\|_X \le 1} |\langle A_2 u_n - A_2 u, \varphi\rangle_X|$$

Hölder

$$\le \sup_{\|\varphi\|_X \le 1} \underbrace{\|u_n - u\|_{L^4}}_{\to 0} \underbrace{\|\nabla u_n\|_{L^2}}_{\text{bdd.}} \underbrace{\|\varphi\|_{L^4}}_{\le C\|\varphi\|_X \le C} + \underbrace{\|u\|_{L^4}}_{\text{bdd.}} \cdot \underbrace{\|u_n - u\|_{L^4}}_{\to 0} \underbrace{\|\nabla \varphi\|_{L^2}}_{\le 1}$$

$$\longrightarrow 0$$

$\Rightarrow$ $A_2 u_n \longrightarrow A_2 u$ in X^*

$\Rightarrow$ A_2 is completely continuous $\qquad f \in H^{-1}$ ✓ $\quad (H_0^1)^* \quad \square$

<u>Theorem 7.2</u> $\forall f \in L^2(\Omega; \mathbb{R}^3)\ \exists u \in X$ solving NSE in the weak setting.

<u>Proof</u>: $A_1 + A_2 : X \to X^*$ are bdd & pseudomonotone

It remains to show coercivity for A_2.

$$\langle A_2 u, u\rangle_X = \int_\Omega \sum_{i,j=1}^3 u^j\, \partial_j u^i\, u^i\, dx$$
$$\overset{\text{chain rule}}{=} \frac{1}{2} \int_\Omega \sum_{j=1}^3 u^j\, \partial_j |u|^2\, dx$$
$$\overset{\text{IBP}}{=} -\frac{1}{2} \int_\Omega \operatorname{div} u \cdot |u|^2\, dx \overset{u \in X}{=} 0$$

$\Rightarrow$ $A_1 + A_2$ are coercive

$\Rightarrow$ Thm. of Brezis $\square$

We know: $\langle F, \varphi\rangle_{H_0^1} := \int \nabla u \cdot \nabla \varphi\, dx + \int [\nabla u] u \cdot \varphi\, dx - \int f \cdot \varphi\, dx \quad \forall \varphi \in X$

We want to find p s.t.

$$\int \nabla u \cdot \nabla \varphi\, dx + \int [\nabla u] u \cdot \varphi\, dx + \int p \operatorname{div} \varphi\, dx = \int f \cdot \varphi\, dx \quad \forall \varphi \in H_0^1$$

Theorem 7.3 (De Rham) Let $F \in H^{-1}(\Omega;\mathbb{R}^3)$. If it holds

$$\langle F, \varphi \rangle_{H_0^1} = 0 \quad \forall \varphi \in X \subseteq H_0^1$$

$\Longrightarrow \exists p \in L^2(\Omega)$ with $\int_\Omega p \, dx = 0$ s.t.

$$\langle F, \varphi \rangle_{H_0^1} = \int p \, \operatorname{div} \varphi \, dx \quad \forall \varphi \in H_0^1$$

Remark: One cannot take $f \in X^*$ to get a pressure

$$X \subseteq H_0^1 \Rightarrow H^{-1} \subseteq X^*$$

no distributional space

8. Evolution problems

$$\boxed{\partial_t}$$

Relevant spaces are Bochner spaces

$$L^p(0,T;X) = \{u : (0,T) \to X \text{ strongly measurable} : \|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty\}$$

$\exists$ step fct. u_n s.t.: $u_n(s) = \sum u_j \chi_j \to u$

$X = L^2(\Omega;\mathbb{R})$:

$$\|u\|^p_{L^p(0,T;X)} = \int_0^T \|u(t)\|^p_{L^2(\Omega)} \, dt$$
$$= \int_0^T \left(\int_\Omega |u(t,x)|^2 dx\right)^{p/2} dt$$

$u : (0,T) \to L^2(\Omega)$

$u(t)(x) := u(t,x)$

$u : (0,T) \times \Omega \to \mathbb{R}$

$X = H_0^1(\Omega)$, $X \hookrightarrow\hookrightarrow L^4(\Omega)$ in 3D but

counterex. $\to$ exercise

$$L^2(0,T;X) \not\hookrightarrow\hookrightarrow L^2(0,T;L^4(\Omega)) \quad ↯$$

Aubin-Lions: $L^2(0,T;X) \cap H^1(0,T;X^*) \hookrightarrow\hookrightarrow L^2(0,T;L^4(\Omega))$

$$= \{ u \in L^2(0,T;X) : \partial_t u \in L^2(0,T;X^*) \}$$

Example: $$\partial_t u - \Delta u = f \quad \text{in} \quad \overset{=(0,T)}{I} \times \Omega$$
$$u = 0 \quad \text{on} \quad I \times \partial\Omega$$
$$u(0) = u_0 \quad \text{in} \quad \Omega$$

Weak form

$$\int_{I\times\Omega} \partial_t u \, \varphi \, d(t,x) + \int_{I\times\Omega} \nabla u \cdot \nabla \varphi \, d(t,x) = \int_{I\times\Omega} f \varphi \, d(t,x)$$
$$\forall \varphi \in L^2(I; H_0^1(\Omega))$$

A priori estimate by formally set $\varphi = u$

$$\int_{I\times\Omega} \partial_t u \, u + \|u\|^2_{L^2(I;H_0^1(\Omega))}$$
$$= \int_{I\times\Omega} f u \overset{\text{Hölder}}{\leq} \|f\|_{L^2(I;L^2(\Omega))} \overbrace{\|u\|_{L^2(I;L^2(\Omega))}}^{\leq C \|u\|_{L^2(0,T;H_0^1(\Omega))}}$$
$$\overset{\text{Young}}{\leq} \frac{C^2}{2} \|f\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2} \|u\|^2_{L^2(0,T;H_0^1(\Omega))}$$

$$= -\int_{I\times\Omega} u \cdot \partial_t u + \int_\Omega (u(t))^2 dx \int (u(0))^2 dx \Rightarrow \int_{I\times\Omega} \partial_t u \, u = \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_0\|^2_{L^2(\Omega)}$$

$$\Longrightarrow \frac{1}{2} \|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2} \|u\|^2_{L^2(I;H_0^1(\Omega))} \leq \frac{C^2}{2} \|f\|^2_{L^2(I,L^2(\Omega))} + \frac{1}{2} \|u_0\|^2_{L^2(\Omega)}$$

$u \in L^\infty(I;L^2(\Omega))$ $\quad$ $u \in L^2(I;H_0^1(\Omega))$ $\quad \longrightarrow u \in L^4(0,T;L^4(\Omega))$

$C(\bar{I};L^2(\Omega))$

$$\& \quad \|\partial_t(eu)\|^2_{L^2(I;H^{-1}(\Omega))} \leq C\left(\|u_0\|^2_{L^2(\Omega)}, \|f\|^2_{L^2(I,L^2(\Omega))}\right)$$

$e: H^1 \to H^{-1}$

$W^{1,p}(\Omega)$

$H^1(\Omega) \subseteq L^2(\Omega)$ $\quad$ $X \overset{d}{\hookrightarrow} Y$

Setting: V Banach space, H Hilbert space,

V is embedded in H ie. $\exists j: V \to H$ injective, linear, cont. & $j(V) \subseteq H$ is dense. ($\overline{j(V)} = H$)

We call (V, H, j) a Gelfand triple ie. (V, H, V^*)

$$\begin{tikzcd} V \arrow[r, hook, "j"] \arrow[rrr, bend right, "e = j^* \circ R \circ j"'] & H \arrow[r, "R", "\cong"'] & H^* \arrow[r, hook, "j^*"] & V^* \end{tikzcd}$$

$\langle Ry, x\rangle_H = (y, x)_H$

$j: V \to H$

$j^*: H^* \to V^*$

$\langle j^* f, v\rangle_V = \langle f, jv\rangle_H$

$\forall f \in H^*, v \in V$

adjoint operator

$\overline{j^*(H^*)} = V^*$ (proof uses the inj. of j)

- Since $j(V) \subseteq H$ is dense, we get that j^* is injective since taking $j^* f = 0$, then

$$0 = \langle j^* f, v\rangle_V = \langle f, jv\rangle_H \quad \forall v \in V$$

$$\Rightarrow \quad f = 0 \text{ in } H^*$$

- $\langle ev, w\rangle_V = \langle j^* \circ R \circ j v, w\rangle_V = \langle Rjv, jw\rangle_H = (jv, jw)_H \quad \forall v, w \in V$
- $\langle ev, w\rangle_V = (jv, jw)_H = (jw, jv)_H = \langle ew, v\rangle_V$

Definition 8.1 Let (V, H, j) be a Gelfand triple. Then $u \in L^p(I; V)$ has a generalized time derivative w.r.t. e if an element $w \in L^{p'}(I; V^*)$ exists ($\frac{1}{p'} + \frac{1}{p} = 1$) s.t. for any $v \in V$, $\varphi \in C_0^\infty(I; \mathbb{R})$ it holds

$\operatorname{supp}\varphi = \overline{\{t \in I : \varphi(t) \neq 0\}}$

$$\int_I \langle w(t), v\rangle_V \, \varphi(t)\, dt = -\int_I \underbrace{(ju(t), jv)_H}_{= \langle eu, v\rangle_V} \varphi'(t)\, dt$$

One sets $w = \partial_t (eu)$.

Remark $j: V \to H$, $e: V \to V^*$ induce embeddings on Bochner space $j: L^p(I; V) \to L^p(I; H)$, $e: L^p(I; V) \to L^p(I; V^*)$ which are defined by

$$\underbrace{(ju)(t)}_{\in H} = j(\underbrace{u(t)}_{\in V}), \quad (eu)(t) = e(u(t)) \quad \text{for a.e. } t \in I$$

Moreover, embedding: $W^{1,1}(0,1) \hookrightarrow C([0,1])$

$W^{1,p,p'}(I;V,V^*) = W = \{ v \in L^p(I;V) : \partial_t(\ell u) \in L^{p'}(I;V^*) \}$

$$\|u\|_W := \|u\|_{L^p(I;V)} + \|\partial_t(\ell u)\|_{L^{p'}(I;V^*)}$$

W is a Banach space & it is reflexive if V is reflexive

Lemma 8.2 V reflexive & separable BS, (V,H,j) Gelfand triple

(a) $W \hookrightarrow C(\bar{I};H)$ i.e. any fct $u \in W$ has a continuous representation in H w.r.t. j i.e. $ju \in C(\bar{I},H)$

(b) $\forall u,v \in W$ & $\forall s,t \in \bar{I}$ it holds

$$\int_s^t \langle \partial_\tau(\ell u)(\tau), v(\tau)\rangle_V + \langle \partial_\tau(\ell v)(\tau), u(\tau)\rangle_V \, d\tau$$

$$= ((ju)(t), (jv)(t))_H - ((ju)(s), (jv)(s))_H$$

Proof Approximations, see book by Etienne Emmerich on operator equations

Remark: $u = v \in W$ in (b)

$$\int_s^t \langle \partial_\tau(\ell u)(\tau), u(\tau)\rangle_V \, d\tau = \frac{1}{2}\|(ju)(t)\|_H^2 - \frac{1}{2}\|(ju)(s)\|_H^2$$

$A = -\Delta u \in H^{-1}$

$$(*) \quad \partial_t(\ell u) + \underset{\in L^p(I;H_0^1)}{\mathcal{A}u} = b \quad \text{in } H^{-1}(\Omega) = V^*$$

$(ju)(0) = u_0$ in $L^2(\Omega) = H$

$u = 0$ on $\partial\Omega$

apply test fct in $H_0^1(\Omega)$

$A: V \to V^*$, then $\mathcal{A}: L^p(I;V) \to L^{p'}(I;V^*)$ s.t. $=: (L^p(I;V))^*$

$$\langle \mathcal{A}u, \varphi\rangle_{L^p(I;V)} = \int_I \langle A(u(t)), \varphi(t)\rangle_V \, dt \quad \forall u,\varphi \in L^p(I;V)$$

Corollary

Theorem 8.3 V sep. & refl. BS, (V,H,j) Gelfand triple, $p \in (1,\infty)$

Let $A: V \to V^*$ be such that

$\mathcal{A}: L^p(I;V) \to (L^p(I;V))^*$

is pseudomonotone & it fulfills the coercivity condition

$$\langle \mathcal{A}u, u\rangle_{L^p(I;V)} \geq c_0 \|u\|^p_{L^p(I;V)} \quad \forall u \in L^p(I;V)$$

$\Rightarrow \forall u_0 \in H,\ b \in L^{p'}(I;V^*)\ \exists$ solution $u \in W$ to $(*)$ ie $u \in W$ fulfills $(ju)(0) = u_0$ in H &

$$\int_I \langle \partial_t(eu)(t) + A(u(t)), \varphi(t)\rangle_V \, dt = \int_I \langle b(t), \varphi(t)\rangle_V \, dt \quad \forall \varphi \in L^p(I;V)$$

<u>Proof</u> Special case of theorem below :)

<u>Lemma 8.4</u> V sep. & ref. BS, $A: V \to V^*$. If A demicontinuous & it fulfills the growth condition

$$\|Au\|_{V^*} \leq c\left(\|u\|_V^{p-1} + 1\right) \quad \forall v \in V,$$

then $\mathcal{A}: L^p(I;V) \to (L^p(I;V))^*$ is bounded.

<u>Proof</u> Special case of lemma below :)

$$\partial_t(eu) - \operatorname{div}(|Du|^{p-2}Du) + g(u) = f \quad \text{in } L^{p'}(I;W^{-1,p'})$$
$$u = 0 \quad \text{on } \partial\Omega$$
$$(ju)(0) = u_0 \quad \text{in } H$$

$V = W_0^{1,p}(\Omega),\ H = L^2(\Omega),\ V^* = W^{-1,p'}(\Omega)$

$\|v\|_V = \|Du\|_p$

(V, H, j) is a Gelfand triple for $p \geq \frac{2d}{d+2}$ (ensures $W_0^{1,p} \hookrightarrow L^2$)

j = id inclusion

$$\langle A_1 u, v\rangle_V = \int_\Omega |Du|^{p-2} Du \cdot Dv \, dx \quad \forall u,v \in V$$
$$\langle A_2 u, v\rangle_V = \int_\Omega g(u)\, v \, dx$$

We have proved that A_1 is <u>bdd</u>, <u>coercive</u>, <u>cont.</u>, <u>strictly mon.</u>

$$\|u\|^p_{L^p(I;W_0^{1,p})} = \int_I \|u(t)\|_V^p \, dt = \int_I \|Du^{(t)}\|_p^p \, dt$$
$$= \int_I \int_\Omega |Du(t,x)|^p \, dx \, dt = \int_{I\times\Omega} |Du(t,x)|^p \, d(t,x)$$

Lemma 8.5 Let $p \geq \frac{2d}{d+2}$. Then the induced operator

$$\mathcal{A}_1 : L^p(I;V) \longrightarrow (L^p(I;V))^*$$

is bdd, cont., strictly mon., & it fulfills the coercivity cond.

$$\langle \mathcal{A}_1 u, u \rangle_{L^p(I;V)} \geq c_0 \|u\|^p_{L^p(I;V)} \quad \forall u \in L^p(I;V)$$

In particular $\mathcal{A}_1$ is pseudomonotone.

Proof Analogous to before but integrate over $I \times \Omega$ (Exercise?)

$A_2 : V \to V^*$ was proved to be bdd & completely continuous if $r < \frac{dp}{d-p}$

What about $\mathcal{A}_2$? Before it was used that $V \overset{\text{compact}}{\hookrightarrow} L^q(\Omega)$, $q < \frac{dp}{d-p}$

but now $L^p(I;V) \overset{\text{cont.}}{\hookrightarrow} L^p(I;L^q(\Omega))$ is not compact anymore.

Counter example: $x \in X$, $f_n(t,x) = \sin(nt)x$ in $L^2(0,T;X)$ $\quad X \overset{\text{cpt}}{\hookrightarrow} H$

We prove that $\sin(nt)x \rightharpoonup 0$ in $L^2(I;X)$ but $\sin(nt)x \not\to 0$ in $L^2(I;H)$

$$\int_I \sin(nt)\,\varphi\,dt \overset{\substack{\varphi \in L^2(0,1) \\ a,b \in I \\ \text{arbitrary}}}{\text{"="}} \int_a^b \sin(nt)\,dt = -\frac{1}{n}\underbrace{(\cos(nb)-\cos(na))}_{\text{bdd.}} \longrightarrow 0$$

$$\Rightarrow \sin(nt) \rightharpoonup 0 \quad \text{in } L^2(0,T) \Rightarrow \sin(nt)x \rightharpoonup 0 \text{ in } L^2(0,T;X)$$

but $\|\sin(n\cdot)\|^2_{L^2(0,T)} = \int_0^T |\sin(nt)|^2\,dt = \int_0^T \frac{1-\cos(2nt)}{2}\,dt$

$$= \frac{T}{2} - \underbrace{\frac{\sin(2nT)}{4n}}_{\to 0} \longrightarrow \frac{T}{2} \neq 0$$

$$\Rightarrow \sin(nt) \not\to 0 \text{ in } L^2(0,T) \Rightarrow \sin(nt)jx \not\to 0 \text{ in } L^2(0,T;H)$$

B, B_0, B_1 Banach spaces, B_0 & B_1 reflexive $\qquad H^1 \hookrightarrow\hookrightarrow L^2 \hookrightarrow (H^1)^*$

$$B_0 \overset{b_0}{\hookrightarrow\hookrightarrow} B \overset{b_1}{\hookrightarrow} B_1$$

$v_n \rightharpoonup v$ in B_0

$\Rightarrow b_0 v_n \to b_0 v$ in B

$\Rightarrow b v_n \to b v$ in B_1

since $b_1 \in \mathcal{L}$

then $b := b_1 \circ b_0 : B_0 \to B_1$ again compact

$$W_0 = \{ u \in L^{p_0}(I;B) : \partial_t(bu) \in L^{p_1}(I;B_1) \}$$

$1 < p_0, p_1 < \infty$ (if $p_0 = p_1 = 1$ then proof is much more difficult and result is called Aubin-Lions-Simon)

NSE: $p_0 = 2$, $p_1 = 4/3, 1$

$$\|u\|_{W_0} = \|u\|_{L^{p_0}(I;B_0)} + \|\partial_t(bu)\|_{L^{p_1}(I;B_1)}$$

Theorem 8.6 (Aubin – Lions) $\qquad W_0 \hookrightarrow\hookrightarrow L^{p_0}(I;B)$

(Aubin: 1963, Lions: 1969)

constant $\qquad \|b_0 v\|_B \le \underbrace{\|b_0\|}_{\text{can be large}} \cdot \|v\|_{B_0}$

Lemma 8.7 (Ehrling, 1954) $\forall \eta>0 \ \exists \, d(\eta)$ s.t.

$$\|b_0 v\|_B \le \eta \|v\|_{B_0} + d(\eta)\, \|bv\|_{B_1} \quad \forall v \in B_0$$

Proof Assume there is a $\eta > 0$ & a seq. $(v_n) \subset B_0$, $(d_n) \subset \mathbb{R}$ with $0 < d_n \to \infty$ s.t.

$$\|b_0 v_n\|_B > \eta \|v_n\|_{B_0} + d_n \|b v_n\|_{B_1}$$

set $w_n := \frac{v_n}{\|v_n\|_{B_0}} \in B_0$ & get. $\qquad \|b_0 w_n\|_B > \eta + \underbrace{d_n}_{\to\infty} \|b w_n\|_{B_1}$

$$\|b_0 w_n\|_B \le \|b_0\| \cdot \underbrace{\|w_n\|_{B_0}}_{=1} = \|b_0\|$$

$$\Rightarrow \underbrace{\frac{\|b_0\|}{d_n}}_{\to 0} > \underbrace{\frac{\eta}{d_n}}_{\to 0} + \|b w_n\|_{B_1} \ge 0 \quad \Rightarrow \quad b w_n \to 0 \text{ in } B_1$$

$\|w_n\|_{B_0} = 1 \quad \Rightarrow \quad w_{n_k} \to w$ in B_0

$\Rightarrow \quad b w_{n_k} \to b w$ in B_1

$\Rightarrow \quad b w = 0$ & $b = b_1 \circ b_0$

$\Rightarrow \quad w = 0$ & $b_0 w = 0$

$\Rightarrow \quad b_0 w_{n_k} \to 0$ in B

$$\underbrace{\|b_0 w_{n_k}\|_B}_{\to 0} > \eta + \overbrace{\underbrace{d_{n_k}}_{\to\infty} \underbrace{\|b w_{n_k}\|_{B_1}}_{\to 0}}^{\ge 0} > \eta \quad ↯$$

Proof (Aubin-Lions) Let (v_k) be a bdd seq. in W_0

$\Rightarrow$ $v_{k_n} \rightharpoonup v$ in W_0

we have to show: $b_0 v_{k_n} \to b_0 v$ in $L^{p_0}(I;B)$

$u_n := v_{k_n} - v$ i.e. $u_n \rightharpoonup 0$ in W_0 show: $b_0 u_n \to 0$ in $L^{p_0}(I;B)$

$\|u_n\|_{W_0} \le C \quad \forall n$

Ehrling-Lemma: $\forall \eta \; \exists d(\eta)$ constant depending on η

$$\|b_0 u_n\|_{L^{p_0}(I;B)} \le \eta \underbrace{\|u_n\|_{L^{p_0}(I;B_0)}}_{\le \|u_n\|_{W_0} \le C} + d(\eta)\, \|b u_n\|_{L^{p_0}(I;B_1)}$$

$\eta = \dfrac{\varepsilon}{2C}$

$$\|b_0 u_n\|_{L^{p_0}(I;B)} \le \frac{\varepsilon}{2} + d(\varepsilon) \underbrace{\|b u_n\|_{L^{p_0}(I;B_1)}}_{\le \frac{\varepsilon}{2d(\varepsilon)}}$$

$\|b u_n\|_{L^{p_0}(I;B_1)} = \int_I \|b u_n(t)\|_{B_1}^{p_0}\,dt$

show: $\to 0$, $n \to \infty$

$\le \varepsilon$

set $p = \min(p_0, p_1)$

we have $b u_n \in W^{1,p}(I;B_1) \hookrightarrow C(\bar I; B_1)$ $\quad$ $X \subseteq Y$, $\|\cdot\|_Y \le C\|\cdot\|_X$

$W_0 \subseteq W^{1,p}(I;B_1) = \{u \in L^p(I;B_1) : \partial_t u \in L^p(I;B_1)\}$

$$\Rightarrow C \ge \|u_n\|_{W_0} \ge C\, \|b u_n\|_{W^{1,p}(I;B_1)} \ge \tilde C\, \|b u_n\|_{C(\bar I;B_1)}$$

$\|b u_n\|_{C(\bar I;B_1)} = \max_{t \in \bar I} \|b u_n(t)\|_{B_1}$

$$\Rightarrow \|b u_n(t)\|_{B_1} \le C \quad \forall t \in \bar I$$

set $w_n(t) := u_n(\lambda t) \quad \forall t \in I$, $\quad \lambda \in (0,1)$ arb. but fixed.

$$b w_n(0) = b u_n(0)$$

$$\|w_n\|_{L^{p_0}(I;B_0)} = \frac{1}{\lambda^{1/p_0}} \|u_n\|_{L^{p_0}(0,\lambda T;B_0)} \le C \lambda^{-1/p_0}$$

$$\|\partial_t (b w_n)\|_{L^{p_1}(I;B_1)} = \frac{\lambda}{\lambda^{1/p_1}} \|\partial_t (b u_n)\|_{L^{p_1}(0,\lambda T;B_1)} \le C\lambda^{1-\frac{1}{p_1}}$$

take $\varphi \in C^1(\bar I)$ with $\underline{\varphi(T) = 0}$, $\varphi(0) = -1$

$I = (0,T)$

$$b w_n(0) = \int_I \left(b w_n(t) \cdot \varphi(t) \right)' \, dt$$

$$= \int_I \underbrace{\varphi(t)}_{\leq \|\varphi\|_\infty} \, \partial_t (b w_n)(t) \, dt + \int_I \varphi'(t) \, b w_n(t) \, dt$$

$$\overset{\|\cdot\|_{B_1}}{\Longrightarrow} \|b w_n(0)\|_{B_1} \leq \|\varphi\|_\infty \cdot \underbrace{\|(b w_n)'\|_{L^1(I;B_1)}}_{\substack{\leq C \, \|\partial_t (b w_n)\|_{L^{p_1}(I;B_1)} \\ \leq C \, \lambda^{1-1/p_1}}} + \left\| \int_I \varphi'(t) \, b w_n(t) \, dt \right\|_{B_1}$$

$\underline{p_1 > 1}$ so choose $\lambda \in (0,1)$ s.t. $\|\varphi\|_\infty C \lambda^{1-\frac{1}{p_1}} \leq \frac{\varepsilon}{2}$

$\forall g \in B_0^*$

$$\left\langle g, \int_I w_n(t) \, \varphi'(t) \, dt \right\rangle_{B_0} = \int_I \langle g, \underbrace{w_n(t)}_{= u_n(\lambda t)} \rangle_{B_0} \, \varphi'(t) \, dt$$

$$= \int_0^{\lambda T} \langle \underbrace{g \, \varphi'\left(\tfrac{s}{\lambda}\right)}_{\in L^{p_0'}(0,\lambda T; B_0^*)}, u_n(s) \rangle_{B_0} \, ds \longrightarrow 0$$

since $u_n \to 0$ in $L^{p_0}(0,\lambda T; B_0)$

$$\Longrightarrow \int_I w_n(t) \, \varphi'(t) \, dt \to 0 \text{ in } B_0$$

$$\overset{B_0 \overset{b_0}{\hookrightarrow} B}{\Longrightarrow} \int_I b_0 w_n(t) \, \varphi'(t) \, dt \to 0 \text{ in } B$$

$$\overset{B \overset{b_1}{\hookrightarrow} B_1}{\Longrightarrow} \int_I b w_n(t) \, \varphi'(t) \, dt \to 0 \text{ in } B_1$$

$$\Longrightarrow b w_n(0) = b u_n(0) \longrightarrow 0 \text{ in } B_1$$

before $s = 0$

analogously do the same for $\tilde{w}_n(t) = u_n(s + \lambda t)$ $\forall s \in I$

$$\Longrightarrow b u_n(s) \to 0 \text{ in } B_1$$

in $*$ $\Rightarrow$ $b_0 u_n \to 0$ in $L^{p_0}(I; B)$ $\square$