On computation of generalized derivatives of the normal-cone mapping and their applications*

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The paper concerns the computation of the graphical derivative and the regular (Fréchet) coderivative of the normal-cone mapping related to C^2 inequality constraints under very weak qualification conditions. This enables us to provide the graphical derivative and the regular coderivative of the solution map to a class of parameterized generalized equations with the constraint set of the investigated type. On the basis of these results we obtain finally a characterization of the isolated calmness property of the mentioned solution map and derive strong stationarity conditions for an MPEC with control constraints.

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1. Introduction

Various generalized derivatives introduced in modern variational analysis represent an efficient tool in stability analysis of multifunctions ([26, 13, 3, 16, 5]). This concerns in particular the so-called solution maps associated with parameter-dependent variational inequalities or generalized equations. Their stability properties have been thoroughly analyzed already in the seventies, above all in the papers by Robinson. A particular attention has been paid to the case of polyhedral constraint sets, independent of the parameter; see, for instance, [21, 22, 24, 4]. An overview of available results in this situation can be found in [5, Chapter 2E]. Concerning non-polyhedral constraint sets, one can find a huge number of works related to constraint sets with nonlinear programming structure, possibly even parameter-dependent. They deal with various types of stability, including the strong regularity of Robinson ([23]) and various other types of "regular" behavior, see [13] and the references therein.

Another group of results concerns the so-called conic constraints, cf. [2]. They are formulated either in the general framework or for special important classes of cones (SDP cones, Lorentz cones etc.).

In the first part of the present paper our main attention is paid to the graphical derivative and the regular coderivative of the normal-cone mapping

$$y \mapsto \widehat{N}_\Gamma(y), \tag{1}$$

where

$$\Gamma = \{y \in \mathbb{R}^m | q_i(y) \le 0, i = 1, 2, \ldots, l\} \tag{2}$$

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with twice continuously differentiable functions $q_i : \mathbb{R}^m \to \mathbb{R}$. In (1), $\widehat{N}_\Gamma$ stands for the regular normal cone to Γ defined at the beginning of the next section. The graphical derivative of (1) has been computed in [26, Cor.13.43(a) and Exercise 13.17] in the case when Γ is a fully amenable set (i.e., locally, the pre-image of a polyhedral set under a constraint qualification). In the case of Γ given by (2) and $\bar{y}^* \in \widehat{N}_\Gamma(\bar{y})$ this formula attains the form

$$D\widehat{N}_\Gamma(\bar{y},\bar{y}^*)(v) = \{\nabla^2(\lambda^T q)(\bar{y})v | \lambda \in \bar{\Lambda}(v)\} + N_{K(\bar{y},\bar{y}^*)}(v), v \in \mathbb{R}^m, \tag{3}$$

where $q(\cdot) = (q_1(\cdot), \ldots, q_m(\cdot))^T$,

$$\bar{\Lambda}(v) = \mathop{\arg\max}_{\substack{\nabla q(\bar{y})^T\lambda = \bar{y}^* \\ \lambda \in N_{\mathbb{R}^l_-}(q(\bar{y}))}} \langle v, \nabla^2(\lambda^T q)(\bar{y})v\rangle$$

and

$$K(\bar{y},\bar{y}^*) := T_\Gamma(\bar{y}) \cap \{\bar{y}^*\}^\perp$$

is the critical cone to Γ at $\bar{y}$ with respect to $\bar{y}^*$. Formula (3) can be viewed as a starting point of two lines of research directed to the relaxation of the full amenability of Γ assumed in [26]. In the first line one does not require the polyhedrality of Γ in order to be able to deal with the problems of second-order or semidefinite cone programming [19]. In the second line, followed in this paper, we will concentrate on Γ given by (2) and relax the constraint qualification associated with amenability which amounts in this case to the classical Mangasarian-Fromovitz constraint qualification (MFCQ).

Concerning the regular coderivative of $\widehat{N}_\Gamma$ for Γ given by (2), it has been studied in [11] and [9] under MFCQ and the Constant Rank constraint qualification (CRCQ) which are also substantially more restrictive than the qualification conditions imposed in this paper.

In the next part of the paper we consider the *solution map* S which assigns to each value of the *parameter* $x \in \mathbb{R}^n$ the corresponding set of solutions to the (parameterized) generalized equation (GE)

$$0 \in F(x,y) + \widehat{N}_\Gamma(y), \tag{4}$$

and a modified solution map $\tilde{S}$ taking into account also possible parameter constraints. In (4), $y \in \mathbb{R}^m$ is the *decision* variable and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuously differentiable mapping. On the basis of the graphical derivative of $\widehat{N}_\Gamma$ it is not difficult to compute the graphical derivative of S (or its outer estimate). On the contrary, the computation of the regular coderivative of S (respectively $\tilde{S}$) requires apart from the regular coderivative of $\widehat{N}_\Gamma$ also the fulfillment of another qualification condition which is typically rather restrictive. In order to relax it we have invoked the idea of *nondegeneracy* which we have extended from the original convex framework (see [2, p.315]) to a nonconvex one. This technique has enabled us to derive a new calculus rule for the regular normal cone to a "set with constraint structure" ([26, Theorem 6.14]) and, eventually, to compute the regular coderivative of S (respectively $\tilde{S}$).

The last part of the paper is devoted to applications. On the basis of the graphical derivative of S we state there a new criterion for the isolated calmness of S (at a given point from the graph of S), which is a valuable stability property and may be used e.g. in postoptimal analysis. Further, on the basis of the regular coderivative of S we have derived sharp necessary optimality conditions for an optimization problem, where (4) arises among the constraints. Such problems are termed mathematical programs with equilibrium constraints (MPECs) and represent a typical application area for new techniques of variational analysis.

The structure of the paper is as follows. In the first of two preparatory sections (Section 2) we provide a background from variational analysis. Apart from standard notions and properties we introduce there in Definition 2 a new stability property for multifunctions which plays a crucial role in further development. In Section 3, the second preparatory section, we fix the notation and state some simple auxiliary results. The main results are collected in Sections 4 and 5. These sections deal with the two mentioned generalized derivatives of $\widehat{N}_\Gamma$ and with the regular coderivatives of S and $\tilde{S}$, respectively. Section 6 is devoted to applications and, finally, in Section 7 we present concluding remarks on the results obtained.

Our notation is basically standard: conv Ω and ri Ω denote the convex hull and the relative interior of the set Ω, respectively, gphΦ stands for the graph of the map Φ and span$\{a,b\}$ signifies the linear subspace generated by vectors a, b. Furthermore, $\stackrel{\Omega}{\to}$ denotes convergence within the set Ω, for a cone K its negative polar is denoted by K°, $\|\cdot\|$ stands for the Euclidean norm and $\ker A$ means the kernel of the matrix A. Finally, $|I|$ is the cardinality of the index set I, $\Omega^\perp$ denotes the orthogonal complement to Ω, $o:\mathbb{R}^+ \to \mathbb{R}$ denotes a function with the property that $o(\lambda)/\lambda \to 0$ when $\lambda \downarrow 0$ and $\mathrm{d}(\cdot,\Omega)$ signifies the (Euclidean) distance function to Ω.

2. Background from variational analysis

In this section we briefly review some generalized differential constructions employed in the paper, confining ourselves only to the settings that appear below. The reader can find more details and extended frameworks in the monographs [16, 26] and in the papers we refer to.

Let us start with geometric objects. Given a set $\Omega \subset \mathbb{R}^d$ and a point $\bar{z} \in \Omega$, define the (Bouligand-Severi) *tangent/contingent cone* to Ω at $\bar{z}$ by

$$T_\Omega(\bar{z}) := \operatorname*{Lim\,sup}_{t\downarrow 0} \frac{\Omega - \bar{z}}{t} = \Big\{u \in \mathbb{R}^d \Big| \, \exists t_k \downarrow 0,\, u_k \to u \ \text{ with } \ \bar{z} + t_k u_k \in \Omega \ \forall\ k\Big\}. \tag{5}$$

Note that one has $T_\Omega(\bar{z}) = \mathbb{R}_+(\Omega - \bar{z})$ when Ω is a convex polyhedron.

The (Fréchet) *regular normal cone* to Ω at $\bar{z} \in \Omega$ can be equivalently defined by

$$\widehat{N}_\Omega(\bar{z}) := \Big\{v^* \in \mathbb{R}^d \Big| \ \limsup_{z \stackrel{\Omega}{\to} \bar{z}} \frac{\langle v^*, z - \bar{z}\rangle}{\|z - \bar{z}\|} \le 0\Big\} = (T_\Omega(\bar{z}))^\circ. \tag{6}$$

Further, the (Mordukhovich) *limiting/basic normal cone* to Ω at $\bar{z} \in \Omega$ is given by

$$N_\Omega(\bar{z}) := \operatorname*{Lim\,sup}_{z \stackrel{\Omega}{\to} \bar{z}} \hat{N}_\Omega(z).$$

The above notation Lim sup stands for the outer set limit in the sense of Painlevé–Kuratowski, see e.g. [26, Chapter 4]. Note that the tangent/contingent cone and the regular normal cone reduce to the classical tangent cone and normal cone of convex analysis, respectively, when the set Ω is convex.

Considering next set-valued (in particular, single-valued) mappings $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$, we define for them the corresponding derivative and coderivative constructions generated by the tangent cone (5) and the normal cone (6), respectively. Given $(\bar{z}, \bar{w}) \in \operatorname{gph}\Psi$, the *graphical derivative* $D\Psi(\bar{z},\bar{w}) : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ of Ψ at $(\bar{z},\bar{w})$ is

$$D\Psi(\bar{z},\bar{w})(u) := \big\{v \in \mathbb{R}^s \big| \, (u,v) \in T_{\operatorname{gph}\Psi}(\bar{z},\bar{w})\big\}, \quad u \in \mathbb{R}^d. \tag{7}$$

From the dual perspective we define the *regular coderivative* $\widehat{D}^*\Psi(\bar{z},\bar{w}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^d$ of Ψ at $(\bar{z},\bar{w}) \in \operatorname{gph}\Psi$ generated by the regular normal cone (6) as

$$\widehat{D}^*\Psi(\bar{z},\bar{w})(v^*) := \big\{u^* \in \mathbb{R}^d \big| \, (u^*, -v^*) \in \widehat{N}_{\operatorname{gph}\Psi}(\bar{z},\bar{w})\big\}, \quad v^* \in \mathbb{R}^s. \tag{8}$$

If Ψ is single-valued at $\bar z$, we drop $\bar w$ in the notation of (7)–(8). In the case of smooth single-valued mappings, for all $u\in\mathbb{R}^d$ and $v^*\in\mathbb{R}^s$ we have the representation

$$D\Psi(\bar z)(u)=\big\{\nabla\Psi(\bar z)u\big\} \text{ and } \widehat{D}^*\Psi(\bar z)(v^*)=\big\{\nabla\Psi(\bar z)^Tv^*\big\}.$$

In variational analysis an important role is played by various stability notions for multifunctions. In the sequel we will be extensively using the following two of them.

Definition 1. Let $\Psi:\mathbb{R}^d\rightrightarrows\mathbb{R}^s$ be a multifunction, let $(\bar u,\bar v)\in\operatorname{gph}\Psi$ and let $\kappa>0$.

1. Ψ is called *metrically regular with modulus* κ near $(\bar u,\bar v)$ if there are neighborhoods U of $\bar u$ and V of $\bar v$ such that

$$\mathrm{d}(u,\Psi^{-1}(v))\le\kappa\mathrm{d}(v,\Psi(u))\ \forall(u,v)\in U\times V. \tag{9}$$

2. Ψ is called *metrically subregular with modulus* κ at $(\bar u,\bar v)$ if there is a neighborhood U of $\bar u$ such that

$$\mathrm{d}(u,\Psi^{-1}(\bar v))\le\kappa\mathrm{d}(\bar v,\Psi(u))\ \forall u\in U. \tag{10}$$

It is well known that metric regularity of the multifunction Ψ near $(\bar u,\bar v)$ is equivalent to the Aubin property (also called Lipschitz-like or pseudo-Lipschitz) of the inverse multifunction Ψ^{-1} and metric subregularity of Ψ at $(\bar u,\bar v)$ is equivalent with the property of *calmness* of its inverse.

For general multifunctions the property of metric regularity is characterized by the so-called Mordukhovich criterion, see e.g. [15], [16, Theorem 4.18].

In the sequel we are dealing mostly with the perturbation mapping $M:\mathbb{R}^m\rightrightarrows\mathbb{R}^l$ associated with (2), which is defined by

$$M(y):=q(y)-\mathbb{R}^l_-.$$

In this case one can ensure the metric regularity or the metric subregularity via the following statements. The first one follows immediately from [26, Exercise 9.44].

Proposition 1. *Given $\bar y\in\Gamma$, M is metrically regular near $(\bar y,0)$ if and only if*

$$\ker(\nabla q(\bar y)^T)\cap\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))=\{0\}. \tag{11}$$

The infimum of the moduli κ for which the metric regularity property holds is equal to

$$\max_{\substack{\lambda\in\widehat{N}_{\mathbb{R}^l_-}(q(\bar y)),\\ \|\lambda\|=1}}\frac{1}{\|\nabla q(\bar y)^T\lambda\|}. \tag{12}$$

It is well known that condition (11) is equivalent to the classical Mangasarian-Fromovitz constraint qualification (MFCQ) at $\bar y$.

For the next statement we need the notion of the *linearized tangent cone* at some point $y\in\Gamma$ defined by

$$T^{\rm lin}_\Gamma(y):=\{v\in\mathbb{R}^m\,|\,\nabla q_i(y)v\le0,\ i\in\mathcal{I}(y)\},$$

where $\mathcal{I}(y):=\{i\in\{1,\dots,l\}\,|\,q_i(y)=0\}$ denotes the index set of constraints, active at y.

Proposition 2 **(Second order sufficient condition for metric subregularity [6, Theorem 6.1])**. *Let $\bar y\in\Gamma$ and assume that for every $0\neq u\in T^{\rm lin}_\Gamma(\bar y)$ one has*

$$\lambda\in\ker(\nabla q(\bar y)^T)\cap\widehat{N}_{\mathbb{R}^l_-}(q(\bar y)),\ u^T\nabla^2(\lambda^Tq)(\bar y)u\ge0\implies\lambda=0.$$

Then M is metrically subregular at $(\bar y,0)$.

We will refer to this condition by using the acronym SOSCMS. In the literature one can find also other sufficient conditions for metric subregularity, see e.g. [10, 12, 27, 28].

It can be easily seen from Proposition 1, that metric regularity of M implies SOSCMS.

In some situations M is not metrically regular near $\bar y\in\Gamma$ but enjoys a weaker property defined below.

DEFINITION 2. Let $\bar y\in\Gamma$, $\kappa>0$. We say that M is *metrically regular (with modulus κ) in the vicinity of $\bar y$*, if there is some neighborhood V of $\bar y$ such that for every $y\in M^{-1}(0)\cap V$, $y\neq\bar y$, the multifunction M is metrically regular near $(y,0)$ with modulus κ.

Since metric regularity is an open property in the sense that it holds in a neighborhood of the point in question, we easily conclude that metric regularity near $(\bar y,0)$ implies metric regularity in the vicinity of $\bar y$.

The following proposition states that by SOSCMS we have an easily applicable criterion for verifying metric regularity in the vicinity of $\bar y$ at hand, when MFCQ does not hold.

PROPOSITION 3. *Let $\bar y\in\Gamma$. Under SOSCMS the mapping M is metrically regular in the vicinity of $\bar y$.*

Proof. The proof follows from [7, Proposition 1] and the observation following Definition 4 therein. □

The following example demonstrates that metric regularity in the vicinity of $\bar y$ holds for a broad class of inequality systems, even when MFCQ is not fulfilled.

EXAMPLE 1. Let $\Gamma\subset\mathbb{R}^2$ be given by

$$q(y)=\begin{pmatrix}-y_2\\ y_2-y_1^d\end{pmatrix}$$

for fixed exponent $d\in\mathbb{N}\setminus\{0\}$ and let $\bar y=0$. Then it is easy to see that the corresponding mapping M is metrically regular near $(\bar y,0)$, i.e., MFCQ holds at $\bar y$ only in case when $d=1$. On the other hand, SOSCMS only holds when $d=2$, because for every $0\neq u\in T_\Gamma^{\rm lin}(\bar y)=\mathbb{R}\times\{0\}$ and every $0\neq\lambda\in\ker(\nabla q(\bar y)^T)\cap\widehat N_{\mathbb{R}^l_-}(q(\bar y))=\{(\lambda_1,\lambda_2)\in\mathbb{R}^2_+\mid\lambda_1=\lambda_2\}$ we have $u^T\nabla^2(\lambda^Tq)(\bar y)u=-2\lambda_2u_1^2<0$ and thus SOSCMS follows. However, M is metrically regular in the vicinity of $\bar y$ for every $d\in\mathbb{N}\setminus\{0\}$. This follows from the fact that at every point $y\in\Gamma\setminus\{\bar y\}$ at most one constraint is active and the gradient of this active constraint is bounded away from 0. △

3. Notation and auxiliary results

Given elements $y\in\Gamma$ and $y^*\in\widehat N_\Gamma(y)$ we define by

$$\Lambda(y,y^*):=\{\lambda\in\widehat N_{\mathbb{R}^l_-}(q(y))\mid\nabla q(y)^T\lambda=y^*\},$$

the set of *Lagrange multipliers* associated with (y,y^*). Moreover

$$K(y,y^*):=T_\Gamma(y)\cap(y^*)^\perp$$

stands for the *critical cone* to Γ at y with respect to y^*.

For a given reference pair $(\bar y,\bar y^*)$, $\bar y\in\Gamma$, $\bar y^*\in\widehat N_\Gamma(\bar y)$, fixed throughout this paper, we shortly set $\bar{\mathcal{I}}:=\mathcal{I}(\bar y)$, $\bar\Lambda:=\Lambda(\bar y,\bar y^*)$ and $\bar K:=K(\bar y,\bar y^*)$. Furthermore we employ the set

$$\bar{\mathcal{N}}:=\{v\in\mathbb{R}^m\mid\nabla q_i(\bar y)v=0,\ i\in\bar{\mathcal{I}}\}$$

(nullspace of gradients of constraints active at $\bar y$). Given a multiplier $\lambda\in\widehat N_{\mathbb{R}^l_-}(q(\bar y))$ we introduce the index sets

$$I^+(\lambda):=\{i\in\{1,\dots,l\}\mid\lambda_i>0\},\ \bar I^0(\lambda):=\bar{\mathcal{I}}\setminus I^+(\lambda),$$

the sets of strongly and weakly active constraints. Apart from them we will be working with

$$\bar I^+ = \bigcup_{\lambda\in\bar\Lambda} I^+(\lambda),\quad \bar I^0 := \bar{\mathcal I}\setminus \bar I^+.$$

With a direction $v\in T_\Gamma^{\rm lin}(\bar y)$ let us now associate the *directional multiplier set*

$$\bar\Lambda(v) := \arg\max_{\lambda\in\bar\Lambda} v^T\nabla^2(\lambda^T q)(\bar y)v.$$

Directly from the definition of metric subregularity one can infer that under metric subregularity of M at $(y,0)$ one has

$$T_\Gamma(y) = T_\Gamma^{\rm lin}(y),$$

cf. also [10, Proposition 1], and thus

$$K(y,y^*) = T_\Gamma^{\rm lin}(y)\cap\{y^*\}^\perp = \{w\in\mathbb R^m\,|\,\nabla q_i(y)w\le 0, i\in\mathcal I(y), y^Tw=0\}. \tag{13}$$

Further it follows that under this condition the regular normal cone to Γ at y amounts to

$$\widehat N_\Gamma(y) = \nabla q(y)^T\widehat N_{\mathbb R^l_-}(q(y))$$

and consequently $\Lambda(y,y^*)\neq\emptyset$. In the rest of this section the metric subregularity of M at $(\bar y,0)$ will be assumed.

Lemma 1. *Let $v\in\bar K$, $\lambda\in\bar\Lambda$ and assume that M is metrically subregular at $(\bar y,0)$. Then*

$$\widehat N_{\bar K}(v) = \{\nabla q(\bar y)^T\mu\,|\,\mu^T\nabla q(\bar y)v = 0, \mu\in T_{\widehat N_{\mathbb R^l_-}(q(\bar y))}(\lambda)\}.$$

Proof. Note that $\widehat N_{\bar K}(v) = \bar K^\circ\cap\{v\}^\perp$ and, by virtue of the Farkas Lemma,

$$\begin{aligned}\bar K^\circ &= \{w\in\mathbb R^m\,|\,\nabla q_i(\bar y)w\le 0, i\in\bar{\mathcal I}, \bar y^{*T}w=0\}^\circ = \{\sum_{i\in\bar{\mathcal I}}\mu_i\nabla q_i(\bar y)+\alpha\bar y^*\,|\,\mu_i\ge 0, i\in\bar{\mathcal I},\alpha\in\mathbb R\}\\ &= \nabla q(\bar y)^T\widehat N_{\mathbb R^l_-}(q(\bar y)) + \mathbb R\bar y^*.\end{aligned}$$

Hence, for every $v^*\in\widehat N_{\bar K}(v)$ there is some $\tilde\mu\in\widehat N_{\mathbb R^l_-}(q(\bar y))$ and some $\alpha\in\mathbb R$ with $v^* = \nabla q(\bar y)^T\tilde\mu + \alpha\bar y^* = \nabla q(\bar y)^T(\tilde\mu+\alpha\lambda)$. Setting $\mu=\tilde\mu+\alpha\lambda$ we have $v^* = \nabla q(\bar y)^T\mu$, $\mu^T\nabla q(\bar y)v = v^{*T}v = 0$ and $\mu\in T_{\widehat N_{\mathbb R^l_-}(q(\bar y))}(\lambda)$, where the last relation follows from the fact that for $t>0$ sufficiently small we have $1+\alpha t>0$ and thus $\lambda+t\mu = (1+\alpha t)\lambda + t\tilde\mu\in\widehat N_{\mathbb R^l_-}(q(\bar y))$.

Conversely, let $\mu\in T_{\widehat N_{\mathbb R^l_-}(q(\bar y))}(\lambda)$ with $\mu^T\nabla q(\bar y)^Tv=0$ be arbitrarily fixed. Then there is some $t>0$ such that $\lambda+t\mu\in\widehat N_{\mathbb R^l_-}(q(\bar y))$ and therefore for all $w\in\bar K = T_\Gamma^{\rm lin}(\bar y)\cap\{\bar y^*\}^\perp\subset T_\Gamma^{\rm lin}(\bar y)$ we have

$$0\ge(\lambda+t\mu)^T\nabla q(\bar y)w = \bar y^{*T}w + t\mu^T\nabla q(\bar y)w = t(\nabla q(\bar y)^T\mu)^Tw$$

showing $\nabla q(\bar y)^T\mu\in\bar K^\circ$. Together with $(\nabla q(\bar y)^T\mu)^Tv=0$ we conclude $(\nabla q(\bar y)^T\mu)\in\widehat N_{\bar K}(v)$. □

Lemma 2. *Assume that M is metrically subregular at $(\bar y,0)$. Then there is some $\tilde\lambda\in\bar\Lambda$ such that $I^+(\tilde\lambda)=\bar I^+$. Further we have*

$$\bar K = \left\{v\in\mathbb R^m\,|\,\begin{array}{l}\nabla q_i(\bar y)v = 0,\ i\in\bar I^+\\ \nabla q_i(\bar y)v\le 0,\ i\in\bar I^0\end{array}\right\}$$

and there is some $v\in\bar K$ satisfying

$$\nabla q_i(\bar y)v = 0,\ i\in\bar I^+,\quad \nabla q_i(\bar y)v<0,\ i\in\bar I^0. \tag{14}$$

Proof. Since $\bar I^+$ is a finite index set, there are $\lambda^i \in \bar\Lambda$, $i=1,\ldots,N$, such that $\bar I^+ = \bigcup_{i=1}^N I^+(\lambda^i)$. Setting $\tilde\lambda := \frac{1}{N}\sum_{i=1}^N \lambda^i$ it easily follows that $\tilde\lambda \in \bar\Lambda$ and $I^+(\tilde\lambda) = \bar I^+$. The second assertion follows from the equivalences

$$v \in \bar K \Leftrightarrow \left(v \in T_\Gamma^{\rm lin}(\bar y) \wedge 0 = \bar y^{*T} v = (\nabla q(\bar y)^T \tilde\lambda)^T v = \tilde\lambda^T \nabla q(\bar y) v = \sum_{i \in I^+(\tilde\lambda)} \tilde\lambda_i \nabla q_i(\bar y) v \right)$$
$$\Leftrightarrow \left(v \in T_\Gamma^{\rm lin}(\bar y) \wedge \nabla q_i(\bar y) v = 0, i \in I^+(\tilde\lambda) = \bar I^+ \right).$$

We prove the last statement by contraposition. Assuming that the system

$$\nabla q_i(\bar y) v = 0,\ i \in \bar I^+, \quad \nabla q_i(\bar y) v \le -1,\ i \in \bar I^0 \tag{15}$$

does not have a solution, by the Farkas lemma there is some $\mu \in \mathbb{R}^l$ such that $\nabla q(\bar y)^T \mu = 0$, $\mu_i = 0,\ i \notin \bar{\mathcal I}$, $\mu_i \ge 0, i \in \bar I^0$ and $\sum_{i \in \bar I^0} \mu_i > 0$. It follows that $\tilde\lambda + t\mu \in \bar\Lambda$ for some $t > 0$ and from $\sum_{i\in \bar I^0} \mu_i > 0$ we conclude that there must be some index $i \in \bar I^0$ with $\tilde\lambda_i + t\mu_i > 0$ implying $i \in \bar I^+$, a contradiction. Hence the system (15) has a solution and this completes the proof. □

Consider for every $v \in \bar K$ the linear optimization problem

$$LP(v) \qquad \min -v^T \nabla^2 (\lambda^T q)(\bar y) v \quad \text{subject to} \quad \lambda \in \bar\Lambda$$

together with its dual program

$$DP(v) \qquad \max \bar y^{*T} z \quad \text{subject to} \quad \nabla q_i(\bar y) z \le -v^T \nabla^2 q_i(\bar y) v,\ i \in \bar{\mathcal I}.$$

Then, by definition, $\bar\Lambda(v)$ is the solution set of $LP(v)$ and, by duality theory of linear programming, $\bar\Lambda(v) \ne \emptyset$ if and only if $DP(v)$ is solvable. Further, given $\lambda \in \bar\Lambda$ and z feasible for $DP(v)$, we have $\lambda \in \bar\Lambda(v)$ and z solves $DP(v)$ if and only if

$$\lambda_i \left(\nabla q_i(\bar y) z + v^T \nabla^2 q_i(\bar y) v \right) = 0, i \in \bar{\mathcal I}. \tag{16}$$

Lemma 3. *For every $\bar v \in \bar K$ there is a neighborhood U of $\bar v$ such that*

$$\bar\Lambda(v) \subset \bar\Lambda(\bar v)\ \forall v \in U \cap \bar K.$$

Proof. If $\bar\Lambda(\bar v) = \emptyset$, we conclude from [1, Theorems 5.4.1,5.4.2] that $\bar\Lambda(v) = \emptyset$ for all v belonging to some neighborhood U of $\bar v$. Now assume $\bar\Lambda(\bar v) \ne \emptyset$. Again by [1, Theorems 5.4.1,5.4.2] for every $\epsilon > 0$ there is some neighborhood U_ϵ such that $\mathrm{d}(\lambda, \bar\Lambda(\bar v)) < \epsilon$ for every $v \in U_\epsilon$ and every $\lambda \in \bar\Lambda(v)$. Further we know that the solution set of a linear optimization problem is a face of the feasible set. Combining both properties we see that for all v near $\bar v$ the solution set $\bar\Lambda(v)$ is a face of $\bar\Lambda(\bar v)$, provided it is not empty. □

In what follows we denote by $\mathcal E$ the set of extreme points of the polyhedron $\bar\Lambda$. The polyhedron $\bar\Lambda$ can be represented as the sum of the convex hull of its extreme points and its recession cone $\mathcal R := \{\lambda \in \widehat N_{\mathbb{R}^l_-}(q(\bar y)) \,|\, \nabla q(\bar y)^T \lambda = 0\}$, i.e., $\bar\Lambda = \operatorname{conv} \mathcal E + \mathcal R$. From the theory of linear programming it is well known that $\bar\Lambda(v) \ne \emptyset$ if and only if

$$v^T \nabla^2 (\lambda^T q)(\bar y) v \le 0\ \forall \lambda \in \mathcal R.$$

In this case the set $\bar\Lambda(v) \cap \mathcal E$ is not empty and contains exactly the extreme points of $\bar\Lambda(v)$. In what follows we denote by $\bar\Lambda^{\mathcal E}(v)$ the compact convex polyhedron $\bar\Lambda^{\mathcal E}(v) := \bar\Lambda(v) \cap \operatorname{conv} \mathcal E$.

4. Graphical derivative and regular coderivative of $\widehat{N}_\Gamma$

We denote by $\mathcal{T}(\bar y,\bar y^*)$ the set

$$\mathcal{T}(\bar y,\bar y^*):=\{(v,v^*)\,|\,\exists\lambda\in\bar\Lambda(v): v^*\in\nabla^2(\lambda^Tq)(\bar y)v+\widehat{N}_{\bar K}(v)\}$$

THEOREM 1. *Assume that M is metrically subregular at $(\bar y,0)$. Then*

$$\mathcal{T}(\bar y,\bar y^*)\subset T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar y,\bar y^*) \tag{17}$$

and equality holds if in addition M is metrically regular in the vicinity of $\bar y$.

Proof. To show (17) let $(v,v^*)\in\mathcal{T}(\bar y,\bar y^*)$ be arbitrarily fixed. In order to show $(v,v^*)\in T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar y,\bar y^*)$ we must prove the existence of sequences $(t_k)\downarrow 0$ and $(y_k)\stackrel{\Gamma}{\to}\bar y$ such that

$$\lim_{k\to\infty}\frac{\bar y+t_kv-y_k}{t_k}=0,\ \lim_{k\to\infty}\frac{\mathrm{d}(\bar y^*+t_kv^*,\widehat{N}_\Gamma(y_k))}{t_k}=0.$$

Let $\lambda\in\bar\Lambda(v)$ such that $v^*\in\nabla^2(\lambda^Tq)(\bar y)v+\widehat{N}_{\bar K}(v)$ and choose $\alpha>0$ so small that $\alpha\|\nabla^2(\lambda^Tq)(\bar y)\|<\frac{1}{2}$. It follows that $2I+\nabla^2((\alpha\lambda)^Tq)(\bar y)$ is positive definite and hence, by applying the standard second order sufficient conditions of nonlinear programming (see, e.g., [2, Proposition 5.48]), we can conclude that there is some positive radius $\rho>0$ such that $\bar y$ is the unique global solution of the problem

$$\min_y\|\bar y+\alpha\bar y^*-y\|^2 \text{ subject to } q(y)\le 0,\ \|y-\bar y\|\le\rho.$$

Since λ solves $LP(v)$, by duality theory of linear programming as discussed in the previous section, there is some vector z solving $DP(v)$ with

$$\nabla q_i(\bar y)z+v^T\nabla^2q_i(\bar y)v\le 0,\ \lambda_i(\nabla q_i(\bar y)z+v^T\nabla^2q_i(\bar y)v)=0,\ i\in\bar{\mathcal{I}}. \tag{18}$$

Since $\lambda_i=0$, $i\not\in\bar{\mathcal{I}}$, we obtain

$$\lambda^T\nabla q(\bar y)z+v^T\nabla^2(\lambda^Tq)(\bar y)v=0. \tag{19}$$

By Lemma 1 there is some $\mu\in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))}(\lambda)\cap(\nabla q(\bar y)v)^\perp$ with $v^*=\nabla^2(\lambda^Tq)(\bar y)v+\nabla q(\bar y)^T\mu$ and thus

$$v^{*T}v=v^T\nabla^2(\lambda^Tq)(\bar y)v+\mu^T\nabla q(\bar y)v=v^T\nabla^2(\lambda^Tq)(\bar y)v. \tag{20}$$

Since $\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))$ is a convex polyhedron and therefore $T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))}(\lambda)=\mathbb{R}_+(\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))-\lambda)$, the condition $\mu\in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))}(\lambda)$ ensures the existence of some $\bar t>0$ such that $\lambda+t\mu\in\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))$ for all $t\in[0,\bar t]$ and since $q_i(\bar y)<0$, $i\not\in\bar{\mathcal{I}}$, we can also assume that for every $t\in[0,\bar t]$ we have $q_i(\bar y+tv+\frac{1}{2}t^2z)<0$, $i\not\in\bar{\mathcal{I}}$. We now consider for each $t\in[0,\bar t]$ a global solution y_t of the optimization problem

$$\min\|\bar y+tv+\frac{1}{2}t^2z+\alpha(\bar y^*+tv^*)-y\|^2 \text{ subject to } y\in\Gamma,\ \|y-\bar y\|\le\rho.$$

In order to show the inclusion $(v,v^*)\in T_{\operatorname{gph}\widehat{N}_\Gamma}(\bar y,\bar y^*)$ it suffices to show

$$\lim_{t\downarrow 0}\frac{\bar y+tv-y_t}{t}=0 \tag{21}$$

because then for all $t>0$ sufficiently small we have $\|y_t-\bar y\|<\rho$ and therefore the standard optimality condition at y_t reads as

$$\alpha(\bar y^*+tv^*)+t(\frac{\bar y+tv-y_t}{t}+\frac{1}{2}tz)\in\widehat{N}_\Gamma(y_t).$$

This, because $\widehat{N}_\Gamma(y_t)$ is a cone, implies that

$$\lim_{t\downarrow 0}\frac{\mathrm{d}(\bar y^*+tv^*,\widehat{N}_\Gamma(y_t))}{t}=0.$$

Our choice of α and ρ guarantees that $y_0=\bar y$ and, by using [2, Proposition 4.4], we conclude $\lim_{t\downarrow 0} y_t=\bar y$. Taking into account (18) and $\nabla q_i(\bar y)v\le 0$, $i\in\bar{\mathcal{I}}$, due to $v\in\bar K$ and (13), we obtain

$$q_i(\bar y+tv+\frac12 t^2 z)=q_i(\bar y)+t\nabla q_i(\bar y)v+\frac12 t^2(\nabla q_i(\bar y)z+v^T\nabla^2 q_i(\bar y)v)+o(t^2)\le o(t^2),\ i\in\bar{\mathcal{I}},$$

and since $q_i(\bar y+tv+\frac12 t^2 z)<0$, $i\not\in\bar{\mathcal{I}}$, and M is assumed to be metrically subregular at $(\bar y,0)$, we can find for every $t\in[0,\bar t]$ some point $\tilde y_t\in\Gamma$ with $\|\bar y+tv+\frac12 t^2 z-\tilde y_t\|=o(t^2)$. Hence

$$\|\bar y+tv+\frac12 t^2 z+\alpha(\bar y^*+tv^*)-y_t\|^2\le\|\bar y+tv+\frac12 t^2 z+\alpha(\bar y^*+tv^*)-\tilde y_t\|^2$$

for all $t\ge 0$ sufficiently small, implying

$$\|\bar y+tv+\frac12 t^2 z-y_t\|^2+2\alpha(\bar y^*+tv^*)^T(\bar y+tv+\frac12 t^2 z-y_t)\le o(t^2). \tag{22}$$

From $\lambda+t\mu\in\widehat{N}_{\mathbb{R}^l_-}(q(\bar y))$, $q(y_t)-q(\bar y)\in T_{\mathbb{R}^l_-}(q(\bar y))$ and $v^*=\nabla^2(\lambda^T q)(\bar y)v+\nabla q(\bar y)^T\mu$ we obtain

$$\begin{aligned}
0 &\ge (\lambda+t\mu)^T(q(y_t)-q(\bar y))\\
&= (\lambda+t\mu)^T\nabla q(\bar y)(y_t-\bar y)+\frac12(y_t-\bar y)^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+\frac{t}{2}(y_t-\bar y)^T\nabla^2(\mu^T q)(\bar y)(y_t-\bar y)\\
&\quad +o(\|y_t-\bar y\|^2)\\
&= (\lambda+t\mu)^T\nabla q(\bar y)(y_t-\bar y)+\frac12(y_t-\bar y)^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+o(\|y_t-\bar y\|^2)\\
&= (\bar y^*+tv^*)^T(y_t-\bar y)-tv^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+\frac12(y_t-\bar y)^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+o(\|y_t-\bar y\|^2)
\end{aligned}$$

and, consequently, by taking into account ${\bar y^*}^T v=0$ and relations (19), (20),

$$\begin{aligned}
&(\bar y^*+tv^*)^T(\bar y+tv+\frac12 t^2 z-y_t)\\
&= (\bar y^*+tv^*)^T(\bar y-y_t)+t{\bar y^*}^T v+\frac12 t^2{\bar y^*}^T z+t^2{v^*}^T v+o(t^2)\\
&\ge tv^T\nabla^2(\lambda^T q)(\bar y)(\bar y-y_t)+\frac12(y_t-\bar y)^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+t{\bar y^*}^T v+\\
&\qquad \frac12 t^2\lambda^T\nabla q(\bar y)z+t^2v^T\nabla^2(\lambda^T q)(\bar y)v+o(t^2)+o(\|y_t-\bar y\|^2)\\
&= tv^T\nabla^2(\lambda^T q)(\bar y)(\bar y-y_t)+\frac12(y_t-\bar y)^T\nabla^2(\lambda^T q)(\bar y)(y_t-\bar y)+\\
&\qquad \frac12 t^2v^T\nabla^2(\lambda^T q)(\bar y)v+o(t^2)+o(\|y_t-\bar y\|^2)\\
&= \frac12(\bar y+tv-y_t)^T\nabla^2(\lambda^T q)(\bar y)(\bar y+tv-y_t)+o(t^2)+o(\|y_t-\bar y\|^2)\\
&\ge -\frac12\|\nabla^2(\lambda^T q)(\bar y)\|\|\bar y+tv-y_t\|^2+o(t^2)+o(\|y_t-\bar y\|^2).
\end{aligned}$$

Since $\alpha\|\nabla^2(\lambda^T q)(\bar y)\|<\frac12$, it follows that

$$\begin{aligned}
&\|\bar y+tv+\frac12 t^2 z-y_t\|^2-\frac12\|\bar y+tv-y_t\|^2\\
&\le \|\bar y+tv+\frac12 t^2 z-y_t\|^2+2\alpha(\bar y^*+tv^*)^T(\bar y+tv+\frac12 t^2 z-y_t)+o(t^2)+o(\|y_t-\bar y\|^2)\\
&\le o(t^2)+o(\|y_t-\bar y\|^2),
\end{aligned}$$

where the last inequality follows from (22), and

$$\frac{1}{2}\|\bar{y}+tv-y_t\|^2 \le o(t^2)+o(\|y_t-\bar{y}\|^2). \tag{23}$$

Hence there is some $\tilde{t}>0$ such that $\frac{1}{2}\|\bar{y}+tv-y_t\|^2 \le \frac{1}{4}(t^2+\|y_t-\bar{y}\|^2)\ \forall t\in[0,\tilde{t}]$. After rearranging we obtain $\frac{1}{4}\|\bar{y}-y_t\|^2+tv^T(\bar{y}-y_t)+\frac{1}{2}t^2\|v\|^2 \le \frac{1}{4}t^2$ showing

$$\frac{1}{4}\|\bar{y}+2tv-y_t\|^2 = \frac{1}{4}\|\bar{y}-y_t\|^2+tv^T(\bar{y}-y_t)+t^2\|v\|^2 \le \frac{1}{4}t^2(1+2\|v\|^2)$$

and $\|\bar{y}-y_t\|\le t(2\|v\|+\sqrt{1+2\|v\|^2})$. From (23) we conclude that (21) holds and therefore $(v,v^*)\in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ follows.

Next we show that the reverse inclusion $\mathcal{T}(\bar{y},\bar{y}^*)\supset T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ is valid under the additional assumption that M is metrically regular in the vicinity of $\bar{y}$. Let $\kappa>0$ denote the modulus of metric regularity according to Definition 2, let $(v,v^*)\in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ and consider sequences $(t_k)\downarrow 0$, $(v_k)\to v$ and $(v_k^*)\to v^*$ such that $y_k^*:=\bar{y}^*+t_kv_k^*\in\widehat{N}_\Gamma(y_k)$, where $y_k:=\bar{y}+t_kv_k$. By passing to a subsequence if necessary we can assume that there is some index set $\tilde{\mathcal{I}}\subset\bar{\mathcal{I}}$ such that $\mathcal{I}(y_k)=\tilde{\mathcal{I}}$ holds for all k and that for every k with $y_k\ne\bar{y}$ the multifunction M is metrically regular with modulus κ near $(y_k,0)$. For every $i\in\bar{\mathcal{I}}$ we have

$$q_i(y_k)=q_i(\bar{y})+t_k\nabla q_i(\bar{y})v_k+o(t_k)=t_k\nabla q_i(\bar{y})v_k+o(t_k)\begin{cases}=0 & \text{if } i\in\tilde{\mathcal{I}},\\ \le 0 & \text{if } i\in\bar{\mathcal{I}}\setminus\tilde{\mathcal{I}}.\end{cases}$$

Dividing by t_k and passing to the limit we obtain

$$\nabla q_i(\bar{y})v\begin{cases}=0 & \text{if } i\in\tilde{\mathcal{I}},\\ \le 0 & \text{if } i\in\bar{\mathcal{I}}\setminus\tilde{\mathcal{I}}.\end{cases} \tag{24}$$

Next consider for each $y^*\in\mathbb{R}^m$ the set

$$\Psi_{\tilde{\mathcal{I}}}(y^*):=\{\lambda\in\mathbb{R}^l\mid\nabla q(\bar{y})^T\lambda=y^*,\lambda_i\ge 0,\ i\in\tilde{\mathcal{I}},\lambda_i=0,i\notin\tilde{\mathcal{I}}\}. \tag{25}$$

By Hoffman's Lemma there is some constant β such that for every $\lambda\in\mathbb{R}^l$ and every $y^*\in\mathbb{R}^m$ with $\Psi_{\tilde{\mathcal{I}}}(y^*)\ne\emptyset$ one has

$$\mathrm{d}(\lambda,\Psi_{\tilde{\mathcal{I}}}(y^*))\le\beta(\|\nabla q(\bar{y})^T\lambda-y^*\|-\sum_{i\in\tilde{\mathcal{I}}}\min\{\lambda_i,0\}+\sum_{i\notin\tilde{\mathcal{I}}}|\lambda_i|). \tag{26}$$

If $y_k\ne\bar{y}$ then, as a consequence of the assumption that M is metrically regular in the vicinity of $\bar{y}$, there is some multiplier $\lambda^k\in\widehat{N}_{\mathbb{R}^l_-}(q(y_k))$ with $y_k^*=\nabla q(y_k)^T\lambda^k$ and using Proposition 1, we have $\lambda^k=0$ when $y_k^*=0$ and

$$\frac{\|\lambda^k\|}{\|y_k^*\|}=\frac{1}{\|\nabla q(y_k)^T\frac{\lambda^k}{\|\lambda^k\|}\|}\le\kappa \quad \text{whenever} \quad y_k^*\ne 0,$$

showing $\|\lambda^k\|\le\kappa\|y_k^*\|$. On the other hand, if $y_k=\bar{y}$, since M is assumed to be metrically subregular at $(\bar{y},0)$, there is also some multiplier $\lambda^k\in\widehat{N}_{\mathbb{R}^l_-}(q(y_k))$ with $y_k^*=\nabla q(y_k)^T\lambda^k$ and by using (26), we can choose λ^k such that $\|\lambda^k\|=\mathrm{d}(0,\Psi_{\tilde{\mathcal{I}}}(y_k^*))\le\beta\|y_k^*\|$. Hence we can assume that the sequence (λ^k) is uniformly bounded by some constant c_1 and, by passing to a subsequence if necessary, we can assume that (λ^k) converges to $\bar\lambda$. Due to the definition of λ^k we have $\nabla q(\bar y)^T\bar\lambda=0$, $\bar\lambda_i\geq 0$ with $i\not\in\tilde{\mathcal I}$, and $\bar\lambda_i=0$ with $i\not\in\tilde{\mathcal I}$, which ensures $\bar\lambda\in\Psi_{\tilde{\mathcal I}}(\bar y^*)$. Since

$$\nabla q(\bar y)^T\lambda^k-\bar y^*=t_kv_k^*+(\nabla q(\bar y)-\nabla q(y_k))^T\lambda^k$$

and $\|\nabla q(\bar y)-\nabla q(y_k)\|\leq c_2\|y_k-\bar y\|=c_2t_k\|v_k\|$ for some constant $c_2\geq 0$, by using (26) once more we can find for each k some $\tilde\lambda^k\in\Psi_{\tilde{\mathcal I}}(\bar y^*)\subset\bar\Lambda$ with $\|\tilde\lambda^k-\lambda^k\|\leq\beta t_k(\|v_k^*\|+c_1c_2\|v_k\|)$. Taking $\mu^k:=(\lambda_k-\tilde\lambda_k)/t_k$, we have that (μ^k) is uniformly bounded. By passing to subsequences if necessary we can assume that both sequences $(\tilde\lambda^k)$ and (μ^k) are convergent to some $\tilde\lambda\in\Psi_{\tilde{\mathcal I}}(\bar y^*)\subset\bar\Lambda$ and some μ. Since $\lambda_i^k=\tilde\lambda_i^k=0$, $i\not\in\tilde{\mathcal I}$, we have ${\mu^k}^T\nabla q(\bar y)v=0\ \forall k$ implying $\mu\in(\nabla q(\bar y)v)^\perp$.

Taking into account ${\tilde\lambda^k}{}^Tq(y_k)=0\ \forall k$, we obtain

$$0=\lim_{k\to\infty}\frac{{\tilde\lambda^k}{}^Tq(y_k)}{t_k}=\lim_{k\to\infty}{\tilde\lambda^k}{}^T\nabla q(\bar y)v_k={\bar y^*}{}^Tv$$

which, together with (24), shows $v\in\bar K$.

Further we have for all $\lambda\in\bar\Lambda$

$$\begin{aligned}0&\leq(\tilde\lambda^k-\lambda)^Tq(y_k)=(\tilde\lambda^k-\lambda)^T(q(\bar y)+t_k\nabla q(\bar y)v_k+\frac{1}{2}t_k^2v_k^T\nabla^2q(\bar y)v^k+o(t_k^2))\\&=(\tilde\lambda^k-\lambda)^T(\frac{1}{2}t_k^2v_k^T\nabla^2q(\bar y)v^k+o(t_k^2)).\end{aligned}$$

Dividing by t_k^2 and passing to the limit we obtain $(\tilde\lambda-\lambda)^Tv^T\nabla^2q(\bar y)v\geq 0\ \forall\lambda\in\bar\Lambda$ and hence $\tilde\lambda\in\bar\Lambda(v)$ follows.

Since

$$y_k^*=\nabla q(\bar y)^T\tilde\lambda^k+t_kv_k^*=\nabla q(y_k)^T\lambda^k,$$

we obtain

$$\begin{aligned}v^*&=\lim_{k\to\infty}v_k^*=\lim_{k\to\infty}\frac{\nabla q(y_k)^T\lambda^k-\nabla q(\bar y)^T\tilde\lambda^k}{t_k}\\&=\lim_{k\to\infty}\frac{(\nabla q(y_k)-\nabla q(\bar y))^T\tilde\lambda^k+\nabla q(y_k)^T(\lambda^k-\tilde\lambda^k)}{t_k}\\&=\nabla^2(\tilde\lambda^Tq)(\bar y)v+\nabla q(\bar y)^T\mu.\end{aligned}$$

If

$$\mu\in T_{\widehat N_{\mathbb R^l_-}(q(\bar y))}(\tilde\lambda)=\left\{\eta\in\mathbb R^l\,|\,\eta_i\begin{cases}\in\mathbb R & \text{if } i\in\bar{\mathcal I},\tilde\lambda_i>0\\ \geq 0 & \text{if } i\in\bar{\mathcal I},\tilde\lambda_i=0\\ =0 & \text{if } i\not\in\bar{\mathcal I}\end{cases}\right\},$$

the assertion is proved by virtue of Lemma 1. Otherwise the set $J:=\{i\in\tilde{\mathcal I}\,|\,\tilde\lambda_i=0,\mu_i<0\}$ is not empty, where we have taken into account that $\tilde{\mathcal I}\subset\bar{\mathcal I}$ and for all $i\not\in\tilde{\mathcal I}$ we have $\tilde\lambda_i^k=0\ \forall k$, $\tilde\lambda_i=0$ and $\mu_i=0$. Let us choose some index $\bar k$ such that $(\lambda_i^{\bar k}-\tilde\lambda_i^{\bar k})/t_{\bar k}\leq\mu_i/2\ \forall i\in J$ and set $\tilde\mu:=\mu+2(\tilde\lambda^{\bar k}-\tilde\lambda)/t_{\bar k}$. Then for all $i\not\in\tilde{\mathcal I}$ we have $\tilde\mu_i=0$, for all $i\in\tilde{\mathcal I}$ with $\tilde\lambda_i=0$ we have $\tilde\mu_i\geq\mu_i$ and for all $i\in J$ we have

$$\tilde\mu_i=\mu_i+2(\tilde\lambda_i^{\bar k}-\tilde\lambda_i)/t_{\bar k}\geq\mu_i+2(\tilde\lambda_i^{\bar k}-\lambda_i^{\bar k})/t_{\bar k}\geq 0$$

and therefore $\tilde\mu\in T_{\widehat N_{\mathbb R^l_-}(q(\bar y))}(\tilde\lambda)$. Observing that $\nabla q(\bar y)^T\tilde\mu=\nabla q(\bar y)^T\mu$ because of $\tilde\lambda,\tilde\lambda^{\bar k}\in\bar\Lambda$ and taking into account Lemma 1 completes the proof. □

Under the assumptions ensuring equality in (17) one has thus the formula

$$D\widehat{N}_\Gamma(\bar y,\bar y^*)(v)=\{\nabla^2(\lambda^T q)(\bar y)v\,|\,\lambda\in\bar\Lambda(v)\}+\widehat{N}_{\bar K}(v),\ \ v\in\mathbb{R}^m.$$

In this way we have recovered formula (3) under substantially weaker assumptions.

Let us turn our attention to the regular coderivative of $\widehat{N}_\Gamma$.

Proposition 4. *Assume that $\bar\Lambda\neq\emptyset$. Then*

$$\mathcal{T}(\bar y,\bar y^*)^\circ=\{(w^*,w)\,|\,w\in\bar K, {w^*}^T v+w^T\nabla^2(\lambda^T q)(\bar y)v\le 0,\ \ v\in\bar K,\lambda\in\bar\Lambda(v)\}.$$

Proof. By definition of the polar cone, we have $(w^*,w)\in\mathcal{T}(\bar y,\bar y^*)^\circ$ if and only if ${w^*}^T v+w^Tv^*\le 0$ $\forall(v,v^*)\in\mathcal{T}(\bar y,\bar y^*)$, i.e.,

$${w^*}^T v+w^T(\nabla^2(\lambda^T q)(\bar y)v+\xi^*)\le 0,\ \ v\in\bar K,\ \ \lambda\in\bar\Lambda(v),\ \ \xi^*\in\widehat{N}_{\bar K}(v).$$

Taking $v=0$, $\lambda\in\bar\Lambda(0)=\bar\Lambda\neq\emptyset$, we obtain $w^Tv^*\le 0$ $\forall v^*\in\widehat{N}_{\bar K}(0)=\bar K^\circ$ showing $w\in\bar K$. Since $\widehat{N}_{\bar K}(v)=\{\xi^*\in\bar K^\circ\,|\,{\xi^*}^T v=0\}\subset\bar K^\circ$ we have $w^T\xi^*\le 0$ $\forall\xi^*\in\widehat{N}_{\bar K}(v)$ and, because of $0\in\widehat{N}_{\bar K}(v)$, we see that $(w^*,w)\in\mathcal{T}(\bar y,\bar y^*)^\circ$ if and only if $w\in\bar K$ and

$${w^*}^T v+w^T\nabla^2(\lambda^T q)(\bar y)v\le 0,\ \ v\in\bar K,\lambda\in\bar\Lambda(v).$$

□

By using Theorem 1 we obtain that $\widehat{N}_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar y,\bar y^*)\subset\mathcal{T}(\bar y,\bar y^*)^\circ$ if M is metrically subregular at $(\bar y,0)$ and this inclusion holds with equality if in addition M is metrically regular in the vicinity of $\bar y$. However, the representation of $\mathcal{T}(\bar y,\bar y^*)^\circ$ by Proposition 4 is not very useful in practise because of the simultaneous appearance of v and $\lambda\in\bar\Lambda(v)$.

We now define for each $v\in\bar{\mathcal{N}}$ the sets

$$\bar{\mathcal{W}}(v):=\{w\in\bar K\,|\,w^T\nabla^2((\lambda^1-\lambda^2)^T q)(\bar y)v=0,\forall\lambda^1,\lambda^2\in\bar\Lambda(v)\},$$

$$\tilde\Lambda^{\mathcal{E}}(v):=\begin{cases}\bar\Lambda^{\mathcal{E}}(v) & \text{if } v\neq 0,\\ \mathrm{conv}\,(\bigcup\limits_{0\neq u\in\bar K}\bar\Lambda^{\mathcal{E}}(u)) & \text{if } v=0,\ \bar K\neq\{0\}\end{cases}$$

and for each $w\in\bar K$

$$\bar L(v;w):=\begin{cases}\{-\nabla^2(\lambda^T q)(\bar y)w\,|\,\lambda\in\tilde\Lambda^{\mathcal{E}}(v)\}+\bar K^\circ & \text{if } \bar K\neq\{0\}\\ \mathbb{R}^m & \text{if } \bar K=\{0\}.\end{cases}$$

Note that $\tilde\Lambda^{\mathcal{E}}(0)$ is a convex compact polyhedron, since there are only finitely many subsets of the finite set $\mathcal{E}$.

Proposition 5. *Assume that $\bar\Lambda(v)\neq\emptyset$ $\forall v\in\bar K$. Then*

$$\mathcal{T}(\bar y,\bar y^*)^\circ\subset\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v),w^*\in\bigcap_{v\in\bar{\mathcal{N}}}\bar L(v;w)\}\tag{27}$$

and equality holds, if either for any $0\neq v_1,v_2\in\bar K$ it holds $\bar\Lambda^{\mathcal{E}}(v_1)=\bar\Lambda^{\mathcal{E}}(v_2)$ or if $\bar I^+=\bar{\mathcal{I}}$.

Proof. If $\bar K=\{0\}$, by Proposition 4 we have $\mathcal{T}(\bar y,\bar y^*)^\circ=\mathbb{R}^m\times\{0\}$ and (27) holds with equality. Now assume that $\bar K\neq 0$, consider $(w^*,w)\in\mathcal{T}(\bar y,\bar y^*)^\circ$ and fix $v\in\bar{\mathcal{N}}\subset\bar K$. Then also $-v\in\bar{\mathcal{N}}$, $\bar\Lambda(v)=\bar\Lambda(-v)$ and by Proposition 4 we obtain ${w^*}^T(\pm v)+w^T\nabla^2(\lambda^Tq)(\bar y)(\pm v)\le 0,\ \lambda\in\bar\Lambda(v)$ and therefore ${w^*}^Tv+w^T\nabla^2(\lambda^Tq)(\bar y)v=0,\ \lambda\in\bar\Lambda(v)$, implying $w\in\bar{\mathcal{W}}(v)$. By Lemma 3 together with the assumption of the proposition there is some compact convex neighborhood U of v such that $\emptyset\neq\bar\Lambda(u)\subset\bar\Lambda(v)\ \forall u\in U\cap\bar K$ and therefore also $\emptyset\neq\bar\Lambda^{\mathcal{E}}(u)\subset\bar\Lambda^{\mathcal{E}}(v)$. We now claim that for every $u\in U\cap\bar K$ there is some $\lambda\in\tilde\Lambda^{\mathcal{E}}(v)$ such that

$$ {w^*}^Tu+w^T\nabla^2(\lambda^Tq)(\bar y)u\le 0. \tag{28}$$

Indeed, consider any $u\in U\cap\bar K$. If $v\neq 0$, then (28) holds for every $\lambda\in\bar\Lambda^{\mathcal{E}}(u)\subset\bar\Lambda^{\mathcal{E}}(v)=\tilde\Lambda^{\mathcal{E}}(v)$ by Proposition 4. If $v=u=0$, then (28) is trivially fulfilled for arbitrary $\lambda\in\tilde\Lambda^{\mathcal{E}}(v)\neq\emptyset$. Finally, if $v=0$ and $u\neq 0$, then again by Proposition 4, for every $\lambda\in\bar\Lambda^{\mathcal{E}}(u)\subset\tilde\Lambda^{\mathcal{E}}(v)$ the inequality (28) holds. Hence our claim is proved and since both $U\cap\bar K$ and $\tilde\Lambda^{\mathcal{E}}(v)$ are nonempty compact convex sets, we obtain

$$0\ge\max_{u\in U\cap\bar K}\min_{\lambda\in\tilde\Lambda^{\mathcal{E}}(v)}{w^*}^Tu+w^T\nabla^2(\lambda^Tq)(\bar y)u=\min_{\lambda\in\tilde\Lambda^{\mathcal{E}}(v)}\max_{u\in U\cap\bar K}{w^*}^Tu+w^T\nabla^2(\lambda^Tq)(\bar y)u.$$

We infer that there is some $\bar\lambda\in\tilde\Lambda^{\mathcal{E}}(v)$ such that $\max_{u\in U\cap\bar K}{w^*}^Tu+w^T\nabla^2(\bar\lambda^Tq)(\bar y)u\le 0$. Together with ${w^*}^Tv+w^T\nabla^2(\bar\lambda^Tq)(\bar y)v=0$ we conclude $(w^*+\nabla^2(\bar\lambda^Tq)(\bar y)w)^T(u-v)\le 0\ \forall u\in U\cap\bar K$ and thus $w^*+\nabla^2(\bar\lambda^Tq)(\bar y)w\in\widehat N_{\bar K}(v)$. Taking into account that for $v\in\bar{\mathcal{N}}$ we have $\widehat N_{\bar K}(v)=\bar K^\circ$, we obtain $w^*\in\bar L(v;w)$. Since $v\in\bar{\mathcal{N}}$ was arbitrarily fixed, inclusion (27) follows.

To prove equality we may also assume that $\bar K\neq\{0\}$ and consider first the case when $\hat\Lambda:=\bar\Lambda^{\mathcal{E}}(v_1)=\bar\Lambda^{\mathcal{E}}(v_2)$ for all $0\neq v_1,v_2\in\bar K$. Then we have $\tilde\Lambda^{\mathcal{E}}(v)=\hat\Lambda\ \forall v\in\bar{\mathcal{N}}$ and the set on the right hand side of the inclusion (27) amounts to

$$\Xi:=\bigcup_{\lambda\in\hat\Lambda}\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v),w^*\in-\nabla^2(\lambda^Tq)(\bar y)w+\bar K^\circ\}.$$

Now assume that there is some element $(w^*,w)\in\Xi\setminus\mathcal{T}(\bar y,\bar y^*)^\circ$, i.e., there is some element $(v,v^*)\in\mathcal{T}(\bar y,\bar y^*)$ with ${w^*}^Tv+w^Tv^*>0$. Then there are multipliers $\hat\lambda\in\hat\Lambda$, $\lambda\in\bar\Lambda(v)$ with $w^*\in-\nabla^2(\hat\lambda^Tq)(\bar y)w+\bar K^\circ$ and $v^*\in\nabla^2(\lambda^Tq)(\bar y)v+\widehat N_{\bar K}(v)$ and hence

$$0<{w^*}^Tv+w^Tv^*\le w^T\nabla^2((\lambda-\hat\lambda)^Tq)(\bar y)v.$$

Thus $v\neq 0$, $\bar\Lambda^{\mathcal{E}}(v)=\hat\Lambda$ and therefore $v^T\nabla^2((\lambda-\hat\lambda)^Tq)(\bar y)v=0$. For $\alpha>0$ with $0<2w^T\nabla^2((\lambda-\hat\lambda)^Tq)(\bar y)v+\alpha w^T\nabla^2((\lambda-\hat\lambda)^Tq)(\bar y)w$ it follows that $0<(v+\alpha w)^T\nabla^2((\lambda-\hat\lambda)^Tq)(\bar y)(v+\alpha w)$ and hence $v+\alpha w\neq 0$ and $\hat\lambda\notin\bar\Lambda(v+\alpha w)$ contradicting $\hat\lambda\in\hat\Lambda=\bar\Lambda^{\mathcal{E}}(v+\alpha w)\subset\bar\Lambda(v+\alpha w)$. Hence $\Xi\subset\mathcal{T}(\bar y,\bar y^*)^\circ$ and equality in (27) is established.

To prove equality in the second case, note that $\bar I^+=\bar{\mathcal{I}}$ implies $\bar K=\bar{\mathcal{N}}$ by Lemma 2. Consider now (w^*,w) belonging to the set on the right hand side of (27) and an arbitrary element $(v,v^*)\in\mathcal{T}(\bar y,\bar y^*)$, i.e., $v\in\bar{\mathcal{N}}$ and $v^*\in\nabla^2(\lambda^Tq(\bar y))w+\widehat N_{\bar K}(v)$ for some $\lambda\in\bar\Lambda(v)$. Because of $v\in\bar{\mathcal{N}}$ we have $\widehat N_{\bar K}(v)=\bar K^\circ$ and, by using the representation $w^*\in-\nabla^2(\bar\lambda^Tq)(\bar y)w+\bar K^\circ$ with $\bar\lambda\in\tilde\Lambda^{\mathcal{E}}(v)$, we obtain

$${w^*}^Tv+w^Tv^*\le w^T\nabla^2((\lambda-\bar\lambda)^Tq)(\bar y)v=0$$

due to $w\in\bar{\mathcal{W}}(v)$. Hence $(w^*,w)\in\mathcal{T}(\bar y,\bar y^*)^\circ$ and equality in (27) follows. □

THEOREM 2. *Assume that M is metrically subregular at $(\bar y,0)$. Then*

$$\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*)\subset\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v),w^*\in\bigcap_{v\in\bar{\mathcal{N}}}\bar L(v;w)\}. \tag{29}$$

Equality holds if, in addition, M is metrically regular in the vicinity of $\bar y$ and either for any $0\neq v_1,v_2\in\bar K$ it holds $\bar\Lambda^{\mathcal{E}}(v_1)=\bar\Lambda^{\mathcal{E}}(v_2)$ or $\bar I^+=\bar{\mathcal{I}}$.

Proof. If M is metrically subregular, then by [6, Theorem 6.1 (2.b)] one has that for every $v \in T_\Gamma^{\rm lin}(\bar y)$ we have $v^T\nabla^2(\lambda^T q)(\bar y)v \le 0$ for all λ belonging to the recession cone $\mathcal{R}$ of $\bar\Lambda$. Hence, $\bar\Lambda(v) \neq \emptyset\ \forall v \in T_\Gamma^{\rm lin}(\bar y) \supset \bar K$ and the assertion of the theorem follows from Theorem 1 and Proposition 5. □

It follows from the definition of the regular coderivative that under metric subregularity of M at $(\bar y, 0)$ one has

$$\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(w)\begin{cases}\subset \bigcap_{v\in\bar{\mathcal{N}}}\bar L(v;-w) & \text{if } w\in\bigcap_{v\in\bar{\mathcal{N}}}-\bar{\mathcal{W}}(v),\\ =\emptyset & \text{else.}\end{cases}\tag{30}$$

Equality holds in this formula provided M is metrically regular in the vicinity of $\bar y$ and either for any $0 \neq v_1, v_2 \in \bar K$ it holds $\bar\Lambda^{\mathcal{E}}(v_1) = \bar\Lambda^{\mathcal{E}}(v_2)$ or $\bar I^+ = \bar{\mathcal{I}}$.

Example 2. The normal cone mapping of the set

$$\Gamma := \left\{ y\in\mathbb{R}^3 \,\middle|\, \begin{array}{r} y_1+\frac12 y_2^2 \le 0\\ y_1+y_2^2-y_2y_3 \le 0\end{array}\right\}$$

is given by

$$\widehat N_\Gamma(y)=\begin{cases}\{(0,0,0)\} & \text{if } y_1<-\max\{\frac12 y_2^2, y_2^2-y_2y_3\},\\ \{(\lambda_1,\lambda_1y_2,0)\,|\,\lambda_1\ge 0\} & \text{if } y_1=-\frac12 y_2^2<-y_2^2+y_2y_3,\\ \{(\lambda_2,\lambda_2(2y_2-y_3),-\lambda_2y_2)\,|\,\lambda_2\ge0\} & \text{if } y_1=-y_2^2+y_2y_3<-\frac12 y_2^2,\\ \{(\lambda_1+\lambda_2,(2\lambda_1+3\lambda_2)y_3,-2\lambda_2y_3)\,|\,\lambda_1,\lambda_2\ge0\} & \text{if } y_1=-y_2^2+y_2y_3=-\frac12y_2^2,\ y_2\neq0,\\ \{(\lambda_1+\lambda_2,-\lambda_2y_3,0)\,|\,\lambda_1,\lambda_2\ge0\} & \text{if } y_1=y_2=0,\\ \emptyset & \text{else.}\end{cases}$$

Note that MFCQ is fulfilled at every point $y \in \Gamma$. The tangent cone and the Fréchet normal cone to gph $\widehat N_\Gamma$ at $\bar y = (0,0,0)$, $\bar y^* = (1,0,0)$ are given by

$$\begin{aligned}T_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) = {}&\{((0,v_2,v_3),(v_1^*,v_2,0))\,|\,(0\le v_2\le 2v_3)\vee(0\ge v_2\ge 2v_3)\}\\ &\cup\{((0,v_2,v_3),(v_1^*,2v_2-v_3,-v_2))\,|\,(v_2\le0\wedge 2v_3\ge v_2)\vee(v_2\ge0\wedge 2v_3\le v_2)\}\\ &\cup\{((0,2v_3,v_3),(v_1^*,(2+\lambda_2)v_3,-2\lambda_2v_3))\,|\,0\le\lambda_2\le1\}\\ &\cup\{((0,0,v_3),(v_1^*,-\lambda_2v_3,0))\,|\,0\le\lambda_2\le1\}\}\end{aligned}$$

and

$$\begin{aligned}\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) = {}&\{((w_1^*,-w_2,0),(0,w_2,w_3))\}\\ &\cap\{((w_1^*,w_3-2w_2,w_2),(0,w_2,w_3))\}\\ &\cap\{((w_1^*,w_2^*,w_3^*),(0,2w_3,w_3))\,|\,2w_2^*+w_3^*+4w_3=0\}\\ &\cap\{((w_1^*,w_2^*,0),(0,0,w_3))\}\\ ={}&\{((w_1^*,0,0),(0,0,0))\}.\end{aligned}$$

Further we have $\bar K = \bar{\mathcal{N}} = \{0\}\times\mathbb{R}\times\mathbb{R}$, $\bar\Lambda = \{(\lambda_1,\lambda_2)\,|\,\lambda_1\ge0,\lambda_2\ge0,\lambda_1+\lambda_2=1\}$ and for $v \in \bar K$ we obtain

$$\bar\Lambda(v)=\begin{cases}\{(1,0)\} & \text{if } v_2^2>2v_2v_3,\\ \bar\Lambda & \text{if } v_2=0 \text{ or } v_2=2v_3,\\ \{(0,1)\} & \text{if } v_2^2<2v_2v_3.\end{cases}$$

For $\tilde v := (0,2,1) \in \bar{\mathcal{N}}$, $\hat v := (0,0,1) \in \bar{\mathcal{N}}$ we have $\bar\Lambda(\tilde v) = \bar\Lambda(\hat v) = \bar\Lambda$ and for every $\lambda^1, \lambda^2 \in \bar\Lambda$ we obtain

$$\begin{aligned}w^T\nabla^2((\lambda^1-\lambda^2)^Tq)(\bar y)\tilde v &= w_2\tilde v_2(\lambda_1^1-\lambda_1^2+2(\lambda_2^1-\lambda_2^2))-(w_2\tilde v_3+w_3\tilde v_2)(\lambda_2^1-\lambda_2^2)\\ &=(\lambda_2^1-\lambda_2^2)(w_2\tilde v_2-w_2\tilde v_3-w_3\tilde v_2)=(\lambda_2^1-\lambda_2^2)(w_2-2w_3)\end{aligned}$$

and, similarly,

$$w^T\nabla^2((\lambda^1-\lambda^2)^Tq)(\bar y)\hat v = -(\lambda^1_2-\lambda^2_2)w_2.$$

Thus $\bar{\mathcal{W}}(\tilde v)=\{(0,w_2,w_3)\,|\,w_2=2w_3\}$, $\bar{\mathcal{W}}(\hat v)=\{(0,0,w_3)\}$ and $\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v)=\{(0,0,0)\}$. Since for every $v\in\bar{\mathcal{N}}$ we have

$$\bar L(v;0)=\bar K^\circ=\mathbb{R}\times\{0\}\times\{0\},$$

we obtain

$$\{(w^*,w)\,|\,w\in\bigcap_{v\in\bar{\mathcal{N}}}\bar{\mathcal{W}}(v), w^*\in\bigcap_{v\in\bar{\mathcal{N}}}\bar L(v;w)\} = \bar K^\circ\times\{(0,0,0)\}=\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*)$$

or $\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(0)=\bar K^\circ$, $\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(w)=\emptyset$, $w\neq 0$ and equality in (29) holds. △

Example 3. The normal cone mapping of the set

$$\Gamma:=\left\{y\in\mathbb{R}^3\;\middle|\;\begin{array}{r}y_1+\frac12 y_2^2\leq 0\\ y_1+y_2^2-y_2y_3\leq 0\\ y_2\leq 0\end{array}\right\}$$

is given by

$$\widehat N_\Gamma(y)=\begin{cases}\{(0,\lambda_3,0)\,|\,\lambda_3\geq 0\} & \text{if } y_1<0,\ y_2=0,\\ \{(\lambda_1+\lambda_2,-\lambda_2y_3+\lambda_3,0)\,|\,\lambda_1,\lambda_2,\lambda_3\geq 0\} & \text{if } y_1=y_2=0,\\ \{(\lambda_1,\lambda_1y_2,0)\,|\,\lambda_1\geq 0\} & \text{if } 2y_3<y_2<0,\ y_1=-\frac12y_2^2,\\ \{(\lambda_2,\lambda_2(2y_2-y_3),-\lambda_2y_2)\,|\,\lambda_2\geq 0\} & \text{if } 2y_3>y_2,\ y_2<0,\ y_1=-y_2^2+y_2y_3,\\ \{(\lambda_1+\lambda_2,(2\lambda_1+3\lambda_2)y_3,-2\lambda_2y_3)\,|\,\lambda_1,\lambda_2\geq 0\} & \text{if } 2y_3=y_2<0,\ y_1=-\frac12y_2^2,\\ \{(0,0,0)\} & \text{if } y_2<0,\ y_1<-\max\{\frac12y_2^2,y_2^2-y_2y_3\},\\ \emptyset & \text{else.}\end{cases}$$

The tangent cone and the Fréchet normal cone to gph $\widehat N_\Gamma$ at $\bar y=(0,0,0)$, $\bar y^*=(1,0,0)$ are given by

$$\begin{aligned}T_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) = {}&\{((0,0,v_3),(v_1^*,v_2^*,0))\,|\,v_2^*\geq\min\{-v_3,0\}\}\\ &\cup\{((0,v_2,v_3),(v_1^*,v_2,0))\,|\,2v_3\leq v_2\leq 0\}\\ &\cup\{((0,v_2,v_3),(v_1^*,2v_2-v_3,-v_2))\,|\,2v_3\geq v_2, v_2\leq 0\}\\ &\cup\{((0,2v_3,v_3),(v_1^*,(2+\lambda_2)v_3,-2\lambda_2v_3))\,|\,v_3\leq 0, 0\leq\lambda_2\leq 1\}\end{aligned}$$

and

$$\begin{aligned}\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*) = {}&\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2\leq w_3^*\leq 0\}\\ &\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2^*+\frac12 w_3^*+w_2\geq 0, w_3^*\geq 0\}\\ &\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,w_2-w_3^*\geq 0, w_2^*+\frac32 w_2+\frac12 w_3^*-w_3\geq 0\}\\ &\cap\{((w_1^*,w_2^*,w_3^*),(0,w_2,w_3))\,|\,2w_2^*+w_3^*+2w_2+\min\{w_2-2w_3,0\}\geq 0\}\\ ={}&\{((w_1^*,w_2^*,0),(0,0,w_3))\,|\,w_2^*\geq\max\{w_3,0\}\}.\end{aligned}$$

Then $\bar K=\{0\}\times\mathbb{R}_-\times\mathbb{R}$, $\bar{\mathcal{N}}=\{0\}\times\{0\}\times\mathbb{R}$, $\bar\Lambda=\{(\lambda_1,\lambda_2,0)\,|\,\lambda_1\geq 0,\lambda_2\geq 0,\lambda_1+\lambda_2=1\}$ and for $v\in\bar K$ we have

$$\bar\Lambda(v)=\begin{cases}\{(1,0,0)\} & \text{if } 0>v_2>2v_3,\\ \bar\Lambda & \text{if } v_2=0 \text{ or } 0>v_2=2v_3,\\ \{(0,1,0)\} & \text{if } v_2<2v_3 \text{ and } v_2<0.\end{cases}$$

For every $0 \neq v = (0,0,v_3) \in \bar{\mathcal{N}}$ we obtain

$$\begin{aligned}\bar{\mathcal{W}}(v) &= \{w \in \bar{K} \mid w^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v = w_2(\lambda_2^2 - \lambda_2^1)v_3 = 0 \forall \lambda^1, \lambda^2 \in \bar{\Lambda}(v) = \bar{\Lambda}\} \\ &= \{0\} \times \{0\} \times \mathbb{R},\end{aligned}$$

$$\tilde{\Lambda}^{\mathcal{E}}(v) = \bar{\Lambda}^{\mathcal{E}}(v) = \bar{\Lambda}$$

and for $w \in \bar{\mathcal{W}}(v)$

$$\bar{L}(v;w) = \{(0, \lambda_2 w_3, 0) \mid 0 \leq \lambda_2 \leq 1\} + \bar{K}^\circ = \mathbb{R} \times [\min\{w_3, 0\}, \infty) \times \{0\}.$$

Hence

$$\begin{aligned}\{(w^*, w) \mid w \in \bigcap_{v \in \bar{\mathcal{N}}} \bar{\mathcal{W}}(v), w^* \in \bigcap_{v \in \bar{\mathcal{N}}} \bar{L}(v;w)\} &= \{((w_1^*, w_2^*, 0), (0,0,w_3)) \mid w_2^* \geq \min\{w_3, 0\}\} \\ &\neq \{((w_1^*, w_2^*, 0), (0,0,w_3)) \mid w_2^* \geq \max\{w_3, 0\}\} \\ &= \widehat{N}_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)\end{aligned}$$

yielding

$$\widehat{D}^* \widehat{N}_\Gamma(\bar{y}, \bar{y}^*)(w) \begin{cases} \subset \{(w_1^*, w_2^*, 0) \mid w_2^* \geq \min\{-w_3, 0\}\} & \text{if } w_1 = w_2 = 0, \\ = \emptyset & \text{else} \end{cases}$$

and equality does not hold. △

In case when the set $\bar{\Lambda}(v)$ remains constant for all $0 \neq v \in \bar{K}$, the formulas for the contingent cone and the regular normal cone of $\operatorname{gph} \widehat{N}_\Gamma$ can be simplified considerably.

Lemma 4. *Assume that $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)\ \forall\ 0 \neq v_1, v_2 \in \bar{K}$. Then for all $v \in \bar{K}$ and all $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$ we have*

$$\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \in \operatorname{span}\{\nabla q_i(\bar{y})^T \mid i \in \bar{I}^+\}.$$

Proof. By contraposition. Assume on the contrary that there are $v \in \bar{K}$ and $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$ such that $\nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \notin \operatorname{span}\{\nabla q_i(\bar{y})^T \mid i \in \bar{I}^+\}$. This is equivalent with the existence of some w satisfying $\nabla q_i(\bar{y})w = 0$, $i \in \bar{I}^+$, and $w^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \neq 0$. By Lemma 2 there is some $\tilde{v}$ with

$$\nabla q_i(\bar{y})\tilde{v} \begin{cases} = 0 & i \in \bar{I}^+, \\ < 0 & i \in \bar{I}^0 \end{cases}$$

and hence $w + \alpha\tilde{v} \in \bar{K}$ for all α sufficiently large. Moreover, we can choose α such that, in addition, $\tilde{w}^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v \neq 0$, where $\tilde{w} = w + \alpha\tilde{v}$. We can assume without loss of generality that $\tilde{w}^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v < 0$, because otherwise we can interchange λ^1 and λ^2. Then we can choose $\beta > 0$ with $v + \beta\tilde{w} \neq 0$ and $\tilde{w}^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(2v + \beta\tilde{w}) < 0$. It follows $v + \beta\tilde{w} \in \bar{K}$ and, together with $v^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})v = 0$ because of $\lambda^1, \lambda^2 \in \bar{\Lambda}(v)$,

$$(v + \beta\tilde{w})^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(v + \beta\tilde{w}) = \beta\tilde{w}^T \nabla^2((\lambda^1 - \lambda^2)^T q)(\bar{y})(2v + \beta\tilde{w}) < 0$$

showing $\lambda^1 \notin \bar{\Lambda}(v + \beta\tilde{w})$. Therefore $\bar{\Lambda}(v + \beta\tilde{w}) \neq \bar{\Lambda}(v)$, a contradiction, since we also have $v \neq 0$. □

Note that the condition $\bar{\Lambda}(v_1) = \bar{\Lambda}(v_2)\ \forall\ 0 \neq v_1, v_2 \in \bar{K}$ implies in particular the corresponding property for the sets $\bar{\Lambda}^{\mathcal{E}}$ used in Theorem 2.

Theorem 3. *Assume that M is metrically subregular at $(\bar y,0)$ and metrically regular in the vicinity of $\bar y$. Further assume that $\bar\Lambda(v_1)=\bar\Lambda(v_2)\ \forall\, 0\neq v_1,v_2\in\bar K$ and let $\bar\lambda$ be an arbitrary multiplier from $\bar\Lambda(v)$ for some $0\neq v\in\bar K$, if $\bar K\neq\{0\}$ and $\bar\lambda\in\bar\Lambda$ otherwise. Then*

$$T_{\operatorname{gph}\widehat N_\Gamma}(\bar y,\bar y^*)=\{(v,v^*)\,|\,v^*\in\nabla^2(\bar\lambda^Tq)(\bar y)v+\widehat N_{\bar K}(v)\}\tag{31}$$

and

$$\widehat N_{\operatorname{gph}\widehat N_\Gamma}(\bar y,\bar y^*)=\{(w^*,w)\,|\,w\in\bar K,w^*\in-\nabla^2(\bar\lambda^Tq)(\bar y)w+\bar K^\circ\}.\tag{32}$$

Consequently,

$$D\widehat N_\Gamma(\bar y,\bar y^*)(v)=\nabla^2(\bar\lambda^Tq)(\bar y)v+\widehat N_{\bar K}(v),\ \ v\in\mathbb R^m,$$
$$\widehat D^*\widehat N_\Gamma(\bar y,\bar y^*)(w)=\begin{cases}\nabla^2(\bar\lambda^Tq)(\bar y)w+\bar K^\circ & \text{if } -w\in\bar K,\\ \emptyset & \text{else.}\end{cases}$$

Proof. By Theorem 1 it is clear that the set on the right hand side of (31) is contained in $T_{\operatorname{gph}\widehat N_\Gamma}(\bar y,\bar y^*)$. To show the reverse inclusion, fix any $(v,v^*)\in T_{\operatorname{gph}\widehat N_\Gamma}(\bar y,\bar y^*)$. Then there is some $\lambda\in\bar\Lambda(v)$ with $v^*\in\nabla^2(\lambda^Tq)(\bar y)v+\widehat N_{\bar K}(v)$ and by Lemma 4 together with the identity $\operatorname{span}\{\nabla q_i(\bar y)^T\,|\,i\in\bar I^+\}+\widehat N_{\bar K}(v)=\widehat N_{\bar K}(v)$ we obtain $v^*\in\nabla^2(\bar\lambda^Tq)(\bar y)v+\operatorname{span}\{\nabla q_i(\bar y)^T\,|\,i\in\bar I^+\}+\widehat N_{\bar K}(v)=\nabla^2(\bar\lambda^Tq)(\bar y)v+\widehat N_{\bar K}(v)$, showing the desired inclusion.

To show (32), note that by our assumptions equality holds in (29). Further, by Lemma 4 and Lemma 2 we obtain $\bar{\mathcal W}(v)=\bar K\ \forall v\in\bar{\mathcal N}$ and, by using the same arguments as above, $L(v;w)=-\nabla^2(\bar\lambda^Tq)(\bar y)w+\bar K^\circ\ \forall w\in\bar K$. Equality (32) follows now from Theorem 2. □

The behavior of the mapping $\bar\Lambda$ required in the above theorem is automatically fulfilled whenever MFCQ and CRCQ [1] hold at $\bar y$, see [9, Corollary 3.2, Remark 3.1]. The following example shows, however, that the requirements of the theorem can very well be satisfied even without CRCQ.

Example 4. Let $\Gamma\subset\mathbb R^2$ be given by

$$q(y)=\begin{pmatrix}-y_1^2+y_2\\-y_1^2-y_2\\y_1\end{pmatrix}.$$

Put $\bar y=0,\bar y^*=0$ and let us compute $T_{\operatorname{gph}\widehat N_\Gamma}(0,0)$. It follows that

$$\bar K=T_\Gamma(0)=\{v|v_1\le 0,\,v_2=0\},$$

$$\bar\Lambda=\{\lambda\in\mathbb R^3_+|\lambda_1=\lambda_2,\lambda_3=0\}$$

and

$$\bar\Lambda(v)=\begin{cases}\{\lambda\in\bar\Lambda|\lambda_1+\lambda_2=0\}=\{0\} & \text{if } 0\neq v\in\bar K,\\ \bar\Lambda & \text{if } v=0.\end{cases}$$

It is easy to show that the second-order sufficient conditions for metric subregularity SOSCMS from Proposition 2 are fulfilled and thus M is metrically subregular and even metrically regular in the vicinity of 0 by Proposition 3. It follows that we can compute $T_{\operatorname{gph}\widehat N_\Gamma}(0,0)$ according to Theorem 3 and obtain

$$T_{\operatorname{gph}\widehat N_\Gamma}(0,0)=\{(v,v^*)|v^*\in\widehat N_{\bar K}(v)\},$$

and, since $\bar K^\circ=\widehat N_\Gamma(0)=\mathbb R_+\times\mathbb R$, one has

$$T_{\operatorname{gph}\widehat N_\Gamma}(0,0)=(\{v\,|\,v_1\le 0,\,v_2=0\}\times\{v^*\,|\,v_1^*=0\})\cup(\{0,0\}\times\mathbb R_+\times\mathbb R).$$

[1] One says that Γ fulfills CRCQ at $\bar y$ provided there exists a neighborhood $\mathcal M$ of $\bar y$ such that for any subsets I of $\bar{\mathcal I}$, the family of gradients $\{\nabla q_i(y)|i\in I\}$ has the same rank for all $y\in\mathcal M$.

Consequently,

$$\widehat{N}_{\mathrm{gph}\,\widehat{N}_\Gamma}(0,0) = (T_{\mathrm{gph}\,\widehat{N}_\Gamma}(0,0))^\circ = \{(v_1^*, v_2^*)\,|\,v_1^* \geq 0\} \times \{(v_1, 0)\,|\,v_1 \leq 0\} = (\bar{K})^\circ \times \bar{K}$$

verifying (32) with $\bar{\lambda} = 0$. △

5. Regular coderivative of the solution map

In the preceding section we have computed (an upper estimate of) the regular coderivative of $\widehat{N}_\Gamma$. To compute the regular coderivative of S we need, in addition, a chain rule for regular normal cones without any convexity assumptions. Such a chain rule is provided in the next statement which is important for its own sake and can be used also in completely different situations.

THEOREM 4. *Let*

$$\Omega := \{x \in \mathbb{R}^n \,|\, G(x) \in D\}$$

for a closed set $D \subset \mathbb{R}^m$ and a mapping $G : \mathbb{R}^n \to \mathbb{R}^m$ continuously differentiable near $\bar{x} \in \Omega$. If the multifunction $x \rightrightarrows G(x) - D$ is metrically subregular at $(\bar{x}, 0)$ and there exists a subspace $L \subset R^m$ such that

$$T_D(G(\bar{x})) + L \subset T_D(G(\bar{x})) \tag{33}$$

and

$$\nabla G(\bar{x})\mathbb{R}^n + L = \mathbb{R}^m, \tag{34}$$

then

$$\widehat{N}_\Omega(\bar{x}) = \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x}).$$

Proof. The inclusion $\widehat{N}_\Omega(\bar{x}) \supset \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x})$ follows immediately from [26, Theorem 6.14]. To show the reverse inclusion, let $x^* \in \widehat{N}_\Omega(\bar{x})$ and consider $h \in \mathcal{S} := \{h \in \mathbb{R}^n \,|\, \nabla G(\bar{x})h \in \mathrm{conv}\, T_D(G(\bar{x}))\}$. Then $\nabla G(\bar{x})h$ can be written as convex combination of elements of $T_D(G(\bar{x}))$:

$$\nabla G(\bar{x})h = \sum_{i=1}^N \alpha_i t_i,\ t_i \in T_D(G(\bar{x})),\ \alpha_i \geq 0,\ i = 1, \ldots, N;\ \sum_{i=1}^N \alpha_i = 1.$$

By the assumptions of the theorem each of the tangents t_i can be represented as $t_i = \nabla G(\bar{x})h_i + l_i$, where $h_i \in \mathbb{R}^n$ and $l_i \in L$. Since L is a subspace we also have $-l_i \in L$ showing $\nabla G(\bar{x})h_i = t_i - l_i \in T_D(G(\bar{x})) + L \subset T_D(G(\bar{x}))$. Because $G(\cdot) - D$ is assumed to be metrically subregular at $(\bar{x}, 0)$, it follows from [10, Proposition 1] that

$$T_\Omega(\bar{x}) = \{u \,|\, \nabla G(\bar{x})u \in T_D(G(\bar{x}))\}. \tag{35}$$

Hence we conclude $h_i \in T_\Omega(\bar{x})$ and $\langle x^*, h_i\rangle \leq 0$. Further we have

$$\nabla G(\bar{x})(h - \sum_{i=1}^N \alpha_i h_i) = \sum_{i=1}^N \alpha_i (t_i - \nabla G(\bar{x})h_i) = \sum_{i=1}^N \alpha_i l_i \in L \subset T_D(G(\bar{x})),$$

and, again by (35), we obtain $h - \sum_{i=1}^N \alpha_i h_i \in T_\Omega(\bar{x})$. Thus $\langle x^*, h - \sum_{i=1}^N \alpha_i h_i\rangle \leq 0$ implying

$$\langle x^*, h\rangle \leq \langle x^*, \sum_{i=1}^N \alpha_i h_i\rangle = \sum_{i=1}^N \alpha_i \langle x^*, h_i\rangle \leq 0.$$

¿From this it follows that $x^* \in \mathcal{S}^\circ$ and, by [25, Corollary 16.3.2], we can conclude $\widehat{N}_\Omega(\bar{x}) \subset \mathcal{S}^\circ = \nabla G(\bar{x})^T(\mathrm{conv}\, T_D(G(\bar{x})))^\circ = \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x}))$, provided there exists some u with $\nabla G(\bar{x})u \in \mathrm{ri}\,\mathrm{conv}\, T_D(G(\bar{x}))$. To show the existence of such an element u, choose any $t \in \mathrm{ri}\,\mathrm{conv}\, T_D(G(\bar{x}))$ and select u and $l \in L$ such that $\nabla G(\bar{x})u + l = t$. Since we also have $\mathrm{conv}\, T_D(\bar{x}) + L \subset \mathrm{conv}\, T_D(\bar{x})$, we obtain from [25, Theorem 6.1] that $\nabla G(\bar{x})u = \frac{1}{2}(t - 2l) + \frac{1}{2}t \in \mathrm{ri}\,\mathrm{conv}\, T_D(G(\bar{x}))$. Hence $\widehat{N}_\Omega(\bar{x}) \subset \nabla G(\bar{x})^T \widehat{N}_D(G(\bar{x})$ holds and this completes the proof. □

If D is convex, then the conditions (33), (34) amount to

$$\nabla G(\bar{x})\mathbb{R}^n + \operatorname{lin} T_D(G(\bar{x})) = \mathbb{R}^m, \tag{36}$$

where lin stands for the lineality space. This condition is known as nondegeneracy [2, Definition 4.70]. In the nonconvex case, however, the formulation (36) cannot be used because of (33).

Consider now the mapping $\tilde{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$\tilde{S}(x) = \begin{cases} \{y \in \mathbb{R}^m \,|\, 0 \in F(x,y) + \widehat{N}_\Gamma(y)\} & \text{if } x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable and $C \subset \mathbb{R}^n$ is a closed set.

Associated with $\tilde{S}$ is the perturbation mapping $\Psi : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ given by

$$\Psi(x,y) := G(x,y) - D,\ G(x,y) := \begin{pmatrix} (y, -F(x,y)) \\ x \end{pmatrix},\ D := \operatorname{gph} \widehat{N}_\Gamma \times C, \tag{37}$$

so that $\operatorname{gph} \tilde{S} = \{(x,y) \,|\, 0 \in \Psi(x,y)\}$.

Theorem 5. *Consider $(\bar{x},\bar{y}) \in \operatorname{gph} \tilde{S}$. Assume that M is metrically subregular at $(\bar{y},0)$ and metrically regular in the vicinity of $\bar{y}$. Further assume that the set-valued mapping Ψ given by (37) is metrically subregular at $((\bar{x},\bar{y})),(0,0,0))$, and suppose that there exists a subspace $P \subset T_C(\bar{x})$ with $T_C(\bar{x}) + P \subset T_C(\bar{x})$ and*

$$\nabla_x F(\bar{x},\bar{y})P + \operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} = \mathbb{R}^m. \tag{38}$$

Then one has, with $\bar{y}^* := -F(\bar{x},\bar{y})$, that

$$\begin{aligned} \widehat{N}_{\operatorname{gph} \tilde{S}}(\bar{x},\bar{y}) &= \nabla G(\bar{x},\bar{y})^T \widehat{N}_D(G(\bar{x},\bar{y})) \\ &= \left\{ \begin{pmatrix} -\nabla_x F(\bar{x},\bar{y})^T w + c^* \\ -\nabla_y F(\bar{x},\bar{y})^T w + w^* \end{pmatrix} \,|\, c^* \in \widehat{N}_C(\bar{x}),\ (w^*,w) \in \widehat{N}_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y},\bar{y}^*) \right\}. \end{aligned} \tag{39}$$

Proof. Set $\Omega := \{(x,y) \,|\, G(x,y) \in D\} = \operatorname{gph} \tilde{S}$. We will invoke Theorem 4 to prove that $\widehat{N}_\Omega(\bar{x},\bar{y}) = \nabla G(\bar{x},\bar{y})^T \widehat{N}_D(G(\bar{x},\bar{y}))$. Using Lemmas 1, 2 we obtain $\tilde{L} := \operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} \subset \widehat{N}_{\bar{K}}(v)\ \forall v \in \bar{K}$. Hence $T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y},\bar{y}^*) + \{0_m\} \times \tilde{L} \subset T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$ by virtue of Theorem 1. Defining the subspace L by $L := (\{0_m\} \times \tilde{L}) \times P$ and taking into account $T_D(G(\bar{x},\bar{y})) = T_{\operatorname{gph} \widehat{N}_\Gamma}(\bar{y},\bar{y}^*) \times T_C(\bar{x})$, we obtain $T_D(G(\bar{x},\bar{y})) + L \subset T_D(G(\bar{x},\bar{y}))$. Next we shall prove that

$$\nabla G(\bar{x},\bar{y})(\mathbb{R}^n \times \mathbb{R}^m) + L = (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n \tag{40}$$

holds true. By the assumptions for any $((v,v^*),u) \in (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ we can choose $p \in P$ and $\tilde{l} \in \tilde{L}$ with $\nabla_x F(\bar{x},\bar{y})p + \tilde{l} = v^* + \nabla_x F(\bar{x},\bar{y})u + \nabla_y F(\bar{x},\bar{y})v$. Then $l = ((0_m,\tilde{l}),p) \in L$ and

$$\nabla G(\bar{x},\bar{y}) \begin{pmatrix} u-p \\ v \end{pmatrix} + l = \begin{pmatrix} (v, -\nabla_x F(\bar{x},\bar{y})(u-p) - \nabla_y F(\bar{x},\bar{y})v + \tilde{l}) \\ u-p+p \end{pmatrix} = \begin{pmatrix} (v,v^*) \\ u \end{pmatrix}.$$

This verifies (40) and we can apply Theorem 4 to obtain the result. □

Corollary 1. *In the setting of Theorem 5 for any $v^* \in \mathbb{R}^m$ one has*

$$\widehat{D}^* \tilde{S}(\bar{x},\bar{y})(v^*) = \{\nabla_x F(\bar{x},\bar{y})^T w + \widehat{N}_C(\bar{x}) \,|\, 0 \in v^* + \nabla_y F(\bar{x},\bar{y})^T w + \widehat{D}^* \widehat{N}_\Gamma(\bar{y},\bar{y}^*)(w)\}. \tag{41}$$

There are various possibilities for verifying the metric subregularity of Ψ at $((\bar x,\bar y),(0,0,0))$. Sometimes one can use even the following simple sufficient condition for metric regularity stated in Proposition 6 below. We think that this criterion is far away from being necessary, but it is easy to verify.

PROPOSITION 6. *Let Ψ be given by (37), and let $0\in\Psi(\bar x,\bar y)$. Assume that M is metrically subregular at $(\bar y,0)$ and metrically regular in the vicinity of $\bar y$. Further assume that for every $\lambda\in\mathcal{E}$, the set of extreme points of $\Lambda(\bar y,-F(\bar x,\bar y))$, one has*

$$\nabla_x F(\bar x,\bar y)^T v\in N_C(\bar x),\ \nabla q_i(\bar x)v=0, i\in I^+(\lambda)\ \Rightarrow v=0.$$

Then Ψ is metrically regular near $((\bar x,\bar y),(0,0,0))$.

Proof. By contraposition. Assuming now on the contrary that Ψ is not metrically regular near $((\bar x,\bar y),(0,0,0))$, by [26, Example 9.44] there is some nonzero $\xi=((v^*,v),x^*)\in N_D(G(\bar x,\bar y))$ such that

$$\nabla G(\bar x,\bar y)^T\xi=\begin{pmatrix}-\nabla_x F(\bar x,\bar y)^T v+x^*\\ -\nabla_y F(\bar x,\bar y)^T v+v^*\end{pmatrix}=0, \tag{42}$$

where G and D are given by (37). It can be easily seen from (42) that $v\neq 0$ since otherwise $\xi=((v^*,v),x^*)$ would be 0. By [26, Proposition 6.41] we have $N_D(G(\bar x,\bar y))=N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*)\times N_C(\bar x)$ with $\bar y^*:=-F(\bar x,\bar y)$, implying $\nabla_x F(\bar x,\bar y)^T v=x^*\in N_C(\bar x)$ and we will now show the existence of some $\lambda\in\mathcal{E}$ satisfying $\nabla q_i(\bar y)v=0$, $i\in I^+(\lambda)$, contradicting the assumption of the proposition.

Since $(v^*,v)\in N_{\mathrm{gph}\,\widehat N_\Gamma}(\bar y,\bar y^*)$, there are sequences $(y_k,y_k^*)\overset{\mathrm{gph}\,\widehat N_\Gamma}{\longrightarrow}(\bar y,\bar y^*)$ and $(v_k^*,v_k)\to(v^*,v)$ with $(v_k^*,v_k)\in\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(y_k,y_k^*)$. For every $(y,y^*)\in\mathrm{gph}\,\widehat N_\Gamma$ near $(\bar y,\bar y^*)$ we have that M is at least metrically subregular at $(y,0)$. Thus, by Theorem 1 the cone $\{0_m\}\times K(y,y^*)^\circ$ is a subset of $T_{\mathrm{gph}\,\widehat N_\Gamma}(y,y^*)$ and this implies

$$(v_k^*,v_k)\in\widehat N_{\mathrm{gph}\,\widehat N_\Gamma}(y_k,y_k^*)=\Big(T_{\mathrm{gph}\,\widehat N_\Gamma}(y_k,y_k^*)\Big)^\circ\subset(\{0_m\}\times K(y_k,y_k^*)^\circ)^\circ=\mathbb{R}^m\times K(y_k,y_k^*)$$

for all k sufficiently large. Using similar arguments as in the second part of the proof of Theorem 1, where we showed equality in (17), we can find a bounded sequence $(\lambda^k)\in\Lambda(y_k,y_k^*)$ and, by passing to a subsequence if necessary, we can assume that it converges to some $\tilde\lambda\in\bar\Lambda$. However, $\bar\Lambda$ is the sum of the convex hull of its extreme points and its recession cone and therefore there is some $\lambda\in\mathcal{E}$ with $I^+(\lambda)\subset I^+(\tilde\lambda)$. Then $I^+(\lambda)\subset I^+(\lambda^k)$ for all k sufficiently large and from $v_k\in K(y_k,y_k^*)$ we deduce $\nabla q_i(y_k)v_k=0$, $i\in I^+(\lambda^k)$. Hence, by passing to the limit we obtain the claimed contradiction $\nabla q_i(\bar y)v=0$, $i\in I^+(\lambda)$, and the proposition is proved. □

The assumption (38) can be considerably weakened, if we strengthen our assumptions imposed on $\bar\Lambda(\cdot)$. To simplify the formulas in the statement below it is reasonable to introduce the Lagrangian associated with our generalized equation, i.e.,

$$\mathcal{L}(x,y,\lambda)=F(x,y)+\nabla q(y)^T\lambda.$$

THEOREM 6. *Let $(\bar x,\bar y)\in\mathrm{gph}\,\tilde S$ and assume that M is metrically subregular at $(\bar y,0)$ and metrically regular in the vicinity of $\bar y$. Further assume that the set-valued mapping Ψ given by (37) is metrically subregular at $((\bar x,\bar y)),(0,0,0))$, that $\bar\Lambda(v_1)=\bar\Lambda(v_2)\ \forall\ 0\neq v_1,v_2\in\bar K$ and suppose that there exists a subspace $P\subset T_C(\bar x)$ with $T_C(\bar x)+P\subset T_C(\bar x)$ and*

$$\nabla_x F(\bar x,\bar y)P+\nabla_y\mathcal{L}(\bar x,\bar y,\bar\lambda)\bar{\mathcal{N}}+\mathrm{span}\,\{\nabla q_i(\bar y)^T\,|\,i\in\bar I^+\}=\mathbb{R}^m, \tag{43}$$

where $\bar\lambda\in\bar\Lambda(v)$ for some $0\neq v\in\bar{\mathcal{N}}$ is chosen arbitrary, if $\bar{\mathcal{N}}\neq\{0\}$, and $\bar\lambda=0$ otherwise. Then (39) holds true and simplifies to

$$\widehat N_{\mathrm{gph}\,\tilde S}(\bar x,\bar y)=\left\{\begin{pmatrix}-\nabla_x F(\bar x,\bar y)^T w\\ -\nabla_y\mathcal{L}(\bar x,\bar y,\bar\lambda)w\end{pmatrix}\ \Big|\ w\in\bar K\right\}+\widehat N_C(\bar x)\times\bar K^\circ, \tag{44}$$

where $\bar{K}, \bar{K}^\circ$ *are computed with* $\bar{y}^* := -F(\bar{x},\bar{y})$. *Moreover, for any* $v^* \in \mathbb{R}^m$ *one has*

$$\widehat{D}^*\tilde{S}(\bar{x},\bar{y})(v^*) = \{\nabla_x F(\bar{x},\bar{y})^T w + \widehat{N}_C(\bar{x}) \,|\, 0 \in v^* + \nabla_y \mathcal{L}(\bar{x},\bar{y},\bar{\lambda})w + \bar{K}^\circ,\ -w \in \bar{K}\}. \tag{45}$$

Proof. The proof follows the same lines as the proof of Theorem 5 with the exception that we choose now $L = \hat{L} \times P$, where $\hat{L} = \{(w, \nabla^2(\bar{\lambda}^T q)(\bar{y})w) \,|\, w \in \bar{\mathcal{N}}\} + \{0_m\} \times \tilde{L}$. In order to prove $T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*) + \hat{L} \subset T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$, choose any $(v,v^*) \in T_{\mathrm{gph}\,\widehat{N}_\Gamma}(\bar{y},\bar{y}^*)$, $w \in \bar{\mathcal{N}}$ and $\xi \in \tilde{L}$. By Theorem 1 we have $v \in \bar{K}$ and there is some $\lambda \in \bar{\Lambda}(v)$ such that $v^* \in \nabla^2(\lambda^T q)(\bar{y})v + \widehat{N}_{\bar{K}}(v)$. By Lemma 4 we have $\nabla^2(\lambda^T q)(\bar{y})v \in \nabla^2(\bar{\lambda}^T q)(\bar{y})v + \tilde{L}$. Since $\widehat{N}_{\bar{K}}(v) + \tilde{L} = \widehat{N}_{\bar{K}}(v)$ because of $\tilde{L} \subset \widehat{N}_{\bar{K}}(v)$, since $\nabla q(\bar{y})w = 0$ because of $w \in \bar{\mathcal{N}}$ and since

$$\begin{aligned}\widehat{N}_{\bar{K}}(v) &= \{\nabla q(\bar{y})^T \mu \,|\, \mu^T \nabla q(\bar{y})v = 0, \mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)\}\\ &= \{\nabla q(\bar{y})^T \mu \,|\, \mu^T \nabla q(\bar{y})(v+w) = 0, \mu \in T_{\widehat{N}_{\mathbb{R}^l_-}(q(\bar{y}))}(\lambda)\} = \widehat{N}_{\bar{K}}(v+w)\end{aligned}$$

by virtue of Lemma 1, we obtain that

$$(v,v^*) + (w, \nabla^2(\bar{\lambda}^T q)(\bar{y})w) + (0_m, \xi) \in (v+w, \nabla^2(\bar{\lambda}^T q)(\bar{y})(v+w)) + \{0_m\} \times \widehat{N}_{\bar{K}}(v+w).$$

This, together with $v + w \in \bar{K}$, shows the desired inclusion. To show (40), fix any $((v,v^*),u) \in (\mathbb{R}^m \times \mathbb{R}^m) \times \mathbb{R}^n$ and choose $p \in P$, $w \in \bar{\mathcal{N}}$ and $\tilde{l} \in \tilde{L}$ with

$$\nabla_x F(\bar{x},\bar{y})p + (\nabla_y F(\bar{x},\bar{y}) + \nabla^2(\bar{\lambda}^T q)(\bar{y}))w + \tilde{l} = v^* + \nabla_x F(\bar{x},\bar{y})u + \nabla_y F(\bar{x},\bar{y})v.$$

Then $l = ((w, \nabla^2(\bar{\lambda}^T q)(\bar{y}))w + \tilde{l}), p) \in L$ and

$$\begin{aligned}&\nabla G(\bar{x},\bar{y}) \begin{pmatrix} u-p \\ v-w \end{pmatrix} + l\\ &= \begin{pmatrix} (v - w + w, -\nabla_x F(\bar{x},\bar{y})(u-p) - \nabla_y F(\bar{x},\bar{y})(v-w) + \nabla^2(\bar{\lambda}^T q)(\bar{y})w + \tilde{l}) \\ u - p + p \end{pmatrix}\\ &= \begin{pmatrix} (v,v^*) \\ u \end{pmatrix}.\end{aligned}$$

This verifies (40) and then again (39) follows from Theorem 4. Theorem 3 yields now the assertion. □

Remark 1. In [11] the authors have derived (45) under the assumptions that $C = \mathbb{R}^n$, $\nabla_x F(\bar{x},\bar{y})$ is surjective and MFCQ and CRCQ are fulfilled at $\bar{y}$. We conclude that this statement follows from Theorem 6 in a straightforward way.

Remark 2. If $\bar{\mathcal{I}} = \bar{I}^+$, then $\operatorname{span}\{\nabla q_i(\bar{y})^T \,|\, i \in \bar{I}^+\} = \bar{\mathcal{N}}^\perp$. If, in addition, $\nabla_y \mathcal{L}(\bar{x},\bar{y},\bar{\lambda})$ is positive definite on $\bar{\mathcal{N}}$, i.e., $v^T \nabla_y \mathcal{L}(\bar{x},\bar{y},\bar{\lambda})v > 0\ \forall\, 0 \neq v \in \bar{\mathcal{N}}$, then for every $v^* \in \mathbb{R}^m$ the generalized equation

$$0 \in \nabla_y \mathcal{L}(\bar{x},\bar{y},\bar{\lambda})v + v^* + \widehat{N}_{\bar{\mathcal{N}}}(v)$$

has a solution, see e.g. [20, Theorem 4.6]. We conclude that in this case assumption (43) is fulfilled with $P = \{0\}$.

6. Applications

6.1. Isolated calmness

A multifunction $\Psi : \mathbb{R}^d \rightrightarrows \mathbb{R}^s$ is said to have the *isolated calmness* property at $(\bar{u},\bar{v}) \in \mathrm{gph}\,\Psi$, provided there exist neighborhoods $\mathcal{U}$ of $\bar{u}$ and $\mathcal{V}$ of $\bar{v}$ and a constant $\kappa \geq 0$ such that

$$\Psi(u) \cap \mathcal{V} \subset \{\bar{v}\} + \kappa\|u - \bar{u}\|\mathbb{B} \text{ when } u \in \mathcal{U}.$$

In [14], it has been proved that Ψ possesses the isolated calmness property at $(\bar{u},\bar{v})$ if and only if

$$D\Psi(\bar{u},\bar{v})(0)=\{0\}, \tag{46}$$

cf. also [5, Theorem 4C.1].

This result has been applied in [9, Theorem 4.1] to the solution map S given by (4). On the basis of Theorem 1 the latter result can be substantially generalized.

Theorem 7. *Let $(\bar{x},\bar{y})\in \operatorname{gph} S$ and assume that M is metrically subregular at $(\bar{y},0)$ and metrically regular in the vicinity of $\bar{y}$. Then S has the isolated calmness property provided for all $v\in\mathbb{R}^m$ one has the implication*

$$\left.\begin{array}{l} 0\in \nabla_y\mathcal{L}(\bar{x},\bar{y},\lambda)v+\widehat{N}_{\bar{K}}(v) \\ \lambda\in\bar{\Lambda}(v)\end{array}\right\}\Rightarrow v=0. \tag{47}$$

Moreover, if $\nabla_x F(\bar{x},\bar{y})$ is surjective, than (47) is not just sufficient but also necessary for S to have the isolated calmness property at $(\bar{x},\bar{y})$.

Proof. By virtue of [26, Theorem 6.31] for all $h\in\mathbb{R}^n$

$$DS(\bar{x},\bar{y})(h)\subset\{v\in\mathbb{R}^m|0\in\nabla_x F(\bar{x},\bar{y})h+\nabla_y F(\bar{x},\bar{y})v+D\widehat{N}_\Gamma(\bar{y},-F(\bar{x},\bar{y}))(v))\}. \tag{48}$$

The first assertion thus follows from the combination of (46), (48) and Theorem 1.

The second assertion follows from the fact that inclusion (48) becomes equality whenever $\nabla_x F(\bar{x},\bar{y})$ is surjective, cf [26, Exercise 6.32]. ☐

Note that in the setting of Theorem 3 condition (47) can be simplified and attains the form:

$$0\in\nabla_y\mathcal{L}(\bar{x},\bar{y},\bar{\lambda})v+\widehat{N}_{\bar{K}}(v)\Rightarrow v=0,$$

where $\bar{\lambda}$ is an arbitrary multiplier from $\bar{\Lambda}(v)$ for some nonzero $v\in\bar{K}$. The case when $\bar{K}=\{0\}$ is, of course, trivial.

Example 5. Consider the GE

$$0\in -x+\widehat{N}_\Gamma(y) \tag{49}$$

with Γ given in Example 4. Let $\bar{x}=(0,1)$ and $\bar{y}=(0,0)$ so that $\bar{y}^*=\bar{x}$, $\bar{K}=T_\Gamma(0)=\{v|v_1\le 0, v_2=0\}$,

$$\bar{\Lambda}=\{\lambda\in\mathbb{R}^3_+|\lambda_1-\lambda_2=1,\lambda_3=0\}$$

and

$$\bar{\Lambda}(v)=\begin{cases}\{\lambda\in\bar{\Lambda}|\lambda_1=1,\lambda_2=0\} & \text{if } \ 0\neq v\in\bar{K}\\ \bar{\Lambda} & \text{if } \ v=0.\end{cases}$$

As in Example 4 all conditions of Theorem 3 are fulfilled and so we may invoke Theorem 7 and conclude that the left-hand side of (47) attains for nonzero v the form of the system

$$0\in\begin{bmatrix}-2v_1\\ 0\end{bmatrix}+\mathbb{R}\begin{bmatrix}0\\1\end{bmatrix}$$

$$v_1\le 0,\quad v_2=0.$$

This system clearly implies that $v=0$ and so the solution map of GE (49) has the isolated calmness property at $(\bar{x},\bar{y})$. △

6.2. S-stationarity conditions for MPECs

Consider the mathematical program with equilibrium constraints

$$\min f(x,y) \quad \text{subject to} \quad 0 \in F(x,y) + \widehat{N}_\Gamma(y), x \in C, \tag{50}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are continuously differentiable and $C \subset \mathbb{R}^n$ is a closed set.

THEOREM 8. *Let $(\bar{x}, \bar{y})$ be a local solution of the MPEC (50). Suppose that the assumptions of Theorem 5 are fulfilled. Then there exists a MPEC multiplier w such that*

$$0 \in \nabla_x f(\bar{x},\bar{y})^T + \nabla_x F(\bar{x},\bar{y})^T w + \widehat{N}_C(\bar{x}) \tag{51}$$
$$0 \in \nabla_y f(\bar{x},\bar{y})^T + \nabla_y F(\bar{x},\bar{y})^T w + \widehat{D}^* \widehat{N}_{\mathrm{gph}\, \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)(w), \tag{52}$$

where $\bar{y}^* := -F(\bar{x},\bar{y})$. In particular we have $w \in -\bigcap_{v \in \bar{\mathcal{N}}} \bar{\mathcal{W}}(v)$ and

$$0 \in \nabla_x f(\bar{x},\bar{y})^T + \nabla_x F(\bar{x},\bar{y})^T w + \widehat{N}_C(\bar{x})$$
$$0 \in \nabla_y f(\bar{x},\bar{y})^T + \nabla_y F(\bar{x},\bar{y})^T w + \bigcap_{v \in \bar{\mathcal{N}}} \bar{L}(v; -w).$$

Proof. Follows from Theorem 5 combined with the standard optimality condition $0 \in \nabla f(x,y) + \widehat{N}_{\mathrm{gph}\, \tilde{S}}(\bar{x},\bar{y})$. □

EXAMPLE 6. Consider the MPEC (50) with $x \in \mathbb{R}^3, y \in \mathbb{R}^3$,

$$f(x,y) = -x_1 - y_1 + \tfrac{1}{2} y_2^2 + y_3^2,$$
$$F(x,y) = x,\ C = \{a \in \mathbb{R}^3 | a_1 \leq -1\}$$

and Γ from Example 2. We claim that the pair $(\bar{x},\bar{y}) = ((-1,0,0),(0,0,0)) \in C \times \Gamma$ is a solution of this MPEC. Indeed, $-\bar{x} \in \widehat{N}_\Gamma(\bar{y})$ by Example 2 and for any feasible pair $(\tilde{x},\tilde{y})$ one has

$$f(\tilde{x},\tilde{y}) \geq \inf_{\substack{x \in C \\ y \in \Gamma}} f(x,y) \geq \inf_{\substack{x_1 \leq -1 \\ (y_2,y_3) \in \mathbb{R}^2}} (-x_1 + \frac{1}{2} y_2^2 + y_3^2) = 1 = f(\bar{x},\bar{y}).$$

Next we verify the assumptions of Theorem 8. The required properties of the corresponding perturbation mapping M hold by virtue of MFCQ. Put $P = \{0\} \times \mathbb{R} \times \mathbb{R}$. Since $T_C(\bar{x}) = \mathbb{R}_- \times \mathbb{R} \times \mathbb{R}$, one has $T_C(\bar{x}) + P \subset T_C(\bar{x})$. Further, $\bar{y}^* = -\bar{x} = (1,0,0), \bar{\Lambda} = \{\lambda \in \mathbb{R}^2_+ | \lambda_1 + \lambda_2 = 1\}$, $\bar{I}^+ = \{1,2\}$ and so

$$\nabla_x F(\bar{x},\bar{y}) P + \mathrm{span}\,\{\nabla q_i(\bar{y}) | i \in \bar{I}^+\} = (\{0\} \times \mathbb{R} \times \mathbb{R}) + (\mathbb{R} \times \{0\} \times \{0\} = \mathbb{R}^3.$$

Note that $\mathcal{E} = \{(0,1),(1,0)\}$ and $N_C(\bar{x}) = \mathbb{R}^+ \times \{0\} \times \{0\}$. Hence for every $\lambda \in \mathcal{E}$ the conditions $\nabla F_x(\bar{x},\bar{y})^T v \in N_C(\bar{x})$, $\nabla q_i(\bar{y}) v = 0$, $i \in I^+(\lambda)$, amount to

$$(v_1, v_2, v_3) = \nabla F_x(\bar{x},\bar{y})^T v \in N_C(\bar{x}) = \mathbb{R}^+ \times \{0\} \times \{0\},\ v_1 = 0$$

implying $v = 0$. Thus all conditions of Proposition 6 and also of Theorem 8 have been verified. As derived in Example 2,

$$\widehat{N}_{\mathrm{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*) = \bar{K}^\circ \times \{(0,0,0)\} = (\mathbb{R} \times \{0\} \times \{0\}) \times (0,0,0),$$

and so we may conclude that conditions (51), (52) are fulfilled with $w = 0$ and with the point $(1,0,0)$ belonging to $\widehat{D}^* \widehat{N}_{\mathrm{gph} \widehat{N}_\Gamma}(\bar{y}, \bar{y}^*)(0)$. △

7. Conclusion

It is well-known that, in contrast to metric regularity and some other stability notions, the property of metric subregularity at a point does not carry over to a neighborhood. This lack of stability does not cause any troubles in the first-order nonsmooth calculus, where qualification conditions based on metric subregularity have been developed for all basic calculus rules, cf. [12]. In the second-order calculus, however, more stable qualification conditions are needed. One uses typically a surjectivity/ nondegeneracy assumption ([17, 18]) or at least MFCQ. In this paper we suggest in this context to require, in addition to the metric subregularity, the metric regularity in the vicinity of the point in question introduced in Definition 2. At the first glance this combination may look somewhat cumbersome, but it turns out that at least in some second-order calculations (like the computation of generalized derivatives of the normal-cone mapping) it can very well be used. Moreover, as shown by examples, there are indeed realistic situations in which this combined property holds.

After completing this paper, some of the results were successfully applied to characterize tilt stability in nonlinear programming in the very recent paper by Gfrerer and Mordukhovich [8]. Thereby it was observed that the assumption of metric regularity in the vicinity of $\bar{y}$ can be weakened by the so-called *bounded extreme point property*, which is not only implied by SOSCMS but also e.g. by CRCQ, see [8, Proposition 3.4]. In fact, by carefully checking the proof of Theorem 1 one can see that the assumption of metric regularity in the vicinity of $\bar{y}$ can be replaced by the requirement, that for all sequences $(t_k) \downarrow 0$, $(v_k) \to v$ and $(v_k^*) \to v^*$ such that $\bar{y}^* + t_k v_k^* \in \widehat{N}_\Gamma(\bar{y} + t_k v_k)$ there exists a bounded sequence of multipliers (λ^k) with $\lambda^k \in \widehat{N}_{\mathbb{R}^l_-}(q(\bar{y} + t_k v_k))$ and $\nabla q(\bar{y} + t_k v_k)^T \lambda^k = \bar{y}^* + t_k v_k^*$ for each k. In [8], this requirement is ensured via the bounded extreme point property.

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