Virtual Element Method for general second-order elliptic problems on polygonal meshes

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Overview

Outline

Assumptions

- Let $\Omega \subset \mathbb{R}^2$ be a bounded, **convex** and polygonal domain with boundary Γ
- Let $\kappa \in L^\infty(\Omega)$ and $\gamma \in L^\infty(\Omega)$ be smooth functions $\Omega \to \mathbb{R}$ with $\kappa(x) \geq \kappa_0 > 0$ for all $x \in \Omega$
- Let $b : \Omega \to \mathbb{R}^2$ be a smooth vector-valued function, $b \in [L^\infty(\Omega)]^2$
- The problem

$$\begin{aligned} \mathcal{L}p := div(-\kappa\nabla p + bp) + \gamma p = f \quad & \text{in } H^{-1}(\Omega), \\ p = 0 \quad & \text{on } \Gamma \end{aligned} \tag{1}$$

is solvable for any $f \in H^{-1}(\Omega)$ and the estimates

$$\|p\|_{1,\Omega} \leq C\|f\|_{-1,\Omega}$$

and

$$\|p\|_{2,\Omega} \leq C\|f\|_{0,\Omega}$$

hold with a constant C independent of f.

Remarks

- Existence and uniqueness is assumed!
- Also existence and uniqueness for the adjoint operator equation
- The following analysis do also hold for κ being a tensor

Outline

Variational Formulation

Definition

Define

$$a(p,q) := \int_\Omega \kappa \nabla p \nabla q \, dx,$$

$$b(p,q) := -\int_\Omega p(b \cdot \nabla q) \, dx,$$

$$c(p,q) := \int_\Omega \gamma p q \, dx,$$

$$B(p,q) := a(p,q) + b(p,q) + c(p,q).$$

Variational formulation

Find $p \in H_0^1(\Omega)$ such that

$$B(p,q) = (f,q)_0 \quad \forall q \in H_0^1(\Omega). \tag{2}$$

Variational Formulation

- The assumptions above imply that $B(\cdot,\cdot)$ fulfills

$$|B(p,q)| \leq M\|p\|_1\|q\|_1 \text{ for } p,q \in H^1(\Omega)$$

  and

$$\sup_{q\in H_0^1} \frac{B(p,q)}{\|q\|_1} \geq C_B\|p\|_1 \text{ for } p \in H_0^1(\Omega)$$

  for some $C_B > 0$ independent of p.

Outline

The Virtual Element Space

- Let $\mathcal{T}_h$ be a decomposition of Ω into star-shaped polygons E, s.t.,
  1. every E is star-shaped w.r.t. every point of a disk with radius $\rho^E h_E$
  2. every edge e of E has length $|e| \geq \rho^E h_E$
- Let $\mathcal{E}_h$ denote the set of edges e of $\mathcal{T}_h$
- $h_E = diam(E)$, $h = \max_{E \in \mathcal{T}_h} h_E$

The Virtual Element Space

Definition (Preliminary local space)

For any integer $k \geq 1$ and any element E we define the preliminary local space by

$$\tilde{\mathcal{Q}}_h^k(E) := \{q \in H^1(E) : q|_e \in \mathbb{P}_k(e) \forall e \in \partial E, \Delta q \in \mathbb{P}_{k-2}(E)\}$$

Definition

For all $q_h \in \tilde{\mathcal{Q}}_h^k(E)$ we define the linear operators

- (D_1) the values $q_h(V_i)$ at the vertices V_i of E

and for $k \geq 2$

- (D_2) the edge moments $\int_e q_h p_{k-2}\, ds$, $p_{k-2} \in \mathbb{P}_{k-2}(e)$, on each e of E,
- (D_3) the internal moments $\int_E q_h p_{k-2}\, dx$, $p_{k-2} \in \mathbb{P}_{k-2}(E)$.

The Virtual Element Space

- For each E and all k the operators $D_1 - D_3$ satisfy

$$\{q \in \mathbb{P}_k(E)\} \wedge \{D_i(q) = 0, i = 1, 2, 3\} \Rightarrow \{q = 0\}$$

- These properties allow to construct a projection $\tilde{\mathcal{Q}}_h^k(E) \to \mathbb{P}_k$
- Set $Dq_h := (D_1 - D_3)(q_h)$ and choose suitable symmetric, positive bilinear form $\mathcal{G}$
- For $q_h \in \tilde{\mathcal{Q}}_h^k(E)$ we define $\Pi_k^{\mathcal{G}} q_h$ as the unique solution of

$$\mathcal{G}(Dq_h - D\Pi_k^{\mathcal{G}} q_h, Dz) = 0 \qquad \forall z \in \mathbb{P}_k.$$

The Virtual Element Space

- Recall Π_k^∇ from the last 2 Seminars:
  For any $q \in H_0^1(\Omega)$ the function $\Pi_k^\nabla q$ on each E is in $\mathbb{P}_k(E)$ and is defined by

  $$(\nabla(\Pi_k^\nabla q - q), \nabla p_k)_{0,E} = 0$$

  and

  $$\int_{\partial E} (\Pi_k^\nabla q - q)\, ds = 0$$

  for all $p_k \in \mathbb{P}_k$.
- For all $q_h \in \tilde{\mathcal{Q}}_h^k(E)$ the polynomial $\Pi_k^\nabla q_h$ can be computed using only the values of D calculated on q_h

The Virtual Element Space

Definition (Virtual element space)

The local virtual element space is defined by

$$\mathcal{Q}_h^k(E) := \left\{ q \in \tilde{\mathcal{Q}}_h^k(E) : \int_E q p_k \, dx = \int_E (\Pi_k^\nabla q) p_k \, dx \quad \forall p_k \in (\mathbb{P}_k \backslash \mathbb{P}_{k-2}(E)) \right\}$$

where $(\mathbb{P}_k \backslash \mathbb{P}_{k-2}(E))$ denotes the polynomials in $\mathbb{P}_k(E)$ that are $L^2(E)$ orthogonal to $P_{k-2}(E)$.

The global virtual element space is

$$\mathcal{Q}_h^k := \left\{ q \in H_0^1(\Omega) : q|_E \in \mathcal{Q}_h^k(E) \;\; \forall E \in \mathcal{T}_h \right\}$$

The Virtual Elemnt Space

- Also recall Π_k^0 the L^2 projection onto $\mathbb{P}_k$ defined by

$$(q - \Pi_k^0 q, p_k)_{0,E} = 0 \qquad \forall p_k \in \mathbb{P}_k$$

- The set of operators (D) is a set of dofs for $\mathcal{Q}_h^k(E)$
- The dofs (D) define an interpolation operator
- The local virtual element space $\mathcal{Q}_h^k(E)$ satisfies
  1. $\mathbb{P}_k(E) \subseteq \mathcal{Q}_h^k(E)$
  2. for all $q_h \in \mathcal{Q}_h^k(E)$
     - the function $\Pi_k^\nabla q_h$,
     - the function $\Pi_k^0 q_h$ and
     - the vector function $\Pi_{k-1}^0 \nabla q_h$

     can be computed from the dofs (D) of q_h

The proofs can be found in [1, 2, 3].

The Discrete Problem

- Denote by $a^E(\cdot,\cdot), b^E(\cdot,\cdot), c^E(\cdot,\cdot)$ and $B^E(\cdot,\cdot)$ the restrictions of the previously defined bilinear forms to E
- Let $S^E(p_h, q_h)$ be a symmetric bilinear form on $\mathcal{Q}_h^k(E) \times \mathcal{Q}_h^k(E)$
- $S^E(\cdot,\cdot)$ scales like $a^E(\cdot,\cdot)$ on the kernel of Π_k^∇, i.e., there exist $0 < \alpha_* \leq \alpha^*$ independent of h such that

$$\alpha_* a^E(q_h, q_h) \leq S^E(q_h, q_h) \leq \alpha^* a^E(q_h, q_h)$$

for all $q_h \in \mathcal{Q}_h^k(E)$ with $\Pi_k^\nabla q_h = 0$.

The Discrete Problem

Definition

Define on each $E \in \mathcal{T}_h$ and for every $p_h, q_h \in \mathcal{Q}_h^k(E)$ the local forms

$$
\begin{aligned}
a_h^E(p_h, q_h) &:= \int_E \kappa \left[\Pi_{k-1}^0 \nabla p_h\right] \cdot \left[\Pi_{k-1}^0 \nabla q_h\right] \, dx + \\
&\qquad\qquad S^E((\mathcal{I} - \Pi_k^\nabla)p_h, (\mathcal{I} - \Pi_k^\nabla)q_h), \\
b_h^E(p_h, q_h) &:= -\int_E \left[\Pi_{k-1}^0 p_h\right] \left[b \cdot \Pi_{k-1}^0 \nabla q_h\right] \, dx, \\
c_h^E(p_h, q_h) &:= \int_E \gamma \left[\Pi_{k-1}^0 p_h\right] \left[\Pi_{k-1}^0 q_h\right] \, dx, \\
(f_h, q_h)_E &:= \int_E f \, \Pi_{k-1}^0 q_h \, dx, \\
B_h^E(p_h, q_h) &:= a_h^E(p_h, q_h) + b_h^E(p_h, q_h) + c_h^E(p_h, q_h).
\end{aligned}
$$

The Discrete Problem

- For $p_h, q_h \in \mathcal{Q}_h^k$ the local forms are summed up

Approximate Problem

Find $p_h \in \mathcal{Q}_h^k$ such that

$$B_h(p_h, q_h) = (f_h, q_h) \quad \forall q_h \in \mathcal{Q}_h^k. \tag{3}$$

The Discrete Problem

Remarks

1. Since Π_k^∇ is a projection

$$S^E((\mathcal{I}-\Pi_k^\nabla)p_k,(\mathcal{I}-\Pi_k^\nabla)q_h)=0 \qquad \forall p_k\in\mathbb{P}_k, \qquad \forall q_h\in\mathcal{Q}_h^k(E)$$

2. The choice

$$a_h^E(p_h,q_h):=\int_E \kappa\left[\nabla\Pi_k^\nabla p_h\right]\cdot\left[\nabla\Pi_k^\nabla q_h\right]\,dx+ \\ S^E((\mathcal{I}-\Pi_k^\nabla)p_h,(\mathcal{I}-\Pi_k^\nabla)q_h),$$

as supposed in [2] yiels heavy losses in the order of convergence for $k\geq 3$, if $\kappa\nabla p$ is a gradient this choice does work.

Outline

Preliminary Results

Lemma (Continuity)

The bilinear form $B_h(\cdot,\cdot)$ is continuous in $\mathcal{Q}_h^k \times \mathcal{Q}_h^k$, that is,

$$|B_h(p_h, q_h)| \leq C_{\kappa,b,\gamma} \|p_h\|_1 \|q_h\|_1 \quad p_h, q_h \in \mathcal{Q}_h^k$$

where $C_{\kappa,b,\gamma} > 0$ does not depend on h.

Proof.

Whiteboard. □

Preliminary Results

Lemma (Consistency)

For all p sufficiently regular and for all $q_h \in \mathcal{Q}_h^k$ it holds

$$|B^E(\Pi_k^0 p, q_h) - B_h^E(\Pi_k^0 p, q_h)| \leq C_{\kappa,b,\gamma} h_E^k \|p\|_{k+1,E} \|q_h\|_{1,E} \quad \forall E \in \mathcal{T}_h.$$

Proof.

For a proof see [3]. □

Preliminary Results

Lemma (Discrete inf-sup condition)

The bilinear form $B_h(\cdot,\cdot)$ *satisfies the following condition: there exists an* $h_0 > 0$ *and a constant* $\overline{C}_B$ *such that, for all* $h < h_0$ *it holds*

$$\sup_{q_h \in \mathcal{Q}_h^k} \frac{B_h(p_h, q_h)}{\|q_h\|_1} \geq \overline{C}_B \|p_h\|_1 \quad \forall p_h \in \mathcal{Q}_h^k.$$

Proof.

For a proof see [3]. □

H^1- estimate

Theorem (H^1-estimate)

For h sufficiently small, the discrete problem (3) has a unique solution $p_h \in \mathcal{Q}_h^k$ *and the error estimate*

$$\|p - p_h\|_1 \le Ch^k(\|p\|_{k+1} + |f|_k)$$

holds with a constant C *depending on* κ, b *and* γ *but independent of* h.

Proof.

$$\|p - p_h\|_1 \le \underbrace{\|p - p_{\mathrm{I}}\|_1}_{\checkmark} + \underbrace{\|p_{\mathrm{I}} - p_h\|_1}_{\textit{Whiteboard}}.$$

□

L^2- estimate

Theorem (L^2-estimate)

For h sufficiently small, the discrete problem (3) has a unique solution $p_h \in \mathcal{Q}_h^k$ and the error estimate

$$\|p - p_h\|_0 \leq Ch^{k+1}(\|p\|_{k+1} + |f|_k)$$

holds with a constant C depending on κ, b and γ but independent of h.

Proof.

For a proof see [3]. □

Outline

- Consider problem (1) with the choice $\Omega = (0,1)^2$ and

$$\kappa = \begin{pmatrix} y^2+1 & -xy \\ -xy & x^2+1 \end{pmatrix}, b = (x,y), \gamma = x^2 + y^3$$

  and with BC and rhs such that the exact solution is

$$p_{ex}(x,y) = x^2 y + sin(2\pi x) sin(2\pi y) + 2.$$

- Comparison of p_{ex} with the L^2-projection of p_h to $\mathbb{P}_k$
- Comparison of $|p_{ex} - \Pi_k^0 p_h|$ at the maximum of p_{ex} which is approx. at $(0.781, 0.766)$

Numerical Experiments

- Four sequences of meshes with $25, 100, 400, 1600$ polygons for each different mesh

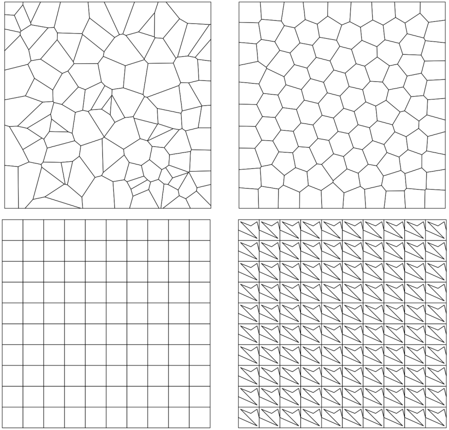

Figure: Considered meshes Lloyd-0, Lloyd-100, square, concave

- Convergence results for $k = 1$

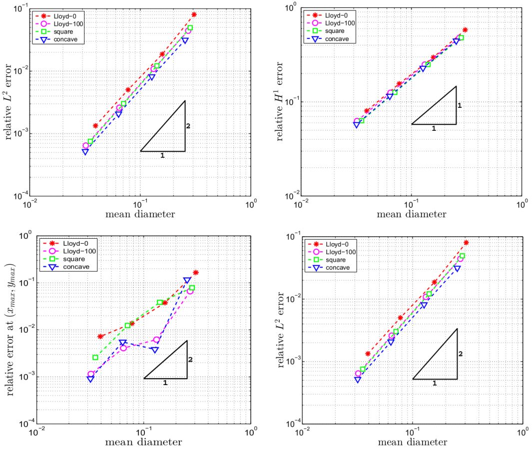

Figure: Convergence rates for the current and the previous choice of $a_h^E(\cdot,\cdot)$, L^2 rate for previous $a_h^E(\cdot,\cdot)$ at bottom right

- Convergence results for $k = 4$

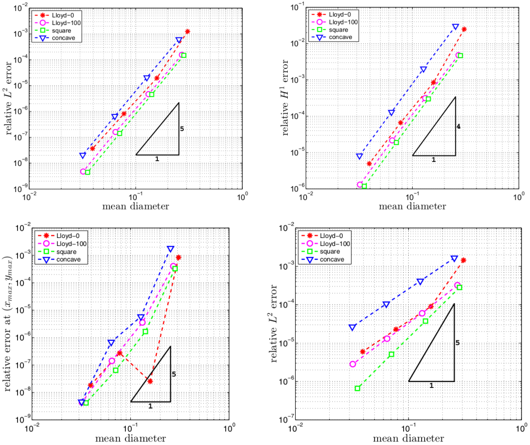

Figure: Convergence rates for the current and the previous choice of $a_h^E(\cdot,\cdot)$, L^2 rate for previous $a_h^E(\cdot,\cdot)$ at bottom right

References

[1] A. Ahmad, A. Alsaedi, F. Brezzi, L. Marini, and A. Russo. Equivalent projectors for virtual element methods. *Computers & Mathematics with Applications*, 66(03):376–391, 2013.

[2] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. Marini, and A. Russo. Basic Principles of virtual element methods. *Mathematical Models and Methods in Applied Sciences*, 23(01):199–214, 2013.

[3] L. Beirão da Veiga, F. Brezzi, L. Marini, and A. Russo. Virtual Element Method for general second-order elliptic problems on polygonal meshes. *Mathematical Models and Methods in Applied Sciences*, 26(04):729–750, 2016.