# DATA OSCILLATION AND CONVERGENCE OF ADAPTIVE FEM

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SIAM J. Numer. Anal., 38(2):466–488,2000

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# The last week's seminar

## Assumptions

- The initial mesh is sufficiently refined to resolve data within a tolerance $\mu\epsilon \ll \epsilon$ (mesh fineness).
- The sum of the local error indicators of elements marked for refinement amounts to a fixed portion of the global error estimator (marking strategy).

# Framework I

- $\Omega \subset \mathbb{R}^d, d \in \{2,3\}$ polygonal/polyhedral bounded domain
- $f \in L^2(\Omega)$
- $(u,v)_{A,G} := (A\nabla u, \nabla v)_{0,G}$ with $(u,v)_{0,G}$ the $L^2(G)$-inner product, $G \subset \Omega$.
- $A$ is piecewise constant positive definite symmetric
- **Continuous:** Seek $u \in H_0^1(\Omega) : (u,v)_{A,\Omega} = (f,v)_{0,\Omega} \qquad v \in H_0^1(\Omega)$
- $\mathcal{T}_H$ conforming regular triangulation of $\Omega$ with piecewise constant mesh-size $H$, i.e., $H|_T = \text{diam}T$ for $T \in \mathcal{T}_H$.
- $V^H$ space of continuous piecewise linear functions over $\mathcal{T}_H$
- $V_0^H \subset V^H$ with vanishing boundary
- **Discrete:** Seek $u_H \in V_0^H : (u_H, \phi)_{A,\Omega} = (f,v)_{0,\Omega} \qquad \phi \in V_0^H$

# Framework II

- $f_H$ piecewise constant function over $\mathcal{T}_H$ that is equal to mean value $f_T$ of $F$ on element $T \in \mathcal{T}_H$.
- $\mathcal{S}_H$ the set of inner sides of $\mathcal{T}_H$
- For $S \in \mathcal{S}_H$ define $\Omega_S$ as union of the two elements in $\mathcal{T}_H$ sharing $S$
- $H_S$ denotes the diameter of $S$

**Assumptions**

- All partitions $\mathcal{T}_H$ match the discontinuities of $A$, i.e., the jumps of $A$ are located on $\mathcal{S}_H$.

# The DIFFERENCE to the last week's seminar I

- Introduce data oscillation,

$$\text{osc}(f, \mathcal{T}_H) := \Big( \sum_{T \in \mathcal{T}_H} \|H(f - f_T)\|_{0,T}^2 \Big)^{1/2}$$

- $\text{osc}(f - \mathcal{T}_H)$ measures intrinsic information missing in the averaging process associated with finite elements, which fail to detect fine structures of $f$.
- The definition of $\text{osc}(.)$ is unrelated to quadrature and quantifies data oscillation with the least amount of information per element, namely one degree of freedom associated with $f_T$.

# The DIFFERENCE to the last week's seminar II

- Last week mesh fineness

$$\Big(\sum_{T\in\mathcal{T}_H} \|Hf\|_H^2\Big)^{1/2} \leq \mu\epsilon.$$

- This week oscillations

$$\Big(\sum_{T\in\mathcal{T}_H} \|H(f - f_T)\|_H^2\Big)^{1/2} \leq \mu\epsilon.$$

# The MAIN result

## Theorem

*Let $(u_k)_k$ be a sequence of FE solution produced by Algorithm C. Then there exist positive constants $C_0$ and $\beta < 1$, depending only on $f$ and the initial grid, such that*

$$\|u - u_k\|_{A,\Omega} \leq C_0 \beta^k,$$

*with* $\|u\|^2_{A,\Omega} := (u,u)_{A,\Omega}$.

## Comparison to PREVIOUS SEMINAR

- Any prescribed error tolerance $\epsilon$ is met in finite steps WITHOUT special tuning of initial mesh
- Theorem does NOT imply that the error decays in every single step: It may be constant for a number of steps due to unresolved data oscillations

# RESIDUAL-TYPE a posteriori error estimator

- Local error indicators

$$\eta_S^2 := \|H_S^{1/2} J_S\|_S^2 + \|Hf\|_{\Omega_S}^2$$

with $J_S := [A\nabla u_H]_S \cdot \nu$.

- Global error estimator

$$\eta_H^2 := \sum_{S\in\mathcal{S}_H} \eta_S^2$$

**Theorem**

$$\|u - u_h\|_{A,\Omega}^2 \leq C_1 \eta_H^2$$
$$\|u - u_h\|_{A,\Omega_S}^2 \geq C_2 \eta_S^2 - C_3 \|H(f - f_h)\|_{0,\Omega_s}^2$$

# Marking I

## Marking Strategy E

Given a parameter $0 < \theta < 1$

1. Construct a subset $\hat{\mathcal{S}}_H \subset \mathcal{S}_H$ such that

$$\Big( \sum_{S \in \hat{\mathcal{S}}_H} \eta_S^2 \Big)^{1/2} \geq \theta \eta_H .$$

2. Let $\hat{\mathcal{T}}_H$ be the set of elements with one side in $\hat{\mathcal{S}}_H$ and mark all these elements.

# Marking II

**Theorem (error reduction)**

*Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ such that each element of $\hat{\mathcal{T}}_H$, as well as each side in $\hat{\mathcal{S}}_H$, contains a node of $\mathcal{T}_h$ in its interior.*
*Then there exist constants $\mu > 0$ and $0 < \alpha < 1$, depending only on the initial triangulation, such that for any $\epsilon > 0$*

$$\textit{osc}(f, \mathcal{T}_H) \leq \mu\epsilon \implies \|u - u_H\|_{A,\Omega} \leq \epsilon \vee \|u - u_h\|_{A,\Omega} \leq \alpha\|u - u_H\|_{A,\Omega}.$$

# Lemmata I

**Lemma (Error reduction $= \|u_H - u_h\|^2_{A,\Omega}$)**

*Let $\mathcal{T}_h$ be a local refinement of $\mathcal{T}_H$ such that $V^H \subset V^h$. Then*

$$\|u - u_h\|^2_{A,\Omega} = \|u - u_H\|^2_{A,\Omega} - \|u_H - u_h\|^2_{A,\Omega}.$$

**Proof.**

Galerkin orthogonality.

$$(u - u_h, v_h)_{A,\Omega} = 0, \forall v_h \in V^h \implies (u - u_h, \underbrace{u_h - u_H}_{=u-u_h+u_h-u_H})_{A,\Omega} = 0$$

□

# Lemmata II

**Lemma ($\|u_H - u_h\|^2_{A,\Omega} \geq ???\|u - u_H\|^2_{A,\Omega}$ proportional error decrease)**

*Let $\mathcal{T}_h$ be a refinement of $\mathcal{T}_H$ satisfying the assumption of the THEOREM. Then there exist constants $C_4, C_5$ depending only on the initial triangulation such that*

$$\eta_S^2 \leq C_4\|u_h - u_H\|^2_{A,\Omega_S} + C_5\|H(f - f_H)\|^2_{0,\Omega_S} \qquad \forall S \in \hat{\mathcal{S}}_H.$$

**Proof.**

CONSTRUCTIVE: Integration by parts, Poincare inequality, triangle inequality.

□

# Lemmata III

**Corollary (GLOBAL lower bound for the error decrease)**

*Assumptions as in THEOREM. Then*

$$\|u_h - u_H\|^2_{A,\Omega} \geq \frac{\theta^2}{2C_4C_1}\|u - u_H\|^2_{A,\Omega} - \frac{C_5}{C_4}\textsf{osc}(f, \mathcal{T}_H)^2.$$

# Lemmata IV

**Proof.**

By previous LEMMA and MARKING STRATEGY E we have

$$\begin{aligned}
\theta^2 \eta_H^2 &\leq \sum_{S \in \hat{\mathcal{S}}_H} \eta_S^2 \\
&\leq C_4 \sum_{S \in \hat{\mathcal{S}}_H} \|u_h - u_H\|_{A,\Omega_S}^2 + C_5 \sum_{S \in \hat{\mathcal{S}}_H} \|H(f - f_H)\|_{0,\Omega_S}^2 \\
&\leq 2C_4 \|u_h - u_H\|_{A,\Omega}^2 + 2C_5 \|H(f - f_H)\|_{0,\Omega}^2 . \\
\Longrightarrow \|u_h - u_H\|_{A,\Omega}^2 &\geq \frac{\theta^2}{2C_4} \eta_H^2 - \frac{C_5}{C_4} \|H(f - f_H)\|_{0,\Omega}^2 .
\end{aligned}$$

Insert error-estimator-LEMMA. □

# Proof of THEOREM

**Proof.**

$$\|u-u_h\|^2_{A,\Omega} = \|u-u_H\|^2_{A,\Omega} - \|u_H-u_h\|^2_{A,\Omega}$$
$$\leq \|u-u_H\|^2_{A,\Omega}\left(1-\frac{\theta^2}{2C_4C_1}\right) + \frac{C_5}{C_4}\text{osc}(f,\mathcal{T}_H)^2.$$

Case $\|u-u_H\|_{A,\Omega} > \epsilon$. Hence

$$\|u-u_h\|^2_{A,\Omega} \leq \|u-u_H\|^2_{A,\Omega} \underbrace{\left(1-\frac{\theta^2}{2C_4C_1}+\frac{C_5}{C_4}\mu^2\right)}_{<1 \text{ for } \mu>0 \text{ sufficiently small}}$$

□

# EXAMPLES: Ingredients for CONVERGERNCE I

**Interior node 1**

**Necessity of creating an interior node inside each refined triangle**

$A = \mathrm{Id}, f \equiv 1, \Omega = (0,1)^2$

"$u_H = (1/12)$"

"$u_h = 1/24(1,1,2,1,1)$"

# EXAMPLES: Ingredients for CONVERGERNCE II

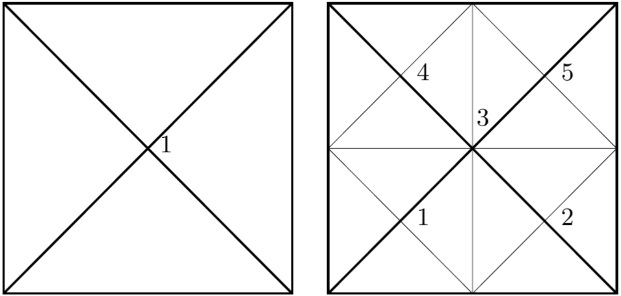



FIG. 3.1. *Refinement by bisecting all triangles twice.*

# EXAMPLES: Ingredients for CONVERGERNCE III

## Interior node 2

**Also happens "later" for osc**$(f, \mathcal{T}_n) = 0$

$f$ is orthogonal to the basis functions of $\mathcal{T}_k, k = 0, 1, 2 \implies u_k \equiv 0, k = 0, 1, 2$.

$u_k = 0, k = 3, 4, \ldots$ on "squares" where $f$ changes sign (symmetry of problem).

$u_3, u_4$ behave like in previous example, ie. $u_3 = u_4$

# EXAMPLES: Ingredients for CONVERGERNCE IV

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

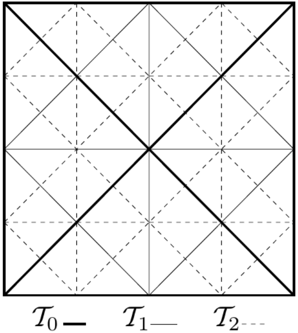



FIG. 3.2. *Values of the function $f$ of Example 3.6 for $n = 3$ (left), and grids $\mathcal{T}_k$ for $k = 0, 1, 2$ (right).*

# EXAMPLES: Ingredients for CONVERGERNCE V

**Data oscillation**

**osc$(f, \mathcal{T}_H)$ has to be small**

See previous example with additional refinement (interior nodes)

# EXAMPLES: Ingredients for CONVERGERNCE VI

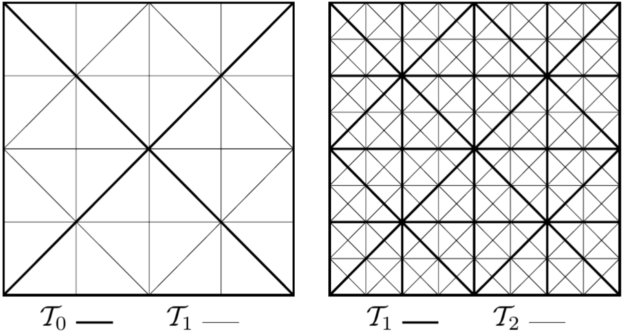



FIG. 3.3. *Resulting grid $\mathcal{T}_1$ (left) and $\mathcal{T}_2$ (right) after performing three bisections on each element of $\mathcal{T}_0$ and $\mathcal{T}_1$, respectively.*

# EXAMPLES: Ingredients for CONVERGERNCE VII

## CONCLUSION

- Interior nodes are necessary for error decrease.
- Interior nodes are not sufficient if mesh does not sufficiently resolve oscillation.
- We must readjust the mesh to resolve $\mathrm{osc}(f, \mathcal{T}_H)$ according to a decreasing tolerance.

# EXAMPLES: Ingredients for CONVERGERNCE VIII

**Lemma**

*Let* $0 < \gamma < 1$ *reduction factor of element size in one refinement step. Let* $0 < \hat{\theta} < 1, \hat{\alpha} := (1-(1-\gamma^2)\hat{\theta}^2)^{1/2}$. *Let* $\hat{\mathcal{T}}_H \subset \mathcal{T}_H$ *such that*

$$\mathit{osc}(f, \hat{\mathcal{T}}_H) \geq \hat{\theta}\,\mathit{osc}(f, \mathcal{T}_H).$$

*Then if* $\mathcal{T}_h$ *is obtained from* $\mathcal{T}_H$ *by refining AT LEAST* $\hat{\mathcal{T}}_H$ *one has*

$$\mathit{osc}(f, \mathcal{T}_h) \leq \hat{\alpha}\,\mathit{osc}(f, \mathcal{T}_H).$$

# EXAMPLES: Ingredients for CONVERGERNCE IX

**Proof.**

Per definition, $f_T = |T|^{-1} \int_T f$ is $L^2$-projection of $f$ onto piecewise constants on $T$. Let $T \in \mathcal{T}_h, \hat{T} \in \hat{\mathcal{T}}_H, T \subset \hat{T}$. Hence $\|f - f_T\|_T \leq \|f - f_{\hat{T}}\|_T$. Per definition $h_T \leq \gamma h_{\hat{T}}$.

$$\begin{aligned}
\text{osc}(f, \mathcal{T}_h)^2 &= \sum_{T \in \mathcal{T}_h} h_T^2 \|f - f_T\|_{0,T}^2 \\
&\leq \gamma^2 \sum_{\hat{T} \in \hat{\mathcal{T}}_H} h_{\hat{T}}^2 \|f - f_{\hat{T}}\|_{0,\hat{T}}^2 + \sum_{T \in \mathcal{T}_H \setminus \hat{\mathcal{T}}_H} h_T^2 \|f - f_T\|_{0,T}^2 \\
&= (\gamma^2 - 1)\text{osc}(f, \hat{\mathcal{T}}_H)^2 + \text{osc}(f, \mathcal{T}_H)^2 \leq \hat{\alpha}^2 \text{osc}(f, \mathcal{T}_H)^2.
\end{aligned}$$

□

# EXAMPLES: Ingredients for CONVERGERNCE X

**Lemma**

*Let $f$ be piecewise $H^s, 0 < s \leq 1$ over initial mesh. Redefine*

$$\text{osc}(f, \mathcal{T}_h) := \Big( \sum_{T \in \mathcal{T}_h} h_T^{2+2s} \|D^s f\|_{0,T}^2 \Big)^{1/2}.$$

*Let $\hat{\alpha} := (1 - (1 - \gamma^{2+2s})\hat{\theta}^2)^{1/2}$. Then $\text{osc}(f, \mathcal{T}_h) \leq \hat{\alpha}\,\text{osc}(f, \mathcal{T}_H)$.*

**Proof.**

Analogous to previous lemma. □

# EXAMPLES: Ingredients for CONVERGERNCE XI

## Marking Strategy D

Given a parameter $0 < \hat{\theta} < 1$ and the subset $\hat{\mathcal{T}}_H \subset \mathcal{T}_H$ produced by **Marking Strategy E**:

1. Enlarge $\hat{\mathcal{T}}_H$ such that
$$\text{osc}(f, \hat{\mathcal{T}}_H) \geq \hat{\theta}\text{osc}(f, \mathcal{T}_H).$$
2. Mark all elements in $\hat{\mathcal{T}}_H$ for refinement.

## Convergent Algorithm C

Choose parameters $0 < \theta, \hat{\theta} < 1$.

1. Pick up any initial mesh $\mathcal{T}_0$ such that $A$ is piecewise constant over $\mathcal{T}_0$.
2. Solve the system on $\mathcal{T}_0$ for the discrete solution $u_0$.
3. Let $k = 0$.
4. Compute the local indicators $\eta_S$.
5. Construct $\hat{\mathcal{T}}_k$ by **Marking Strategy D** and parameter $\hat{\theta}$.
6. Let $\mathcal{T}_{k+1}$ be a refinement of $\mathcal{T}_k$ such that each element of $\hat{\mathcal{T}}_k$, as well as each of its sides, contains a node of $\mathcal{T}_{k+1}$ in its interior.
7. Solve the system on $\mathcal{T}_{k+1}$ for the discrete solution $u_{k+1}$.
8. Let $k = k + 1$ and go to 4.

# The MAIN RESULT I

## Theorem (CONVERGENCE)

*For $0 < \theta, \hat{\theta} < 1$, let $0 < \alpha < 1, \mu > 0$ be given by the "error decreas theorem" and $0 < \hat{\alpha} < 1$ by the previous lemmata.* ***Algorithm C*** *produces a convergent sequence $(u_k)_{k \in \mathbb{N}_0}$ with*

$$\|u - u_k\|_{A,\Omega} \leq C_0 \beta^k,$$

$$\beta = \max\{\alpha, \hat{\alpha}\},$$

$$C_0 = \max\{\|u - u_0\|_{A,\Omega}, \frac{\mathit{osc}(f, \mathcal{T}_0)}{\alpha\mu}\}.$$

# The MAIN RESULT II

**Proof.**

INDUCTION. IA $k = 0$ ✓.

IS Case study

1. $\|u - u_k\|_{A,\Omega} > C_0 \beta^{k+1}$
2. $\|u - u_k\|_{A,\Omega} \leq C_0 \beta^{k+1}$.

□

# The MAIN RESULT III

**Proof continued.**

1. **Marking Strategy D** gives

$$\text{osc}(f, \mathcal{T}_k) \leq \hat{\alpha}^k \text{osc}(f, \mathcal{T}_0) \leq \beta^k \text{osc}(f, \mathcal{T}_0)$$

Hence for $\epsilon := C_0 \beta^{k+1}$

$$\text{osc}(f, \mathcal{T}_k) \leq \mu C_0 \alpha \beta^k \leq \mu C_0 \beta^{k+1} = \mu\epsilon.$$

Since, per assumption, $\|u - u_k\|_{A,\Omega} > \epsilon$ use IH and Error Reduction THEOREM

$$\|u - u_{k+1}\|_{A,\Omega} \leq \beta \|u - u_k\|_{A,\Omega} \leq C_0 \beta^{k+1}.$$

□

# The MAIN RESULT IV

**Proof continued.**

2. Since $\mathcal{T}_{k+1}$ is refinement of $\mathcal{T}_k$, error cannot increase,

$$\|u - u_{k+1}\|_{A,\Omega} \leq \|u - u_k\|_{A,\Omega} \leq C_0 \beta^{k+1}$$

$\square$

**Practical method?**

**Algorithm C** only needs $\theta, \hat{\theta}$. The unknown constants $\alpha, \hat{\alpha}, \mu$ are not needed (but give convergence rate).

# EXAMPLE: Crack problem

- $\Omega = \{|x| + |y| < 1\} \setminus \{0 \leq x \leq 1, y = 0\}$
- $u(r, \theta) = r^{1/2} \sin \frac{\theta}{2} - \frac{1}{4} r^2$.
- $A = I, f = 1$.

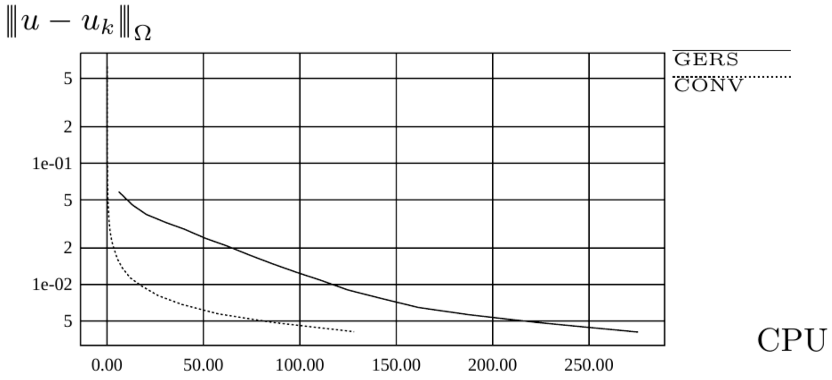



FIG. 5.2. *Comparison of CPU time for GERS and CONV.*

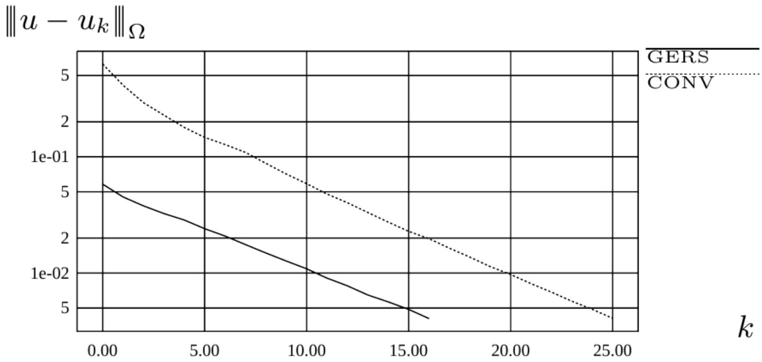



FIG. 5.3. *Comparison of reduction rate $\alpha^k$ for GERS, CONV.*

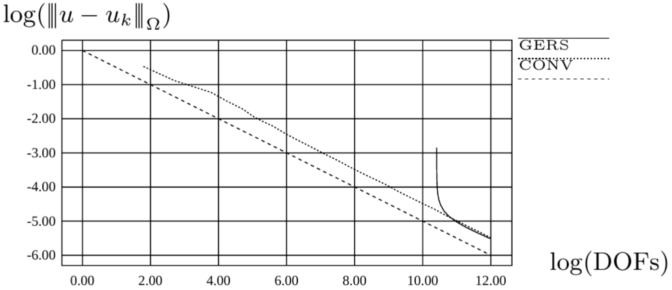
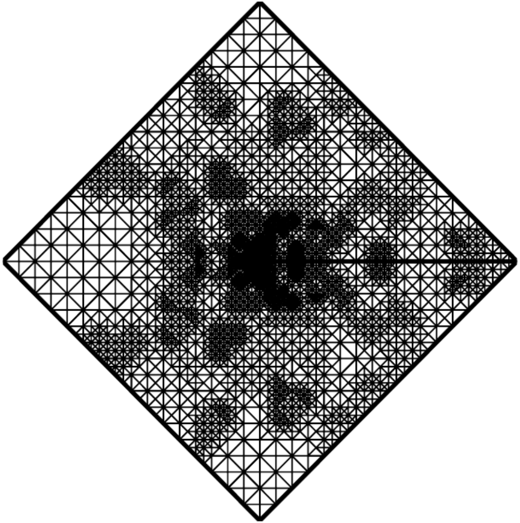
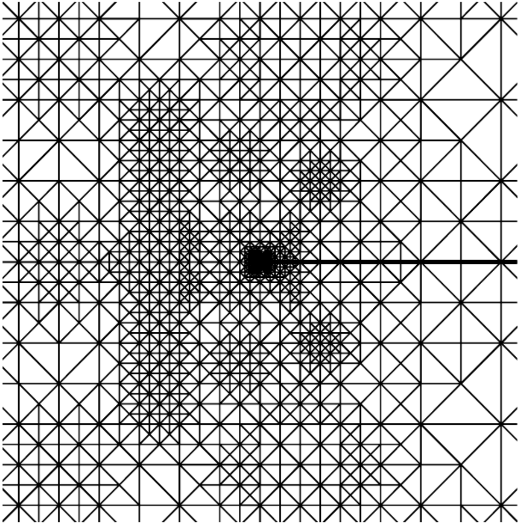
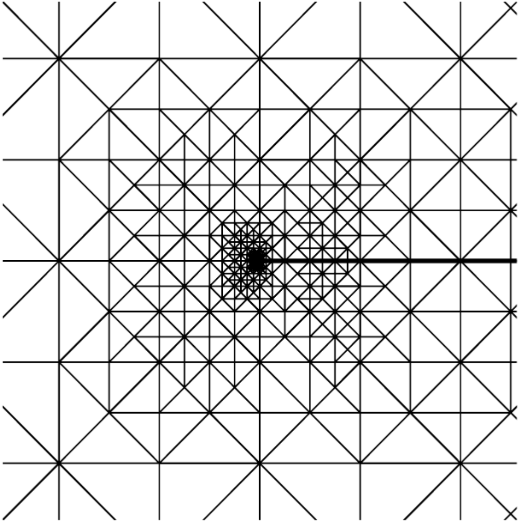
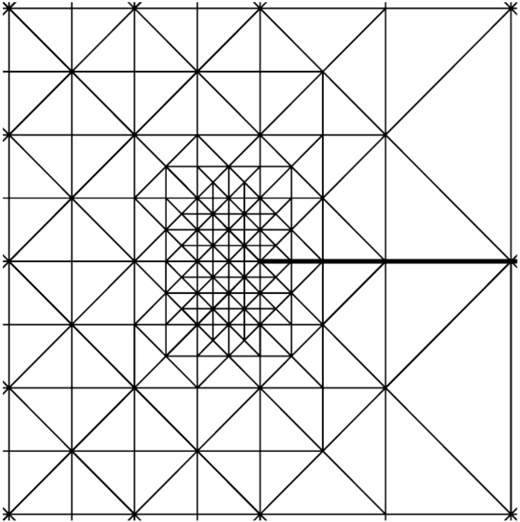



FIG. 5.4. *Quasioptimality of GERS and CONV. The optimal decay is indicated by the dashed line with slope* $-1/2$.









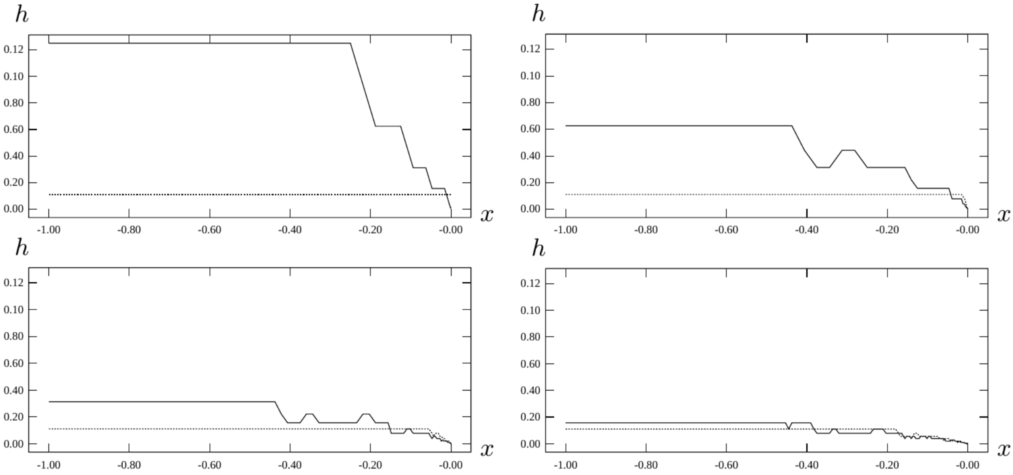



FIG. 5.6. *Comparison of local meshsizes h on the line $y = 0$ for GERS (dotted line) and CONV (solid line) on meshes with approximately same errors* $\|u - u_k\|_\Omega$.

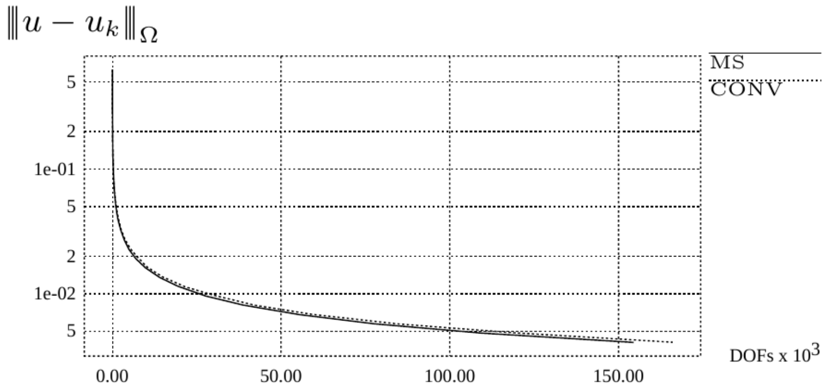



FIG. 5.7. *Comparison of CONV and MS.*

# EXAMPLE: Discontinuous coefficients

- $\Omega = (-1, 1)^2$
- $A = a_1 I$ in the first and third quadrants
- $A = a_2 I$ in the second and fourth quadrants
- Exact weak solution of $u$ for $f \equiv 0$ is given by $u(r, \theta) = r^\gamma \mu(\theta)$ with

$$\mu(\theta) = \begin{cases} \cos((\pi/2 - \sigma)\gamma) \cdot \cos((\theta - \pi/2 + \rho)\gamma) & \text{if } 0 \leq \theta \leq \pi/2, \\ \cos(\rho\gamma) \cdot \cos((\theta - \pi + \sigma)\gamma) & \text{if } \pi/2 \leq \theta \leq \pi, \\ \cos(\sigma\gamma) \cdot \cos((\theta - \pi - \rho)\gamma) & \text{if } \pi \leq \theta < 3\pi/2, \\ \cos((\pi/2 - \rho)\gamma) \cdot \cos((\theta - 3\pi/2 - \sigma)\gamma) & \text{if } 3\pi/2 \leq \theta \leq 2\pi \end{cases}$$

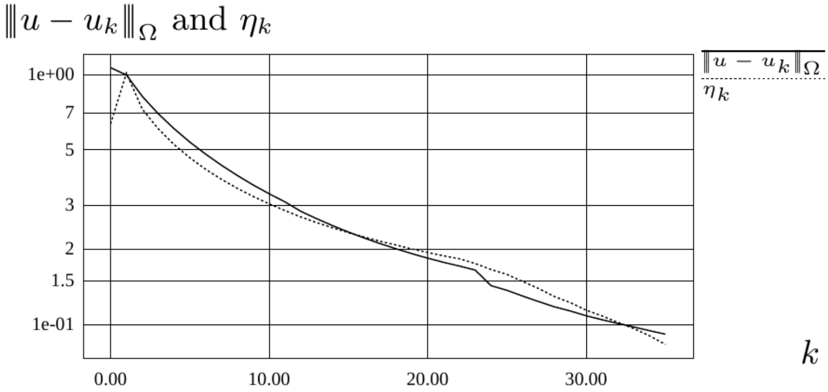



FIG. 5.8. *Error reduction: estimate and true error.*

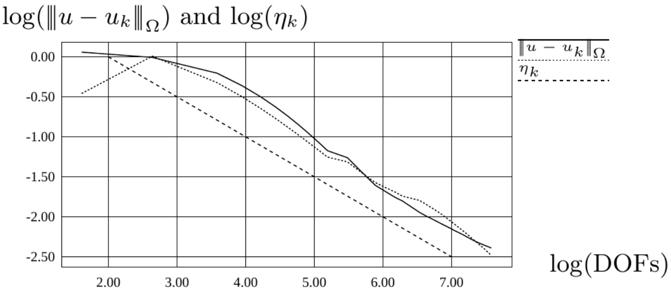
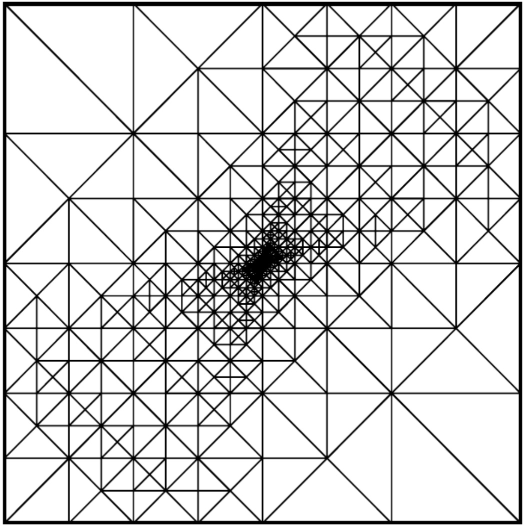
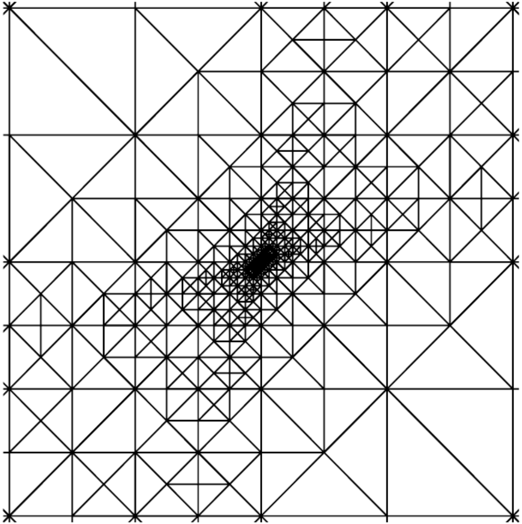
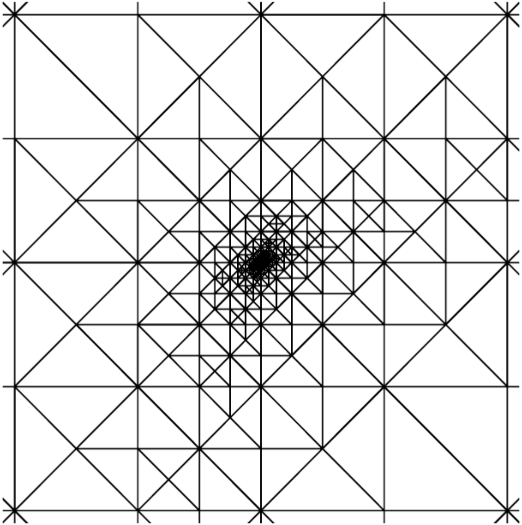
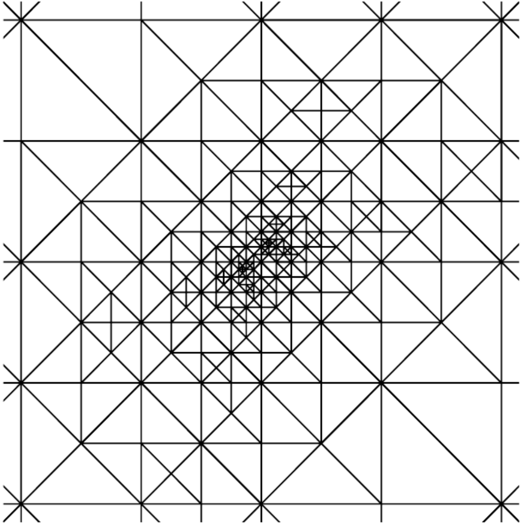



FIG. 5.9. *Quasioptimality of CONV: estimate and true error. The optimal decay is indicated by the dashed line with slope* $-1/2$.









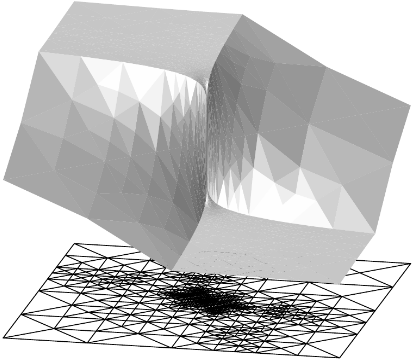


FIG. 5.11. *Graph of the discrete solution and underlying grid.*

# EXAMPLE: Variable source

- $\Omega = (-1,1)^d, d = 2,3$
- $u(x) = e^{-10|x|^2}$
- $A = I$ and nonconstant $f = -\Delta u$.
- $f$ exhibits large variations in $\Omega$, forcing "additional" refinement due to oscillation.

TABLE 5.1

*Total number and number of marked elements per iteration in two dimensions (left) and three dimensions (right): est.: marked elements due to error estimator, osc.: additionally marked elements to data oscillation.*

| iter. | elements | est. | osc. |
|---|---|---|---|
| 0 | 4 | 8 | 0 |
| 1 | 64 | 16 | 16 |
| 2 | 704 | 56 | 8 |
| 3 | 2256 | 80 | 0 |
| 4 | 4208 | 96 | 8 |
| 5 | 6624 | 112 | 24 |
| 6 | 8752 | 344 | 0 |
| 7 | 17512 | 432 | 0 |
| 8 | 28368 | 608 | 0 |
| 9 | 42896 | 768 | 16 |
| 10 | 60216 | 2192 | 0 |
| 11 | 113040 | 2296 | 24 |
| 12 | 160592 | 3816 | 24 |

| iter. | elements | est. | osc. |
|---|---|---|---|
| 0 | 6 | 6 | 0 |
| 1 | 384 | 48 | 0 |
| 2 | 7776 | 48 | 48 |
| 3 | 15936 | 576 | 0 |
| 4 | 112320 | 5040 | 0 |
| 5 | 860592 | 5136 | 720 |
| 6 | 1693536 | 30144 | 0 |

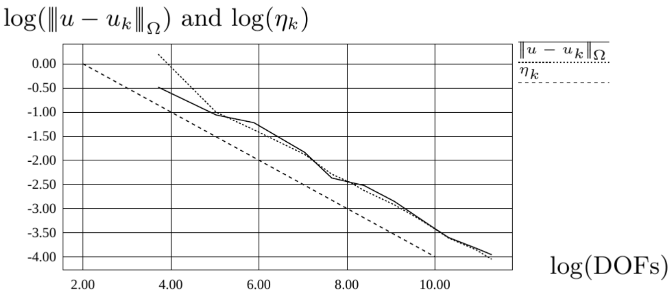



FIG. 5.12. *Quasioptimality of CONV: estimate and true error in two dimensions. The optimal decay is indicated by the line with slope* $-1/2$.

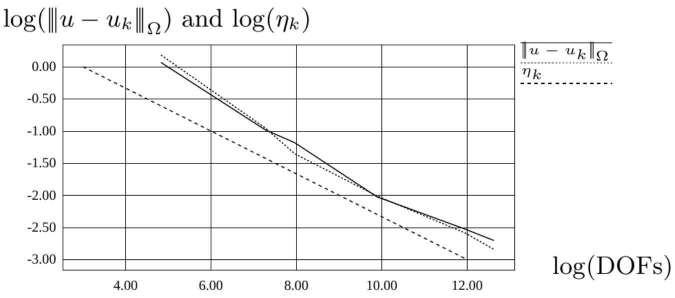



FIG. 5.13. *Quasioptimality of CONV: estimate and true error in three dimensions. The optimal decay is indicated by the line with slope* $-1/3$.

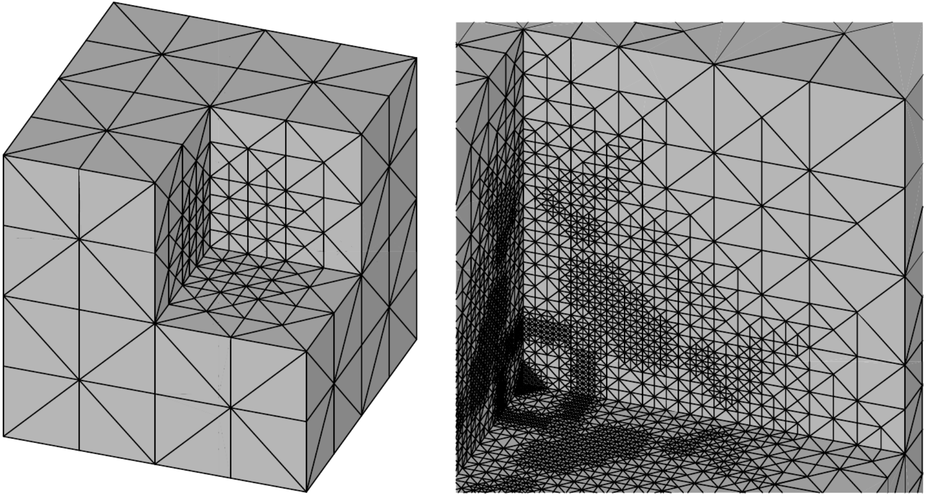


FIG. 5.14. *Adaptive grids of the three-dimensional simulation on* $\partial((-1,1)^3\backslash(0,1)^3)$*: full grid of the 2nd iteration (left), zoom into the grid of the 4th iteration (right).*