Lipschitz and Hölder stability of optimization problems and generalized equations

Helmut Gfrerer · **Diethard Klatte**

Abstract This paper studies stability aspects of solutions of parametric mathematical programs and generalized equations, respectively, with disjunctive constraints. We present sufficient conditions that, under some constraint qualifications ensuring metric subregularity of the constraint mapping, continuity results of upper Lipschitz and upper Hölder type, respectively, hold. Furthermore, we apply the above results to parametric mathematical programs with equilibrium constraints and demonstrate, how some classical results for the nonlinear programming problem can be recovered and even improved by our theory.

Keywords Mathematical programs with disjunctive constraints · stationarity · metric subregularity · variational analysis · upper Lipschitz stability · upper Hölder stability

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1 Introduction

Consider the optimization problem

$$P(\omega) \qquad \min_x f(x, \omega) \quad \text{subject to} \quad q(x, \omega) \in P, \tag{1}$$

depending on the parameter vector ω belonging to some topological space Ω. In (1), $f : \mathbb{R}^n \times \Omega \to \mathbb{R}$ and $q : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ are continuous mappings and $P \subset \mathbb{R}^m$ is the union of finitely many convex polyhedra P_i, $i = 1, \ldots, p$, having the representation

$$P_i = \{y \,|\, a_{ij}^T y \leq b_{ij}, j = 1, \ldots, m_i\} \tag{2}$$

with $a_{ij} \in \mathbb{R}^m$ and $b_{ij} \in \mathbb{R}$. We will study stability results of Lipschitz or Hölder type for stationary and optimal solutions of $P(\omega)$ if ω varies near some reference parameter $\bar{\omega}$.

Of course, the parameter dependent nonlinear programming problem

$$NLP(\omega) \qquad \min f(x, \omega) \quad \text{subject to} \quad g(x, \omega) \leq 0, \ h(x, \omega) = 0,$$

where $f(x, \omega) : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$ and $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$, is a special case of (1) with $q(x, \omega) := (g(x, \omega), h(x, \omega))$ and $P := \mathbb{R}_-^{m_I} \times \{0\}^{m_E}$. Let us consider some more involved examples.

Helmut Gfrerer

Institute of Computational Mathematics, Johannes Kepler University Linz, A-4040 Linz, Austria, E-mail: helmut.gfrerer@jku.at

Diethard Klatte

Department of Business Administration, Professorship Mathematics for Economists, University of Zurich, Switzerland, E-mail: diethard.klatte@business.uzh.ch

Example 1 Consider the parameter dependent MPEC

$$MPEC(\omega) \qquad \min f(x,\omega) \quad \text{subject to} \qquad \begin{aligned} & g(x,\omega) \leq 0,\ h(x,\omega) = 0, \\ & \left. \begin{aligned} & G_i(x,\omega) \geq 0,\ H_i(x,\omega) \geq 0 \\ & G_i(x,\omega)H_i(x,\omega) = 0 \end{aligned} \right\} i = 1,\ldots,m_C, \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$, $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$ and $G,H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_C}$.

The problem $MPEC(\omega)$ fits into our setting (1) with

$$q := (g,h,-(G_1,H_1),\ldots-(G_{m_C},H_{m_C})),$$

$$P := \mathbb{R}^{m_I}_- \times \{0\}^{m_E} \times Q^{m_C}_{EC},$$

where $Q_{EC} := \{(a,b) \in \mathbb{R}^2_- \,|\, ab = 0\}$. Since Q_{EC} is the union of the convex polyhedra $\mathbb{R}_- \times \{0\}$ and $\{0\} \times \mathbb{R}_-$, P is the union of 2^{m_C} polyhedra.

Example 2 Another prominent example is the mathematical program with vanishing constraints (MPVC)

$$MPVC(\omega) \qquad \min f(x,\omega) \quad \text{subject to} \quad \begin{aligned} & g(x,\omega) \leq 0,\ h(x,\omega) = 0, \\ & \left. \begin{aligned} & G_i(x,\omega) \geq 0 \\ & G_i(x,\omega)H_i(x,\omega) \leq 0 \end{aligned} \right\} i = 1,\ldots,m_V, \end{aligned}$$

where $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I}$, $h : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_E}$ and $G,H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_V}$. For more details on MPVCs we refer the reader to [1,15].

Again, the problem $MPVC(\omega)$ can be written in the form (1) with

$$q := (g,h,(G_1,H_1),\ldots(G_{m_V},H_{m_V})),$$

$$P := \mathbb{R}^{m_I}_- \times \{0\}^{m_E} \times Q^{m_V}_{VC},$$

where $Q_{VC} := \{(a,b) \in \mathbb{R}_+ \times \mathbb{R} \,|\, ab \leq 0\}$ is the union of the two convex polyhedra $\mathbb{R}_+ \times \mathbb{R}_-$ and $\{0\} \times \mathbb{R}_+$.

If f is partially differentiable with respect to x, then the first order optimality conditions at a local minimizer x for $P(\omega)$ can be written as a generalized equation

$$0 \in \nabla_x f(x,\omega) + \hat{N}(x;\mathscr{F}(\omega)),$$

where $\hat{N}(x;\mathscr{F}(\omega))$ stands for the *Fréchet normal cone* to the set $\mathscr{F}(\omega)$ at x and

$$\mathscr{F}(\omega) := \{x \in \mathbb{R}^n \,|\, q(x,\omega) \in P\} \tag{3}$$

denotes the feasible region of the problem $P(\omega)$. We also consider the generalized equation

$$GE(\omega) \qquad 0 \in F(x,\omega) + \hat{N}(x;\mathscr{F}(\omega)), \tag{4}$$

for arbitrary continuous mappings $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$.

Throughout this paper we will make the following assumption:

Assumption 1 *There are neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ such that f and q are twice partially differentiable with respect to x, F is partially differentiable with respect to x on $U \times W$, $F(x,\omega)$, $q(x,\omega)$, $\nabla_x f(x,\omega)$, $\nabla_x q(x,\omega)$, $\nabla_x F(x,\omega)$, $\nabla_x^2 f(x,\omega)$ and $\nabla_x^2 q(x,\omega)$ are continuous at $(\bar{x},\bar{\omega})$ and $\nabla_x^2 f(\cdot,\omega)$, $\nabla_x^2 q(\cdot,\omega)$ are continuous on U for every $\omega \in W$.*

Given a fixed parameter $\bar{\omega}$ and a solution $\bar{x}$ of $P(\bar{\omega})$ respectively $GE(\bar{\omega})$, we are interested in estimates of the distance of solutions x of problem $P(\omega)$ respectively $GE(\omega)$ to $\bar{x}$ for parameters ω belonging to some neighborhood of $\bar{\omega}$.

We will present such estimates in terms of the mappings $e_l, \tau_l, \hat{\tau}_l : \Omega \to \mathbb{R}$, $l = 1,2$, given by

$$e_l(\omega) = \|\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega})\| + \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\|^{\frac{1}{l}}$$

and

$$\tau_l(\omega) := \|\nabla_x f(\bar{x},\omega) - \nabla_x f(\bar{x},\bar{\omega})\| + e_l(\omega), \quad \hat{\tau}_l(\omega) := \|F(\bar{x},\omega) - F(\bar{x},\bar{\omega})\| + e_l(\omega).$$

The quantities $\tau_l(\omega)$ and $\hat{\tau}_l(\omega)$ measure how much the problem data at the reference point $\bar{x}$ for the perturbed problem $P(\omega)$ and $GE(\omega)$, respectively, differ from that for the unperturbed problem $P(\bar{\omega})$ and $GE(\bar{\omega})$, respectively.

In case that we can bound the distance of a solution x of $P(\omega)$ $(GE(\omega))$ to $\bar{x}$ by the estimate $L\tau_1(\omega)$ $(L\hat{\tau}_1(\omega))$, where L denotes some constant, we speak of *upper Lipschitz stability* of the solutions. We speak of *upper Hölder stability* when a bound of the form $\|x - \bar{x}\| \leq L\tau_2(\omega)$ (or $\|x - \bar{x}\| \leq L\hat{\tau}_2(\omega)$) is available. This notation is motivated by the situation that Ω is a metric space equipped with the metric d, and $q(\bar{x}, \cdot)$, $\nabla_x q(\bar{x}, \cdot)$ and $\nabla_x f(\bar{x}, \cdot)$ (or $F(\bar{x}, \cdot)$) are Lipschitz near $\bar{\omega}$, because in this circumstance the bounds are of the form $\|x - \bar{x}\| \leq Ld(\omega, \bar{\omega})$ and $\|x - \bar{x}\| \leq L\sqrt{d(\omega, \bar{\omega})}$, respectively. Note that in this case the property of upper Lipschitz stability is also called *isolated calmness* in the literature (see, e.g.,[6]).

Many quantitative stability results are known for the parameter dependent nonlinear programming problem $NLP(\omega)$. We refer to the monographs [4,6,20] and the references therein. Compared with the huge amount of stability results for NLP, very little research has been done with the stability of $P(\omega)$. Most of the results are known for $MPEC(\omega)$, see e.g. [3,16,18,27,29]. Stability of M-stationarity solutions was characterized in the recent paper [5] for a special type of problems with complementarity constraints. Sensitivity and stability results for MPVC are given in [17]. In the recent paper [12], Guo, Lin and Ye presented various stability results for more general problems. In particular, they proved upper Lipschitz stability for stationary pairs consisting of stationary solutions and associated multipliers under the structural assumption that the graph of the limiting normal cone mapping to the set P is the union of finitely many convex polyhedra.

In contrary to the stability results of [12], we focus our interest on the stability of solutions of (4) on its own and not of stationary pairs. This has the advantage that our theory is also applicable in case when multipliers do not exist or the multipliers do not behave continuous, cf. Examples 3 and 4 below.

Our results are mainly based on characterizations of metric subregularity as introduced in [8–11]. The main constraint qualifications used in this paper are that, at the reference point $\bar{x}$, either the first order or the second order sufficient conditions for metric subregularity are fulfilled for the problem $P(\bar{\omega})$. Although the property of metric subregularity is not stable in general, we will see that the sufficient conditions of order l, $l = 1, 2$, for metric subregularity guarantee some stability. In particular, we will prove that there is some constant γ such that for all points x feasible for the problem $P(\omega)$ and satisfying $\|x - \bar{x}\| > \gamma e_l(\omega)$, the constraints of $P(\omega)$ are metrically regular near $(x, 0)$ with some uniform modulus. This result allows us to divide the solution sets of $P(\omega)$, similarly for $GE(\omega)$, into two parts: one part is contained in a ball around $\bar{x}$ with radius $\gamma e_l(\omega)$ and behaves upper Lipschitz $(l = 1)$ or Hölder $(l = 2)$ stable by the definition, whereas the other part is outside this ball and we can assume metric regularity. Moreover, we can show that locally optimal solutions for the perturbed problems $P(\omega)$ exist, provided ω is sufficiently close to $\bar{\omega}$. The obtained results are partially new even in case of $NLP(\omega)$.

The rest of the paper is organized as follows: In section 2 we recall the basic definitions of metric (sub)regularity and their directional versions, together with the characterization of these properties by objects from generalized differentiation. In section 3 we give some stability results for the feasible point mapping $\mathscr{F}$. Section 4 is devoted to the stability behavior of solutions of the generalized equation $GE(\omega)$ and the optimization problem $P(\omega)$, respectively. In section 5 we apply the obtained results to the special problem $MPEC(\omega)$ by explicitly calculating all objects from generalized differentiation. Moreover we present some examples.

2 Preliminaries

We start by recalling several definitions and results from variational analysis. Let $\Gamma \subset \mathbb{R}^n$ be an arbitrary closed set and $x \in \Gamma$. The *contingent* (also called *tangent* or *Bouligand*) *cone* to Γ at x, denoted by $T(x; \Gamma)$, is given by

$$T(x; \Gamma) := \{u \in \mathbb{R}^n \mid \exists (u_k) \to u, (t_k) \downarrow 0 : x + t_k u_k \in \Gamma\}.$$

We denote by

$$\hat{N}(x; \Gamma) = \{\xi \in \mathbb{R}^n \mid \limsup_{x' \xrightarrow{\Gamma} x} \frac{\xi^T (x' - x)}{\|x' - x\|} \leq 0\} \tag{5}$$

the *regular* (or *Fréchet*) *normal cone* to Γ. Finally, the *limiting* (or *basic/Mordukhovich*) *normal cone* to Γ at x is defined by

$$N(x;\Gamma) := \{\xi \mid \exists(x_k) \xrightarrow{\Gamma} x, \ (\xi_k) \to \xi : \xi_k \in \hat{N}(x_k;\Gamma) \forall k\}.$$

If $x \notin \Gamma$, we put $T(x;\Gamma) = \emptyset$, $\hat{N}(x;\Gamma) = \emptyset$ and $N(x;\Gamma) = \emptyset$.

The limiting normal cone is generally nonconvex, whereas the regular normal cone is always convex. In the case of a convex set Γ, both the regular normal cone and the limiting normal cone coincide with the standard normal cone from convex analysis and, moreover, the contingent cone is equal to the tangent cone in the sense of convex analysis.

Note that $\xi \in \hat{N}(x;\Gamma) \Leftrightarrow \xi^T u \leq 0 \ \forall u \in T(x;\Gamma)$, i.e., $\hat{N}(x;\Gamma)$ is the polar cone of $T(x;\Gamma)$.

Given a multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M := \{(x,y) \in X \times Y \mid y \in M(x)\}$ from its graph, the *coderivative* of M at $(\bar{x}, \bar{y})$ is a multifunction $D^*M(\bar{x},\bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ with the values $D^*M(\bar{x},\bar{y})(\eta) := \{\xi \in \mathbb{R}^n \mid (\xi, -\eta) \in N((\bar{x},\bar{y}); \mathrm{gph}\, M)\}$, i.e., $D^*M(\bar{x},\bar{y})(\eta)$ is the collection of all $\xi \in \mathbb{R}^n$ for which there are sequences $(x_k, y_k) \to (\bar{x}, \bar{y})$ and $(\xi_k, \eta_k) \to (u, v)$ with $(\xi_k, -\eta_k) \in \hat{N}((x_k, y_k); \mathrm{gph}\, M)$.

For more details we refer to the monographs [23, 26]

The following directional versions of these limiting constructions were introduced in [9], see also [11] for the finite dimensional setting. Given a direction $u \in \mathbb{R}^n$, the limiting normal cone to a subset $\Gamma \subset \mathbb{R}^n$ in direction u at $x \in \Gamma$ is defined by

$$N(x;\Gamma;u) := \{\xi \in \mathbb{R}^n \mid \exists(t_k) \downarrow 0, \ (u_k) \to u, \ (\xi_k) \to \xi : \xi_k \in \hat{N}(x+t_ku_k;\Gamma) \forall k\}.$$

For a multifunction $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a direction $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$, the coderivative of M in direction (u,v) at $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$ is defined as the multifunction $D^*M((\bar{x},\bar{y}); (u,v)) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ given by $D^*M((\bar{x},\bar{y}); (u,v))(\eta) := \{\xi \in \mathbb{R}^n \mid (\xi, -\eta) \in N((\bar{x},\bar{y}); \mathrm{gph}\, M; (u,v))\}$.

Note that, by the definition, we have $N(x;\Gamma;0) = N(x;\Gamma)$ and $D^*M((\bar{x},\bar{y}); (0,0)) = D^*M(\bar{x},\bar{y})$. Further, $N(x;\Gamma;u) \subset N(x;\Gamma)$ for all u and $N(x;\Gamma;u) = \emptyset$ if $u \notin T(x;\Gamma)$.

We now turn our attention to the set P from problem (1). For every $y \in P$, we denote by $\mathscr{P}(y) := \{i \in \{1,\ldots,p\} \mid y \in P_i\}$ the index set of polyhedra containing y, and for each $i \in \mathscr{P}(y)$, we denote by $\mathscr{A}_i(y) := \{j \in \{1,\ldots,m_i\} \mid a_{ij}^T y = b_{ij}\}$ the index set of active constraints. Then for every $y \in P$ we have

$$T(y;P) = \bigcup_{i \in \mathscr{P}(y)} T(y;P_i) = \bigcup_{i \in \mathscr{P}(y)} \{z \in \mathbb{R}^m \mid a_{ij}^T z \leq 0, j \in \mathscr{A}_i(y)\},$$

$$\hat{N}(y;P) = \bigcap_{i \in \mathscr{P}(y)} \hat{N}(y;P_i) = \bigcap_{i \in \mathscr{P}(y)} \{\sum_{j \in \mathscr{A}_i(y)} \mu_{ij} a_{ij} \mid \mu_{ij} \geq 0, j \in \mathscr{A}_i(y)\}.$$

Some formulas for the limiting normal cone respectively its directional counterpart can be found in [10].

The following lemma will be useful for applications.

Lemma 1 *Let* $\Gamma = \Gamma_1 \times \ldots \times \Gamma_l \subset \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_l}$ *be the Cartesian product of the closed sets* Γ_i *and* $y = (y_1,\ldots,y_l) \in \Gamma$. *Then*

$$T(y;\Gamma) \subset T(y_1;\Gamma_1) \times \ldots \times T(y_l;\Gamma_l), \tag{6}$$

and for every $u = (u_1,\ldots,u_l) \in T(y;\Gamma)$ *one has*

$$N(y;\Gamma;u) \subset N(y_1;\Gamma_1;u_1) \times \ldots \times N(y_l;\Gamma_l;u_l). \tag{7}$$

Furthermore, equality holds in both inclusions if Γ *is the union of finitely many convex polyhedra.*

Proof The inclusion (6) can be found in [26, Proposition 6.41], and the inclusion (7) follows immediately from the formula for the regular normal cone from this proposition and the definition of the directional normal cone. To show equality, assume that Γ coincides with the set P from (1), and let $(u_1,\ldots,u_l) \in T(y_1;\Gamma_1) \times \ldots \times T(y_l;\Gamma_l)$ and $(\xi_1,\ldots,\xi_l) \in N(y_1;\Gamma_1;u_1) \times \ldots \times N(y_l;\Gamma_l;u_l)$. Then there are sequences $(t_{ik}) \downarrow 0, (u_{ik}) \to u_i, (\xi_{ik}) \to \xi_i, i = 1,\ldots,l$, with $z_k := (y_1 + t_{1k}u_{1k},\ldots,y_l + t_{lk}u_{lk}) \in \Gamma_1 \times \ldots \times \Gamma_l$ and $(\xi_{1k},\ldots,\xi_{lk}) \in \hat{N}(y_1 + t_{1k}u_{1k};\Gamma_1) \times \ldots \times \hat{N}(y_l + t_{lk}u_{lk};\Gamma_l)$ for all k. By passing to subsequences we can assume that there are index sets $\mathscr{P} \subset \{1,\ldots,p\}$ and $\mathscr{A}_i, i \in \mathscr{P}$, such that $\mathscr{P} = \mathscr{P}(z_k)$ and $\mathscr{A}_i = \mathscr{A}_i(z_k), i \in \mathscr{P}$, hold for all k. Furthermore, we can assume that for each $i \notin \mathscr{P}$ there is an index j_i with $a_{ij_i} z_k > b_{ij_i}$ for all k.

Since the convex polyhedra P_j are closed, for each $j \in \mathscr{P}$ we also have $\lim_k z_k = y \in P_j$ and therefore $(1-\alpha)y + \alpha z_k \in P_j \; \forall \alpha \in [0,1], \forall k$. Further, for every $\alpha \in (0,1)$ and every k we have $\mathscr{A}_i = \mathscr{A}_i((1-\alpha)y + \alpha z_k)$ and $a_{ij_i}((1-\alpha)y + \alpha z_k) > b_{ij_i}$, $i \notin \mathscr{P}$, showing $\hat{N}((1-\alpha)y + \alpha z_k; \Gamma) = \hat{N}(z_k; \Gamma)$ and, together with [26, Proposition 6.41],

$$\hat{N}(z_k; \Gamma) = \hat{N}(y_1 + t_{1k}u_{1k}; \Gamma_1) \times \ldots \times \hat{N}(y_l + t_{lk}u_{lk}; \Gamma_l)$$
$$= \hat{N}((1-\alpha)y + \alpha z_k; \Gamma) = \hat{N}(y_1 + \alpha t_{1k}u_{1k}; \Gamma_1) \times \ldots \times \hat{N}(y_l + \alpha t_{lk}u_{lk}; \Gamma_l).$$

Now let $t_k := \min_i t_{ik}$. Then for each $i = 1, \ldots, l$ we have

$$(1 - \frac{t_k}{t_{ik}})y + \frac{t_k}{t_{ik}}z_k = (y_1 + t_{1k}\frac{t_k}{t_{ik}}u_{1k}, \ldots, y_i + t_k u_{ik}, \ldots, y_l + t_{lk}\frac{t_k}{t_{ik}}u_{lk}) \in \Gamma_1 \times \ldots \times \Gamma_l,$$

$$(\xi_{1k}, \ldots, \xi_{lk}) \in \hat{N}(z_k, \Gamma) = \hat{N}(y_1 + t_{1k}\frac{t_k}{t_{ik}}u_{1k}; \Gamma_1) \times \ldots \times \hat{N}(y_i + t_k u_{ik}; \Gamma_i) \times \ldots \times \hat{N}(y_l + t_{lk}\frac{t_k}{t_{ik}}u_{lk}; \Gamma_l),$$

showing $y_i + t_k u_{ik} \in \Gamma_i$ and $\xi_{ik} \in \hat{N}(y_i + t_k u_{ik}; \Gamma_i)$. Hence $\tilde{z}_k := (y_1 + t_k u_{1k}, \ldots, y_l + t_k u_{lk}) \in \Gamma_1 \times \ldots \times \Gamma_l = \Gamma$ and, by using [26, Proposition 6.41] again, $(\xi_{1k}, \ldots, \xi_{lk}) \in \hat{N}(y_1 + t_k u_{1k}; \Gamma_1) \times \ldots \times \hat{N}(y_l + t_k u_{lk}; \Gamma_l) = \hat{N}(\tilde{z}_k; \Gamma)$, showing $(u_1, \ldots, u_l) \in T(x; \Gamma)$ and $(\xi_1, \ldots, \xi_l) \in N(y; \Gamma; u)$. $\square$

In this paper we are mainly concerned with multifunctions $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ of the form $M(x) = q(x) - \Gamma$, where $q : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and single valued. For this special type of multifunctions, we now give a formula for the directional limiting coderivative of M.

Lemma 2 *Let the multifunction $M : \mathbb{R}^n \rightrightarrows R^m$ be given by $M(x) := q(x) - \Gamma$, where $q : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable and $\Gamma \subset \mathbb{R}^m$ is a closed set. Further, let $\bar{x} \in q^{-1}(\Gamma)$ and $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$ be given. Then*

$$D^*M((\bar{x};0);(u,v))(\lambda) = \{\nabla q(\bar{x})^T \lambda \mid \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u - v)\}. \tag{8}$$

Proof Let $x^* \in D^*M((\bar{x};0);(u,v))(\lambda)$, that is $(x^*, -\lambda) \in N((\bar{x},0); \mathrm{gph}\,M; (u,v))$ by the definition, and consider corresponding sequences $(t_k) \downarrow 0$, $(u_k, v_k) \to (u,v)$ and $(x_k^*, \lambda_k) \to (x^*, \lambda)$ with $(x_k^*, -\lambda_k) \in \hat{N}((\bar{x} + t_k u_k, t_k v_k); \mathrm{gph}\,M)$. Hence $(\bar{x} + t_k u_k, t_k v_k) \in \mathrm{gph}\,M$ implying $\gamma_k := q(\bar{x} + t_k u_k) - t_k v_k \in \Gamma$. Since for all x and all $\gamma \in \Gamma$

$$\hat{N}((x, q(x) - \gamma); \mathrm{gph}\,M) = \{(\nabla q(x)^T \eta, -\eta) \mid \eta \in \hat{N}(\gamma; \Gamma)\}, \tag{9}$$

we obtain $\lambda_k \in \hat{N}(\gamma_k; \Gamma)$ and $x_k^* = \nabla q(\bar{x} + t_k u_k)^T \lambda_k$. Hence $x^* = \nabla q(\bar{x})^T \lambda$ and $\lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u - v)$, since $\lim_{k \to \infty}(\gamma_k - q(\bar{x}))/t_k = \nabla q(\bar{x})u - v$, and $D^*M((\bar{x};0);(u,v))(\lambda) \subset \{\nabla q(\bar{x})^T \lambda \mid \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u - v)\}$ follows.

To show the converse inclusion, we fix an arbitrary element $\lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u - v)$ and consider sequences $(t_k) \downarrow 0$, $(\gamma_k) \to \gamma$ and $(\lambda_k) \to \lambda$ with $\lambda_k \in \hat{N}(\gamma_k; \Gamma)$ and $\lim_{k \to \infty}(\gamma_k - q(\bar{x}))/t_k = \nabla q(\bar{x})u - v$. By setting $u_k := u$, $v_k := (q(\bar{x} + t_k u_k) - \gamma_k)/t_k$ we have $(\bar{x} + t_k u_k, t_k v_k) \in \mathrm{gph}\,M$ and $(\nabla q(\bar{x} + t_k u_k)^T \lambda_k, -\lambda_k) \in \hat{N}((\bar{x} + t_k u_k, q(\bar{x} + t_k u_k) - \gamma_k); \mathrm{gph}\,M) = \hat{N}((\bar{x} + t_k u_k, t_k v_k); \mathrm{gph}\,M)$ by (9). Passing to the limit and taking into account that

$$\lim_{k \to \infty} v_k = \lim_{k \to \infty} \frac{q(\bar{x} + t_k u_k) - q(\bar{x}) - (\gamma_k - q(\bar{x}))}{t_k} = \nabla q(\bar{x})u - (\nabla q(\bar{x})u - v) = v,$$

we obtain $(\nabla q(\bar{x})^T \lambda, -\lambda) \in N((\bar{x},0); \mathrm{gph}\,M; (u,v))$, and, by the definition of the directional limiting coderivative, we can conclude $\nabla q(\bar{x})^T \lambda \in D^*M((\bar{x};0);(u,v))(\lambda)$. This completes the proof. $\square$

Now we consider the notions of metric regularity and subregularity, respectively, and its characterization by coderivatives and limiting normal cones.

Definition 1 Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction, let $(\bar{x}, \bar{y}) \in \mathrm{gph}\,M$ and let $\kappa > 0$.

1. M is called *metrically regular with modulus κ* around $(\bar{x}, \bar{y})$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathrm{d}(x, M^{-1}(y)) \leq \kappa \mathrm{d}(y, M(x)) \; \forall (x,y) \in U \times V. \tag{10}$$

2. M is called *metrically subregular with modulus* κ at $(\bar{x}, \bar{y})$ if there is a neighborhood U of $\bar{x}$ such that

$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \kappa \mathrm{d}(\bar{y}, M(x)) \ \forall x \in U. \tag{11}$$

It is well known that metric regularity of the multifunction M around $(\bar{x}, \bar{y})$ is equivalent to the Aubin property of the inverse multifunction M^{-1}. A multifunction $S : R^m \rightrightarrows R^n$ has the *Aubin property* with modulus $L \geq 0$ around some point $(\bar{y}, \bar{x}) \in \mathrm{gph}\, S$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$S(y_1) \cap U \subset S(y_2) + L\|y_1 - y_2\|\mathscr{B}_{\mathbb{R}^n} \quad \forall y_1, y_2 \in V,$$

where $\mathscr{B}_{\mathbb{R}^n}$ denotes the unit ball in $\mathbb{R}^n$ in the underlying norm.

Metric subregularity of M at $(\bar{x}, \bar{y})$ is equivalent to the property of *calmness* of the inverse multifunction M^{-1}. A multifunction $S : R^m \rightrightarrows R^n$ is called *calm* with modulus $L \geq 0$ at $(\bar{y}, \bar{x}) \in \mathrm{gph}\, S$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$S(y) \cap U \subset S(\bar{y}) + L\|y - \bar{y}\|\mathscr{B}_{\mathbb{R}^n} \quad \forall y \in V.$$

To introduce directional versions of metric (sub)regularity it is convenient to define the following neighborhoods of a direction: Given a direction $u \in \mathbb{R}^n$ and positive numbers $\rho, \delta > 0$, the set $V_{\rho,\delta}(u)$ is given by

$$V_{\rho,\delta}(u) := \{z \in \rho\mathscr{B}_{\mathbb{R}^n} \mid \big| \|u\| z - \|z\| u \big| \leq \delta \|z\| \, \|u\| \}. \tag{12}$$

This can also be written in the form

$$V_{\rho,\delta}(u) = \begin{cases} \{0\} \cup \left\{ z \in \rho\mathscr{B}_{\mathbb{R}^n} \setminus \{0\} \,\middle|\, \left\| \frac{z}{\|z\|} - \frac{u}{\|u\|} \right\| \leq \delta \right\} & \text{if } u \neq 0, \\ \rho\mathscr{B}_{\mathbb{R}^n} & \text{if } u = 0. \end{cases}$$

Definition 2 Let $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction, let $(\bar{x}, \bar{y}) \in \mathrm{gph}\, M$ and let $\kappa > 0$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$.

1. M is called *metrically regular with modulus* κ *in direction* $w := (u, v)$ at $(\bar{x}, \bar{y})$ if there are positive real numbers ρ and δ such that
$$\mathrm{d}(x, M^{-1}(y)) \leq \kappa \mathrm{d}(y, M(x)) \tag{13}$$
   holds for all $(x, y) \in (\bar{x}, \bar{y}) + V_{\rho,\delta}(w)$ with $\|w\|\mathrm{d}((x, y), \mathrm{gph}\, M) \leq \delta\|w\|\|(x, y) - (\bar{x}, \bar{y})\|$.
2. M is called *metrically subregular with modulus* κ *in direction* u at $(\bar{x}, \bar{y})$ if there are positive real numbers ρ and δ such that
$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \kappa \mathrm{d}(\bar{y}, M(x)) \ \forall x \in \bar{x} + V_{\rho,\delta}(u). \tag{14}$$

Note that metric regularity in direction $(0,0)$ and metric subregularity in direction 0 are equivalent to the properties of metric regularity and metric subregularity, respectively. Further, metric regularity in direction $(u, 0)$ implies metric subregularity in direction u, cf. [9, Lemma 1].

Theorem 1 *Let the multifunction $M : \mathbb{R}^n \rightrightarrows R^m$ be given by $M(x) := q(x) - \Gamma$, where $q : R^n \to \mathbb{R}^m$ is continuously differentiable and $\Gamma \subset \mathbb{R}^m$ is a closed set. Further let $(\bar{x}, 0) \in \mathrm{gph}\, M$, $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ be given.*

1. *(Mordukhovich criterion): M is metrically regular around $(\bar{x}, 0)$ if and only if*
$$\nabla q(\bar{x})^T \lambda = 0, \lambda \in N(q(\bar{x}); \Gamma) \implies \lambda = 0.$$

2. *M is metrically regular in direction (u, v) at $(\bar{x}, 0)$ if and only if*
$$\nabla q(\bar{x})^T \lambda = 0, \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u - v) \implies \lambda = 0.$$

3. *Assume that q is twice Fréchet differentiable at $\bar{x}$, that Γ is the union of finitely many convex polyhedra and the condition*
$$\nabla q(x)^T \lambda = 0, \ \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u), \ u^T \nabla^2(\lambda^T q)(\bar{x})u \geq 0 \implies \lambda = 0$$

   is fulfilled. Then M is metrically subregular in direction u at $(\bar{x}, 0)$.

Proof The first statement is a specialization of the more general statement [23, Theorem 4.18] and can be found e.g. in [26, Example 9.44]. Similarly, the second statement follows from [9, Theorem 5] by taking into account that the involved spaces are finite dimensional and (8). Finally, the last statement follows from [11, Theorem 2.6] □

Taking into account that in finite dimensions a multifunction is metrically subregular if and only if it is metrically subregular in every nonzero direction, see [11, Lemma 2.7], and $N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u) = \emptyset$ if $\nabla q(\bar{x})u \notin T(q(\bar{x}); \Gamma)$, we obtain the following consequence of Theorem 1:

Corollary 1 *Let the multifunction $M : \mathbb{R}^n \rightrightarrows R^m$ be given by $M(x) := q(x) - \Gamma$, where $q : R^n \to \mathbb{R}^m$ is continuously differentiable and $\Gamma \subset \mathbb{R}^m$ is a closed set. Then M is metrically subregular at $(\bar{x}, 0)$ if one of the following conditions is fulfilled:*

1. First order sufficient condition for metric subregularity (FOSCMS)*: For every $0 \neq u \in \mathbb{R}^n$ with $\nabla q(\bar{x})u \in T(q(\bar{x}); \Gamma)$ one has*

$$\nabla q(\bar{x})^T \lambda = 0, \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u) \implies \lambda = 0.$$

2. Second order sufficient condition for metric subregularity (SOSCMS)*: q is twice Fréchet differentiable at $\bar{x}$, Γ is the union of finitely many convex polyhedra, and for every $0 \neq u \in \mathbb{R}^n$ with $\nabla q(\bar{x})u \in T(q(\bar{x}); \Gamma)$ one has*

$$\nabla q(x)^T \lambda = 0, \ \lambda \in N(q(\bar{x}); \Gamma; \nabla q(\bar{x})u), \ u^T \nabla^2 (\lambda^T q)(\bar{x})u \geq 0 \implies \lambda = 0.$$

Next we consider optimality conditions for the problem

$$\min_x f(x) \quad \text{subject to} \quad q(x) \in \Gamma \tag{15}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $q : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable and $\Gamma \subset \mathbb{R}^m$ is closed. We denote the feasible region of (15) by $\mathscr{F}$. Given a feasible point $\bar{x} \in \mathscr{F}$, we define the *linearized cone* by

$$T^{\text{lin}}(\bar{x}) := \{u \in \mathbb{R}^n \,|\, \nabla q(\bar{x})u \in T(q(\bar{x}); \Gamma)\}$$

and the *critical cone* by

$$\mathscr{C}(\bar{x}) = \{u \in T^{\text{lin}}(\bar{x}) \,|\, \nabla f(\bar{x})u \leq 0\}.$$

Definition 3 Let $\bar{x} \in \mathscr{F}$ be feasible for the problem (15).

1. We say that $\bar{x}$ is *B-stationary* if

$$0 \in \nabla f(\bar{x}) + \hat{N}(\bar{x}; \mathscr{F}).$$

2. We say that $\bar{x}$ is *M-stationary* if

$$0 \in \nabla f(\bar{x}) + \nabla q(\bar{x})^T N(q(\bar{x}); \Gamma).$$

Since $\hat{N}(\bar{x}; \mathscr{F})$ is the polar cone of $T(\bar{x}; \mathscr{F})$, B-stationarity can be equivalently written as $\nabla f(\bar{x})u \geq 0$ $\forall u \in T(\bar{x}; \mathscr{F})$. Hence, B-stationarity means that there does not exist feasible descent directions at $\bar{x}$, which is a first-order necessary condition for $\bar{x}$ being a local minimizer.

Usually B-stationarity is not very useful in practice, since the regular normal cone of $\mathscr{F}$ at $\bar{x}$ is difficult to compute, in general. Hence, M-stationarity conditions are used as first-order necessary condition, which, however, are only valid under some constraint qualification condition. Indeed, the following lemma even shows: Under some weak constraint qualification, M-stationarity is not only necessary for local minimizers, but also for B-stationarity.

Lemma 3 *Let $\bar{x} \in \mathscr{F}$ be B-stationary for the problem* (15)*, and assume that either $\mathscr{C}(\bar{x}) = \{0\}$ and the multifunction $\tilde{M}(u) := \nabla q(\bar{x})u - T(q(\bar{x}); \Gamma)$ is metrically subregular at $(0, 0)$ or there exists $\bar{u} \in \mathscr{C}(\bar{x})$ such that the mapping $M(x) := q(x) - \Gamma$ is metrically subregular in direction $\bar{u}$ at $(\bar{x}, 0)$. Then $\bar{x}$ is M-stationary.*

Proof If $\mathscr{C}(\bar{x}) = \{0\}$, then 0 is solution of the problem

$$\min_{u \in \mathbb{R}^n} \nabla f(\bar{x})u \quad \text{subject to} \quad 0 \in \nabla q(\bar{x})u - T(q(\bar{x}); \Gamma). \tag{16}$$

Since the mapping $\tilde{M}(u)$ is assumed to be metrically subregular at $(0,0)$, by [9, Corollary 2] there is some λ such that $0 \in \nabla f(\bar{x}) + D^*\tilde{M}(0,0)(\lambda)$. By [26, Example 9.44, Proposition 6.27] we conclude $-\nabla f(\bar{x}) \in D^*\tilde{M}(0,0)(\lambda) = \{\nabla q(\bar{x})^T \lambda\}$ and $\lambda \in N(0; T(q(\bar{x}); \Gamma)) \subset N(q(\bar{x}); \Gamma)$, showing M-stationarity of $\bar{x}$. In the second case, since $-\nabla f(\bar{x}) \in \hat{N}(\bar{x}; \mathscr{F})$, by [26, Theorem 6.11] there is some smooth function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla \tilde{f}(\bar{x}) = \nabla f(\bar{x})$ and $\bar{x}$ is a global minimizer of the problem

$$\min \tilde{f}(x) \quad \text{subject to} \quad x \in \mathscr{F} = \{x \mid 0 \in M(x)\}.$$

Now, again by [9, Corollary 2], we obtain that there is some multiplier λ such that $-\nabla f(\bar{x}) = -\nabla \tilde{f}(\bar{x}) \in D^*M(\bar{x}; 0)(\lambda)$, and from [26, Example 9.44] we obtain $D^*M(\bar{x}; 0)(\lambda) = \{\nabla q(\bar{x})^T \lambda\}$ and $\lambda \in N(q(\bar{x}); \Gamma)$. $\qquad\square$

Remark 1 If Γ is the union of finitely many convex polyhedra, then so is $T(q(\bar{x}); \Gamma)$, and hence the multifunction $\tilde{M}$ is a polyhedral multifunction and consequently metrically subregular by Robinson's result [25].

Given any element $g \in \hat{N}(\bar{x}; \mathscr{F})$, then, by the definition, $\bar{x}$ is a B-stationary solution of the problem

$$\min -g^T x \quad \text{subject to} \quad q(x) \in \Gamma.$$

If M is metrically subregular in $(\bar{x}, 0)$, then it is also metrically subregular in every direction, and further, $\tilde{M}$ is metrically subregular by [8, Proposition 2.1]. Hence $\bar{x}$ is also M-stationary, and we obtain $g \in \nabla q(\bar{x})^T N(q(\bar{x}); \Gamma)$. Thus we rediscover the following formula, which would also follow from [13, Theorem 4.1]:

Corollary 2 *Let $q : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable, $\Gamma \subset \mathbb{R}^m$ be closed and let $\bar{x} \in q^{-1}(\Gamma)$. If the multifunction $x \rightrightarrows q(x) - \Gamma$ is metrically subregular at $(\bar{x}, 0)$, then*

$$\hat{N}(\bar{x}; q^{-1}(\Gamma)) \subset \nabla q(\bar{x})^T N(q(\bar{x}); \Gamma).$$

3 Stability properties of the feasible set mapping $\mathscr{F}$

In this section, we study conditions for metric regularity as well as Hölder and Lipschitz stability of the feasible set mapping $\mathscr{F}$. In particular, we introduce and discuss two regularity properties which imply metric regularity around points *near* some reference point.

We start with the following technical lemma, where the functions q and F are supposed to satisfy the basic Assumption 1:

Lemma 4 *For every $\varepsilon > 0$ there are neighborhoods $\hat{U}_\varepsilon$ of $\bar{x}$ and $\hat{W}_\varepsilon$ of $\bar{\omega}$ such that*

$$\|q(x, \omega) - q(x, \bar{\omega})\| + \|\nabla_x q(x, \omega) - \nabla_x q(x, \bar{\omega})\| \leq e_1(\omega) + \varepsilon \|x - \bar{x}\|, \tag{17}$$

$$\|F(x, \omega) - F(x, \bar{\omega})\| + \|q(x, \omega) - q(x, \bar{\omega})\| + \|\nabla_x q(x, \omega) - \nabla_x q(x, \bar{\omega})\| \leq \hat{\tau}_1(\omega) + \varepsilon \|x - \bar{x}\|, \tag{18}$$

$$\begin{aligned}
\|q(x, \omega) - q(x, \bar{\omega})\| &\leq \|q(\bar{x}, \omega) - q(\bar{x}, \bar{\omega})\| + e_2(\omega)\|x - \bar{x}\| + \frac{\varepsilon}{2}\|x - \bar{x}\|^2 \\
&\leq e_2(\omega)^2 + e_2(\omega)\|x - \bar{x}\| + \frac{\varepsilon}{2}\|x - \bar{x}\|^2
\end{aligned} \tag{19}$$

hold for all $(x, \omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$.

Proof Let $\varepsilon > 0$ be fixed, and choose a ball $\hat{U}_\varepsilon$ around $\bar{x}$ and a neighborhood $\hat{W}_\varepsilon$ of $\bar{\omega}$ such that

$$\|\nabla_x F(x,\omega) - \nabla_x F(\bar{x},\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(\bar{x},\bar{\omega})\| + \|\nabla_x^2 q(x,\omega) - \nabla_x^2 q(\bar{x},\bar{\omega})\| \leq \frac{\varepsilon}{2},$$

and consequently

$$\|\nabla_x F(x,\omega) - \nabla_x F(x,\bar{\omega})\| + \|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| + \|\nabla_x^2 q(x,\omega) - \nabla_x^2 q(x,\bar{\omega})\| \leq \varepsilon$$

hold for all $(x,\omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$.

For arbitrarily fixed $(x,\omega) \in \hat{U}_\varepsilon \times \hat{W}_\varepsilon$ we can find $u, \mu \in \mathscr{B}_{\mathbb{R}^n}$, $\xi, \lambda \in \mathscr{B}_{\mathbb{R}^m}$ such that

$$\|F(x,\omega) - F(x,\bar{\omega})\| = \mu^T(F(x,\omega) - F(x,\bar{\omega})),$$
$$\|q(x,\omega) - q(x,\bar{\omega})\| = \xi^T(q(x,\omega) - q(x,\bar{\omega})),$$
$$\|\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega})\| = \lambda^T(\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega}))u.$$

Then there is a point z belonging to the line segment $[\bar{x},x] \subset U_\varepsilon$ such that

$$\begin{aligned}
\xi^T&(q(x,\omega) - q(x,\bar{\omega})) + \lambda^T(\nabla_x q(x,\omega) - \nabla_x q(x,\bar{\omega}))u \\
&= \xi^T(q(\bar{x},\omega) - q(\bar{x},\bar{\omega})) + \lambda^T(\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega}))u + \big(\xi^T(\nabla_x q(z,\omega) - \nabla_x q(z,\bar{\omega})) \\
&\quad + u^T(\nabla_x^2(\lambda^T q)(z,\omega) - \nabla_x^2(\lambda^T q)(z,\bar{\omega}))\big)(x-\bar{x}) \\
&\leq e_1(\omega) + \big(\|\nabla_x q(z,\omega) - \nabla_x q(z,\bar{\omega})\| + \|\nabla_x^2 q(z,\omega) - \nabla_x^2 q(z,\bar{\omega})\|\big)\|x - \bar{x}\|,
\end{aligned}$$

and (17) follows. The estimate (18) follows analogously.

To show (19) note that there is some point $\tilde{z}$ belonging to the line segment $[\bar{x},x]$ such that

$$\begin{aligned}
\xi^T(q(x,\omega) - q(x,\bar{\omega})) &= \xi^T(q(\bar{x},\omega) - q(\bar{x},\bar{\omega})) + \xi^T(\nabla_x q(\bar{x},\omega) - \nabla_x q(\bar{x},\bar{\omega}))(x-\bar{x}) \\
&\quad + \frac{1}{2}(x-\bar{x})^T(\nabla_x^2(\xi^T q)(\tilde{z},\omega) - \nabla_x^2(\xi^T q)(\tilde{z},\bar{\omega}))(x-\bar{x}) \\
&\leq \|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\| + e_2(\omega)\|x - \bar{x}\| \\
&\quad + \frac{1}{2}\|\nabla_x^2 q(\tilde{z},\omega) - \nabla_x^2 q(\tilde{z},\bar{\omega})\|\|x - \bar{x}\|^2,
\end{aligned}$$

and from this inequality (19) follows. $\qquad\square$

For every $\omega \in \Omega$ we define the linearized cone

$$T_\omega^{\mathrm{lin}}(x) := \{u \in \mathbb{R}^n \,|\, \nabla_x q(x;\omega)u \in T(q(x,\omega);P)\}$$

as well as the multifunction $M_\omega : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $M_\omega(x) := q(x,\omega) - P$. By [14, Proposition 1] and Corollary 2, respectively, we have

$$T(x;\mathscr{F}(\omega)) = T_\omega^{\mathrm{lin}}(x), \quad \hat{N}(x;\mathscr{F}(\omega)) \subset \nabla_x q(x,\omega)^T N(q(x,\omega);P)$$

for every $x \in \mathscr{F}(\omega)$ such that M_ω is metrically subregular at $(x,0)$.

It is well known that the property of metric regularity is stable under small Lipschitzian perturbations, see e.g. [20, Section 4.1], [23, Section 4.2.3]. We state here the following result:

Theorem 2 *Assume that $M_{\bar{\omega}}$ is metrically regular around $(\bar{x},0)$. Then there are neighborhoods U of $\bar{x}$, V of 0, W of $\bar{\omega}$ and a constant $\kappa > 0$ such that*

$$\mathrm{d}(x, M_\omega^{-1}(y)) \leq \kappa \mathrm{d}(y, M_\omega(x)) = \kappa \mathrm{d}(q(x,\omega) - y, P) \quad \forall (x,y,\omega) \in U \times V \times W. \tag{20}$$

In particular we have $\mathscr{F}(\omega) \neq \emptyset$, $\mathrm{d}(\bar{x}, \mathscr{F}(\omega)) \leq \kappa\|q(\bar{x},\omega) - q(\bar{x},\bar{\omega})\| \leq \kappa e_1(\omega)$ for all $\omega \in W$ and for every $x \in \mathscr{F}(\omega) \cap U$ with $\omega \in W$ the multifunction M_ω is metrically regular with modulus κ around $(x,0)$.

Proof A similar result was stated in [12, Lemma 3.1]. However, the given proof appears to be not correct, since the variable x is used in an ambiguous way and therefore, in the notation of [12], the claimed equation $\mathrm{dist}(x, S_p^{-1}(u) = \mathrm{dist}(x, S_{p^*}^{-1}(u + F(x, p^*) - F(x, p)))$ does not hold true in general. Thus we present a different proof. Let $\delta > 0$, $\kappa' > 0$ be chosen such that

$$\mathrm{d}(x, M_{\bar{\omega}}^{-1}(y)) \le \kappa' \mathrm{d}(y, M_{\bar{\omega}}(x)) \quad \forall (x, y) \in (\bar{x} + \delta \mathscr{B}_{\mathbb{R}^n}) \times \delta \mathscr{B}_{\mathbb{R}^m}.$$

Setting $\varepsilon := \frac{1}{24(\kappa'+1)}$, we denote by $\hat{U}_\varepsilon$ and $\hat{W}_\varepsilon$ the neighborhoods according to Lemma 4, and we choose some radius $r \in (0, \min\{\frac{\delta}{2}, 1\})$ with $\bar{x} + r \mathscr{B}_{\mathbb{R}^n} \subset \hat{U}_\varepsilon$. Then we choose some positive radius $\bar{r} \le \frac{1}{2} r$ and some neighborhood $W \subset \hat{W}_\varepsilon$ such that

$$\|q(x, \omega) - q(\bar{x}, \bar{\omega})\| < \tfrac{1}{2} \varepsilon r \; \forall (x, \omega) \in (\bar{x} + \bar{r} \mathscr{B}_{\mathbb{R}^n}) \times W \; \text{ and } \; e_1(\omega) < \varepsilon \bar{r} \; \forall \omega \in W.$$

We now show that the assertion of the theorem holds with $U := \bar{x} + \bar{r} \mathscr{B}_{\mathbb{R}^n}$, $V := \frac{1}{4} \varepsilon r \mathscr{B}_{\mathbb{R}^m}$ and $\kappa = 2(\kappa' + 1)$. Consider arbitrarily fixed elements $(\xi, y, \omega) \in U \times V \times W$ and define the functions

$$g'(x) := q(x, \bar{\omega}) - q(x, \omega) + y \; \text{ and } \; g(x) := q(x, \bar{\omega}) - q(x, \omega) + \zeta,$$

where $\zeta \in M_\omega(\xi)$ is chosen such that

$$\|y - \zeta\| = \mathrm{d}(y, M_\omega(\xi)) = \mathrm{d}(y, q(\xi, \omega) - P) \le \|y - (q(\xi, \omega) - q(\bar{x}, \bar{\omega}))\|$$

and, consequently, $\|\zeta\| \le 2\|y\| + \|q(\xi, \omega) - q(\bar{x}, \bar{\omega})\| < \varepsilon r$. Then g and g' are Lipschitz on $\bar{x} + r \mathscr{B}_{\mathbb{R}^n}$ with constant less than or equal to

$$\sup\{\|\nabla_x q(x, \omega) - \nabla_x q(x, \bar{\omega})\| \,|\, x \in \bar{x} + r \mathscr{B}_{\mathbb{R}^n}\} \le e_1(\omega) + \varepsilon r < 2\varepsilon r \le 2\varepsilon < \frac{1}{2(\kappa' + 1)},$$

where the first inequality follows from (17). Further we have

$$\sup\{\|g'(x)\| \,|\, x \in \bar{x} + r \mathscr{B}_{\mathbb{R}^n}\} < e_1(\omega) + \varepsilon r + \|y\| < 3\varepsilon r < \frac{r}{8(\kappa' + 1)}$$

and

$$\sup\{\|g(x)\| \,|\, x \in \bar{x} + r \mathscr{B}_{\mathbb{R}^n}\} \le e_1(\omega) + \varepsilon r + \|\zeta\| < 3\varepsilon r = \frac{r}{8(\kappa' + 1)}.$$

Since $g(\xi) \in M_{\bar{\omega}}(\xi)$, we can apply [20, Theorem 4.3] to find some ξ' such that $g'(\xi') \in M_{\bar{\omega}}(\xi')$ and $\|\xi - \xi'\| \le 2(\kappa' + 1)\|g'(\xi) - g(\xi)\| = 2(\kappa' + 1)\|y - \zeta\|$. It follows that $y \in q(\xi', \omega) - P$ and thus $\xi' \in M_\omega^{-1}(y)$ and

$$\mathrm{d}(\xi, M_\omega^{-1}(y)) \le 2(\kappa' + 1)\|y - \zeta\| = 2(\kappa' + 1)\mathrm{d}(y, M_\omega(\xi)),$$

showing (20). Taking $\xi = \bar{x}$, $y = 0$, we obtain $\mathrm{d}(\bar{x}, M_\omega^{-1}(0)) = \mathrm{d}(\bar{x}, \mathscr{F}(\omega)) \le \kappa \mathrm{d}(0, M_\omega(\bar{x})) \le \kappa\|q(\bar{x}, \omega) - q(\bar{x}, \bar{\omega})\| \le \kappa e_1(\omega)$ and thus $\mathscr{F}(\omega) \ne \emptyset$. To complete the proof, note that metric regularity of M_ω around $(x, 0)$, where $x \in \mathscr{F}(\omega) \cap U$, is a simple consequence of (20). $\qquad\square$

When we do not assume metric regularity of $M_{\bar{\omega}}$ around $(\bar{x}, 0)$, then we cannot expect in general that the multifunctions M_ω, for ω near $\bar{\omega}$, are metrically regular around all points $(x, 0)$ with $x \in \mathscr{F}(\omega)$ close to $\bar{x}$. To handle also this situation, we give the following definition.

Definition 4 Let $l \in \{1, 2\}$. We say that property R_l holds if there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and constants $\kappa > 0$ and $\gamma > 0$ such that for every $\omega \in W$ and every $x \in \mathscr{F}(\omega) \cap U$ with $\|x - \bar{x}\| > \gamma e_l(\omega)$, the multifunction M_ω is metrically regular with modulus κ around $(x, 0)$.

In particular, properties R_1 and R_2 imply that $M_{\bar{\omega}}$ is metrically regular with some uniform modulus around every point $(x, 0)$ with $x \in \mathscr{F}(\bar{\omega}) \setminus \{\bar{x}\}$ close to $\bar{x}$.

We will now show that property R_1 or property R_2 holds if $M_{\bar{\omega}}$ fulfills at $(\bar{x}, 0)$ the condition FOSCMS or SOSCMS, respectively.

Proposition 1 *1. Assume that FOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$, i.e., for every direction $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ we have*

$$\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0, \ \lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u) \Longrightarrow \lambda = 0.$$

Then property R_1 holds.

2. If for every direction $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ we have

$$\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0, \ \lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u), \ u^T \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega})u = 0 \Longrightarrow \lambda = 0, \qquad (21)$$

then property R_2 holds. In particular, SOSCMS implies property R_2.

Proof To prove the first part, assume on the contrary that for every k we can find $(x^k, \omega^k) \in \hat{U}_{1/k} \times \hat{W}_{1/k}$ with $x^k \in \mathscr{F}(\omega^k) \cap (\bar{x} + \frac{1}{k}\mathscr{B}_{R^n})$, $\|\nabla_x q(x^k, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})\| \leq \frac{1}{k}$, $\|x^k - \bar{x}\| > ke_1(\omega^k)$ such that M_{ω^k} is not metrically regular around $(x^k, 0)$ with some modulus less than or equal to k, where $\hat{U}_{1/k}$, $\hat{W}_{1/k}$ are as in Lemma 4. By [26, Example 9.44], there is some $\lambda^k \in N(q(x^k, \omega^k); P)$ with $\|\lambda^k\| = 1$ and $\|\nabla_x q(x^k, \omega^k)^T \lambda^k\| \leq \frac{1}{k}$. According to the definition of the limiting normal cone, for each k we can find elements $q^k \in (q(x^k, \omega^k) + \frac{\|x^k - \bar{x}\|}{k}\mathscr{B}_{\mathbb{R}^n}) \cap P$ and $\xi^k \in \hat{N}(q^k, P) \cap (\lambda^k + \frac{\|x^k - \bar{x}\|}{k}\mathscr{B}_{\mathbb{R}^n})$. By passing to a subsequence if necessary, we can assume that $u^k := (x^k - \bar{x})/\|x^k - \bar{x}\| \to u$ and $\lim_{k \to \infty} \lambda^k = \lim_{k \to \infty} \xi^k = \lambda \neq 0$. Because of Lemma 4 we have $\|q(x^k, \omega^k) - q(x^k, \bar{\omega})\|/\|x^k - \bar{x}\| \leq e_1(\omega^k)/\|x^k - \bar{x}\| + 1/k \leq 2/k$ and therefore

$$\begin{aligned}
\lim_{k \to \infty} \frac{q^k - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} &= \lim_{k \to \infty} \frac{q(x^k, \omega^k) - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} \\
&= \lim_{k \to \infty} \left(\frac{q(x^k, \omega^k) - q(x^k, \bar{\omega})}{\|x^k - \bar{x}\|} + \frac{q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} \right) \\
&= \nabla_x q(\bar{x}, \bar{\omega})u,
\end{aligned}$$

showing $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ and $\lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$. Since we also have

$$0 = \lim_{k \to \infty} \nabla_x q(x^k, \omega^k)^T \lambda^k = \nabla_x q(\bar{x}, \bar{\omega})^T \lambda,$$

we obtain a contradiction to the assumption. Hence the first part is proved.

To prove the second part, assume on the contrary that for every k we can find $(x^k, \omega^k) \in \hat{U}_{1/k} \times \hat{W}_{1/k}$ with $x^k \in \mathscr{F}(\omega^k) \cap (\bar{x} + \frac{1}{k}\mathscr{B}_{R^n})$, $\|\nabla_x q(x^k, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})\| \leq \frac{1}{k}$, $\|x^k - \bar{x}\| > ke_2(\omega^k)$ such that M_{ω^k} is not metrically regular around $(x^k, 0)$ with some modulus less than or equal to than k. Then $\|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\|^{\frac{1}{2}} \leq e_2(\omega^k) \leq 1/k^2 \leq 1$ and $e_2(\omega^k) \geq e_1(\omega^k)$ follows. Hence we can proceed similarly as in the first part of the proof to find the sequences (λ^k), (ξ^k) and (q^k) together with the limits $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ and $0 \neq \lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$ satisfying $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$, with the only difference that we now require $q^k \in (q(x^k, \omega^k) + \frac{\|x^k - \bar{x}\|^2}{k}\mathscr{B}_{\mathbb{R}^n}) \cap P$. By passing to a subsequence if necessary, we can assume that there are index sets $\mathscr{P} \subset \{1, \ldots, p\}$ and $\mathscr{A}_i \subset \{1, \ldots, m_i\}$, $i \in \mathscr{P}$, such that $\mathscr{P}(q^k) = \mathscr{P}$ and $\mathscr{A}_i(q^k) = \mathscr{A}_i$, $i \in \mathscr{P}$, holds for all k. Further there are numbers $\mu_{ij}^k \geq 0$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, such that

$$\xi^k - \sum_{j \in \mathscr{A}_i} \mu_{ij}^k a_{ij} = 0, \ i \in \mathscr{P},$$

and $\sum_{j \in \mathscr{A}_i} \mu_{ij}^k \leq \beta_i \|\xi^k\|$ for some constant β_i, $i \in \mathscr{P}$. By passing to a subsequence once more, we can assume that the sequences μ_{ij}^k converge to $\mu_{ij} \geq 0$ for each $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, and it follows that $\lambda - \sum_{j \in \mathscr{A}_i} \mu_{ij} a_{ij} = 0$, $i \in \mathscr{P}$. Since $\mathscr{P} \subset \mathscr{P}(q(\bar{x}, \bar{\omega}))$ and $\mathscr{A}_i \subset \mathscr{A}_i(q(\bar{x}, \bar{\omega}))$, $i \in \mathscr{P}$, we obtain for each $i \in \mathscr{P}$

$$\lambda^T q^k = \sum_{j \in \mathscr{A}_i} \mu_{ij} a_{ij}^T q^k = \sum_{j \in \mathscr{A}_i} \mu_{ij} b_{ij} = \sum_{j \in \mathscr{A}_i} \mu_{ij} a_{ij}^T q(\bar{x}, \bar{\omega}) = \lambda^T q(\bar{x}, \bar{\omega}) \ \forall k.$$

Using Lemma 4 we have

$$\|q(x^k, \omega^k) - q(x^k, \bar{\omega})\| \leq e_2(\omega^k)^2 + e_2(\omega^k)\|x^k - \bar{x}\| + \frac{1}{2k}\|x^k - \bar{x}\|^2 \leq \left(\frac{1}{k^2} + \frac{1}{k} + \frac{1}{2k}\right)\|x^k - \bar{x}\|^2,$$

and so, together with $\|q^k - q(x^k, \omega^k)\| \leq \frac{1}{k}\|x^k - \bar{x}\|^2$ and $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$, we obtain

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\lambda^T(q^k - q(\bar{x}, \bar{\omega}))}{\|x^k - \bar{x}\|^2} = \lim_{k\to\infty} \frac{\lambda^T(q^k - q(x^k, \omega^k) + q(x^k, \omega^k) - q(x^k, \bar{\omega}) + q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega}))}{\|x^k - \bar{x}\|^2} \\
&= \lim_{k\to\infty} \frac{\lambda^T(q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega}))}{\|x^k - \bar{x}\|^2} = \lim_{k\to\infty} \frac{\lambda^T(\nabla_x q(\bar{x}, \bar{\omega})(x^k - \bar{x}) + \frac{1}{2}(x^k - \bar{x})^T \nabla_x^2 q(\bar{x}, \bar{\omega})(x^k - \bar{x}))}{\|x^k - \bar{x}\|^2} \\
&= \frac{1}{2} u^T \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega}) u,
\end{aligned}
$$

contradicting (21). The last statement follows from the observation that SOSCMS implies (21). $\qquad\square$

Proposition 2 *Let $\bar{x} \in \mathscr{F}(\bar{\omega})$.*

1. *Assume that there is some $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ such that*

$$\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0, \ \lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u) \implies \lambda = 0.$$

   Then there is a constant $\kappa > 0$ and a neighborhood W of $\bar{\omega}$ such that

$$\mathrm{d}(\bar{x}, \mathscr{F}(\omega)) \leq \kappa\|q(\bar{x}, \omega) - q(\bar{x}, \bar{\omega})\| \leq \kappa e_1(\omega) \ \forall \omega \in W.$$

2. *Assume that there is some $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ such that*

$$\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0, \ \lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u), \ u^T \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u \geq 0 \implies \lambda = 0.$$

   Then there is a constant $\kappa > 0$ and a neighborhood W of $\bar{\omega}$ such that

$$\mathrm{d}(\bar{x}, \mathscr{F}(\omega)) \leq \kappa e_2(\omega) \ \forall \omega \in W.$$

Proof We can assume without loss of generality that $\|u\| = 1$. In both cases, the assumption ensures that $M_{\bar{\omega}}$ is metrically subregular at $(\bar{x}, 0)$ in direction u. Now we make some preliminary considerations, before proving the assertions of the proposition. For every k we can find a neighborhood W^k of $\bar{\omega}$ and a radius $\rho^k > 0$ such that $(\bar{x} + \rho^k \mathscr{B}_{\mathbb{R}^n}) \times W^k \subset \hat{U}_{1/k} \times \hat{W}_{1/k}$,

$$\sup\{\|\nabla_x^2 q(x, \omega) - \nabla_x^2 q(x, \bar{\omega})\| \,|\, x \in \bar{x} + \rho^k \mathscr{B}_{\mathbb{R}^n}, \omega \in W^k\} < \frac{1}{9k^3},$$

$$\sup\{\|\nabla_x q(x, \omega) - \nabla_x q(\bar{x}, \bar{\omega})\| \,|\, x \in \bar{x} + \rho^k \mathscr{B}_{\mathbb{R}^n}, \omega \in W^k\} < \frac{1}{k},$$

$$\sup\{e_2(\omega) \,|\, \omega \in W^k\} \leq \min\{\frac{1}{16k^3}, \frac{\rho^k}{16k^2}\}.$$

Now consider sequences $(t^k) \downarrow 0$ and (ω^k) such that $t^k < \rho^k/2$ and $\omega^k \in W^k$ hold for all k. Since $\nabla_x q(\bar{x}, \bar{\omega})u \in T(q(\bar{x}, \bar{\omega}); P)$ and P is the union of finitely many convex polyhedral sets, we have $q(\bar{x}, \bar{\omega}) + t^k \nabla_x q(\bar{x}, \bar{\omega})u \in P$ for all k sufficiently large. Thus there is some constant L such that $\mathrm{d}(q(\bar{x} + t^k u, \bar{\omega}), P) \leq L(t^k)^2$ and, by metric subregularity of $M_{\bar{\omega}}$ in direction u, there is some $\kappa' > 0$ such that for every k there is some point $\bar{x}^k \in \mathscr{F}(\bar{\omega}) \cap (\bar{x} + t^k u + \kappa' L(t^k)^2 \mathscr{B}_{\mathbb{R}^n})$. For all k sufficiently large we also have $t^k L \kappa' < \frac{1}{2}$, implying

$$\frac{t^k}{2} < \|\bar{x}^k - \bar{x}\| < \frac{3t^k}{2} < \rho^k,$$

and we obtain

$$
\begin{aligned}
\mathrm{d}(q(\bar{x}^k, \omega^k), P) &\leq \mathrm{d}(q(\bar{x}^k, \bar{\omega}), P) + \|q(\bar{x}^k, \omega^k) - q(\bar{x}^k, \bar{\omega})\| \\
&\leq 0 + \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega}))\| + \|\bar{x}^k - \bar{x}\|\|\nabla_x q(\bar{x}, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})\| \\
&\quad + \frac{\|\bar{x}^k - \bar{x}\|^2}{2} \sup\{\|\nabla_x^2 q(x, \omega^k) - \nabla_x^2 q(x, \bar{\omega})\| \,|\, \|x - \bar{x}\| \leq \|\bar{x}^k - \bar{x}\|\} \\
&\leq \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega}))\| + \frac{3}{2} t^k \|\nabla_x q(\bar{x}, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})\| + \frac{(t^k)^2}{8k^3} =: \varepsilon^k.
\end{aligned}
$$

By using Ekeland's variational principle, we can find for every k some point $x^k \in \bar{x}^k + k\varepsilon^k \mathscr{B}_{\mathbb{R}^n}$ such that $\mathrm{d}(q(x^k, \omega^k), P) \le \mathrm{d}(q(\bar{x}^k, \omega^k), P)$ and

$$\mathrm{d}(q(x^k, \omega^k), P) \le \mathrm{d}(q(x, \omega^k), P) + \frac{1}{k}\|x - x^k\| \; \forall x \in \mathbb{R}^n.$$

Since P is closed, there is for every k some $q^k \in P$ with $\mathrm{d}(q(x^k, \omega^k), P) = \|q(x^k, \omega^k) - q^k\|$, and we conclude

$$\|q(x^k, \omega^k) - q^k\| \le \|q(x, \omega^k) - y\| + \frac{1}{k}\|x - x^k\| \; \forall (x, y) \in \mathbb{R}^n \times P,$$

i.e., (x^k, q^k) is a solution of the optimization problem

$$\min_{x,y} \|q(x, \omega^k) - y\| + \frac{1}{k}\|x - x^k\| \quad \text{subject to} \quad y \in P.$$

By applying first-order optimality conditions, it follows that $\xi^k := (q(x^k, \omega^k) - q^k)/\|q(x^k, \omega^k) - q^k\| \in \hat{N}(q^k; P)$ and $\|\nabla_x q(x^k, \omega^k)^T \xi^k\| \le \frac{1}{k}$, provided that $q(x^k, \omega^k) - q^k \ne 0$.

Now let us prove the first assertion by contraposition. We assume on the contrary that for every k there is some $\omega^k \in W^k$ with $\mathrm{d}(\bar{x}, \mathscr{F}(\omega^k)) > 12k^2 \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\|$. Note that $\|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\| = 0$ or $e_2(\omega^k) = 0$ implies $\bar{x} \in \mathscr{F}(\omega^k)$ and thus $\|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\| > 0$. By taking $t^k := 4k^2 \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\| \le 4k^2 e_1(\omega^k) \le 4k^2 e_2(\omega^k) \le \min\{\frac{1}{4k}, \frac{\rho^k}{4}\}$ we obtain

$$\varepsilon^k \le \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\|(1 + 6k^2 \|\nabla_x q(\bar{x}, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})\| + \frac{t^k}{2k})$$

$$\le \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\|(1 + \frac{3}{8k} + \frac{t^k}{2k}) \le 2\|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\| = \frac{1}{2k^2}t^k < \frac{\|\bar{x}^k - \bar{x}\|}{k^2}$$

and therefore $\|x^k - \bar{x}^k\| \le k\varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k}$, showing

$$\lim_{k \to \infty} \frac{x^k - \bar{x}}{\|x^k - \bar{x}\|} = \lim_{k \to \infty} \frac{\bar{x}^k - \bar{x}}{\|\bar{x}^k - \bar{x}\|} = u, \tag{22}$$

$$(1 - \frac{1}{k})\|\bar{x}^k - \bar{x}\| < \|x^k - \bar{x}\| < (1 + \frac{1}{k})\|\bar{x}^k - \bar{x}\|$$
$$< \frac{3}{2}(1 + \frac{1}{k})t^k \le 12k^2 \|q(\bar{x}, \omega^k) - q(\bar{x}, \bar{\omega})\| < \mathrm{d}(\bar{x}, \mathscr{F}(\omega^k)), \tag{23}$$

and

$$\|q(x^k, \omega^k) - q^k\| \le \varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k^2} \le \frac{\|x^k - \bar{x}\|}{k} \; \forall k \ge 2. \tag{24}$$

From (23) we can conclude $q(x^k, \omega^k) - q^k \ne 0$, since otherwise $\mathrm{d}(\bar{x}, \mathscr{F}(\omega^k)) \le \|\bar{x} - x^k\|$ would hold. Hence ξ^k is well defined, and we can proceed as in the proof of Proposition 1 to obtain the contradiction that every limit point λ of the sequence (ξ^k) fulfills $\|\lambda\| = 1$, $\lambda \in N(q(\bar{x}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$ and $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$.

We prove the second part similarly. Assuming on the contrary that for every k there is some $\omega^k \in W^k$ with $e_2(\omega^k) > 0$ and $\mathrm{d}(\bar{x}, \mathscr{F}(\omega^k))/e_2(\omega^k) > 24k^2$ and setting $t^k := 8k^2 e_2(\omega^k) \le \min\{\frac{1}{2k}, \frac{\rho^k}{2}\}$, we obtain $\|\bar{x}^k - \bar{x}\| < \frac{3}{2}t^k \le \frac{3}{4}$ and, together with $\frac{1}{2}t^k < \|\bar{x}^k - \bar{x}\|$,

$$\varepsilon^k \le e_2(\omega^k)^2(1 + 12k^2 + 8k) = (t^k)^2 \frac{1 + 12k^2 + 8k}{64k^4} < \frac{\|\bar{x}^k - \bar{x}\|^2}{k^2} \le \frac{\|\bar{x}^k - \bar{x}\|}{k^2} \; \forall k \ge 2.$$

As before we obtain $\|x^k - \bar{x}^k\| \le k\varepsilon^k < \frac{\|\bar{x}^k - \bar{x}\|}{k}$, and thus (22) as well as

$$(1 - \frac{1}{k})\|\bar{x}^k - \bar{x}\| < \|x^k - \bar{x}\| < (1 + \frac{1}{k})\|\bar{x}^k - \bar{x}\| < \frac{3}{2}(1 + \frac{1}{k})t^k \le 24k^2 e_2(\omega^k) < \mathrm{d}(\bar{x}, \mathscr{F}(\omega^k))$$

and (24) hold. This implies again that (ξ^k) is well defined; using the same arguments as in the proof of the second part of Proposition 1, we obtain that every limit point λ of the sequence (ξ^k) fulfills $\|\lambda\| = 1$, $\lambda \in N(q(\bar{x}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$, $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$ and $u^T \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega})u = 0$, a contradiction. $\qquad\square$

4 Lipschitz and Hölder stability of solution mappings

In what follows we denote by $X : \Omega \rightrightarrows \mathbb{R}^n$ the mapping which assigns to every ω the set of local minimizers for the problem $P(\omega)$ in (1), and by $S : \Omega \rightrightarrows \mathbb{R}^n$ and $S_M : \Omega \rightrightarrows \mathbb{R}^n$ the mappings which assign to every $\omega \in \Omega$ the sets of B-stationary points and M-stationary points, respectively, for the problem $P(\omega)$, i.e.,

$$S(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in \nabla_x f(x, \omega) + \hat{N}(x; \mathscr{F}(\omega))\},$$

$$S_M(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in \nabla_x f(x, \omega) + \nabla_x q(x, \omega)^T N(q(x, \omega); P)\}.$$

Then we have $X(\omega) \subset S(\omega) \,\forall \omega \in \Omega$.

Further we consider the solution mapping $\hat{S} : \Omega \rightrightarrows \mathbb{R}^n$ of the generalized equation (4),

$$\hat{S}(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in F(x, \omega) + \hat{N}(x; \mathscr{F}(\omega))\},$$

and the related mapping

$$\hat{S}_M(\omega) := \{x \in \mathbb{R}^n \,|\, 0 \in F(x, \omega) + \nabla_x q(x, \omega)^T N(q(x, \omega); P)\}.$$

The first result of this section are sufficient conditions such that estimates of the form

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_l(\omega) \mathscr{B}_{\mathbb{R}^n} \,\forall \omega \in W$$

or

$$(\hat{S}(\omega) \cup \hat{S}_M(\omega)) \cap U \subset \bar{x} + L\hat{\tau}_l(\omega) \mathscr{B}_{\mathbb{R}^n} \,\forall \omega \in W$$

hold, where $L > 0$ and U and W are neighborhoods of $\bar{x}$ and $\bar{\omega}$. The following result is only stated for the solution mappings $\hat{S}(\omega) \cup \hat{S}_M(\omega)$ for $GE(\omega)$, the corresponding statement for $P(\omega)$ follows immediately by taking $F(x, \omega) := \nabla_x f(x, \omega)$.

Theorem 3 *Let* $\bar{x} \in \mathscr{F}(\bar{\omega})$.

1. *If property* R_1 *holds and there does not exist a triple* $(u, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ *satisfying*

$$0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x}), \tag{25}$$

$$\lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u) \tag{26}$$

$$\mu \in T(\lambda; N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)) \tag{27}$$

$$F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0 \tag{28}$$

$$\nabla_x F(\bar{x}, \bar{\omega})u + \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u + \nabla_x q(\bar{x}, \bar{\omega})^T \mu = 0, \tag{29}$$

   then there are neighborhoods U *of* $\bar{x}$, W *of* $\bar{\omega}$ *and a constant* $L > 0$ *such that*

$$(\hat{S}(\omega) \cup \hat{S}_M(\omega)) \cap U \subset \bar{x} + L\hat{\tau}_1(\omega) \mathscr{B}_{\mathbb{R}^n} \,\forall \omega \in W.$$

2. *If property* R_2 *holds and there does not exist a quadruple* $(u, \lambda, \mu, v) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n$ *such that* (u, λ, μ) *fulfills* (25)-(29) *and*

$$\nabla_x q(\bar{x}, \bar{\omega})v + u^T \nabla_x^2 q(\bar{x}, \bar{\omega})u \in T(q(\bar{x}, \bar{\omega}); \bigcup_{i \in \mathscr{I}(u)} P_i), \tag{30}$$

$$u^T \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u = F(\bar{x}, \bar{\omega})^T v, , \tag{31}$$

   where $\mathscr{I}(u) := \{i \,|\, \nabla_x q(\bar{x}, \bar{\omega})u \in T(q(\bar{x}, \bar{\omega}); P_i)\}$, *then there are neighborhoods* U *of* $\bar{x}$, W *of* $\bar{\omega}$ *and a constant* $L > 0$ *such that*

$$(\hat{S}(\omega) \cup \hat{S}_M(\omega)) \cap U \subset \bar{x} + L\hat{\tau}_2(\omega) \mathscr{B}_{\mathbb{R}^n} \,\forall \omega \in W.$$

Proof We prove the first part by contraposition. Let W denote the neighborhood according to the definition of property R_1. Assume on the contrary that for every k we can find $(x^k, \omega^k) \in (\hat{U}_{1/k} \cap (\bar{x} + \frac{1}{k}\mathscr{B}_{\mathbb{R}^n})) \times (\hat{W}_{1/k} \cap W)$ with $(x^k) \in \hat{S}_M(\omega^k) \cup \hat{S}(\omega^k)$ and $\|x^k - \bar{x}\| > k\hat{\tau}_1(\omega^k)$. Because of $e_1(\omega^k) \le \hat{\tau}_1(\omega^k)$ it follows that $e_1(\omega^k)/\|x^k - \bar{x}\| \le \hat{\tau}_1(\omega^k)/\|x^k - \bar{x}\| < \frac{1}{k}$, and, due to property R_1, we have that M_{ω^k} is metrically regular with modulus κ around $(x^k, 0)$ for all k sufficiently large. Hence, $x^k \in \hat{S}_M(\omega^k)$ and there exists a vector $\lambda^k \in N(q(x^k, \omega^k); P)$ with $F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \lambda^k = 0$. From [26, Example 9.44] we conclude $\|\lambda^k\| \le \kappa \|F(x^k, \omega^k)\|$, showing that (λ^k) is bounded. By the definition of the limiting normal cone, we can find for each k elements $q^k \in (q(x^k, \omega^k) + \frac{\|x^k - \bar{x}\|^2}{k}\mathscr{B}_{\mathbb{R}^n}) \cap P$ and $\xi^k \in \hat{N}(q^k, P) \cap (\lambda^k + \frac{\|x^k - \bar{x}\|}{k}\mathscr{B}_{\mathbb{R}^n})$. By passing to a subsequence if necessary, we can assume that $u^k := (x^k - \bar{x})/\|x^k - \bar{x}\| \to u$ and $\lambda^k \to \lambda$. Because of Lemma 4 we have $\|q(x^k, \omega^k) - q(x^k, \bar{\omega})\|/\|x^k - \bar{x}\| \le \hat{\tau}_1(\omega^k)/\|x^k - \bar{x}\| + 1/k \to 0$ and therefore

$$\lim_{k \to \infty} \frac{q^k - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} = \lim_{k \to \infty} \frac{q(x^k, \omega^k) - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} = \lim_{k \to \infty} \frac{q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} = \nabla_x q(\bar{x}, \bar{\omega})u,$$

showing $0 \ne u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ and $\lambda \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$. Using Lemma 4 again, we similarly obtain

$$\lim_{k \to \infty} \frac{\nabla_x q(x^k, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} := \nabla_x^2 q(\bar{x}, \bar{\omega})u, \quad \lim_{k \to \infty} \frac{F(x^k, \omega^k) - F(\bar{x}, \bar{\omega})}{\|x^k - \bar{x}\|} = \nabla_x F(\bar{x}, \bar{\omega})u. \quad (32)$$

Further

$$0 = \lim_{k \to \infty} (F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \lambda^k) = F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \lambda. \quad (33)$$

By passing to a subsequence once more, we can assume that there are index sets $\mathscr{P} \subset \{1, \ldots, p\}$ and $\mathscr{A}_i \subset \{1, \ldots, m_i\}$, $i \in \mathscr{P}$, such that $\mathscr{P}(q^k) = \mathscr{P}$ and $\mathscr{A}_i(q^k) = \mathscr{A}_i$, $i \in \mathscr{P}$, holds for all k. Further, by the generalized Farkas lemma, cf. [4, Proposition 2.201], there are numbers $\zeta_{ij}^k \ge 0$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, such that

$$\xi^k - \sum_{j \in \mathscr{A}_i} \zeta_{ij}^k a_{ij} = 0, \ i \in \mathscr{P},$$

and $\sum_{j \in \mathscr{A}_i} \zeta_{ij}^k \le \beta_i \|\xi^k\|$, $i \in \mathscr{P}$, for some constants β_i. By passing to a subsequence once more, we can assume that the sequences ζ_{ij}^k converge to $\zeta_{ij} \ge 0$ for each $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, and it follows that $\lambda - \sum_{j \in \mathscr{A}_i} \zeta_{ij} a_{ij} = 0$, $i \in \mathscr{P}$. Further, $F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \lambda = 0$ and thus, by Hoffman's lemma, there is some real $\bar{\gamma} > 0$ such that for each k we can find $\bar{\xi}^k \in \mathbb{R}^m$ and nonnegative numbers $\bar{\zeta}_{ij}^k$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, satisfying

$$\|\bar{\xi}^k - \xi^k\| + \sum_{i \in \mathscr{P}} \sum_{j \in \mathscr{A}_i} |\zeta_{ij}^k - \bar{\zeta}_{ij}^k| \le \bar{\gamma} \|F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \xi^k\|,$$

$$\bar{\xi}^k - \sum_{j \in \mathscr{A}_i} \bar{\zeta}_{ij}^k a_{ij} = 0, \ i \in \mathscr{P},$$

$$F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \bar{\xi}^k = 0.$$

Taking into account $F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \lambda^k = 0$ and $\|\nabla_x q(\bar{x}, \bar{\omega})^T (\lambda^k - \xi^k)\| \le (\|\nabla_x q(\bar{x}, \bar{\omega})\| \|x_k - \bar{x}\|)/k$, we obtain

$$\limsup_{k \to \infty} \frac{\|F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \xi^k\|}{\|x^k - \bar{x}\|}$$

$$= \limsup_{k \to \infty} \frac{\|F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \xi^k - F(x^k, \omega^k) - \nabla_x q(x^k, \omega^k)^T \lambda^k\|}{\|x^k - \bar{x}\|}$$

$$\le \limsup_{k \to \infty} \frac{\|F(\bar{x}, \bar{\omega}) - F(x^k, \omega^k) + (\nabla_x q(\bar{x}, \bar{\omega}) - \nabla_x q(x^k, \omega^k))^T \lambda^k\| + \|\nabla_x q(\bar{x}, \bar{\omega})^T (\lambda^k - \xi^k)\|}{\|x^k - \bar{x}\|}$$

$$= \|\nabla_x F(\bar{x}, \bar{\omega})u + \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega})u\| < \infty,$$

where the last equation follows from (32). Hence for each $j \in \mathscr{A}_i$, $i \in \mathscr{P}$, the sequence $(\zeta_{ij}^k - \bar{\zeta}_{ij}^k)/\|x^k - \bar{x}\|$ is bounded, and we can assume that it converges to some v_{ij}, where we have eventually passed to a subsequence. Now consider the set $\mathscr{I} := \{(i, j) \mid i \in \mathscr{P}, j \in \mathscr{A}_i, \zeta_{ij} = 0\}$. If $v_{ij} \ge 0$ $\forall (i, j) \in \mathscr{I}$ we set

$\bar{v}_{ij} = v_{ij}$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$. Otherwise we choose an index $\bar{k}$ such that $(\zeta_{ij}^{\bar{k}} - \bar{\zeta}_{ij}^{\bar{k}})/\|x^{\bar{k}} - \bar{x}\| < v_{ij}/2$ hold for all $(i,j) \in \mathscr{I}$ with $v_{ij} < 0$ and set $\bar{v}_{ij} = v_{ij} + 2(\bar{\zeta}_{ij}^{\bar{k}} - \zeta_{ij})/\|x^{\bar{k}} - \bar{x}\|$, $j \in \mathscr{A}_i$, $i \in \mathscr{P}$. Then we have for every $(i,j) \in \mathscr{I}$,

$$\bar{v}_{ij} = v_{ij} + 2\bar{\zeta}_{ij}^{\bar{k}}/\|x^{\bar{k}} - \bar{x}\| \geq v_{ij} + 2(\bar{\zeta}_{ij}^{\bar{k}} - \zeta_{ij}^{\bar{k}})/\|x^{\bar{k}} - \bar{x}\| > 0,$$

and hence in any case there is some $\bar{t} > 0$ such that $\zeta_{ij} + t\bar{v}_{ij} \geq 0$ holds for all $j \in \mathscr{A}_i$, $i \in \mathscr{P}$ and $t \in [0,\bar{t}]$. Setting $\mu := \sum_{j \in \mathscr{A}_i} \bar{v}_{ij} a_{ij}$ for an arbitrarily chosen $i \in \mathscr{P}$, we either have $\mu = \lim_{k \to \infty} \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|}$ or $\mu = \lim_{k \to \infty} \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|} + 2\frac{\bar{\xi}^{\bar{k}} - \lambda}{\|x^{\bar{k}} - \bar{x}\|}$, and therefore $\mu = \sum_{j \in \mathscr{A}_i} \bar{v}_{ij} a_{ij}$ holds for all $i \in \mathscr{P}$. Hence $\lambda + t\mu \in \hat{N}(q^k; P)$ $\forall k$ and $\lambda + t\mu \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$, and $\mu \in T(\lambda; N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u))$ follows. Taking into account $\nabla_x q(\bar{x}, \bar{\omega})^T \lambda = \nabla_x q(\bar{x}, \bar{\omega})^T \bar{\xi}^{\bar{k}} = -F(\bar{x}, \bar{\omega})$, we obtain

$$\begin{aligned}
0 &= \lim_{k \to \infty} \frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \lambda^k}{\|x^k - \bar{x}\|} = \lim_{k \to \infty} \frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \xi^k}{\|x^k - \bar{x}\|} \\
&= \lim_{k \to \infty} \left(\frac{F(x^k, \omega^k) + \nabla_x q(x^k, \omega^k)^T \bar{\xi}^k}{\|x^k - \bar{x}\|} + \nabla_x q(x^k, \omega^k)^T \frac{\xi^k - \bar{\xi}^k}{\|x^k - \bar{x}\|} \right) \\
&= \nabla_x q(\bar{x}, \bar{\omega})^T \mu + \lim_{k \to \infty} \frac{F(x^k, \omega^k) - F(\bar{x}, \bar{\omega}) + (\nabla_x q(x^k, \omega^k) - \nabla_x q(\bar{x}, \bar{\omega}))^T \bar{\xi}^k}{\|x^k - \bar{x}\|} \\
&= \nabla_x q(\bar{x}, \bar{\omega})^T \mu + \nabla_x F(\bar{x}, \bar{\omega})u + \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u.
\end{aligned}$$

Thus the triple (u, λ, μ) fulfills conditions (25)-(29), a contradiction, and the first part is proved.

We also prove the second assertion by contraposition. Let W now denote the neighborhood according to the definition of property R_2. Assume on the contrary that for every k we can find $(x^k, \omega^k) \in (\hat{U}_{1/k} \cap (\bar{x} + \frac{1}{k}\mathscr{B}_{\mathbb{R}^n})) \times (\hat{W}_{1/k} \cap W)$ with $x^k \in \hat{S}_M(\omega^k) \cup \hat{S}(\omega^k)$ and $\|x^k - \bar{x}\| > k\hat{\tau}_2(\omega^k)$. Then $\hat{\tau}_2(\omega^k) \leq 1/k^2$ and consequently $\hat{\tau}_1(\omega^k)/\|x^k - \bar{x}\| \leq \hat{\tau}_2(\omega^k)/\|x^k - \bar{x}\| \leq 1/k$ for all k, and we can proceed as in the first part to find (u, λ, μ). Thus, in order to prove the second assertion, it remains to show that there is some v such that (30)-(31) holds. Since $\mathscr{P} \subset \mathscr{P}(q(\bar{x}, \bar{\omega}))$ and $\mathscr{A}_i \subset \mathscr{A}_i(q(\bar{x}, \bar{\omega}))$, $i \in \mathscr{P}$, we have $\lambda^T(q^k - q(\bar{x}, \bar{\omega})) = (\sum_{j \in \mathscr{A}_i} \zeta_{ij} a_{ij})^T(q^k - q(\bar{x}, \bar{\omega})) = \sum_{j \in \mathscr{A}_i} \zeta_{ij}(b_{ij} - b_{ij}) = 0$ for all k. Now fix any $\bar{i} \in \mathscr{P}$ and consider an arbitrarily fixed vector $\xi \in \mathbb{R}^m$ such that $F(\bar{x}, \bar{\omega}) + \nabla_x q(\bar{x}, \bar{\omega})^T \xi = 0$ and there exist nonnegative numbers $\tau_{\bar{i}j} \geq 0$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$ with $\xi = \sum_{j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))} \tau_{\bar{i}j} a_{\bar{i}j}$. Since $q^k \in P_{\bar{i}}$ we have $a_{\bar{i}j}^T q^k \leq b_{\bar{i}j} = a_{\bar{i}j}^T q(\bar{x}, \bar{\omega})$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$ and therefore $\xi^T(q^k - q(\bar{x}, \bar{\omega})) \leq 0$. Hence,

$$\begin{aligned}
0 &\geq (\xi - \lambda)^T(q^k - q(\bar{x}, \bar{\omega})) \\
&= (\xi - \lambda)^T \left(q^k - q(x^k, \omega^k) + q(x^k, \omega^k) - q(x^k, \bar{\omega}) \right. \\
&\qquad \left. + \nabla_x q(\bar{x}, \bar{\omega})^T(x^k - \bar{x}) + \frac{1}{2}(x^k - \bar{x})^T \nabla_x^2 q(\bar{x}, \bar{\omega})(x^k - \bar{x}) \right) + r^k,
\end{aligned}$$

where $r^k := (\xi - \lambda)^T \left(q(x^k, \bar{\omega}) - q(\bar{x}, \bar{\omega}) - \nabla_x q(\bar{x}, \bar{\omega})(x^k - \bar{x}) - \frac{1}{2}(x^k - \bar{x})^T \nabla_x^2 q(\bar{x}, \bar{\omega})(x^k - \bar{x}) \right)$. By Lemma 4 we have $(q(x^k, \omega^k) - q(x^k, \bar{\omega}))/\|x^k - \bar{x}\|^2 \to 0$. Since $(\xi - \lambda)^T \nabla_x q(\bar{x}, \bar{\omega}) = 0$, $(q^k - q(x^k, \omega^k))/\|x^k - \bar{x}\|^2 \to 0$ and $r^k/\|x^k - \bar{x}\|^2 \to 0$, by dividing by $\|x^k - \bar{x}\|^2$ and taking the limit $k \to \infty$, we obtain

$$0 \geq \frac{1}{2} u^T \nabla_x^2((\xi - \lambda)^T q)(\bar{x}, \bar{\omega})u.$$

Setting $\zeta_{\bar{i}j} = 0$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega})) \setminus \mathscr{A}_{\bar{i}}$, we obtain that $\zeta_{\bar{i}j}$, $j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))$, is a solution of the linear optimization problem

$$\begin{aligned}
\min \quad & \sum_{j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))} -\tau_{\bar{i}j} a_{\bar{i}j}^T(u^T \nabla_x^2 q(\bar{x}, \bar{\omega})u) \\
\text{subject to} \quad & \sum_{j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega}))} \tau_{\bar{i}j} \nabla_x q(\bar{x}, \bar{\omega})^T a_{\bar{i}j} = -F(\bar{x}, \bar{\omega}) \\
& \tau_{\bar{i}j} \geq 0, \ j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega})).
\end{aligned}$$

Then, by duality theory of linear optimization, the dual program also has a solution $v \in \mathbb{R}^n$ and

$$-F(\bar{x}, \bar{\omega})^T v = -u^T \nabla_x^2 (\lambda^T q)(\bar{x}, \bar{\omega}) u$$
$$a_{\bar{i}j}^T \nabla_x q(\bar{x}, \bar{\omega}) v \leq -a_{\bar{i}j}^T (u^T \nabla_x^2 q(\bar{x}, \bar{\omega}) u), \ j \in \mathscr{A}_{\bar{i}}(q(\bar{x}, \bar{\omega})).$$

Hence $\nabla_x q(\bar{x}, \bar{\omega}) v + u^T \nabla_x^2 q(\bar{x}, \bar{\omega}) u \in T(q(\bar{x}, \bar{\omega}), P_{\bar{i}})$, and since $\bar{i} \in \mathscr{P} \subset \mathscr{I}(u)$, the quadruple (u, λ, μ, v) fulfills (25)-(31), a contradiction, and the second part is also proved. $\qquad\square$

Now we consider the solution mappings for the the problem (1). It turns out that the assumptions of Theorem 3 can be partially replaced by a second-order sufficient condition and a quadratic growth condition, respectively. Further we can guarantee the existence of locally optimal solutions.

Recall that a point $\bar{x} \in \mathscr{F}(\bar{\omega})$ is an *essential local minimizer of second order* for the problem $P(\bar{\omega})$ if there is a neighborhood U of $\bar{x}$ and a constant $\eta > 0$ such that

$$\max\{f(x, \bar{\omega}) - f(\bar{x}, \bar{\omega}), \ \mathrm{d}(q(x, \bar{\omega}), P)\} \geq \eta \|x - \bar{x}\|^2 \ \forall x \in U. \tag{34}$$

This implies that the *quadratic growth condition* for the problem $P(\bar{\omega})$ holds at $\bar{x}$, i.e., $f(x, \bar{\omega}) - f(\bar{x}, \bar{\omega}) \geq \eta \|x - \bar{x}\|^2 \ \forall x \in U \cap \mathscr{F}(\bar{\omega})$. The opposite direction is true if $M_{\bar{\omega}}$ is metrically subregular at $(\bar{x}, \bar{\omega})$. To see this one could use similar arguments as in [7, Section 3] by noting that convexity of P is not needed and the assumption of metric regularity used in [7] can be replaced by assuming metric subregularity.

Note that the proof of the following proposition does not use the polyhedral form of P, it suffices to suppose that P is a nonempty closed set.

Proposition 3 *Let $\bar{x}$ be an essential local minimizer of second order for the problem $P(\bar{\omega})$. Then there are constants $\gamma_1, \gamma_2 > 0$ and a neighborhood W of $\bar{\omega}$ such that for all $\omega \in W$ with $\mathrm{d}(\bar{x}, \mathscr{F}(\omega)) < \gamma_1$ one has*

$$\mathrm{d}(\bar{x}, X(\omega)) \leq \gamma_2 \left(\mathrm{d}(\bar{x}, \mathscr{F}(\omega))^{\frac{1}{2}} + \tau_2(\omega) \right)$$

Proof Let $\eta > 0$ and $\rho > 0$ be such that (34) hold for all $x \in \bar{x} + \rho \mathscr{B}_{\mathbb{R}^n}$, and define $L := \sup\{\|\nabla_x f(x, \bar{\omega})\| \mid x \in \bar{x} + \rho \mathscr{B}_{\mathbb{R}^n}\} + 1$, $\gamma_2 := \max\{\frac{1 + \sqrt{2\eta + 1}}{\eta}, 2\sqrt{L/\eta}\}$. Then we choose $0 < \bar{\rho} \leq \rho$ and $W \subset \hat{W}_{\eta/2}$ such that $\bar{x} + \bar{\rho} \mathscr{B}_{\mathbb{R}^n} \subset \hat{U}_{\eta/2}$,

$$\|\nabla_x^2 f(x, \omega) - \nabla_x^2 f(\bar{x}, \bar{\omega})\| \leq \frac{\eta}{4} \ \forall (x, \omega) \in (\bar{x} + \bar{\rho} \mathbb{R}_{\mathscr{B}^n}) \times W,$$

and finally, that $\tau_2(\omega) < \min\{L/2, \bar{\rho}/(2\gamma_2)\} \ \forall \omega \in W$. Further, we set

$$\gamma_1 := \min\{2L/\eta, \bar{\rho}^2/(4\gamma_2^2)\} \leq \min\{2L/\eta, \bar{\rho}^2/(16L/\eta)\} \leq \bar{\rho}.$$

Now let $\omega \in W$ with $\mathrm{d}(\bar{x}, \mathscr{F}(\omega)) < \gamma_1$ be arbitrarily fixed, and choose $y \in \mathscr{F}(\omega)$ with $\|y - \bar{x}\| = \mathrm{d}(\bar{x}, \mathscr{F}(\omega))$. For $x \in \bar{x} + \bar{\rho} \mathscr{B}_{\mathbb{R}^n}$ we obtain

$$|f(x, \omega) - f(x, \bar{\omega}) - (f(y, \omega) - f(y, \bar{\omega})) - (\nabla_x f(\bar{x}, \omega) - \nabla_x f(\bar{x}, \bar{\omega}))(x - y)|$$
$$\leq \frac{\eta}{4} (\|x - \bar{x}\|^2 + \|y - \bar{x}\|^2)$$

and, together with Lemma 4,

$$|f(x, \omega) - f(x, \bar{\omega}) - (f(y, \omega) - f(\bar{x}, \bar{\omega}))| + \|q(x, \omega) - q(x, \bar{\omega})\|$$
$$\leq e_2(\omega)^2 + \tau_2(\omega)(\|x - \bar{x}\| + \|y - \bar{x}\|) + \frac{\eta}{2} \|x - \bar{x}\|^2 + \frac{\eta}{4} \|y - \bar{x}\|^2 + |f(y, \bar{\omega}) - f(\bar{x}, \bar{\omega})|$$
$$\leq \tau_2(\omega)^2 + \tau_2(\omega)\|x - \bar{x}\| + \frac{\eta}{2} \|x - \bar{x}\|^2 + (\tau_2(\omega) + \frac{\eta}{4}\|y - \bar{x}\| + L)\|y - \bar{x}\|$$
$$\leq \tau_2(\omega)^2 + \tau_2(\omega)\|x - \bar{x}\| + \frac{\eta}{2} \|x - \bar{x}\|^2 + 2L\|y - \bar{x}\|,$$

implying

$$\begin{aligned}
\alpha(x,y,\omega) &:= \max\{f(x,\omega) - f(y,\omega), \mathrm{d}(q(x,\omega),P)\} \\
&\geq \max\{f(x,\bar{\omega}) - f(\bar{x},\bar{\omega}), \mathrm{d}(q(x,\bar{\omega}),P)\} \\
&\quad - |f(x,\omega) - f(x,\bar{\omega}) - (f(y,\omega) - f(\bar{x},\bar{\omega}))| - \|q(x,\omega) - q(x,\bar{\omega})\| \\
&\geq \frac{\eta}{2}\|x - \bar{x}\|^2 - \tau_2(\omega)^2 - \tau_2(\omega)\|x - \bar{x}\| - 2L\|y - \bar{x}\|.
\end{aligned}$$

Now assume $\|x - \bar{x}\| > \gamma_2(\|y - \bar{x}\|^{\frac{1}{2}} + \tau_2(\omega))$. Then

$$\|x - \bar{x}\| > 2\sqrt{\frac{L}{\eta}}\|y - \bar{x}\|^{\frac{1}{2}} + \frac{1 + \sqrt{2\eta + 1}}{\eta}\tau_2(\omega),$$

and therefore

$$\begin{aligned}
\frac{\eta}{2}\left(\|x - \bar{x}\| - \frac{\tau_2(\omega)}{\eta}\right)^2 &> \frac{\eta}{2}\left(2\sqrt{\frac{L}{\eta}}(\|y - \bar{x}\|^{\frac{1}{2}} + \frac{\sqrt{2\eta + 1}}{\eta}\tau_2(\omega)\right)^2 \\
&\geq 2L\|y - \bar{x}\| + \frac{2\eta + 1}{2\eta}\tau_2(\omega)^2,
\end{aligned}$$

showing $\alpha(x,y,\omega) > 0$. Hence we conclude that for every $x \in \mathscr{F}(\omega)$ with $\gamma_2(\|y - \bar{x}\|^{\frac{1}{2}} + \tau_2(\omega)) < \|x - \bar{x}\| \leq \bar{\rho}$ we have $f(x,\omega) > f(y,\omega)$, showing, together with $\gamma_2(\|y - \bar{x}\|^{\frac{1}{2}} + \tau_2(\omega)) < \bar{\rho}$, that there is a global solution $\bar{x}_\omega$ of the problem

$$\min \ f(x,\omega) \text{ subject to } q(x,\omega) \in P, \ x \in \bar{x} + \bar{\rho}B_{\mathbb{R}^n},$$

with $\bar{x}_\omega \in X(\omega)$ and $\|\bar{x}_\omega - \bar{x}\| \leq \gamma_2(\|y - \bar{x}\|^{\frac{1}{2}} + \tau_2(\omega)) < \bar{\rho}$. $\square$

Given $x \in \mathscr{F}(\omega)$, we denote by $\mathscr{C}_\omega(x) := \{u \in T^{\mathrm{lin}}_\omega(x) \,|\, \nabla_x f(x,\omega)u \leq 0\}$ the cone of critical directions at x. Further we introduce the Lagrangian $\mathscr{L} : \mathbb{R}^n \times \mathbb{R}^m \times \Omega \to \mathbb{R}$,

$$\mathscr{L}(x,\lambda,\omega) := f(x,\omega) + \lambda^T q(x,\omega), \tag{35}$$

and for every $\omega \in \Omega$, every $x \in \mathscr{F}(\omega)$ and every $u \in \mathscr{C}_\omega(x)$ we define the set of multipliers

$$\Lambda^1_\omega(x;u) := \{\lambda \in N(q(x,\omega);P;\nabla_x q(x,\omega)u) \,|\, \nabla_x \mathscr{L}(x,\lambda,\omega) = 0\}$$

Definition 5 Let $\bar{x} \in \mathscr{F}(\bar{\omega})$. We say that the *refined strong second-order sufficient condition* (RSSOSC) holds at $\bar{x}$ for the problem $P(\bar{\omega})$ if for every nonzero critical direction $0 \neq u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ one has

$$u^T \nabla^2_x \mathscr{L}(\bar{x},\lambda,\bar{\omega})u > 0 \ \ \forall \lambda \in \Lambda^1_{\bar{\omega}}(\bar{x};u).$$

Note that RSSOSC is sufficient for $\bar{x}$ being an essential local minimizer of second order only under some additional first-order optimality condition, e.g., if $\bar{x}$ is an extended M-stationary point in the sense of [11], i.e. $\Lambda^1_{\bar{\omega}}(\bar{x};u) \neq \emptyset \ \forall 0 \neq u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$.

Recall that the multifunction M_ω is defined by $M_\omega(x) = q(x,\omega) - P$. In what follows, the first- and second order sufficient conditions for metric subregularity FOSCMS and SOSCMS are used as in Proposition 1.

Theorem 4 *Let $\bar{x} \in \mathscr{F}(\bar{\omega})$, and assume that Assumption 1 is fulfilled.*

1. *If FOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ and RSSOSC holds at $\bar{x}$ for $P(\bar{\omega})$, then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that*

$$(S_M(\omega) \cup S(\omega)) \cap U \subset \bar{x} + L\tau_1(\omega)\mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W. \tag{36}$$

In addition, if $\bar{x}$ is an essential local minimizer for $P(\bar{\omega})$ and either $T^{\mathrm{lin}}_{\bar{\omega}}(\bar{x}) \neq \{0\}$ or $M_{\bar{\omega}}$ is metrically regular around $(\bar{x},0)$, then $X(\omega) \cap U \neq \emptyset \ \forall \omega \in W$.

2. *If SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ and $\bar{x}$ is an essential local minimizer of second order for $P(\bar{\omega})$, then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that*

$$(S_M(\omega) \cup S(\omega)) \cap U \subset \bar{x} + L\tau_2(\omega)\mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W. \tag{37}$$

In addition, if either $T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x}) \neq \{0\}$ or $M_{\bar{\omega}}$ is metrically regular around $(\bar{x}, 0)$, then $X(\omega) \cap U \neq \emptyset$ $\forall \omega \in W$.

Proof We show the first part by contraposition. If the assertion does not hold, by virtue of Theorem 3 with $F = \nabla_x f$ together with Proposition 1, there is some triple (u, λ, μ) satisfying $\lambda \in \Lambda_{\bar{\omega}}^1(\bar{x}; u)$, (25), (27) and $0 = \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega})u + \nabla_x q(\bar{x}; \bar{\omega})^T \mu$. By (27), there are sequences $(\alpha^k) \downarrow 0$ and $(\mu^k) \to \mu$ with $\lambda + \alpha^k \mu^k \in N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega})u)$, and by using [11, Lemma 2.1] we obtain $\lambda^T \nabla_x q(\bar{x}, \bar{\omega})u = (\lambda + \alpha^k \mu^k)^T \nabla_x q(\bar{x}, \bar{\omega})u = 0 \ \forall k$, showing $\lambda^T \nabla_x q(\bar{x}, \bar{\omega})u = \mu^T \nabla_x q(\bar{x}, \bar{\omega})u = 0$. Hence $u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ because of $0 = \nabla_x \mathscr{L}(\bar{x}, \lambda, \bar{\omega})u = \nabla_x f(\bar{x}, \bar{\omega})u$. It follows $0 = u^T \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega})u$, contradicting RSSOSC.

We also prove the second part by contraposition. Assuming on the contrary that the assertion does not hold, by applying Theorem 3 with $F = \nabla_x f$ together with Proposition 1, we find (u, λ, μ, v) satisfying $\lambda \in \Lambda_{\bar{\omega}}^1(\bar{x}; u)$, (25), (27) and $0 = \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega})u + \nabla_x q(\bar{x}; \bar{\omega})^T \mu$, (30) and $u^T \nabla_x^2(\lambda^T q)(\bar{x}, \bar{\omega})u = \nabla_x f(\bar{x}, \bar{\omega})^T v$. Proceeding as before, we obtain $u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ and $0 = u^T \nabla_x^2 \mathscr{L}(\bar{x}, \lambda, \bar{\omega})u = u^T \nabla_x^2 f(\bar{x}, \bar{\omega})u + \nabla_x f(\bar{x}, \bar{\omega})^T v \in T(\nabla_x f(\bar{x}, \bar{\omega})u; \mathbb{R}_-)$. Further, we have $T(q(\bar{x}, \bar{\omega}); P_i) \subset T(\nabla_x q(\bar{x}, \bar{\omega})u; T(q(\bar{x}, \bar{\omega}); P_i))$ for each $i \in \mathscr{I}(u)$ and thus

$$\begin{aligned}
\nabla_x q(\bar{x}, \bar{\omega})v + u^T \nabla_x^2 q(\bar{x}, \bar{\omega})u \in T(q(\bar{x}, \bar{\omega}); \bigcup_{i \in \mathscr{I}(u)} P_i) &= \bigcup_{i \in \mathscr{I}(u)} T(q(\bar{x}, \bar{\omega}); P_i) \\
&\subset \bigcup_{i \in \mathscr{I}(u)} T(\nabla_x q(\bar{x}, \bar{\omega})u; T(q(\bar{x}, \bar{\omega}); P_i)) \\
&= T(\nabla_x q(\bar{x}, \bar{\omega})u; T(q(\bar{x}, \bar{\omega}); \bigcup_{i \in \mathscr{I}(u)} P_i)) \\
&= T(\nabla_x q(\bar{x}, \bar{\omega})u; T(q(\bar{x}, \bar{\omega}); P)).
\end{aligned}$$

Hence we can conclude from [11, Lemma 3.16] that $\bar{x}$ is not an essential local minimizer of second order for $P(\bar{\omega})$ contradicting our assumption.

Finally, in both cases the assertion that $X(\omega) \cap U \neq \emptyset \ \forall \omega \in W$ follows from Proposition 2 respectively Theorem 2 and Proposition 3. $\qquad\square$

Remark 2 The statements (36) and (37), respectively, remain valid if we replace FOSCMS and SOSCMS by the formally weaker assumption that properties R_1 and R_2, respectively, hold. In order that the statement $X(\omega) \cap U \neq \emptyset$ holds in case that $\bar{x}$ is an essential local minimizer, we must ensure that $\mathscr{F}(\omega) \neq \emptyset$ for ω close to $\bar{\omega}$, e.g. by the assumption that there is a direction $0 \neq u \in T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x})$ fulfilling the assumptions of Proposition 2 or by the assumption of metric regularity of $M_{\bar{\omega}}$ around $(\bar{x}, 0)$.

Remark 3 If $p = 1$, i.e. if P is a polyhedron, then it follows from [8, Proposition 3.9] that the conditions FOSCMS for $M_{\bar{\omega}}$ and $T_{\bar{\omega}}^{\mathrm{lin}}(\bar{x}) \neq \{0\}$ imply metric regularity of $M_{\bar{\omega}}$ around $(\bar{x}, 0)$.

5 Application to MPECs

In this section we consider the special case $MPEC(\omega)$ as given in Example 1. In what follows we denote by $\bar{x}$ a point feasible for the problem $MPEC(\bar{\omega})$. Recall that the set P is given by $P = \mathbb{R}_-^{m_I} \times \{0\}^{m_E} \times Q_{EC}^{m_C}$. Hence, by [26, Proposition 6.41] and by Lemma 1, for every $(\tilde{g}, \tilde{h}, \tilde{a}) \in P \subset \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times (\mathbb{R}^2)^{m_C}$ we have

$$T((\tilde{g}, \tilde{h}, \tilde{a}); P) = T(\tilde{g}; \mathbb{R}_-^{m_I}) \times \{0\}^{m_E} \times \prod_{i=1}^{m_C} T(\tilde{a}_i; Q_{EC}),$$

$$N((\tilde{g}, \tilde{h}, \tilde{a}); P) = N(\tilde{g}; \mathbb{R}_-^{m_I}) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} N(\tilde{a}_i; Q_{EC}),$$

$$\hat{N}((\tilde{g}, \tilde{h}, \tilde{a}); P) = \hat{N}(\tilde{g}; \mathbb{R}_-^{m_I}) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} \hat{N}(\tilde{a}_i; Q_{EC}).$$

Further, for every direction $(u,v,w) \in T((\tilde{g},\tilde{h},\tilde{a});P)$ we have

$$N((\tilde{g},\tilde{h},\tilde{a});P;(u,v,w)) = N(\tilde{g};\mathbb{R}^{m_I}_-;u) \times \mathbb{R}^{m_E} \times \prod_{i=1}^{m_C} N(\tilde{a}_i;Q_{EC};w_i).$$

For the inequality constraints we obviously have

$$T(\tilde{g};\mathbb{R}^{m_I}_-) = \{u \in \mathbb{R}^{m_I} \,|\, u_i \leq 0 \;\forall i : \tilde{g}_i = 0\},$$
$$N(\tilde{g};\mathbb{R}^{m_I}_-) = \hat{N}(\tilde{g};\mathbb{R}^{m_I}_-) = \{\lambda \in \mathbb{R}^{m_I}_+ \,|\, \lambda_i \tilde{g}_i = 0,\; i = 1,\ldots,m_I\},$$
$$N(\tilde{g};\mathbb{R}^{m_I}_-;u) = \{\lambda \in \hat{N}(\tilde{g};\mathbb{R}^{m_I}_-) \,|\, \lambda_i u_i = 0,\; i = 1,\ldots,m_I\}.$$

By straightforward calculation we can obtain the formulas for the regular normal cone, the limiting normal cone and the contingent cone of the set Q_{EC} as follows: For all $a = (a_1,a_2) \in Q_{EC}$ we have

$$\hat{N}(a;Q_{EC}) = \left\{ (\xi_1,\xi_2) \,\middle|\, \begin{array}{ll} \xi_2 = 0 & \text{if } 0 = a_1 > a_2 \\ \xi_1 \geq 0, \xi_2 \geq 0 & \text{if } a_1 = a_2 = 0 \\ \xi_1 = 0 & \text{if } a_1 < a_2 = 0 \end{array} \right\},$$

$$N(a;Q_{EC}) = \begin{cases} \hat{N}(a;Q_{EC}) & \text{if } a \neq (0,0) \\ \{(\xi_1,\xi_2) \,|\, \text{either } \xi_1 > 0,\; \xi_2 > 0 \text{ or } \xi_1\xi_2 = 0\} & \text{if } a = (0,0), \end{cases}$$

$$T(a;Q_{EC}) = \left\{ (u_1,u_2) \,\middle|\, \begin{array}{ll} u_1 = 0 & \text{if } 0 = a_1 > a_2 \\ u_1 \leq 0, u_2 \leq 0, u_1 u_2 = 0 & \text{if } a_1 = a_2 = 0 \\ u_2 = 0 & \text{if } a_1 < a_2 = 0 \end{array} \right\},$$

and for all $u = (u_1,u_2) \in T(a;Q_{EC})$ we have

$$T(u;T(a;Q_{EC})) = \begin{cases} T(a;Q_{EC}) & \text{if } a \neq (0,0) \\ T(u;Q_{EC})) & \text{if } a = (0,0), \end{cases}$$

$$\hat{N}(u;T(a;Q_{EC})) = \begin{cases} \hat{N}(a;Q_{EC}) & \text{if } a \neq (0,0) \\ \hat{N}(u;Q_{EC}) & \text{if } a = (0,0), \end{cases}$$

$$N(a;Q_{EC};u) = \begin{cases} N(a;Q_{EC}) & \text{if } a \neq (0,0) \\ N(u;Q_{EC}) & \text{if } a = (0,0). \end{cases}$$

Denoting

$$\bar{I}_g := \{i \in \{1,\ldots,m_I\} \,|\, g_i(\bar{x},\bar{\omega}) = 0\},$$
$$\bar{I}^{+0} := \{i \in \{1,\ldots,m_C\} \,|\, G_i(\bar{x},\bar{\omega}) > 0 = H_i(\bar{x},\bar{\omega})\},$$
$$\bar{I}^{0+} := \{i \in \{1,\ldots,m_C\} \,|\, G_i(\bar{x},\bar{\omega}) = 0 < H_i(\bar{x},\bar{\omega})\},$$
$$\bar{I}^{00} := \{i \in \{1,\ldots,m_C\} \,|\, G_i(\bar{x},\bar{\omega}) = 0 = H_i(\bar{x},\bar{\omega})\},$$

the cone $T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ is given by

$$T^{\text{lin}}_{\bar{\omega}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \nabla_x g_i(\bar{x},\bar{\omega})u \leq 0,\; i \in \bar{I}_g, \\ \nabla_x h_i(\bar{x},\bar{\omega})u = 0,\; i = 1,\ldots,m_E, \\ \nabla_x G_i(\bar{x},\bar{\omega})u = 0,\; i \in \bar{I}^{0+}, \\ \nabla_x H_i(\bar{x},\bar{\omega})u = 0,\; i \in \bar{I}^{+0}, \\ -(\nabla_x G_i(\bar{x},\bar{\omega})u, \nabla_x H_i(\bar{x},\bar{\omega})u) \in Q_{EC},\; i \in \bar{I}^{00} \end{array} \right\}.$$

Given $u \in T^{\text{lin}}_{\bar{\omega}}(\bar{x})$ we define

$$I_g(u) := \{i \in \bar{I}_g \,|\, \nabla_x g_i(\bar{x},\bar{\omega})u = 0\}$$
$$I^{+0}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u > 0 = \nabla_x H_i(\bar{x},\bar{\omega})u\},$$
$$I^{0+}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u = 0 < \nabla_x H_i(\bar{x},\bar{\omega})u\},$$
$$I^{00}(u) := \{i \in \bar{I}^{00} \,|\, \nabla_x G_i(\bar{x},\bar{\omega})u = 0 = \nabla_x H_i(\bar{x},\bar{\omega})u\}.$$

For given $u \in T_{\bar{\omega}}^{\lin}(\bar{x})$, the set $N(q(\bar{x}, \bar{\omega}); P; \nabla_x q(\bar{x}, \bar{\omega}) u)$ consists of all $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying

$$\lambda_i^g \geq 0, \ i \in I_g(u), \quad \lambda_i^g = 0, \ i \notin I_g(u), \tag{38}$$

$$\lambda_i^H = 0, \ i \in \bar{I}^{0+} \cup I^{0+}(u), \quad \lambda_i^G = 0, \ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{39}$$

$$\text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \ i \in I^{00}(u) \tag{40}$$

Then we have the following characterizations of metric subregularity: FOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T_{\bar{\omega}}^{\lin}(\bar{x})$ the only multiplier $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying (38)-(40) and

$$\sum_{i=1}^{m_I} \lambda_i^g \nabla_x g_i(\bar{x}, \bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x h_i(\bar{x}, \bar{\omega}) - \sum_{i=1}^{m_C} (\lambda_i^G \nabla_x G_i(\bar{x}, \bar{\omega}) + \lambda_i^H \nabla_x H_i(\bar{x}, \bar{\omega})) = 0 \tag{41}$$

is the trivial multiplier $\lambda = 0$.

Similarly, SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T_{\bar{\omega}}^{\lin}(\bar{x})$ the only multiplier $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ satisfying (38)-(41) and

$$u^T \left(\sum_{i=1}^{m_I} \lambda_i^g \nabla_x^2 g_i(\bar{x}, \bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x^2 h_i(\bar{x}, \bar{\omega}) - \sum_{i=1}^{m_C} (\lambda_i^G \nabla_x^2 G_i(\bar{x}, \bar{\omega}) + \lambda_i^H \nabla_x^2 H_i(\bar{x}, \bar{\omega})) \right) u \geq 0 \tag{42}$$

is $\lambda = 0$.

Further, $M_{\bar{\omega}}$ is metrically regular around $(\bar{x}, 0)$ if and only if in case $u = 0$ the only multiplier λ fulfilling (38)-(41) is $\lambda = 0$. Note that $I_g(0) = \bar{I}_g$, $I^{+0}(0) = I^{0+}(0) = \emptyset$ and $I^{00}(0) = \bar{I}^{00}$.

We now translate Theorem 3 into terms of $MPEC(\omega)$. To do this it is convenient to consider the (extended) Lagrangian

$$\mathscr{L}^{\lambda_0}(x, \lambda, \omega) := \lambda_0 f(x, \omega) + \sum_{i=1}^{m_I} \lambda_i^g g_i(\bar{x}, \omega) + \sum_{i=1}^{m_E} \lambda_i^h h_i(\bar{x}, \omega) - \sum_{i=1}^{m_C} (\lambda_i^G G_i(\bar{x}, \omega) + \lambda_i^H H_i(\bar{x}, \omega))$$

where $\lambda_0 \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ and $\omega \in \Omega$. The Lagrangian $\mathscr{L}^1$ corresponds to the Lagrangian $\mathscr{L}$ as defined in (35).

Corollary 3 *Let $\bar{x}$ be feasible for $MPEC(\bar{\omega})$.*

1. *If property R_1 holds and there does not exist $0 \neq u \in T_{\bar{\omega}}^{\lin}(\bar{x})$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$ and $\mu = (\mu^g, \mu^h, \mu^G, \mu^H)$ satisfying (38)-(40),*

$$\mu_i^g \geq 0, \ i \in I_g(u) : \lambda_i^g = 0, \quad \mu_i^g = 0, \ i \notin I_g(u), \tag{43}$$

$$\mu_i^H = 0, \ i \in \bar{I}^{0+} \cup I^{0+}(u), \quad \mu_i^G = 0, \ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{44}$$

$$\text{either } \mu_i^G > 0, \mu_i^H > 0 \text{ or } \mu_i^G \mu_i^H = 0, \ i \in I^{00}(u) : \lambda_i^G = \lambda_i^H = 0 \tag{45}$$

$$\mu_i^G \geq 0, \ i \in I^{00}(u) : \lambda_i^G = 0, \lambda_i^H > 0, \quad \mu_i^G = 0, \ i \in I^{00}(u) : \lambda_i^G = 0, \lambda_i^H < 0, \tag{46}$$

$$\mu_i^H \geq 0, \ i \in I^{00}(u) : \lambda_i^H = 0, \lambda_i^G > 0, \quad \mu_i^H = 0, \ i \in I^{00}(u) : \lambda_i^H = 0, \lambda_i^G < 0, \tag{47}$$

$$\nabla_x \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) = 0, \tag{48}$$

$$\nabla_x^2 \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) u + \nabla_x \mathscr{L}^0(\bar{x}, \mu, \bar{\omega}) = 0, \tag{49}$$

then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the stationary solution mappings $S(\omega)$ and $S_M(\omega)$ of $MPEC(\omega)$ one has

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_1(\omega) \mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W.$$

2. *If property R_2 holds and there does not exist $0 \neq u \in T_{\bar{\omega}}^{\lin}(\bar{x})$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H)$, $\mu = (\mu^g, \mu^h, \mu^G, \mu^H)$ and $v \in \mathbb{R}^n$ satisfying (38)-(40), (43)-(49) and*

$$\nabla_x g_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 g_i(\bar{x},\bar{\omega})u \leq 0, \ i \in \bar{I}_g, \tag{50}$$

$$\nabla_x h_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 h_i(\bar{x},\bar{\omega})u = 0, \ i = 1,\dots,m_E, \tag{51}$$

$$\nabla_x G_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 G_i(\bar{x},\bar{\omega})u = 0, \ i \in \bar{I}^{0+} \cup I^{0+}(u), \tag{52}$$

$$\nabla_x H_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 H_i(\bar{x},\bar{\omega})u = 0, \ i \in \bar{I}^{+0} \cup I^{+0}(u), \tag{53}$$

$$-(\nabla_x G_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 G_i(\bar{x},\bar{\omega})u, \nabla_x H_i(\bar{x},\bar{\omega})v + u^T \nabla_x^2 H_i(\bar{x},\bar{\omega})u) \in T((0,0),Q_{EC}), \ i \in I^{00}(u), \tag{54}$$

$$u^T \nabla_x^2 \mathscr{L}^0(\bar{x},\lambda,\bar{\omega})u = \nabla_x f(\bar{x},\bar{\omega})v, \tag{55}$$

then there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the stationary solution mappings $S(\omega)$ and $S_M(\omega)$ of MPEC(ω) one has

$$(S(\omega) \cup S_M(\omega)) \cap U \subset \bar{x} + L\tau_2(\omega)\mathscr{B}_{\mathbb{R}^n} \ \forall \omega \in W.$$

In the following table we specify for each condition of Theorem 3 its corresponding counterpart in Corollary 3:

Thm.3(1.)	Cor.3(1.)	Thm.3(2.)	Cor.3(2.)
(26)	(38)-(40)	(30)	(50)-(54)
(27)	(43)-(47)	(31)	(55)
(28)	(48)		
(29)	(49)		

For verifying RSSOSC we also need the sets $\Lambda_{\bar{\omega}}^1(\bar{x};u)$. For given $u \in \mathscr{C}(\bar{x})$, this set is the collection of all multipliers $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C}$ fulfilling (38)-(40) and $\nabla_x \mathscr{L}^1(x,\lambda,\bar{\omega}) = 0$.

Finally let us mention that the definition of M-stationary solutions for MPEC(ω) is the same as coined by Scholtes [28], i.e., the set of MPEC-multipliers associated with x,

$$\Lambda_{MPEC}^M(x,\omega)$$
$$:= \left\{ \lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}_+^{m_I} \times \mathbb{R}^{m_E} \times \mathbb{R}^{m_C} \times \mathbb{R}^{m_C} \ \middle| \ \begin{array}{l} \nabla_x \mathscr{L}^1(x,\lambda,\omega) = 0, \\ \lambda^{gT} g(\bar{x},\bar{\omega}) = 0, \\ \lambda_i^H = 0, \ i \in \bar{I}^{0+}, \quad \lambda_i^G = 0, \ i \in \bar{I}^{+0}, \\ \text{either } \lambda_i^G > 0, \lambda_i^H > 0 \text{ or } \lambda_i^G \lambda_i^H = 0, \ i \in I^{00} \end{array} \right\},$$

is not empty.

Now we demonstrate our results in the following examples:

Example 3 Consider the following MPEC depending on the parameter $\omega \in \Omega := \mathbb{R}_+$,

$$MPEC(\omega) \qquad \min_x -2x_1 + x_2$$
$$\text{subject to} \qquad g_1(x,\omega) := x_1 - x_2 - \omega \leq 0,$$
$$g_2(x,\omega) := \omega \phi(x_1) - x_2 \leq 0,$$
$$-(G_1(x,\omega), H_1(x,\omega)) := -(x_1,x_2) \in Q_{EC},$$

where $\phi(x) := x^6(1 - \cos\frac{1}{x})$ for $x \neq 0$ and $\phi(0) := 0$, with $\bar{x} = (0,0)$ and $\bar{\omega} = 0$. Then we have $\bar{I}_g = \{1,2\}$, $\bar{I}^{0+} = \bar{I}^{+0} = \emptyset$, $\bar{I}^{00} = \{1\}$,

$$T_{\bar{\omega}}^{\lin}(\bar{x}) = \{(u_1,u_2) \mid u_1 - u_2 \leq 0, \ -u_2 \leq 0, \ -(u_1,u_2) \in Q_{EC}\} = \{(0,u_2) \mid u_2 \geq 0\}$$

and we obtain, that $I_g(u) = I^{+0}(u) = I^{00}(u) = \emptyset$, $I^{0+}(u) = \{1\}$ for every $0 \neq u = (0,u_2) \in T_{\bar{\omega}}^{\lin}(\bar{x})$ and the set of multipliers $\lambda = (\lambda_1^g, \lambda_2^g, \lambda^G, \lambda^H)$ fulfilling (38)-(40) is given by the relations

$$\lambda_1^g = 0, \ \lambda_2^g = 0, \ \lambda^H = 0. \tag{56}$$

Since (41) reads as

$$\lambda_1^g(1,-1) + \lambda_2^g(0,-1) - \lambda^G(1,0) - \lambda^H(0,1) = (0,0),$$

we conclude that FOSCMS is fulfilled and therefore property R_1 holds. But for $u = (0,0)$ we see that $\lambda = (0,1,0,-1)$ satisfies (38)-(41) and therefore metric regularity around $(\bar{x},0)$ does not hold.

It is easy to see that $\mathscr{C}_{\bar{\omega}}(\bar{x}) = \{0\}$. Hence RSSOSC is fulfilled, $\bar{x}$ is an extended M-stationary point in the sense of [11] and therefore $\bar{x}$ is an essential local minimizer of second order by [11, Theorem 3.21]. Hence we can apply Theorem 4(1.) to obtain

$$(S(\omega) \cup S_M(\omega)) \cap U \subset L\omega \mathscr{B}_{\mathbb{R}^2} \ \forall \omega \in W$$

for some neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ and some constant $L > 0$. Further we have $X(\omega) \cap U \neq \emptyset$ $\forall \omega \in W$, since $(0,1) \in T_{\bar{\omega}}^{\lin}(\bar{x})$. Further, the set of MPEC-multipliers is given by

$$\Lambda_{MPEC}^M(\bar{x},\bar{\omega}) = \{(0,1,-2,0)\} \cup \{(2,\lambda_2^g,0,\lambda^H) \,|\, \lambda_2^g \geq 0, \lambda_2^g + \lambda^H = -1\}.$$

Now let us compute the sets $S(\omega)$, $S_M(\omega)$ and $X(\omega)$. For every $\omega > 0$ the feasible set $\mathscr{F}(\omega)$ consists of the union of the nonnegative x_2 axis, $\{(0,x_2) \,|\, x_2 \geq 0\}$, and the set $\tilde{X}(\omega) := \{(x_1,0) \,|\, 0 < x_1 \leq \omega, \phi(x_1) = 0\} = \{(\frac{1}{2k\pi},0) \,|\, k \in \mathbb{N}, 2k\pi\omega \geq 1\}$ consisting of a sequence of isolated points with limit $\bar{x}$. It is easily checked that $(0,0)$ is M-stationary with unique MPEC-multiplier $(0,1,-2,0)$ for every $\omega > 0$, but not a local minimizer. Further, every element of $\tilde{X}(\omega)$ is a local minimizer and consequently B-stationary, but all these local minimizers are not M-stationary, because the constraints are degenerated; the only possible exception being the point $(\omega,0)$ if $\frac{1}{2\pi\omega} \in \mathbb{N}$, where the multipliers are

$$\Lambda_{MPEC}^M((\omega,0),\omega) = \{(2,\lambda_2^g,0,\lambda^H) \,|\, \lambda_2^g \geq 0, \lambda_2^g + \lambda^H = -1\}.$$

Summarizing all, we have

$$S(\omega) = X(\omega) = \tilde{X}(\omega), S_M(\omega) = \begin{cases} \{(0,0)\} & \text{if } \frac{1}{2\pi\omega} \notin \mathbb{N} \\ \{(0,0),(\omega,0)\} & \text{else} \end{cases}$$

for $\omega > 0$ and $S(0) = S_M(0) = X(0) = \{0,0\}$. It is quite surprising that we could prove existence and upper Lipschitz continuity of solutions in the absence of metric regularity of the constraints, a situation which is not possible in case of NLP (see,e.g. [20, Lemma 8.31]).

For the sake of completeness we also formulate explicitly the sufficient conditions for upper Lipschitz continuity of Corollary 3, although we know by the proof of Theorem 4 that they are implied by RSSOSC. The conditions of Corollary 3 are, that there are no elements $u = (0,u_2)$ with $u_2 > 0$ and $\lambda = (\lambda_1^g, \lambda_2^g, \lambda^G, \lambda^H)$, $\mu = (\mu_1^g, \mu_2^g, \mu^G, \mu^H)$ fulfilling (56), $\mu_1^g = \mu_2^g = \mu^H = 0$,

$$(-2,1) + \lambda_1^g(1,-1) + \lambda_2^g(0,-1) - \lambda^G(1,0) - \lambda^H(0,1) = (0,0) \tag{57}$$

and

$$(0,0) + \mu_1^g(1,-1) + \mu_2^g(0,-1) - \mu^G(1,0) - \mu^H(0,1) = (0,0).$$

It is easy to see that this holds true, because (56) and (57) are inconsistent.

For this example let us now compare our results with those of Guo,Lin and Ye [12, Theorem 5.5], who considered a somewhat different setting, namely the calmness of pairs of M-stationary solutions together with their associated multipliers. Application of [12, Theorem 5.5] yields that, for every $\bar{\lambda} \in \Lambda_{MPEC}(\bar{x},\bar{\omega})$, there are neighborhoods U of $(\bar{x},\bar{\lambda})$ and W of $\bar{\omega}$ and a constant $\kappa > 0$ such that for every $(x,\lambda,\omega) \in U \times W$ with $x \in S_M(\omega)$, $\lambda \in \Lambda_{MPEC}^M(x,\omega)$ one has

$$\|x - \bar{x}\| + \mathrm{d}(\lambda, \Lambda_{MPEC}(\bar{x},\bar{\omega})) \leq \kappa \|\omega - \bar{\omega}\|.$$

In particular, by applying this result with $\bar{\lambda} = (0,1,-2,0)$ and $\bar{\lambda} = (2,0,0,-1)$, respectively, we obtain that $S_M(\omega)$ behaves upper Lipschitz stable. But in our example we also have local minimizers in $X(\omega)$, which are not M-stationary, and [12, Theorem 5.5] does not give any information for these local minimizers, whereas our theory establishes upper Lipschitz continuity. Moreover, if we slightly change the data of this example by setting $\phi(x) := x^6(0.9 - \cos\frac{1}{x})$ for $x \neq 0$ and $\phi(0) := 0$, then one can show that for $\omega > 0$

all local minimizers in $X(\omega)$ are also M-stationary, but for all local minimizers different from $(\omega,0)$ one has that the corresponding MPEC multiplier tends to infinity when ω tends to $\bar{\omega}$. Also in this case [12, Theorem 5.5] does not give any information due to the discontinuity of the multipliers in contrast to our results.

Example 4 [see [18]] Consider the problem

$$MPEC(\omega) \qquad \min_{x} -x_1 - (x_2 + \omega)^2$$
$$\text{subject to} \qquad -(G_1(x,\omega), H_1(x,\omega)) := -(x_1, x_2) \in Q_{EC},$$

depending on the parameter $\omega \in \Omega := \mathbb{R}$. This example was used in [18] to demonstrate that M-stationary points are not strongly stable in the sense of Kojima [21], but the C-stationary points are strongly stable. Straightforward analysis yields that

$$X(\omega) = S(\omega) = S_M(\omega) = \begin{cases} \emptyset & \text{if } \omega > 0, \\ (0, -\omega) & \text{if } \omega < 0 \end{cases}$$

and $X(0) = S(0) = \emptyset$, $S_M(0) = (0,0)$. Let us now choose $\bar{x} = (0,0)$ and $\bar{\omega} = 0$ as reference point and apply Corollary 3(1.). Obviously, the constraint mapping $M_{\bar{\omega}}$ is metrically regular around $(\bar{x}, 0)$ and thus property R_1 is fulfilled. Since the cone $T_{\bar{\omega}}^{\text{lin}}(\bar{x})$ is generated by the 2 directions $u^1 = (1,0)$ and $u^2 = (0,1)$, in order to establish upper Lipschitz stability of $S(\omega) \cup S_M(\omega)$ it suffices to show that there are no multipliers $\lambda = (\lambda^G, \lambda^H)$ and $\mu = (\mu^G, \mu^H)$ fulfilling either

$$\lambda^G = \mu^G = 0,$$
$$\nabla_x \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) = (-1,0) - \lambda^G(1,0) - \lambda^H(0,1) = 0,$$
$$\nabla_x^2 \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega})u^1 + \nabla_x \mathscr{L}^0(\bar{x}, \mu, \bar{\omega}) = (0,0) - \mu^G(1,0) - \mu^H(0,1) = 0$$

or

$$\lambda^H = \mu^H = 0,$$
$$\nabla_x \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega}) = (-1,0) - \lambda^G(1,0) - \lambda^H(0,1) = 0,$$
$$\nabla_x^2 \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega})u^2 + \nabla_x \mathscr{L}^0(\bar{x}, \mu, \bar{\omega}) = (0,-2) - \mu^G(1,0) - \mu^H(0,1) = 0,$$

which is obviously the case. Hence we obtain

$$(S(\omega) \cup S_M(\omega)) \cap U \subset L\omega \mathscr{B}_{\mathbb{R}^2} \ \forall \omega \in W$$

for some neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ and some constant $L > 0$.

Note that the assumptions of [12, Theorem 5.5] are not fulfilled, and therefore it cannot be applied.

Corollary 3 recovers well-known results for standard nonlinear programs in the case of C^2 data. Consider the parametric nonlinear optimization problem

$$NLP(\omega) \qquad \min f(x,\omega) \quad \text{subject to} \quad g(x,\omega) \le 0, \ h(x,\omega) = 0, \tag{58}$$

where the function $f : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, and $q = (g,h) : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}$ satisfy the Basic Assumption 1 and, in addition, that $f(\bar{x}, \cdot)$, $\nabla_x f(\bar{x}, \cdot)$, $q(\bar{x}, \cdot)$ and $\nabla_x q(\bar{x}, \cdot)$ are Lipschitzian near $\bar{\omega}$.

We use the same notation as above in this section, the results and representations reduce to forms which omit the complementarity data G_i and H_i. In particular, the cone $T_{\bar{\omega}}^{\text{lin}}(\bar{x})$ is given by

$$T_{\bar{\omega}}^{\text{lin}}(\bar{x}) = \left\{ u \in \mathbb{R}^n \mid \begin{array}{l} \nabla_x g_i(\bar{x}, \bar{\omega})u \le 0, \ i \in \bar{I}_g, \\ \nabla_x h_i(\bar{x}, \bar{\omega})u = 0, \ i = 1, \dots, m_E \end{array} \right\},$$

with $\bar{I}_g = \{i \in \{1, \dots, m_I\} \mid g_i(\bar{x}, \bar{\omega}) = 0\}$, and the extended Lagrangian defined above reduces to

$$\mathscr{L}^{\lambda_0}(x, \lambda^g, \lambda^h, \omega) := \lambda_0 f(x,\omega) + \sum_{i=1}^{m_I} \lambda_i^g g_i(x,\omega) + \sum_{i=1}^{m_E} \lambda_i^h h_i(x,\omega).$$

Then a point $x \in \mathscr{F}(\omega)$ feasible for $NLP(\omega)$ is called *stationary in the Karush-Kuhn-Tucker (KKT) sense*, if the set of multipliers associated with x,

$$\Lambda(x,\omega) := \{(\lambda^g, \lambda^h) \in \mathbb{R}^{m_I}_+ \times \mathbb{R}^{m_E} \mid \nabla_x \mathscr{L}^1(x, \lambda^g, \lambda^h, \omega) = 0, \lambda^{gT} g(\bar{x}, \bar{\omega}) = 0\}$$

is not empty. Further recall that under some constraint qualification (CQ), e.g. the Mangasarian-Fromovitz CQ (MFCQ), the three concepts of M-stationarity, B-stationarity and KKT-stationarity coincide. Since MFCQ persists under small perturbations in our setting, hence $S(\omega) \cap U$ and $S_M(\omega) \cap U$ coincide for some neighborhood U of $\bar{x} \in S(\bar{\omega})$ and all ω close to $\bar{\omega}$, provided that $\bar{x}$ satisfies MFCQ.

We show how to prove two classical results by means of Corollary 3 or Theorem 4, respectively.

Corollary 4 *(Klatte, Kummer [20, Theorem 8.24]) Given a stationary solution $\bar{x}$ of $NLP(\bar{\omega})$ in the KKT sense, we suppose that MFCQ is satisfied at $(\bar{x}, \bar{\omega})$. Then S is locally upper Lipschitz at $(\bar{\omega}, \bar{x})$ if for every $\lambda = (\lambda^g, \lambda^h) \in \Lambda(\bar{x}, \bar{\omega})$, the system*

$$\begin{aligned}
&u \in T^{\mathrm{lin}}_{\bar{\omega}}(\bar{x}), \quad (\alpha, \beta) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}, \\
&\nabla^2_x \mathscr{L}^1(\bar{x}, \lambda^g, \lambda^h, \bar{\omega}) u + \nabla_x \mathscr{L}^0(\bar{x}, \alpha, \beta, \bar{\omega}) = 0, \\
&\nabla_x g_i(\bar{x}, \bar{\omega}) u = 0, \qquad \text{if } i \in \bar{I}_g : \lambda^g_i > 0, \\
&\alpha_i \geq 0, \ \alpha_i \nabla_x g_i(\bar{x}, \bar{\omega}) u = 0, \text{ if } i \in \bar{I}_g : \lambda^g_i = 0, \\
&\alpha_i = 0, \qquad \text{if } i \notin \bar{I}_g.
\end{aligned} \qquad (59)$$

has no solution (u, α, β) with $u \neq 0$.

Proof Take any (u, λ, μ) which satisfies $u \in T^{\mathrm{lin}}_{\bar{\omega}}(\bar{x})$, (38), (43), (48) and (49). It is sufficient to show that (u, λ, μ) also satisfies both $\lambda \in \Lambda(\bar{x}, \bar{\omega})$ and (59) when putting $\alpha = \mu^g$ and $\beta = \mu^h$. Indeed, then the assumption of the corollary says that $u = 0$. Since MFCQ at $(\bar{x}, \bar{\omega})$ implies property R_1, Corollary 3 thus immediately gives in the special case $NLP(\omega)$

$$S(\omega) \cap U \subset \bar{x} + L\tau_1(\omega) \ \forall \omega \in W.$$

By our assumptions for f and $q = (g, h)$, we have for all ω sufficiently close to $\bar{\omega}$,

$$\begin{aligned}
\tau_1(\omega) &= \|\nabla_x f(\bar{x}, \omega) - \nabla_x f(\bar{x}, \bar{\omega})\| + \|\nabla_x q(\bar{x}, \omega) - \nabla_x q(\bar{x}, \bar{\omega})\| + \|q(\bar{x}, \omega) - q(\bar{x}, \bar{\omega})\| \\
&\leq const. \|\omega - \bar{\omega}\|
\end{aligned}$$

and so we are done.

It remains to show that (u, λ, μ) satisfies both $\lambda \in \Lambda(\bar{x}, \bar{\omega})$ and (59) when putting $\alpha = \mu^g$ and $\beta = \mu^h$. First, we observe that (38) says that $\lambda^g_i = 0$ if $i \notin \bar{I}_g$ or if $i \in \bar{I}_g \setminus I_g(u)$, and $\lambda^g_i \geq 0$ if $i \in I_g(u)$. Together with (48), this gives $\lambda \in \Lambda(\bar{x}, \bar{\omega})$. Second, $u \in T^{\mathrm{lin}}_{\bar{\omega}}(\bar{x})$ and (49) directly appear in the first two lines of (59). Moreover, (43) says (i) $\alpha_i = \mu^g_i = 0$ if $i \notin \bar{I}_g$ or if $i \in \bar{I}_g : \nabla_x g_i(\bar{x}, \bar{\omega}) u < 0$, and (ii) $\alpha_i = \mu^g_i \geq 0$ if $i \in \bar{I}_g : \lambda^g_i = 0$, which implies that (u, α) fulfills the last two lines of (59). Finally, assuming that for some $i \in \bar{I}_g : \lambda^g_i > 0$ one has $\nabla_x g_i(\bar{x}, \bar{\omega}) u \neq 0$, i.e., $i \notin I_g(u)$, this contradicts (38). Therefore, also the third line of (59) is satisfied. $\qquad \square$

Remark 4 The opposite direction of Corollary 4 is true if the parametric problem (58) includes canonical perturbations, i.e. if f and $q = (g, h)$ are defined by $f(x, \omega) = \tilde{f}(t, x) - \langle a, x \rangle$ and $(g, h)(x, \omega) = (\tilde{g}, \tilde{h})(t, x) - b$ for varying $\omega = (t, a, b) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{m_I + m_L}$, see [20, Thm. 8.24].

The following Hölder stability result is a generalization of Proposition 4.41 in Bonnans and Shapiro [4], where global minimizing sets of $NLP(\omega)$ are considered instead of the sets of stationary solutions or local minimizers as here. Recall that $\bar{x} \in \mathscr{F}(\bar{\omega})$ satisfies the quadratic growth condition for $NLP(\bar{\omega})$ if $f(x, \bar{\omega}) - f(\bar{x}, \bar{\omega}) \geq \eta \|x - \bar{x}\|^2 \ \forall x \in V \cap \mathscr{F}(\bar{\omega})$ holds for some $\eta > 0$ and some neighborhood V of $\bar{x}$.

Corollary 5 *(upper Hölder continuity of local minimizers)*
Suppose that $\bar{x} \in \mathscr{F}(\bar{\omega})$ satisfies the quadratic growth condition for NLP($\bar{\omega}$), MFCQ is satisfied at $\bar{x}$, and the functions f, g, and h are twice continuously differentiable near $(\bar{x}, \bar{\omega})$. Then X and S are locally nonempty-valued and upper Hölder of order $\frac{1}{2}$ at $(\bar{\omega}, \bar{x})$, i.e., there are neighborhoods U of $\bar{x}$, W of $\bar{\omega}$ and a constant $L > 0$ such that for the solution mapping $X(\omega)$ of $P(\omega)$ one has

$$\emptyset \neq X(\omega) \cap U \subset S(\omega) \cap U \subset \{\bar{x}\} + L \|\omega - \bar{\omega}\|^{\frac{1}{2}} \quad \forall \omega \in W.$$

Proof The corollary is an immediate consequence of the second part of Theorem 4 by taking into account that MFCQ is equivalent to metric regularity of the multifunction $M_{\bar{\omega}}$ around $(\bar{x}, 0)$ implying SOSCMS by virtue of Theorem 1, and that under MFCQ the quadratic growth condition is equivalent to the property that $\bar{x}$ is an essential local minimizer of second-order.   □

Results of this type are classical, but concern global or so-called complete local minimizing (CLM) sets, see e.g. [2–4,19]. Corollary 5 extends this by providing even existence and upper Hölder continuity for stationary solution sets and the sets of *all* local minimizers, while part 2 of Theorem 4 allows us in addition to weaken the assumption MFCQ by assuming SOSCMS (which implies R_2) instead. Note that upper Hölder stability of the global minimizing or CLM set mapping even holds if $\bar{x}$ is not locally isolated (see e.g. [4,19]). Recently, Kummer [22] presented an alternative characterization of upper Hölder stability of KKT- stationary solutions via convergence properties of suitable iteration procedures.

Remark 5 For nonlinear programs, a standard second order sufficient optimality condition at $\bar{x} \in \mathscr{F}(\bar{\omega})$ (see for example [4, Prop.5.48]) is equivalent to the property (34) (i.e., $\bar{x}$ is an essential local minimizer of second order for the problem $NLP(\bar{\omega})$). This is true without any constraint qualification, see the remarks following Thm. 5.11 in [7].

Further note that in the case (58), SOSCMS is fulfilled for $M_{\bar{\omega}}$ at $\bar{x}$ if and only if for every direction $0 \neq u \in T_{\bar{\omega}}^{\text{lin}}(\bar{x})$, the only multiplier $\lambda = (\lambda^g, \lambda^h) \in \mathbb{R}^{m_I} \times \mathbb{R}^{m_E}$ satisfying

$$\sum_{i=1}^{m_I} \lambda_i^g \nabla g_i(\bar{x}, \bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla h_i(\bar{x}, \bar{\omega}) = 0, \ \lambda_i^g \geq 0, \ i \in I_g(u), \ \lambda_i^g = 0, \ i \notin I_g(u),$$

$$u^T \left(\sum_{i=1}^{m_I} \lambda_i^g \nabla_x^2 g_i(\bar{x}, \bar{\omega}) + \sum_{i=1}^{m_E} \lambda_i^h \nabla_x^2 h_i(\bar{x}, \bar{\omega}) \right) u \geq 0$$

is $\lambda = 0$, where as above $I_g(u) = \{i \in \bar{I}_g \mid \nabla_x g_i(\bar{x}, \bar{\omega}) u = 0\}$.

Example 5 This example shows, as mentioned above, that part 2 of Theorem 4 gives a stronger result than the (more classical) Corollary 5. Consider the parameter dependent program

$$P(\omega) \quad \min x_1^2 - x_2^2 \ \text{ s.t. } -x_2 + \omega \leq 0, \ x_2 - x_1^2 \leq 0,$$

with $\Omega = \mathbb{R}$, $\bar{\omega} = 0$ and $\bar{x} = (0,0)$. $M_{\bar{\omega}}$ is not metrically regular near $(\bar{x}, 0)$. One easily sees that SOSCMS holds for $M_{\bar{\omega}}$ at $\bar{x}$: one has to check only the directions $u = (\pm 1, 0) \in T_{\bar{\omega}}^{\text{lin}}(\bar{x})$. Obviously, $\bar{x}$ is an an essential local minimizer of second order for the problem $P(\bar{\omega})$, and one has

$$S(\omega) = S_M(\omega) = \begin{cases} \{(0,0), (0,\omega)\} & \text{if } \omega < 0, \\ \{(\pm\sqrt{\omega}, \omega)\} & \text{if } 0 \leq \omega \leq \frac{1}{2}, \end{cases}$$

$$X(\omega) = \begin{cases} \{(0,\omega)\} & \text{if } \omega < 0, \\ \{(\pm\sqrt{\omega}, \omega)\} & \text{if } 0 \leq \omega \leq \frac{1}{2}. \end{cases}$$

Indeed, the assumptions and the statement of part 2 of Theorem 4 are fulfilled.

At the end of this paper let us mention, that our results incorporate the combinatorial structure of the problem and are more far reaching than those results which one could obtain by decomposing into different branches and applying direct arguments to the subproblems. One simple reason is that (globally optimal) solutions of the subproblems may not be M-stationary solutions of the overall problem, as the following example demonstrates.

Example 6 Consider the problem

$$MPEC(\omega) \qquad \min_x \quad x_2$$
$$\text{subject to} \qquad g_1(x,\omega) := x_1^2 - x_2 \leq 0$$
$$-(G_1(x,\omega), H_1(x,\omega)) := -(x_2 + x_3 + \omega, x_3) \in Q_{EC},$$

with $\bar{x} = (0,0,0)$, $\bar{\omega} = 0$. Then we have

$$X(\omega) = S(\omega) = S_M(\omega) = \begin{cases} (0,0,0) & \text{if } \omega \geq 0, \\ (0,0,-\omega) & \text{if } \omega < 0. \end{cases}$$

Since the only multiplier $\lambda = (\lambda^g, \lambda^G, \lambda^H)$ fulfilling

$$\lambda^g \geq 0, \text{ either } \lambda^G > 0, \lambda^H > 0 \text{ or } \lambda^G \lambda^H = 0, \ \lambda^g(0,-1,0) - \lambda^G(0,1,1) - \lambda^H(0,0,1) = 0$$

is $\lambda = (0,0,0)$, the constraint mapping $M_{\bar{\omega}}$ is metrically regular around $(\bar{x},0)$ by Theorem 1. Consequently FOSCMS and property R_1 are fulfilled. Now let us verify that RSSOSC also holds. The condition $u = (u_1, u_2, u_3) \in \mathscr{C}_{\bar{\omega}}(\bar{x})$ amounts to

$$u_2 \leq 0, \ -u_2 \leq 0, -(u_2 + u_3) \leq 0, \ -u_3 \leq 0, (u_2 + u_3)u_3 = 0$$

or equivalently $u_2 = u_3 = 0$. Hence, for every $u \in \mathscr{C}_{\bar{\omega}}(\bar{x})$, the set $\Lambda^1_{\bar{\omega}}(\bar{x}; u)$ is the collection of all $\lambda = (\lambda^g, \Lambda^G, \lambda^H)$ fulfilling

$$(0,1,0) + \lambda^g(0,-1,0) - \lambda^G(0,1,1) - \lambda^H(0,0,1) = 0$$
$$\lambda^g \geq 0, \text{ either } \lambda^G, \lambda^H > 0 \text{ or } \lambda^G \lambda^H = 0,$$

resulting in $\Lambda^1_{\bar{\omega}}(\bar{x}; u) = \{(1,0,0)\}$. Hence, $u^T \nabla^2_x \mathscr{L}^1(\bar{x}, \lambda, \bar{\omega})u = 2u_1^2 > 0 \ \forall 0 \neq u \in \mathscr{C}_{\bar{\omega}}(\bar{x}), \lambda \in \Lambda^1_{\bar{\omega}}(\bar{x}; u)$, showing the validity of RSSOSC and thus, by virtue of Theorem 4,

$$(S(\omega) \cup S_M(\omega)) \cap U \subset L\omega \mathscr{B}_{\mathbb{R}^3} \ \forall \omega \in W$$

for some neighborhoods U of $\bar{x}$ and W of $\bar{\omega}$ and some constant $L > 0$. We see that by our results we can establish upper Lipschitz stability of B-stationary and M-stationary solutions and hence, in particular, also of local minimizers.

Now we want to compare this result with the one, which one would obtain by decomposing into different branches. Since $Q_{EC} = (\mathbb{R}_- \times \{0\}) \cup (\{0\} \times \mathbb{R}_-)$, the MPEC decomposes into the two nonlinear convex programs

$$NLP^1(\omega) \qquad \min_x \quad x_2$$
$$\text{subject to} \qquad g_1(x,\omega) := x_1^2 - x_2 \leq 0,$$
$$g_2(x,\omega) := -(x_2 + x_3 + \omega) \leq 0,$$
$$h_1(x,\omega) := -x_3 = 0.$$

and

$$NLP^2(\omega) \qquad \min_x \quad x_2$$
$$\text{subject to} \qquad g_1(x,\omega) := x_1^2 - x_2 \leq 0,$$
$$g_2(x,\omega) := -x_3 \leq 0,$$
$$h_1(x,\omega) := -(x_2 + x_3 + \omega) = 0.$$

For the first problem the set of minimizers is given by

$$X^1(\omega) = \begin{cases} \{(0,0,0)\} & \text{if } \omega \geq 0, \\ \{(x_1, -\omega, 0) \mid |x_1| \leq \sqrt{-\omega}\} & \text{if } \omega < 0 \end{cases}$$

and for the second problem we obtain

$$X^2(\omega) = \begin{cases} \emptyset & \text{if } \omega > 0, \\ \{(0,0,-\omega)\} & \text{if } \omega \leq 0 \end{cases}$$

Hence, for the union $X^1(\omega) \cup X^2(\omega)$ we only have upper Hölder continuity and this demonstrates that in a straightforward way we cannot derive the stability property of solutions of the MPEC from the continuity properties of solutions of the different branches.

Acknowledgements

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