A posteriori error estimates for the VEM and AVEM

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The Model Problem

The Problem

Find $u \in H_0^1(\Omega)$ such that

$$-div(\kappa \nabla u) + \beta \cdot \nabla u + \gamma u = f \qquad \text{in } \Omega \subset \mathbb{R}^d.$$

- $f, \gamma \in L^\infty(\Omega)$
- $\beta \in [W_\infty^1(\Omega)]^d$
- $\kappa \in [L^\infty(\Omega)]^{d \times d}$ and symmetric and uniformly positive definite

The Model Problem

The weak form

Find $u \in H_0^1(\Omega)$ such that

$$(\kappa \nabla u, \nabla v) + (\beta \cdot \nabla u, v) + (\gamma u, v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

The weak form (split)

$$A(u, v) := a(u, v) + b(u, v) = (f, v),$$

where

$$a(u, v) := (\kappa \nabla u, \nabla v) + (\mu u, v),$$

$$b(u, v) := \frac{1}{2}[(\beta \cdot \nabla u, v) - (\beta \cdot \nabla v, u)]$$

The Model Problem

We assume

$$\exists \kappa^*, \kappa_* : \qquad \kappa_* |\xi|^2 \leq \xi.\kappa(x)\xi \leq \kappa^* |\xi|^2,$$

$$\exists \mu_0 : \qquad \mu(x) := \gamma(x) - \frac{1}{2}\nabla \cdot \beta(x) \geq \mu_0 > 0,$$

for all $\xi \in \mathbb{R}^d$ and for almost all $x \in \Omega$.

VEM

Assumptions on the Mesh

Assumption 1 (Mesh regularity)

1. $\forall E \in \mathcal{T}_h$ E is star-shaped w.r.t a ball with radius ρh_E
2. $\forall E \in \mathcal{T}_h$ all interfaces s of E should fulfill $h_s \geq \rho h_E$
3. for $d = 3$, the two assumptions from above should also hold for every interface

- $\mathcal{T}_h$ Partition
- E an Element

VEM Spaces

- **The usual VEM space**
  $\tilde{V}_h^E := \{v \in H^1(E) : v_{|\partial E} \in P_p(E), \Delta v \in P_{p-2}(E)\}$
- **Auxiliary VEM space**
  $W_h^E := \{v \in H^1(E) : v_{|\partial E} \in P_p(E), \Delta v \in P_{p}(E)\}$

VEM Spaces

DOFs

- $\mathcal{N}^\omega$ nodal values.
  For a vertex z of ω, $\mathcal{N}^\omega_z(v) := v(z)$ and
  $\mathcal{N}^\omega := \{\mathcal{N}^\omega_z : z \text{ is a vertex }\}$
- $\mathcal{M}^\omega_l$ polynomial moments up to order $l \geq 0$.
  For a multi-index α,
  $\mathcal{M}^\omega_\alpha(v) := \frac{1}{|\omega|}(v, m_\alpha)_\omega,$ with $m_\alpha := \left(\frac{x - x_\omega}{h_\omega}\right)^\alpha,$
  where x_ω is the barycentre of ω.
  $\mathcal{M}^\omega_l := \{\mathcal{M}^\omega_\alpha : |\alpha| \leq l\}$

- $DOFs(W^E_h) =$
  $\mathcal{N}^E \cup \mathcal{M}^E_p \cup \{\mathcal{M}^s_{p-2}$ for all edge interfaces $s \in \partial E\}$

VEM Spaces

Definition

The operator Π_p^0 is the L^2 -projection onto the space $P_p(E)$.

Definition (Computability)

We say a term is computable if it may be evaluated using

- *data,*
- *DOFs*
- *and the polynomial component of the VEM space only.*

The VEM Spaces, $\tilde{V}_h^E, W_h^E$

- $P_p(E) \subseteq \tilde{V}_h^E$, $P_p(E) \subseteq W_h^E$
- Π_p^0 and Π_{p-1}^0 are not computable in $\tilde{V}_h^E$ but in W_h^E.
- $\mathcal{M}_p^\omega \setminus \mathcal{M}_{p-2}^\omega$ are redundant in $P_p(E)$

It is possible to construct an L^2 stable projection Π_p^* which only depend on the reduced set of degrees of freedoms
$DOFs(\tilde{V}_h^E) =$
$\mathcal{N}^E \cup \mathcal{M}_{p-2}^E \cup \{\mathcal{M}_{p-2}^s$ for all edge interfaces $s \in \partial E\}$.

The VEM Space

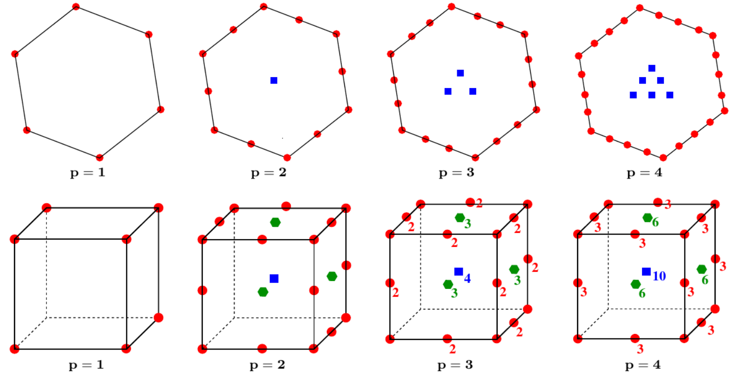
$$V_h^E := \{v_h \in W_h^E : (v_h, m_\alpha)_E = (\Pi_p^* v_h, m_\alpha)_E \forall m_\alpha \text{ with } p-1 \leq |\alpha| \leq p\}$$

$$m_\alpha := \left(\frac{x - x_\omega}{h_\omega}\right)^\alpha$$

$$V_h := \{v_h \in H_0^1(\Omega) : v_{h|E} \in V_h^E \forall E \in \mathcal{T}_h\}$$

- Π_p^0 and Π_{p-1}^0 are computable in V_h^E using Π_p^*.

DOFs



Figures taken from

A. Cangiani, E. H. Georgoulis T. Pryer, and O.J Sutton.
A posteriori error estimates for the virtual element method
Numerische Mathematik 137(4): 857-893

Admissible Stabilizing Forms

Definition

Let $E \in \mathcal{T}_h$ *and* $S_1^E, S_0^E : V_h^E \setminus P_p(E) \times V_h^E \setminus P_p(E) \mapsto \mathbb{R}$ *be symmetric and positive definite computable bilinear forms. We say* S_1^E, S_0^E *are admissible stabilizing forms if there exist a constant* $C_{Stab} > 0$, *indep. of* E *and* h *such that*

$$C_{Stab}^{-1} \int_E |\sqrt{\kappa} \nabla v_h|^2 \leq S_1^E(v_h, v_h) \leq C_{Stab} \int_E |\sqrt{\kappa} \nabla v_h|^2,$$

and

$$C_{Stab}^{-1} \int_E \mu v_h^2 \leq S_0^E(v_h, v_h) \leq C_{Stab} \int_E \mu v_h^2,$$

for all $v_h \in V_h^E \setminus P_p(E)$.

The Local Bilinear Forms

$$A_h^E(u_h, v_h) := a_h^E(u_h, v_h) + b_h^E(u_h, v_h),$$

$$\begin{aligned} a_h^E(u_h, v_h) :=& (\kappa \Pi_{p-1}^0 \nabla u_h, \Pi_{p-1}^0 \nabla v_h)_E + (\mu \Pi_p^0 u_h, \Pi_p^0 v_h)_E \\ &+ S^E((I - \Pi_p^0) u_h, (I - \Pi_p^0) v_h) \\ S^E :=& s_1 S_1^E + s_0 S_0^E \qquad s_1, s_0 > 0 \end{aligned}$$

$$b_h^E(u_h, v_h) := \frac{1}{2}\left[(\beta \cdot \Pi_{p-1}^0 \nabla u_h, \Pi_p^0 v_h)_E - (\beta \cdot \Pi_{p-1}^0 \nabla v_h, \Pi_p^0 u_h)_E\right]$$

The Computable Discrete Bilinear Form

The Model Problem in the discrete space

Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f_h, v_h) \qquad \forall v_h \in V_h,$$

with

$$A_h(u_h, v_h) := \sum_{E \in \mathcal{T}_h} A_h^E(u_h, v_h), \qquad f_h := \Pi_{p-1}^0 f.$$

Approximation Properties

Polynomial Approximation

Theorem (Approximation using polynomials)

Let Assumption 1 be fulfilled, $E \in \mathcal{T}_h$, $\Pi_l^0 : L^2(E) \mapsto P_l(E)$ *be the orth. Project. onto* $P_l(E)$, $l \geq 0$. *Then for all* $w \in H^m(E)$, *with* $1 \leq m \leq l+1$ *it holds*

$$\|w - \Pi_l^0 w\|_{0,E} + h_E |w - \Pi_l^0 w|_{1,E} \leq C_{proj} h_E^m |w|_{m,E},$$

with C_{proj} *only depending on* l *and the mesh regularity.*

Clément Interpolation for FEM

Theorem

Let $T \in \hat{\mathcal{T}}_h$ be an element in the globally shape regular finite element sub triangulation $\hat{\mathcal{T}}_h$ of $\mathcal{T}_h$ and $\tilde{T}$ the patch around T. Then for a classical Clement interpolant v_c of degree p holds

$$\|v - v_c\|_{0,T} + h|v - v_c|_{1,T} \leq \hat{C}_{Cl\acute{e}m} h |v|_{1,\tilde{T}},$$

for all $v \in H^1(\Omega)$.

"Clément Interpolation" for VEM

Theorem

Let $T \in \mathcal{T}_h$ and $\tilde{E}$ the patch around E. Then for a classical Clement interpolant v_I of degree p holds

$$\|v - v_I\|_{0,E} + h|v - v_I|_{1,E} \leq C_{Clém} h |v|_{1,\tilde{E}},$$

for all $v \in H^1(\Omega)$ and v_I solves the problem

$$\begin{aligned} -\Delta v_I &= -\Delta \Pi_p^0 v_c \quad && \text{in } E, \\ v_I &= v_c \quad && \text{on } \partial E. \end{aligned}$$

Proof.

See

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A Posteriori Error Analysis

Energy Norm

We define the energy norm on the domain $\omega \subseteq \Omega$ and $v \in H^1(\Omega)$ as follows

$$|||v|||_\omega^2 := \|\sqrt{\kappa}\nabla v\|_{0,\omega}^2 + \|\sqrt{\mu} v\|_{0,\omega}^2.$$

We have

$$(C_{eqiv})^{-1}\|v\|_{1,\Omega} \leq |||v|||_\Omega \leq \hat{C}_{eqiv}\|v\|_{1,\Omega},$$

with $C_{eqiv} := \sqrt{(1 + C_{PF})/\kappa_*}$ and $\hat{C}_{eqiv} := \sqrt{max(\kappa^*, \|\mu\|_\infty)}$.

The Residual Equation

Let $e := u - u_h \in H^1(\Omega)$ and $v \in H^1(\Omega)$. Then we have for $\chi \in V_h$ that

$$A(e, v) = (f, v) - A(u_h, v)$$

The Residual Equation

Let $e := u - u_h \in H^1(\Omega)$ and $v \in H^1(\Omega)$. Then we have for $\chi \in V_h$ that

$$A(e, v) = (f, v) - A(u_h, \chi) - A(u_h, v - \chi)$$

The Residual Equation

Let $e := u - u_h \in H^1(\Omega)$ and $v \in H^1(\Omega)$. Then we have for $\chi \in V_h$ that

$$\begin{aligned} A(e, v) &= (f, v) - A(u_h, \chi) - A(u_h, v - \chi) \\ &= (f, v) - (f_h, \chi) + A_h(u_h, \chi) - A(u_h, v - \chi) - A(u_h, \chi) \end{aligned}$$

The Residual Equation

Let $e := u - u_h \in H^1(\Omega)$ and $v \in H^1(\Omega)$. Then we have for $\chi \in V_h$ that

$$\begin{aligned} A(e, v) &= (f, v) - A(u_h, \chi) - A(u_h, v - \chi) \\ &= (f, v) - (f_h, \chi) + A_h(u_h, \chi) - A(u_h, v - \chi) - A(u_h, \chi) \\ &= (f - f_h, \chi) - (f, v - \chi) + A_h(u_h, \chi) - A(u_h, v - \chi) - A(u_h, \chi). \end{aligned}$$

Piecewise Approximations of the Coefficients

We approximate the coefficients by piecewise polynomials on the elements:

$$\kappa_h \approx \kappa,$$

$$\beta_h \approx \beta,$$

$$\gamma_h \approx \gamma.$$

The Residual Equation

$$
\begin{aligned}
A(e,v) = \sum_{E\in\mathcal{T}_h} &\big((R_E, v-\chi) + (\theta_E, v-\chi) + B^E(u_h, v-\chi)\big) \\
&- \sum_{s\in S_h} \big((J_s, v-\chi)_{0,s} + (\theta_s, v-\chi)_{0,s}\big) \\
&+ (f - f_h, \chi) + A_h(u_h,\chi) - A(u_h,\chi),
\end{aligned}
$$

where

$$
\begin{aligned}
R_E &:= (f_h + \nabla\cdot\kappa_h \Pi^0_{p-1}\nabla u_h - \beta_h\cdot\Pi^0_{p-1}\nabla u_h - \gamma_h\Pi^0_p u_h)_{|E}, \\
J_s &:= [\![\kappa_h\Pi^0_{p-1}\nabla u_h]\!]_{|s}, \\
\theta_E &:= (f - f_h + \nabla\cdot(\kappa-\kappa_h)\Pi^0_{p-1}\nabla u_h - (\beta-\beta_h)\cdot\Pi^0_{p-1}\nabla u_h - (\gamma-\gamma_h)\Pi^0_p u_h)_{|E}, \\
\theta_s &:= [\![(\kappa-\kappa_h)\Pi^0_{p-1}\nabla u_h]\!]_{|s}, \\
B_E(w_h, v) &:= (\kappa(I-\Pi^0_{p-1})\nabla w_h, \nabla v)_E + (\beta\cdot(I-\Pi^0_{p-1})\nabla w_h, v)_E + (\gamma(I-\Pi^0_p)w_h, v)_E.
\end{aligned}
$$

Theorem (Upper Bound for the Error)

Let u_h be the virtual element solution, then there exist a constant C independent of h, u and u_h such that

$$|||u-u_h|||^2 \le C \sum_{E\in\mathcal{T}_h} \left(\eta^E + \Theta^E + \mathfrak{S}^E + \Psi^E\right),$$

where

$$\begin{aligned}
\eta^E &:= h_E^2\|R_E\|_{0.E}^2 + \sum_{s\in\partial E} h_s\|J_s\|_{0.s}^2,\\
\Theta^E &:= h_E^2\|\theta_E\|_{0.E}^2 + h_E^2\|f-f_h\|_{0.E}^2 + \sum_{s\in\partial E} h_s\|\theta_s\|_{0.s}^2,\\
\mathfrak{S}^E &:= S^E((I-\Pi_p^0)u_h,(I-\Pi_p^0)u_h),\\
\Psi^E &:= \|(I-\Pi_{p-1}^0)(\kappa\Pi_{p-1}^0\nabla u_h)\|_{0,E}^2 + h_E^2\|(I-\Pi_p^0)(\beta\cdot\Pi_{p-1}^0\nabla u_h)\|_{0,E}^2\\
&\quad + \|(I-\Pi_{p-1}^0)(\beta\cdot\Pi_p^0 u_h)\|_{0,E}^2 + \|(I-\Pi_p^0)(\mu\Pi_p^0 u_h)\|_{0,E}^2.
\end{aligned}$$

Proof.

Sketch on Black- or White-board ☐

Theorem (Upper Bound for the Projection)

Let u_h be the virtual element solution, then there exist a constant C independent of h, u and u_h such that

$$|||u - \Pi_p^0 u_h|||^2 \leq C \sum_{E \in \mathcal{T}_h} \left(\eta^E + \Theta^E + \mathfrak{S}^E + \Psi^E \right).$$

Proof.

See

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□

Theorem (A Local Lower Bound)

Let u_h be the virtual element solution, then there exist a constant C independent of h, u and u_h such that

$$\eta^E \leq C \sum_{E' \in \omega_E} \left(|||u - u_h|||_{E'}^2 + \Theta^{E'} + \mathfrak{S}^{E'} \right),$$

where ω_E describes the set of elements, which have a common face with E.

Proof.

See

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Proof.

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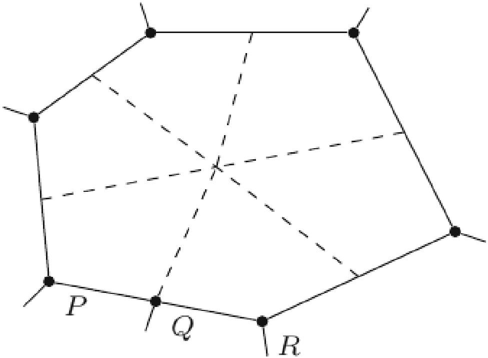
Numerical Results

The Adaptive Refinement Procedure



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Kellogg Problem I

We consider on the model problem with $\Omega = (0,1)^2$ with the following coefficients:

$$\kappa(x,y) := \begin{cases} b & (x-a)(y-a) \geq 0 \\ 1 & \text{else} \end{cases},$$

$$\beta = 0,$$

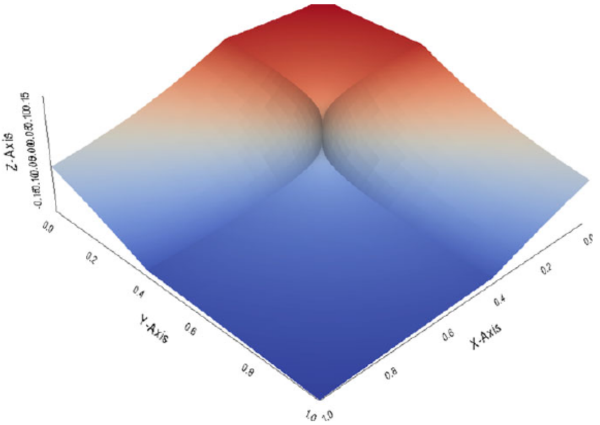
$$\gamma = 0,$$

$$a = 0.4.$$

However as boundary condition

$$u = r^{\frac{1}{4}} g(\theta).$$

Approximate Solution of the Kellog Problem I



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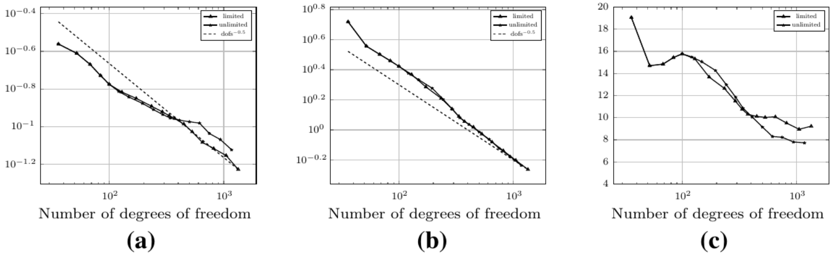

(a) The error $\|\nabla(u - \Pi_1^0 u_h)\|$, (b) The estimated error, (c) effetivity index
Screenshot taken from

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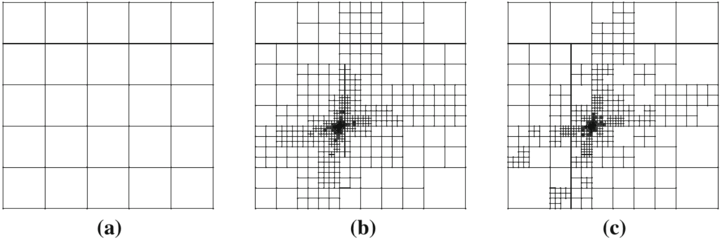

(a) initial mesh (b) mesh with limited number of maximum nodes (1) ,
(c) mesh with unlimited number of maximum nodes
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Kellogg Problem II

We consider on the model problem with $\Omega = (0,1)^2$ with the following coefficients:

$$\kappa(x,y) := \begin{cases} b & (x-a)(y-a) \geq 0 \\ 1 & \text{else} \end{cases},$$

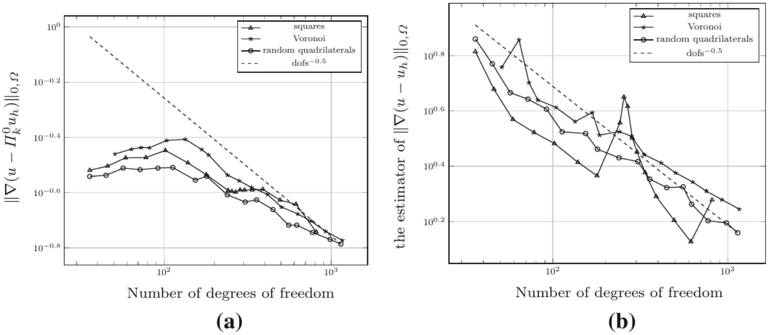
$$\beta = 0,$$

$$\gamma = 0,$$

$$a = 0.4 * \sqrt{2}.$$

However as boundary condition

$$u = r^{\frac{1}{4}} g(\theta).$$



(a) The error $\|\nabla(u - \Pi_1^0 u_h)\|$, (b) The estimated error
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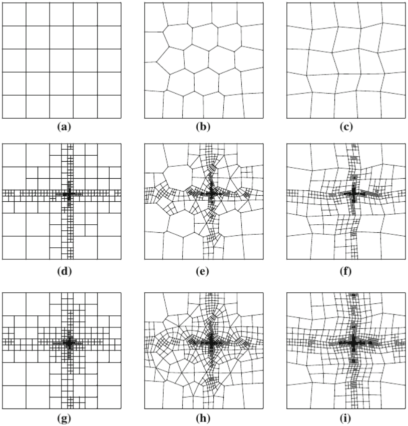

(a,b,c) initial meshes (d,e,f) meshes with unlimited number of hanging nodes, (g,h,i) meshes with a limited number of nodes

Screenshot taken from

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Thank you for your Attention!