

A Robust Multigrid Method for Isogeometric Analysis using Boundary Correction

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Abstract The fast solution of linear systems arising from an isogeometric discretization of a partial differential equation is of great importance for the practical use of Isogeometric Analysis. For classical finite element discretizations, multigrid methods are well known to be fast solvers showing optimal convergence behavior. However, if a geometric multigrid solver is naively applied to a linear system arising from an isogeometric discretization, the convergence rates deteriorate significantly if the spline degree is increased. Recently, a robust approximation error estimate and a corresponding inverse inequality for B-splines of maximum smoothness have been shown, both with constants independent of the spline degree. In the present paper, we use these results to construct a multigrid solver for discretizations based on B-splines with maximum smoothness which exhibits robust convergence rates.

1 Introduction

Isogeometric Analysis (IGA), introduced by Hughes et al. [14], is an approach to the discretization of boundary value problems for partial differential equations (PDEs) which aims to bring geometrical modeling and numerical simulation closer together. The fundamental idea is to use the spaces of B-splines or non-uniform rational B-splines (NURBS) which are commonly used to describe geometries in CAD systems also as discretization spaces for the numerical solution of PDEs. As for classical finite element methods (FEM), this leads to large, sparse matrices with condition number that grows as the discretization is refined. A good approximation of the solution of the PDE requires sufficient refinement, which causes both the

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dimension and the condition number of the stiffness matrix to grow. At least for problems on three-dimensional domains or the space-time cylinder, the use of direct solvers does not seem feasible. The development of efficient linear solvers or preconditioners for such linear systems is therefore essential.

For classical FEM, it is well-known that hierarchical methods, like multigrid methods or multilevel methods, are very efficient and show optimal complexity, that is, the required number of iterations for reaching a fixed accuracy goal is independent of the grid size. In this case, the overall computational complexity of the method grows only linearly with the number of unknowns.

It therefore seems natural to extend these methods to IGA, and several results in this direction can be found in the literature; cf. [9, 12, 11, 13] for classical geometric multigrid approaches, [7, 6, 8] for a symbol-based approach to multigrid and [1] for multilevel methods. However, the question how exactly multigrid (or multilevel) methods for IGA should be realized has not been fully answered yet.

While it has been shown early on that a standard approach to constructing geometric multigrid solvers for IGA leads to methods which are robust in the grid size [9], it has been equally observed that the resulting convergence rates deteriorate significantly when the spline degree is increased. Even for moderate choices like spline degree 4, the methods typically require too many iterations for practical purposes.

In recent works [11], some progress has been made by using a Richardson method preconditioned with the mass matrix as a smoother (*mass-Richardson smoother*). The idea behind this kind of smoothers is to carry over the concept of operator preconditioning to multigrid smoothing: here the (inverse of) the mass matrix can be understood as a Riesz isomorphism representing the standard L^2 -norm, the Hilbert space where the classical multigrid convergence analysis is developed. Local Fourier analysis (and similar concepts) indicate that a multigrid method equipped with such a smoother should show convergence rates that are independent of both the grid size and the spline degree. However, numerical results indicate that the proposed method is not robust in the spline degree in practice. This is due to boundary effects, which cannot be captured by local Fourier analysis.

In the present paper, we take a closer look at the origin of these boundary effects and introduce a boundary correction that deals with these effects. We prove that the multigrid method equipped with a properly corrected mass-Richardson smoother converges robustly both in the grid size and the spline degree for one- and two-dimensional problems. We give numerical results indicating the same. For the proof, we make use of the results of the recent paper [15], where robust approximation error estimates and robust inverse estimates for splines of maximum smoothness have been shown. Throughout we restrict ourselves to the case of splines with maximum continuity, which is of particular interest in IGA.

The bulk of our analysis is first carried out in the one-dimensional setting. While multigrid solvers are typically not interesting in this setting from a practical point of view, the tensor product structure of the spline spaces commonly used in IGA lends itself very well to first analyzing the one-dimensional case and then extending the results into higher dimensions. In the present work, we extend the ideas from the one-dimensional to the two-dimensional case and obtain a robust and efficiently realizable smoother also in that case.

The paper is organized as follows. In Section 2, we introduce the spline spaces used for the discretization and the Poisson model problem. A general multigrid

framework and the concept of basis-independent Richardson smoothers are introduced in Section 3. This concept of smoothers requires a proper choice of the Hilbert space, which will be discussed in detail for one-dimensional domains in Section 4. In Section 5, we will make use of the tensor product structure to extend these results to the case of two-dimensional domains. Some details on the numerical realization of the proposed smoother as well as numerical experiments illustrating the theory are given in Section 6. In Section 7, we close with some concluding remarks.

2 Preliminaries

2.1 B-splines and tensor product B-splines

Let first $\Omega = (0, 1)$. We introduce for any $\ell \in \mathbb{N}_0$ a uniform subdivision (grid) \mathcal{M}_ℓ by splitting Ω into $n_\ell = n_0 2^\ell$ subintervals (elements) of length $h_\ell := \frac{1}{n_\ell} = \frac{1}{n_0} 2^{-\ell}$ by uniform dyadic refinement. On these grids we introduce spaces of spline functions as follows. Note that we restrict ourselves to splines of maximum continuity.

Definition 1 $S_{p,\ell}(\Omega)$ is the space of all functions in $C^{p-1}(\Omega)$ which are polynomials of degree p (spline degree) on each element of the subdivision \mathcal{M}_ℓ .

As usual, for $m \geq 0$, $C^m(\Omega)$ is the space of all continuous functions mapping $\Omega \rightarrow \mathbb{R}$ that are m times continuously differentiable. Note that, by construction, the spaces are nested for fixed spline degree p , that is, $S_{p,\ell-1} \subset S_{p,\ell}$, and the number of degrees of freedom roughly doubles in each refinement step. The parameter ℓ will play the role of the grid level in the construction of our multigrid algorithm.

As a basis for $S_{p,\ell}(\Omega)$, we choose the normalized B-splines as described by, e.g., de Boor [5]. To this end, we introduce an open knot vector

$$\underbrace{(0, \dots, 0)}_{p+1 \text{ times}}, h_\ell, 2h_\ell, \dots, (n_\ell - 1)h_\ell, \underbrace{1, \dots, 1}_{p+1 \text{ times}}$$

and define the B-spline basis over this knot vector in the standard way. We denote the B-spline basis functions by

$$\{\varphi_{p,\ell}^{(1)}, \dots, \varphi_{p,\ell}^{(m_\ell)}\}, \quad (1)$$

where $m_\ell = \dim S_{p,\ell}(\Omega) = n_\ell + p$. Note that they satisfy the partition of unity property $\sum_{j=1}^{m_\ell} \varphi_{p,\ell}^{(j)}(x) = 1$ for all $x \in \Omega$.

In higher dimensions, we will assume $\Omega = (0, 1)^d$ with $d > 1$ and introduce tensor product B-spline basis functions of the form $(x, y) \mapsto \varphi_{p,\ell}^{(j_1)}(x) \varphi_{p,\ell}^{(j_2)}(y)$. The space spanned by them will again be denoted by $S_{p,\ell}(\Omega)$. The extension to the case where $\Omega \subset \mathbb{R}^d$ is the tensor product of d arbitrary bounded and open intervals is straightforward. Similarly, it is no problem to have different spline degrees, grid sizes, and/or number of subintervals for each dimension, as long as the grid sizes are approximately equal in all directions. However, for the sake of simplicity of the notation, we will always assume that the discretization is identical in each coordinate direction.

2.2 Model problem

For the sake of simplicity, we restrict ourselves to the following model problem. Let $\Omega = (0, 1)^d$ and assume $f \in L^2(\Omega)$ to be a given function. Find a function $u : \Omega \rightarrow \mathbb{R}$ such that

$$-\Delta u = f \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

In variational form, this problem reads: find $u \in H^1(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \quad (2)$$

To guarantee existence and uniqueness of the solution, we have to assume $\int_{\Omega} f \, dx = \int_{\Omega} u \, dx = 0$. Here and in what follows, $L^2(\Omega)$ is the standard Lebesgue space of square integrable functions and $H^1(\Omega)$ denotes the standard Sobolev space of weakly differentiable functions with derivatives in $L^2(\Omega)$.

Applying a Galerkin discretization using spline spaces, we obtain the following discrete problem: find $u_{\ell} \in S_{p,\ell}(\Omega)$ such that

$$(\nabla u_{\ell}, \nabla v_{\ell})_{L^2(\Omega)} = (f, v_{\ell})_{L^2(\Omega)} \quad \forall v_{\ell} \in S_{p,\ell}(\Omega).$$

Still, we have to guarantee that $\int_{\Omega} u_{\ell} \, dx = 0$, which can be handled as in classical FEM, cf. Remark 21.1.5 in [10].

Using the B-spline basis, the discretized problem can be rewritten in matrix-vector notation,

$$K_{\ell} \underline{u}_{\ell} = \overline{f}_{\ell}, \quad (3)$$

where K_{ℓ} is the B-spline stiffness matrix. Underlined quantities like \underline{u}_{ℓ} will always denote the coefficient vector representing the function u_{ℓ} with respect to the B-spline basis. Overlined quantities, like \overline{f}_{ℓ} , denote the vector of L^2 scalar products of the function f_{ℓ} and the basis functions of the chosen basis. With M_{ℓ} being the B-spline mass matrix, the identity $M_{\ell} \underline{f}_{\ell} = \overline{f}_{\ell}$ holds.

Remark 1 Our model problem may seem rather restrictive in the sense that we only allow for tensor product domains. In practical IGA problems, one often uses such domains as parameter domains and introduces a geometry mapping, usually given in the same basis as used for the discretization, which maps the parameter domain to the actual physical domain of interest. As long as the geometry mapping is well-behaved, a good solver on the parameter domain can be used as a preconditioner for the problem on the physical domain, and our model problem therefore captures all essential difficulties that arise in the construction of multigrid methods in this more general setting. Alternatively, our construction can be directly applied in a straightforward way to the discretization on the physical domain, although we do not perform the analysis for this case. Geometry mappings with singularities as well as multi-patch domains are beyond the scope of the present paper.

3 A multigrid solver framework

3.1 Description of the multigrid algorithm

The multigrid algorithm for solving the discretized equation (3) on grid level ℓ reads as follows. Starting from an initial approximation $\underline{u}_\ell^{(0)}$, one iteration of the multigrid method to obtain the next iterate $\underline{u}_\ell^{(1)}$ is given by the following two steps:

- *Smoothing procedure:* For some fixed number ν of smoothing steps, compute

$$\underline{u}_\ell^{(0,m)} := \underline{u}_\ell^{(0,m-1)} + \tau \widehat{K}_\ell^{-1} \left(\bar{f}_\ell - K_\ell \underline{u}_\ell^{(0,m-1)} \right) \quad \text{for } m = 1, \dots, \nu, \quad (4)$$

where $\underline{u}_\ell^{(0,0)} := \underline{u}_\ell^{(0)}$. The choice of the matrix \widehat{K}_ℓ^{-1} and the damping parameter $\tau > 0$ will be discussed below.

- *Coarse-grid correction:*

- Compute the defect and restrict it to grid level $\ell - 1$ using a restriction matrix $I_\ell^{\ell-1}$:

$$\bar{r}_{\ell-1}^{(1)} := I_\ell^{\ell-1} \left(\bar{f}_\ell - K_\ell \underline{u}_\ell^{(0,\nu)} \right).$$

- Compute the update $\underline{p}_{\ell-1}^{(1)}$ by solving the coarse-grid problem

$$K_{\ell-1} \underline{p}_{\ell-1}^{(1)} = \bar{r}_{\ell-1}^{(1)} \quad (5)$$

approximately.

- Prolongate $\underline{p}_{\ell-1}^{(1)}$ to the grid level ℓ using a prolongation matrix $I_{\ell-1}^\ell$ and add the result to the previous iterate:

$$\underline{u}_\ell^{(1)} := \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^\ell \underline{p}_{\ell-1}^{(1)}.$$

As we have assumed nested spaces, the intergrid transfer matrices $I_{\ell-1}^\ell$ and $I_\ell^{\ell-1}$ can be chosen in a canonical way: $I_\ell^{\ell-1}$ is the canonical embedding and the restriction $I_{\ell-1}^\ell$ is its transpose. In the IGA setting, the prolongation matrix $I_{\ell-1}^\ell$ can be computed by means of knot insertion algorithms.

If the problem on the coarser grid is solved exactly (two-grid method), the coarse-grid correction is given by

$$\underline{u}_\ell^{(1)} := \underline{u}_\ell^{(0,\nu)} + I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} \left(\bar{f}_\ell - K_\ell \underline{u}_\ell^{(0,\nu)} \right). \quad (6)$$

In practice the problem (5) is approximately solved by recursively applying one step (V-cycle) or two steps (W-cycle) of the multigrid method. On the coarsest grid level ($\ell = 0$) the problem (5) is solved exactly by means of a direct method.

The only remaining step is the choice of the smoother, i.e., of the matrix \widehat{K}_ℓ . For multigrid methods for elliptic problems with a FEM discretization, it is typically sufficient to use a damped Jacobi iteration or a Gauss-Seidel iteration as the smoother. This is also possible if an isogeometric discretization is used. However, numerical results show that the convergence rates deteriorate significantly if p is increased ([9, 13]).

It is therefore necessary to have a closer look at the convergence analysis and possible smoothers. In Subsection 3.2, we will introduce the general framework for the multigrid convergence analysis as introduced by Hackbusch. In Subsection 3.3, we will discuss the concept of basis-independent Richardson smoothers. This kind of smoothers requires a proper choice of the Hilbert space. We will discuss this first in Section 4 for one-dimensional domains. In Section 5 we will see how this can be extended to two-dimensional domains using the tensor product structure of the discretization.

3.2 Multigrid convergence theory

As the two-grid method is a linear iteration scheme, its action on the error is easily described by the iteration matrix. Let \underline{u}_ℓ^* denote the exact solution of (3), then the initial error $\underline{u}_\ell^* - \underline{u}_\ell^{(0)}$ and the error after one two-grid cycle $\underline{u}_\ell^* - \underline{u}_\ell^{(1)}$ are related by the equation

$$\underline{u}_\ell^* - \underline{u}_\ell^{(1)} = T_\ell S_\ell^\nu (\underline{u}_\ell^* - \underline{u}_\ell^{(0)}),$$

where

$$S_\ell = I - \tau \widehat{K}_\ell^{-1} K_\ell$$

is the iteration matrix of the smoother and

$$T_\ell = I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell$$

is the iteration matrix of the coarse-grid correction.

For showing convergence in a certain norm $\|\cdot\|_{L_\ell}$, where the matrix L_ℓ is to be specified, it is sufficient to show that

$$q := \|T_\ell S_\ell^\nu\|_{L_\ell} < 1, \quad (7)$$

which obviously implies the q -linear convergence property

$$\|\underline{u}_\ell^* - \underline{u}_\ell^{(1)}\|_{L_\ell} \leq q \|\underline{u}_\ell^* - \underline{u}_\ell^{(0)}\|_{L_\ell}.$$

The discrete norms are defined as follows. For a symmetric and positive definite matrix A and a vector \underline{x}_ℓ , we define $\|\underline{x}_\ell\|_A^2 := (A\underline{x}_\ell, \underline{x}_\ell) = \|A^{1/2}\underline{x}_\ell\|^2$, where $\|\cdot\|$ and (\cdot, \cdot) are the Euclidean norm and scalar product, respectively. Applied to matrices, the norms $\|\cdot\|_A$ and $\|\cdot\|$ are the corresponding operator norms.

To analyze $\|T_\ell S_\ell^\nu\|_{L_\ell}$, we use semi-multiplicity of norms and obtain

$$\|T_\ell S_\ell^\nu\|_{L_\ell} = \|L_\ell^{1/2} T_\ell S_\ell^\nu L_\ell^{-1/2}\| \leq \|L_\ell^{1/2} T_\ell K_\ell^{-1} L_\ell^{1/2}\| \|L_\ell^{-1/2} K_\ell S_\ell^\nu L_\ell^{-1/2}\|.$$

For showing (7), it therefore suffices to prove the two conditions

$$\|L_\ell^{1/2} T_\ell K_\ell^{-1} L_\ell^{1/2}\| \leq C_A, \quad (\text{approximation property}) \quad (8)$$

$$\|L_\ell^{-1/2} K_\ell S_\ell^\nu L_\ell^{-1/2}\| \leq C_S \nu^{-1}, \quad (\text{smoothing property}) \quad (9)$$

cf. equation (6.1.5) in [10] for this splitting. With these, (7) follows immediately for $\nu > C_A C_S$, that is, if sufficiently many smoothing steps are applied. As we are interested in robust convergence, the constants C_A and C_S should be independent of both the grid size and the spline degree.

In classical proofs by Hackbusch [10], convergence is shown in the L^2 -norm (or in a properly scaled L^2 -like norm). This suggests the choice of the mass matrix for L_ℓ , or more precisely, the properly scaled version $L_\ell := h_\ell^{-2}M_\ell$ of the mass matrix.

Classical proofs for showing the convergence of multigrid solvers, equipped with damped Jacobi iteration or Gauss-Seidel iteration as smoothers, use the assumption that the mass matrix M_ℓ , which is taken to define the matrix L_ℓ , is spectrally equivalent to its diagonal. This is true not only for the Courant element, but also for B-splines. However, in the spline case, the equivalence deteriorates if the degree p is increased. We point out that this issue is closely related to the so-called condition number of the B-spline basis (see, e.g., [3]). The growth of the condition number with p explains why standard Jacobi iteration and Gauss-Seidel iteration do not work well for B-splines.

Having this in mind, the idea arises to develop a smoother that does not require the spectral equivalence of the mass matrix M_ℓ and its diagonal. One possibility to do so is to set up a basis-independent smoother, i.e., a smoother which is based on the continuous setting.

3.3 Basis-independent Richardson smoother

The simplest choice of a smoother is the Richardson iteration (gradient method). Here, we compute at each smoothing step the residual (gradient) $\bar{r}_\ell^{(0,m-1)} := \bar{f}_\ell - K_\ell \bar{u}_\ell^{(0,m-1)}$ and use it to compute the update $\underline{p}_\ell^{(0,m-1)}$ used to construct the next iterate $\bar{u}_\ell^{(0,m)} := \underline{u}_\ell^{(0,m-1)} + \underline{p}_\ell^{(0,m-1)}$.

The idea of Richardson iteration is to use the residual as update. However, having the continuous setting in mind, one observes that the residual $\bar{r}_\ell^{(0,m-1)}$ is a dual variable, while the update $\underline{p}_\ell^{(0,m-1)}$ is a primal variable. So, by choosing these two vectors to be equal, we would obtain a basis-dependent smoother. To avoid this, we can require update and residual to be equal in a continuous setting. To do so, we use the Riesz mapping to transform the residual $\bar{r}_\ell^{(0,m-1)}$ into a primal variable, which is then used as the update $\underline{p}_\ell^{(0,m-1)}$.

In the Hilbert space $(\mathbb{R}^{m_\ell}, (\cdot, \cdot)_{L_\ell})$, the Riesz mapping is L_ℓ^{-1} , the inverse of the matrix defining the inner product. The Richardson smoother based on this Hilbert space is then given by (4) with $\hat{K}_\ell := L_\ell$. We now show that for this choice, the smoothing property is always satisfied when τ is chosen properly.

Lemma 1 *Let K_ℓ and L_ℓ be symmetric and positive definite matrices. Assume that $\hat{K}_\ell = L_\ell$ and that τ is chosen such that*

$$0 < \tau \leq \frac{1}{\|L_\ell^{-1/2} K_\ell L_\ell^{-1/2}\|}. \quad (10)$$

Then the smoothing property (9) is satisfied for $C_S = \tau^{-1}$. Moreover, the smoother satisfies the non-expansivity results $\|S_\ell\|_{L_\ell} \leq 1$ and $\|S_\ell\|_{K_\ell} \leq 1$.

Proof The proof is based on Lemma 6.2.1 in [10]. To keep the paper self-contained, we give the full proof.

Observe that

$$L_\ell^{-1/2} K_\ell (I - \tau \widehat{K}_\ell^{-1} K_\ell)^\nu L_\ell^{-1/2} = \widehat{K}_\ell^{-1/2} K_\ell (I - \tau \widehat{K}_\ell^{-1} K_\ell)^\nu \widehat{K}_\ell^{-1/2} = \bar{K}_\ell (I - \tau \bar{K}_\ell)^\nu$$

with $\bar{K}_\ell := \widehat{K}_\ell^{-1/2} K_\ell \widehat{K}_\ell^{-1/2}$ and that $\bar{K}_\ell (I - \tau \bar{K}_\ell)^\nu$ is symmetric. Furthermore, the spectral radius bound $\rho(\tau \bar{K}_\ell) \leq 1$ holds by construction. Thus we obtain

$$\begin{aligned} & \|L_\ell^{-1/2} K_\ell (I - \tau \widehat{K}_\ell^{-1} K_\ell)^\nu L_\ell^{-1/2}\| \\ &= \rho(\bar{K}_\ell (I - \tau \bar{K}_\ell)^\nu) = \sup_{\lambda \in \sigma(\bar{K}_\ell)} \lambda(1 - \tau\lambda)^\nu \leq \tau^{-1} \sup_{\mu \in [0,1]} \mu(1 - \mu)^\nu \\ &= \tau^{-1} \left(\frac{\nu}{1 + \nu} \right)^\nu \frac{1}{1 + \nu} \leq \frac{\tau^{-1}}{\nu}, \end{aligned}$$

which finishes the proof of the smoothing property. The estimates $\|S_\ell\|_{L_\ell} \leq 1$ and $\|S_\ell\|_{K_\ell} \leq 1$ follow by similar arguments. \square

In view of this lemma, it is required to choose a proper Hilbert space or, in other words, a proper matrix L_ℓ . Specifically, we should choose L_ℓ such that

- L_ℓ can be inverted efficiently,
- τ can be chosen independently of the grid size and the spline degree such that (10) holds, and
- the approximation property (8) can be shown with a constant independent of the grid size and the spline degree.

Remark 2 A robust convergence result could also be obtained if we could only show that $q = C_S C_A = \tau^{-1} C_A$ can be bounded robustly from above. However, if this is possible, we can always achieve that τ and C_A are independent of the grid size and the spline degree by scaling the matrix L_ℓ properly. Allowing C_A or τ to depend on the grid size or the spline degree does therefore not yield any additional insight.

The next two sections are mainly dedicated to the construction of the matrix L_ℓ for one- and two-dimensional domains. In both sections, the construction of the matrix L_ℓ is followed by convergence proofs.

4 Robust multigrid for one-dimensional domains

4.1 Motivation and construction of the Hilbert space setting

In this subsection, we will discuss the introduction of a proper Hilbert space setting such that the above-mentioned conditions are satisfied. The standard convergence analysis for the multigrid method, as introduced by Hackbusch [10], gives convergence in a (properly scaled) L^2 -norm, which corresponds to the choice $L_\ell := h_\ell^{-2} M_\ell$. Following the idea of basis-independent Richardson smoothers, this would suggest the choice $\widehat{K}_\ell := L_\ell = h_\ell^{-2} M_\ell$ for the smoother. This is the aforementioned mass-Richardson smoother already studied in [11]. Local Fourier analysis indicates that such an approach should lead to robust convergence of the overall multigrid solver. However, the numerical experiments given in [11] show that this is not the case.

With the choice $\widehat{K}_\ell := L_\ell := h_\ell^{-2} M_\ell$, condition (10) reads

$$\tau \leq \frac{1}{h_\ell^2 \|M_\ell^{-1/2} K_\ell M_\ell^{-1/2}\|}$$

or equivalently

$$\sup_{\underline{u}_\ell \in \mathbb{R}^{m_\ell} \setminus \{0\}} \frac{\|\underline{u}_\ell\|_{K_\ell}}{h_\ell^{-1} \|\underline{u}_\ell\|_{M_\ell}} = \sup_{u_\ell \in S_{p,\ell}(\Omega) \setminus \{0\}} \frac{|u_\ell|_{H^1(\Omega)}}{h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)}} \leq \tau^{-1/2}.$$

In other words, it is required that the spline space satisfies an inverse inequality with a constant that is independent of the grid size and the spline degree. However, such a robust inverse inequality does not hold, cf. the counterexample given in [15]: for all $p \in \mathbb{N}$ and all $\ell \in \mathbb{N}_0$, there exists a function $u_\ell \in S_{p,\ell}(\Omega)$ with

$$\frac{|u_\ell|_{H^1(\Omega)}}{h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)}} \geq p. \quad (11)$$

The functions u_ℓ in this counterexample are non-periodic and have support which is localized close to the boundary. Therefore, they and their effect on the spectrum of the stiffness matrix cannot be captured by local Fourier analysis (and similar tools) which considers only the periodic setting.

The estimate (11) indicates that we have to choose $\tau \leq Cp^{-2}$, and because of $\nu > C_S C_A = \tau^{-1} C_A$, we have to increase the number of smoothing steps with the order p^2 in order to guarantee $q < 1$. This reduces the efficiency of the method. The numerical experiments in [11] confirm that it is required to decrease τ and increase ν for robust convergence if p is increased.

We remark that with a choice $\tau > 2\|L_\ell^{-1/2} K_\ell L_\ell^{-1/2}\|^{1/2}$, the smoother is a divergent iteration scheme. In this case it has to be expected that also the multigrid method diverges, and indeed this can be observed experimentally.

In [15] it was shown that a robust inverse estimate does hold for a large subspace of $S_{p,\ell}(\Omega)$. This makes it clear that the above-mentioned counterexample only describes the effect of a few outliers (cf. also [2] on the topic of spectral outliers in IGA). Thus it seems that if we could eliminate these outliers, it should be possible to choose τ independently of p and obtain robust convergence. The mentioned robust inverse estimate reads as follows.

Theorem 1 ([15]) *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$|u_\ell|_{H^1(0,1)} \leq 2\sqrt{3}h_\ell^{-1} \|u_\ell\|_{L^2(0,1)}$$

is satisfied for all $u_\ell \in \widetilde{S}_{p,\ell}(0,1)$, where $\widetilde{S}_{p,\ell}(0,1)$ is the space of all $u_\ell \in S_{p,\ell}(0,1)$ whose odd derivatives vanish at the boundary,

$$\frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(0) = \frac{\partial^{2l+1}}{\partial x^{2l+1}} u_\ell(1) = 0 \text{ for all } l \in \mathbb{N}_0 \text{ with } 2l+1 < p.$$

In [15] it was discussed that the space $\widetilde{S}_{p,\ell}(\Omega)$ is almost as large as the space $S_{p,\ell}(\Omega)$: their sizes differ by p (for p even) or $p-1$ (for p odd) dimensions. Using a

standard B-spline basis, it is not easy to characterize the space $\tilde{S}_{p,\ell}(\Omega)$ succinctly. However, this is not necessary. It suffices to decompose

$$S_{p,\ell}(\Omega) = S_{p,\ell}^{(I)}(\Omega) + S_{p,\ell}^{(T)}(\Omega),$$

where $S_{p,\ell}^{(T)}(\Omega)$ is spanned by the first p basis functions $\varphi_{p,\ell}^{(1)}, \dots, \varphi_{p,\ell}^{(p)}$ and the last p basis functions $\varphi_{p,\ell}^{(m_\ell-p+1)}, \dots, \varphi_{p,\ell}^{(m_\ell)}$. The space $S_{p,\ell}^{(I)}(\Omega)$ is spanned by all the remaining basis functions. By construction, the space $S_{p,\ell}^{(I)}(\Omega)$ consists of all functions that vanish on the boundary together with all derivatives up to order $p-1$. It is clear then that $S_{p,\ell}^{(I)}(\Omega) \subseteq \tilde{S}_{p,\ell}(\Omega)$. Based on this separation of the degrees of freedom, we can prove the following generalized inverse inequality.

Lemma 2 *For any $u_\ell \in S_{p,\ell}(\Omega)$ with $u_\ell = u_{\Gamma,\ell} + u_{I,\ell}$, where $u_{\Gamma,\ell} \in S_{p,\ell}^{(T)}(\Omega)$ and $u_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega)$, the generalized inverse inequality*

$$|u_\ell|_{H^1(\Omega)} \leq \sqrt{2}(1 + 4\sqrt{6})(h_\ell^{-2} \|u_\ell\|_{L^2(\Omega)}^2 + |\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)}^2)^{1/2}$$

holds. Here, $\mathcal{H}_{\Gamma,\ell} : S_{p,\ell}^{(T)}(\Omega) \rightarrow S_{p,\ell}(\Omega)$ is the discrete harmonic extension, i.e., $\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell} = u_{\Gamma,\ell} + s_{I,\ell}$ where $s_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega)$ is chosen such that the energy $|\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)}$ is minimized.

As we need some preliminary results first, we postpone the proof of this lemma to Subsection 4.3.

Based on the splitting $S_{p,\ell} = S_{p,\ell}^{(T)} + S_{p,\ell}^{(I)}$, we can reorder the degrees of freedom accordingly and write the matrix K_ℓ and the vector \underline{u}_ℓ in block structure as follows:

$$K_\ell = \begin{pmatrix} K_{\Gamma\Gamma,\ell} & K_{\Gamma I,\ell}^T \\ K_{I\Gamma,\ell} & K_{II,\ell} \end{pmatrix}, \quad \underline{u}_\ell = \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ \underline{u}_{I,\ell} \end{pmatrix}. \quad (12)$$

Using this decomposition, the discrete harmonic extension and its energy norm can be determined explicitly by the use of the Schur complement,

$$|\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)} = \left\| \begin{pmatrix} \underline{u}_{\Gamma,\ell} \\ -K_{II,\ell}^{-1} K_{I\Gamma,\ell} \underline{u}_{\Gamma,\ell} \end{pmatrix} \right\|_{K_\ell} = \|\underline{u}_{\Gamma,\ell}\|_{K_{\Gamma\Gamma,\ell} - K_{\Gamma I,\ell}^T K_{II,\ell}^{-1} K_{I\Gamma,\ell}}. \quad (13)$$

Using this equality, the statement of Lemma 2 in matrix-vector notation reads

$$\|\underline{u}_\ell\|_{K_\ell} \leq \sqrt{2}(1 + 4\sqrt{6}) \|\underline{u}_\ell\|_{L_\ell}$$

with

$$\hat{K}_\ell := L_\ell := h_\ell^{-2} M_\ell + \underbrace{\begin{pmatrix} K_{\Gamma\Gamma,\ell} - K_{\Gamma I,\ell}^T K_{II,\ell}^{-1} K_{I\Gamma,\ell} & 0 \\ 0 & 0 \end{pmatrix}}_{\tilde{K}_\ell}. \quad (14)$$

Hence we obtain $\|L_\ell^{-1/2} K_\ell L_\ell^{-1/2}\| \leq 2(1 + 4\sqrt{6})^2 \leq 234$, which guarantees that the choice $\tau = 1/234$ satisfies (10) for all grid levels and all choices of p . We note that in practice, larger choices for τ are admissible, as we will see in the numerical experiments given in Subsection 6.2. Having shown (10), Lemma 1 implies the following result.

Lemma 3 *With $\hat{K}_\ell := L_\ell$ defined as in (14), there is a $\tau > 0$ which does not depend on the grid size h_ℓ and the spline degree p such that the smoothing property (9) is satisfied with $C_S = \tau^{-1}$.*

4.2 Proof of approximation property

The next step is to show the approximation property. To this end we first prove an error estimate for the \widehat{H}^1 -orthogonal projection $\widetilde{\Pi}_\ell : H^1(\Omega) \rightarrow \widetilde{S}_{p,\ell}(\Omega)$, which is given as follows. For any function $u \in H^1(\Omega)$, the \widehat{H}^1 -orthogonal projection $\widetilde{\Pi}_\ell u$ is the solution of the following problem: find $\widetilde{u}_\ell \in \widetilde{S}_{p,\ell}(\Omega)$ such that

$$(\widetilde{u}_\ell, \widetilde{s}_\ell)_{\widehat{H}^1(\Omega)} = (u, \widetilde{s}_\ell)_{\widehat{H}^1(\Omega)} \quad \forall \widetilde{s}_\ell \in \widetilde{S}_{p,\ell}(\Omega), \quad (15)$$

where

$$(u, v)_{\widehat{H}^1(\Omega)} := (\nabla u, \nabla v)_{L^2(\Omega)} + \frac{1}{|\Omega|} \left(\int_{\Omega} u(x) dx \right) \left(\int_{\Omega} v(x) dx \right).$$

The following error estimate for this projector is a slightly improved version of Theorem 1 in [15].

Theorem 2 *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$\|u - \widetilde{\Pi}_\ell u\|_{L^2(0,1)} \leq 2\sqrt{2}h_\ell |u|_{H^1(0,1)}$$

is satisfied for all $u \in H^1(0,1)$.

Proof We let $u \in H^1(0,1)$ be fixed and extend it to a periodic function $w \in H^{1,per}(-1,1)$ by setting $w(x) := u(|x|)$. Let $S_{p,\ell}^{per}(-1,1)$ denote the subspace of $S_{p,\ell}(-1,1)$ whose members are in C^{p-1} when extended periodically to the real line. Using Lemma 6 in [15], we obtain

$$\|w - \Pi_\ell^{per} w\|_{L^2(-1,1)} \leq 2\sqrt{2} h_\ell |w|_{H^1(-1,1)}, \quad (16)$$

where Π_ℓ^{per} is the \widehat{H}^1 -orthogonal projector to $S_{p,\ell}^{per}(-1,1)$.

The function $w_\ell := \Pi_\ell^{per} w$ is symmetric, i.e., $w_\ell(x) = w_\ell(-x)$, as can be seen by the following argument. Let $\widehat{w}_\ell(x) := w_\ell(-x)$. As w is symmetric, we have

$$\|w - w_\ell\|_{\widehat{H}^1(-1,1)} = \|w - \widehat{w}_\ell\|_{\widehat{H}^1(-1,1)}.$$

Now we recall that the \widehat{H}^1 -orthogonal projection minimizes the \widehat{H}^1 -norm of the error $w - w_\ell$ or $w - \widehat{w}_\ell$, respectively. As the norm is equal for both of them and the minimization problem has a unique solution, we obtain $w_\ell = \widehat{w}_\ell$. This shows that $w_\ell = \Pi_\ell^{per} w$ is symmetric.

By restricting w_ℓ to $(0,1)$, we obtain a function $u_\ell \in \widetilde{S}_{p,\ell}(0,1)$. Due to the symmetry of w and w_ℓ , we have

$$|w|_{H^1(-1,1)} = \sqrt{2}|u|_{H^1(0,1)}, \quad \|w - w_\ell\|_{L^2(-1,1)} = \sqrt{2}\|u - u_\ell\|_{L^2(0,1)}.$$

Therefore, it follows from (16) that u_ℓ satisfies the desired error estimate

$$\|u - u_\ell\|_{L^2(0,1)} \leq 2\sqrt{2}h_\ell |u|_{H^1(0,1)}.$$

It only remains to show that $u_\ell = \widetilde{\Pi}_\ell u$, i.e., that u_ℓ is the \widehat{H}^1 -orthogonal projection of u to $\widetilde{S}_{p,\ell}(0,1)$. By definition, this means that we have to show that

$$(u - u_\ell, \widetilde{u}_\ell)_{\widehat{H}^1(0,1)} = 0 \quad \forall \widetilde{u}_\ell \in \widetilde{S}_{p,\ell}(0,1). \quad (17)$$

Define $\tilde{w}_\ell \in S_{p,\ell}^{per}(-1,1)$ by $\tilde{w}_\ell(x) := \tilde{u}_\ell(|x|)$ and observe

$$2(u - u_\ell, \tilde{w}_\ell)_{\hat{H}^1(0,1)} = (w - w_\ell, \tilde{w}_\ell)_{\hat{H}^1(-1,1)}.$$

Because w_ℓ is the \hat{H}^1 -orthogonal projection of w into $S_{p,\ell}^{per}(-1,1)$, we know that $(w - w_\ell, \tilde{w}_\ell)_{\hat{H}^1(0,1)} = 0$. This shows (17) and therefore $u_\ell = \tilde{\Pi}_\ell u$. \square

A similar result is satisfied for Π_ℓ , the \hat{H}^1 -orthogonal projection from $H^1(\Omega)$ to $S_{p,\ell}(\Omega)$.

Theorem 3 *Let $\ell \in \mathbb{N}_0$ and $p \in \mathbb{N}$. Then*

$$\|u - \Pi_\ell u\|_{L^2(0,1)} \leq 4\sqrt{2}h_\ell |u|_{H^1(0,1)}$$

is satisfied for all $u \in H^1(0,1)$.

Proof As $\tilde{S}_{p,\ell}(\Omega) \subseteq S_{p,\ell}(\Omega)$, the identity $\tilde{\Pi}_\ell = \tilde{\Pi}_\ell \Pi_\ell$ holds. Using the triangle inequality, this identity, Theorem 2, and the stability of the \hat{H}^1 -orthogonal projection, we obtain

$$\begin{aligned} \|u - \Pi_\ell u\|_{L^2(0,1)} &\leq \|(I - \tilde{\Pi}_\ell)u\|_{L^2(0,1)} + \|(I - \tilde{\Pi}_\ell)\Pi_\ell u\|_{L^2(0,1)} \\ &\leq 2\sqrt{2}h_\ell(|u|_{H^1(0,1)} + |\Pi_\ell u|_{H^1(0,1)}) \leq 4\sqrt{2}h_\ell |u|_{L^2(0,1)}. \end{aligned}$$

\square

Using this result, we can now prove the approximation property for L_ℓ as defined as in (14).

Lemma 4 *With L_ℓ defined as in (14), the approximation property (8) is satisfied with a constant C_A which does not depend on the grid size h_ℓ and the spline degree p .*

Proof In matrix-vector notation, the application of Theorem 3 to $u_\ell \in S_{p,\ell}(\Omega)$ reads

$$\|(I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) \underline{u}_\ell\|_{M_\ell} \leq 4\sqrt{2}h_{\ell-1} \|\underline{u}_\ell\|_{K_\ell} = 8\sqrt{2}h_\ell \|\underline{u}_\ell\|_{K_\ell} \quad \forall \underline{u}_\ell \in \mathbb{R}^{m_\ell}, \quad (18)$$

which would be sufficient to show the approximation property for the choice $L_\ell := h_\ell^{-2} M_\ell$. However, we want to show it for the stronger norm $L_\ell := h_\ell^{-2} M_\ell + \tilde{K}_\ell$.

Using (13) and the definition of the discrete harmonic extension, we obtain for all $u_\ell = u_{\Gamma,\ell} + u_{I,\ell} \in S_{p,\ell}(\Omega)$:

$$\|\underline{u}_\ell\|_{\tilde{K}_\ell} = \inf_{s_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega)} |u_{\Gamma,\ell} + s_{I,\ell}|_{H^1(\Omega)} \leq |u_\ell|_{H^1(\Omega)} = \|\underline{u}_\ell\|_{K_\ell}. \quad (19)$$

Together with the stability of the Galerkin projection,

$$\|(I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) \underline{u}_\ell\|_{K_\ell}^2 \leq \|\underline{u}_\ell\|_{K_\ell}^2,$$

equation (19) yields the statement

$$\|(I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) \underline{u}_\ell\|_{\tilde{K}_\ell}^2 \leq \|\underline{u}_\ell\|_{K_\ell}^2. \quad (20)$$

The combination of (18) and (20) and the definition of L_ℓ , i.e., (14), yields

$$\|(I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) \underline{u}_\ell\|_{L_\ell}^2 \leq (1 + 128) \|\underline{u}_\ell\|_{K_\ell}^2,$$

which can be written in matrix notation as

$$\|X_\ell\| := \|L_\ell^{1/2} (I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) K_\ell^{-1/2}\| \leq \sqrt{129},$$

and therefore $\|X_\ell X_\ell^T\| \leq 129$. We have

$$\begin{aligned} X_\ell X_\ell^T &= L_\ell^{1/2} (I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) K_\ell^{-1} (I - K_\ell I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1}) L_\ell^{1/2} \\ &= L_\ell^{1/2} (I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell)^2 K_\ell^{-1} L_\ell^{1/2}, \end{aligned}$$

and using the fact that the matrix $(I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell)$ is a projector, we obtain

$$\|L_\ell^{1/2} (I - I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) K_\ell^{-1} L_\ell^{1/2}\| \leq 129,$$

i.e., the approximation property. This argument is variant of the Aubin-Nitsche duality trick. \square

4.3 Proof of generalized inverse inequality

We will now prove Lemma 2, where Theorem 2 from the previous subsection will again be used as a tool.

Proof (Proof of Lemma 2) Assume $u_\ell \in S_{p,\ell}(\Omega)$ to be given. This function can be decomposed as $u_\ell = u_{\Gamma,\ell} + u_{I,\ell}$ with $u_{\Gamma,\ell} \in S_{p,\ell}^{(\Gamma)}(\Omega)$ and $u_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega)$.

Using the triangle inequality, we obtain

$$|u_\ell|_{H^1(\Omega)} \leq \inf_{\tilde{s}_\ell \in \tilde{S}_{p,\ell}(\Omega)} (|u_{I,\ell} + \tilde{s}_\ell|_{H^1(\Omega)} + |u_{\Gamma,\ell} - \tilde{s}_\ell|_{H^1(\Omega)}).$$

As $u_{I,\ell} + \tilde{s}_\ell \in \tilde{S}_{p,\ell}(\Omega)$, Theorem 1 and the triangle inequality yield

$$\begin{aligned} |u_\ell|_{H^1(\Omega)} &\leq \inf_{\tilde{s}_\ell \in \tilde{S}_{p,\ell}(\Omega)} (2\sqrt{3}h_\ell^{-1} \|u_{I,\ell} + \tilde{s}_\ell\|_{L^2(\Omega)} + |u_{\Gamma,\ell} - \tilde{s}_\ell|_{H^1(\Omega)}) \\ &\leq 2\sqrt{3}h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} \\ &\quad + \inf_{\tilde{s}_\ell \in \tilde{S}_{p,\ell}(\Omega)} (2\sqrt{3}h_\ell^{-1} \|u_{\Gamma,\ell} - \tilde{s}_\ell\|_{L^2(\Omega)} + |u_{\Gamma,\ell} - \tilde{s}_\ell|_{H^1(\Omega)}). \end{aligned}$$

Theorem 2 implies

$$\|(I - \tilde{\Pi}_\ell) u_{\Gamma,\ell}\|_{L^2(\Omega)} = \|(I - \tilde{\Pi}_\ell)^2 u_{\Gamma,\ell}\|_{L^2(\Omega)} \leq 2\sqrt{2}h_\ell |(I - \tilde{\Pi}_\ell) u_{\Gamma,\ell}|_{H^1(\Omega)}.$$

With the choice $\tilde{s}_\ell := \tilde{\Pi}_\ell u_{\Gamma,\ell}$, we obtain

$$\begin{aligned} |u_\ell|_{H^1(\Omega)} &\leq 2\sqrt{3}h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} + (4\sqrt{6} + 1) |(I - \tilde{\Pi}_\ell) u_{\Gamma,\ell}|_{H^1(\Omega)} \\ &\leq (4\sqrt{6} + 1) (h_\ell^{-1} \|u_\ell\|_{L^2(\Omega)} + |(I - \tilde{\Pi}_\ell) u_{\Gamma,\ell}|_{H^1(\Omega)}) \\ &\leq \sqrt{2} (4\sqrt{6} + 1) (h_\ell^{-2} \|u_\ell\|_{L^2(\Omega)}^2 + |(I - \tilde{\Pi}_\ell) u_{\Gamma,\ell}|_{H^1(\Omega)}^2)^{1/2}. \end{aligned}$$

As $\tilde{\Pi}_\ell$ is the \widehat{H}^1 -orthogonal projection, we obtain

$$\begin{aligned} |(I - \tilde{\Pi}_\ell)u_{\Gamma,\ell}|_{H^1(\Omega)} &= \inf_{\tilde{s}_\ell \in \tilde{S}_{p,\ell}(\Omega)} |u_{\Gamma,\ell} + \tilde{s}_\ell|_{H^1(\Omega)} \\ &\leq \inf_{s_{I,\ell} \in S_{p,\ell}^{(I)}(\Omega)} |u_{\Gamma,\ell} + s_{I,\ell}|_{H^1(\Omega)} = |\mathcal{H}_{\Gamma,\ell} u_{\Gamma,\ell}|_{H^1(\Omega)}, \end{aligned}$$

which finishes the proof. \square

4.4 Two-grid convergence

Combining the approximation property (8) and the smoothing property (9), we obtain the convergence of the two-grid method in the L_ℓ -norm.

Theorem 4 *Using $\widehat{K}_\ell := L_\ell$ as defined in (14), we obtain*

$$\|\underline{u}_\ell^* - \underline{u}_\ell^{(1)}\|_{L_\ell} \leq \frac{C_A C_S}{\nu} \|\underline{u}_\ell^* - \underline{u}_\ell^{(0)}\|_{L_\ell}$$

with constants C_A and C_S which do not depend on the grid size h_ℓ and the spline degree p . Thus, the two-grid method converges in the norm $\|\cdot\|_{L_\ell}$ if $\nu > C_A C_S$ smoothing steps are applied.

Proof Follows directly from the combination of Lemma 3 and Lemma 4. \square

Lemma 1 states that the proposed smoother is non-expansive, therefore also the two-grid method with ν pre- and ν post-smoothing steps converges, i.e.,

$$\|S_\ell^\nu T_\ell S_\ell^\nu\|_{L_\ell} \leq \frac{C_A C_S}{\nu}$$

holds. As $S_\ell^\nu T_\ell S_\ell^\nu$ is self-adjoint in the scalar product $(\cdot, \cdot)_{K_\ell}$, this implies

$$\|S_\ell^\nu T_\ell S_\ell^\nu\|_{K_\ell} \leq \frac{C_A C_S}{\nu}. \quad (21)$$

Thus we have shown the following corollary.

Corollary 1 *Using $\widehat{K}_\ell := L_\ell$ as defined in (14), the two-grid method with ν pre- and ν post-smoothing steps converges in the energy norm $\|\cdot\|_{K_\ell}$ if $\nu > C_A C_S$.*

4.5 Multigrid W-cycle convergence

The analysis can be easily carried over to W-cycle multigrid methods, following the classical path of the analysis, cf. [10]. Note that there the analysis is shown in the L^2 -norm. This would mean that in our case, the analysis shall be shown in the norm $\|\cdot\|_{L_\ell}$. However, it is not obvious how to show the convergence of the multigrid method for this case as the relation between the matrices L_ℓ and $L_{\ell-1}$ is non-trivial, since the ‘‘boundary layer’’ affected by the boundary correction grows with the grid size.

However, we can show convergence in the energy norm (which is, as always, weaker than convergence in any other norm).

Theorem 5 Using $\widehat{K}_\ell := L_\ell$ as defined in (14), the W -cycle multigrid method with ν pre- and ν post-smoothing steps converges in the energy norm $\|\cdot\|_{K_\ell}$ if $\nu > 4C_A C_S$.

Proof The iteration matrix of the multigrid method is given by

$$W_\ell = S_\ell^\nu (I - I_{\ell-1}^\ell (I - W_{\ell-1}^2) K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) S_\ell^\nu,$$

for $\ell > 0$ and $W_0 = 0$ (because we solve the problem exactly on the coarsest grid level). Using triangular inequality and semi-multiplicativity of norms, we obtain the following bound for the convergence rate q_ℓ :

$$\begin{aligned} q_\ell &= \|W_\ell\|_{K_\ell} = \|S_\ell^\nu (I - I_{\ell-1}^\ell (I - W_{\ell-1}^2) K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell) S_\ell^\nu\|_{K_\ell} \\ &\leq \|S_\ell^\nu T_\ell S_\ell^\nu\|_{K_\ell} + \|S_\ell^\nu I_{\ell-1}^\ell W_{\ell-1}^2 K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell S_\ell^\nu\|_{K_\ell} \\ &\leq \|S_\ell^\nu T_\ell S_\ell^\nu\|_{K_\ell} + \|S_\ell\|_{K_\ell}^{2\nu} \|K_\ell^{1/2} I_{\ell-1}^\ell K_{\ell-1}^{-1/2}\|^2 \|W_{\ell-1}\|_{K_{\ell-1}}^2. \end{aligned}$$

Observe that using the Galerkin principle, we obtain $K_{\ell-1} = I_\ell^{\ell-1} K_\ell I_{\ell-1}^\ell$ and thus $\|K_\ell^{1/2} I_{\ell-1}^\ell K_{\ell-1}^{-1/2}\|^2 = \rho(I_\ell^{\ell-1} K_\ell I_{\ell-1}^\ell K_{\ell-1}^{-1}) = 1$. Lemma 1 states that $\|S_\ell\|_{K_\ell} \leq 1$. Using these two statements, (21) and $q_{\ell-1} = \|W_{\ell-1}\|_{K_{\ell-1}}$, we obtain that

$$q_\ell \leq \frac{C_A C_S}{\nu} + q_{\ell-1}^2$$

holds. By choosing $\nu \geq 4C_A C_S$, we obtain $q_\ell \leq \frac{1}{4} + q_{\ell-1}^2$. Using $q_0 = 0$ we obtain by induction that $q_\ell \leq \frac{1}{2}$, which finishes the proof. \square

Note that this proof is similar to the analysis presented in [10], however the proof here is easier, as we only show the convergence in the energy norm.

5 Robust multigrid for two-dimensional domains

In this section, we extend the theory presented in the previous section to the (more relevant) case of two-dimensional domains. As outlined in Remark 1, we restrict ourselves to problems without geometry mapping, i.e., problems on the unit square only. However, these approaches can be used as preconditioners for problems on general geometries if there is a regular geometry mapping. In the following, we are interested in solving the problem (3) with $d = 2$, which we now write as

$$\mathcal{K}_\ell \underline{u}_\ell = \bar{f}_\ell, \quad (22)$$

where calligraphic letters will refer to matrices for the two-dimensional domain, whereas standard letters refer to matrices for the one-dimensional domain. The multigrid framework from Section 3 applies unchanged, only replacing standard letters by calligraphic letters.

Using the tensor product structure of the problem and the discretization, the mass matrix has the tensor product structure

$$\mathcal{M}_\ell = M_\ell \otimes M_\ell,$$

where \otimes denotes the Kronecker product. To keep the notation simple, we assume that we have the same matrix M_ℓ for both directions of the two-dimensional domain. However, this is not needed for the analysis.

The stiffness matrix for two dimensions is the sum of two Kronecker products,

$$\mathcal{K}_\ell = K_\ell \otimes M_\ell + M_\ell \otimes K_\ell, \quad (23)$$

which reflects $(\nabla \cdot, \nabla \cdot)_{L^2(\Omega)} = (\partial_x \cdot, \partial_x \cdot)_{L^2(\Omega)} + (\partial_y \cdot, \partial_y \cdot)_{L^2(\Omega)}$.

Based on the structure \mathcal{K}_ℓ , the idea of setting up the smoother is as follows. Choose

$$\underline{u}_\ell^{(0,m)} = \underline{u}_\ell^{(0,m-1)} + \tau \widehat{\mathcal{K}}_{\ell,orig}^{-1} (f_\ell - \mathcal{K}_\ell \underline{u}_\ell^{(0,m-1)}),$$

where the matrix $\widehat{\mathcal{K}}_{\ell,orig}$ is constructed by taking the matrix \mathcal{K}_ℓ and replacing K_ℓ by \widehat{K}_ℓ . Thus, we have

$$\widehat{\mathcal{K}}_{\ell,orig} := \widehat{K}_\ell \otimes M_\ell + M_\ell \otimes \widehat{K}_\ell.$$

The problem is, of course, that the inversion of $\widehat{\mathcal{K}}_{\ell,orig}$ is as complicated as the inversion of the stiffness matrix \mathcal{K}_ℓ itself.

However, we can make use of the particular structure of \widehat{K}_ℓ , being the mass matrix plus some correction term \widetilde{K}_ℓ and obtain

$$\begin{aligned} \widehat{\mathcal{K}}_{\ell,orig} &= (h_\ell^{-2} M_\ell + \widetilde{K}_\ell) \otimes M_\ell + M_\ell \otimes (h_\ell^{-2} M_\ell + \widetilde{K}_\ell) \\ &= 2h_\ell^{-2} M_\ell \otimes M_\ell + \widetilde{K}_\ell \otimes M_\ell + M_\ell \otimes \widetilde{K}_\ell. \end{aligned}$$

Here, the formula can be simplified further if the factor 2 is dropped. Doing so, we obtain a spectrally equivalent matrix, i.e., $\sigma(\widehat{\mathcal{K}}_\ell^{-1} \widehat{\mathcal{K}}_{\ell,orig}) \subseteq [1, 2]$, where σ is the spectrum and

$$\widehat{\mathcal{K}}_\ell := \mathcal{L}_\ell := h_\ell^{-2} M_\ell \otimes M_\ell + \widetilde{K}_\ell \otimes M_\ell + M_\ell \otimes \widetilde{K}_\ell = h_\ell^2 \widehat{K}_\ell \otimes \widehat{K}_\ell - h_\ell^2 \widetilde{K}_\ell \otimes \widetilde{K}_\ell.$$

Using the grouping of the degrees of freedom used in (12), we obtain further

$$\widehat{\mathcal{K}}_\ell = \mathcal{L}_\ell = h_\ell^2 \widehat{K}_\ell \otimes \widehat{K}_\ell - h_\ell^2 [P_\ell^T \otimes P_\ell^T] [Q_\ell \otimes Q_\ell] [P_\ell \otimes P_\ell], \quad (24)$$

where

$$P_\ell := \begin{pmatrix} I \\ 0 \end{pmatrix} \quad \text{and} \quad Q_\ell := K_{\Gamma\Gamma,\ell} - K_{\Gamma\Gamma,\ell}^T K_{II,\ell}^{-1} K_{\Gamma\Gamma,\ell}. \quad (25)$$

We point out that $\widehat{K}_\ell \otimes \widehat{K}_\ell$ can be efficiently inverted by exploiting its tensor product structure using the ideas from [4]. The second summand, $h_\ell^2 [P_\ell^T \otimes P_\ell^T] [Q_\ell \otimes Q_\ell] [P_\ell \otimes P_\ell]$, is a low-rank correction that lives only on the four corners of the domain. We can thus invert the smoother using the Sherman-Morrison-Woodbury formula and obtain

$$\widehat{\mathcal{K}}_\ell^{-1} = h_\ell^{-2} (I + ([\widehat{K}_\ell^{-1} P_\ell] \otimes [\widehat{K}_\ell^{-1} P_\ell]) \mathcal{R}_\ell^{-1} (P_\ell^T \otimes P_\ell^T)) (\widehat{K}_\ell^{-1} \otimes \widehat{K}_\ell^{-1}),$$

where P_ℓ and Q_ℓ are defined as in (25),

$$\mathcal{R}_\ell := Q_\ell^{-1} \otimes Q_\ell^{-1} - W_\ell^{-1} \otimes W_\ell^{-1}, \quad (26)$$

$$W_\ell := Q_\ell + h_\ell^{-2} (M_{\Gamma\Gamma,\ell} - M_{\Gamma\Gamma,\ell}^T M_{II,\ell}^{-1} M_{\Gamma\Gamma,\ell}), \quad (27)$$

and $M_{II,\ell}$, $M_{\Gamma\Gamma,\ell}$ and $M_{\Gamma\Gamma,\ell}$ are the blocks of the matrix M_ℓ if the degrees of freedom are grouped as in (12). For this choice, we can carry over the results from the previous section to the case of two-dimensional domains.

Lemma 5 For $\widehat{\mathcal{K}}_\ell = \mathcal{L}_\ell$ defined as in (24) there is a $\tau > 0$, which does not depend on the grid size h_ℓ and the spline degree p , such that the smoothing property

$$\|\mathcal{L}_\ell^{-1/2} \mathcal{K}_\ell (I - \tau \widehat{\mathcal{K}}_\ell^{-1} \mathcal{K}_\ell) \mathcal{L}_\ell^{-1/2}\| \leq \frac{C_S}{\nu}$$

is satisfied with $C_S = \tau^{-1}$.

Proof Lemma 2 states that the matrices in the one-dimensional setting satisfy

$$\|\underline{w}_\ell\|_{\mathcal{K}_\ell} \leq \sqrt{2}(1 + 4\sqrt{6}) \|\underline{w}_\ell\|_{\widehat{\mathcal{K}}_\ell}$$

for all $\underline{w}_\ell \in \mathbb{R}^{m_\ell}$. Using the tensor product structure of both \mathcal{K}_ℓ and $\widehat{\mathcal{K}}_{\ell,orig}$, this implies that we have

$$\|\underline{u}_\ell\|_{\mathcal{K}_\ell} \leq \sqrt{2}(1 + 4\sqrt{6}) \|\underline{u}_\ell\|_{\widehat{\mathcal{K}}_{\ell,orig}}$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. As $\widehat{\mathcal{K}}_{\ell,orig} \leq 2\widehat{\mathcal{K}}_\ell$, we obtain

$$\|\underline{u}_\ell\|_{\mathcal{K}_\ell} \leq 2(1 + 4\sqrt{6}) \|\underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell} = 2(1 + 4\sqrt{6}) \|\underline{u}_\ell\|_{\mathcal{L}_\ell}$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$ and finally

$$\|\mathcal{L}_\ell^{-1/2} \mathcal{K}_\ell \mathcal{L}_\ell^{-1/2}\| \leq 4(1 + 4\sqrt{6})^2.$$

The smoothing property then immediately follows from Lemma 1. \square

The next step is to show the approximation property. The proof again relies on results from the previous section.

Lemma 6 With \mathcal{L}_ℓ defined as in (24), the approximation property

$$\|\mathcal{L}_\ell^{1/2} (I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \mathcal{K}_\ell^{-1} \mathcal{L}_\ell^{1/2}\| \leq C_A,$$

where $\mathcal{I}_{\ell-1}^\ell := I_{\ell-1}^\ell \otimes I_{\ell-1}^\ell$ and $\mathcal{I}_\ell^{\ell-1} := I_\ell^{\ell-1} \otimes I_\ell^{\ell-1}$, is satisfied with a constant C_A which does not depend on the grid size h_ℓ and the spline degree p .

Proof Equation (18) in the proof of Lemma 4 states

$$\|(I - P_\ell) \underline{u}_\ell\|_{M_\ell} \leq 8\sqrt{2} h_\ell \|\underline{u}_\ell\|_{\mathcal{K}_\ell} \quad (28)$$

with $P_\ell := I_{\ell-1}^\ell K_{\ell-1}^{-1} I_\ell^{\ell-1} K_\ell$ for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell}$. Moreover, as the coarse-grid correction is K_ℓ -orthogonal (Galerkin principle), we have stability, i.e.,

$$\|(I - P_\ell) \underline{u}_\ell\|_{\mathcal{K}_\ell} \leq \|\underline{u}_\ell\|_{\mathcal{K}_\ell} \quad (29)$$

holds for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell}$. Combining (28) and (29), we obtain

$$\begin{aligned} \|(I - P_\ell \otimes P_\ell) \underline{u}_\ell\|_{M_\ell \otimes K_\ell}^2 &\leq 128 h_\ell^2 \|\underline{u}_\ell\|_{K_\ell \otimes K_\ell}^2, \\ \|(I - P_\ell \otimes P_\ell) \underline{u}_\ell\|_{K_\ell \otimes M_\ell}^2 &\leq 128 h_\ell^2 \|\underline{u}_\ell\|_{K_\ell \otimes K_\ell}^2 \end{aligned}$$

and therefore also

$$\|(I - P_\ell \otimes P_\ell) \underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \leq 256 h_\ell^2 \|\underline{u}_\ell\|_{K_\ell \otimes K_\ell}^2$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. As

$$\mathcal{K}_\ell \mathcal{M}_\ell^{-1} \mathcal{K}_\ell = K_\ell M_\ell^{-1} K_\ell \otimes M_\ell + 2K_\ell \otimes K_\ell + M_\ell \otimes K_\ell M_\ell^{-1} K_\ell \geq 2K_\ell \otimes K_\ell,$$

we obtain

$$\|(I - P_\ell \otimes P_\ell) \underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \leq 128 h_\ell^2 \|\underline{u}_\ell\|_{\mathcal{K}_\ell \mathcal{M}_\ell^{-1} \mathcal{K}_\ell}^2$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. Using the Galerkin principle, we obtain

$$\|(I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \leq 128 h_\ell^2 \|\underline{u}_\ell\|_{\mathcal{K}_\ell \mathcal{M}_\ell^{-1} \mathcal{K}_\ell}^2$$

and further

$$\|(I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \underline{u}_\ell\|_{h_\ell^{-2} \mathcal{M}_\ell}^2 \leq 128 \|\underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \quad (30)$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. Moreover, we know that the Galerkin projection is stable. Thus,

$$\|(I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \leq \|\underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \quad (31)$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. The combination of (30) and (31) yields

$$\|(I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \underline{u}_\ell\|_{h_\ell^{-2} \mathcal{M}_\ell + \mathcal{K}_\ell}^2 \leq 129 \|\underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2,$$

and further

$$\|(I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \underline{u}_\ell\|_{\widehat{\mathcal{K}}_\ell}^2 \leq 129 \|\underline{u}_\ell\|_{\mathcal{K}_\ell (h_\ell^{-2} \mathcal{M}_\ell + \mathcal{K}_\ell)^{-1} \mathcal{K}_\ell}^2 \quad (32)$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$. Using the definition of $\widehat{\mathcal{K}}_{\ell, \text{orig}}$ and (19), we obtain

$$\|\underline{u}_\ell\|_{\mathcal{K}_\ell (h_\ell^{-2} \mathcal{M}_\ell + \mathcal{K}_\ell)^{-1} \mathcal{K}_\ell}^2 \leq 2 \|\underline{u}_\ell\|_{\mathcal{K}_\ell \widehat{\mathcal{K}}_{\ell, \text{orig}}^{-1} \mathcal{K}_\ell}^2 \leq 2 \|\underline{u}_\ell\|_{\mathcal{K}_\ell \mathcal{L}_\ell^{-1} \mathcal{K}_\ell}^2,$$

for all $\underline{u}_\ell \in \mathbb{R}^{m_\ell^2}$, which implies in combination with (32) the estimate

$$\|\mathcal{K}_\ell^{1/2} (I - \mathcal{I}_{\ell-1}^\ell \mathcal{K}_{\ell-1}^{-1} \mathcal{I}_\ell^{\ell-1} \mathcal{K}_\ell) \mathcal{K}_\ell^{-1} \mathcal{L}_\ell^{1/2}\| \leq \sqrt{258}.$$

By squaring, the desired result follows using the Aubin-Nitsche duality trick analogously to the proof of Lemma 4. \square

Combining these results, we obtain the following theorem.

Theorem 6 *Using $\widehat{\mathcal{K}}_\ell = \mathcal{L}_\ell$ as defined in (24), we obtain*

$$\|\underline{u}_\ell^* - \underline{u}_\ell^{(1)}\|_{\mathcal{L}_\ell} \leq \frac{C_A C_S}{\nu} \|\underline{u}_\ell^* - \underline{u}_\ell^{(0)}\|_{\mathcal{L}_\ell}$$

with constants C_A and C_S which do not depend on the grid size h_ℓ and the spline degree p . Thus, the two-grid method for (22) converges if $\nu > C_A C_S$ smoothing steps are applied.

Proof Follows directly from the combination of Lemma 5 and Lemma 6. \square

The convergence of the two-grid method in the energy norm and of the multi-grid W-cycle can be shown as in the one-dimensional case.

Corollary 2 Using $\widehat{\mathcal{K}}_\ell = \mathcal{L}_\ell$ as defined in (24), the two-grid method for (22) with ν pre- and ν post-smoothing steps converges in the energy norm $\|\cdot\|_{\mathcal{K}_\ell}$ if $\nu > C_A C_S$.

Theorem 7 Using $\widehat{\mathcal{K}}_\ell = \mathcal{L}_\ell$ as defined in (24), the W-cycle multigrid method for (22) with ν pre- and ν post-smoothing steps converges in the energy norm $\|\cdot\|_{\mathcal{K}_\ell}$ if $\nu > 4C_A C_S$.

We omit the proofs for these two statements as they are completely analogous to the corresponding proofs in Subsections 4.4 and 4.5.

6 Numerical realization and results

6.1 Numerical realization

As the smoothing procedure seems to be rather complex, we give a pseudocode describing how the two-dimensional smoother based on the matrix $\widehat{\mathcal{K}}_\ell$ as defined in (24) can be implemented efficiently such that the overall multigrid method achieves optimal complexity.

function smoother()

- **given matrices:** one-dimensional mass matrix $M_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$, one-dimensional stiffness matrix $K_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$
- **input:** function value $\underline{u}_\ell^{(0,n)} \in \mathbb{R}^{m_\ell^2}$, corresponding residual $\bar{r}_\ell^{(0,n)} \in \mathbb{R}^{m_\ell^2}$
- Preparatory steps, only done at first call:
 1. Compute the Cholesky factorization of $K_{II,\ell}$.
 2. Determine $Q_\ell \in \mathbb{R}^{2p \times 2p}$ as defined in (25).
 3. Compute the Cholesky factorization of $M_{II,\ell}$.
 4. Determine $W_\ell \in \mathbb{R}^{2p \times 2p}$ as defined in (27).
 5. Determine $\mathcal{R}_\ell \in \mathbb{R}^{4p^2 \times 4p^2}$ as defined in (26).
 6. Compute the Cholesky factorization of \mathcal{R}_ℓ .
 7. Determine the sparse matrix $\widehat{\mathcal{K}}_\ell \in \mathbb{R}^{m_\ell \times m_\ell}$ as defined in (14).
 8. Compute the Cholesky factorization of $\widehat{\mathcal{K}}_\ell$.
- For each smoothing step do:
 1. Determine $\underline{q}_\ell^{(0)} := h_\ell^{-2}(\widehat{\mathcal{K}}_\ell^{-1} \otimes \widehat{\mathcal{K}}_\ell^{-1})\bar{r}_\ell$.
 2. Determine $\underline{q}_\ell^{(1)} := (P_\ell^T \otimes P_\ell^T)\underline{q}_\ell^{(0)}$.
 3. Determine $\bar{q}_\ell^{(2)} := \mathcal{R}_\ell^{-1}\underline{q}_\ell^{(1)}$ using the Cholesky factorization of \mathcal{R}_ℓ .
 4. Determine $\underline{q}_\ell^{(3)} := ([\widehat{\mathcal{K}}_\ell^{-1}P_\ell] \otimes [\widehat{\mathcal{K}}_\ell^{-1}P_\ell])\bar{q}_\ell^{(2)}$ using the Cholesky factorization of $\widehat{\mathcal{K}}_\ell$.
 5. Set $\underline{p}_\ell = \underline{q}_\ell^{(0)} + \underline{q}_\ell^{(3)}$.
 6. Update the function $\underline{u}_\ell^{(0,n+1)} := \underline{u}_\ell^{(0,n)} + \tau\underline{p}_\ell$.
 7. Update the residual $\bar{r}_\ell^{(0,n+1)} := \bar{r}_\ell^{(0,n)} - \tau\mathcal{K}_\ell\underline{p}_\ell$.
- **output:** function value $\underline{u}_\ell^{(0,n+1)}$, corresponding residual $\bar{r}_\ell^{(0,n+1)}$

The overall costs of the preparatory steps are $\mathcal{O}(m_\ell p^2 + p^6)$ floating point operations, where $m_\ell > p$ is the number of degrees of freedom in one dimension and m_ℓ^2 is the overall number of degrees of freedom.

Here, for the preparatory steps 1 and 3, $\mathcal{O}(m_\ell p^2)$ operations are required, as the dimension of the matrices $K_{II,\ell}$ and $M_{II,\ell}$ is $m_\ell - 2p = \mathcal{O}(m_\ell)$ and the bandwidth is $\mathcal{O}(p)$. For the preparatory steps 2 and 4, it is required to solve $\mathcal{O}(p)$ linear systems involving this factorization, which requires again $\mathcal{O}(m_\ell p^2)$ operations. The preparatory steps 5 and 6 just live on the vertices and require $\mathcal{O}(p^6)$ operations. The preparatory step 7 costs $\mathcal{O}(m_\ell p)$ operations just for adding K_ℓ and \tilde{K}_ℓ . The Cholesky factorization to be performed in the preparatory step 8 has – as in the steps 1 and 3 – a computational complexity of $\mathcal{O}(m_\ell p^2)$ operations.

The steps 1 and 4 of the smoother itself require $\mathcal{O}(m_\ell^2 p)$ operations each if the tensor product structure $(\hat{K}_\ell^{-1} \otimes \hat{K}_\ell^{-1}) = (I \otimes \hat{K}_\ell^{-1})(\hat{K}_\ell^{-1} \otimes I)$ is used (see [4] for the algorithmic idea). Step 2 of the smoother requires $\mathcal{O}(m_\ell p^2)$ operations. In step 3, only $\mathcal{O}(p^4)$ operations are required as \mathcal{R}_ℓ is a dense $4p^2 \times 4p^2$ -matrix. The steps 5 and 6, which are only adding up, can be completed with $\mathcal{O}(m_\ell^2)$ operations. Step 7 can be computed using the decomposition

$$\mathcal{K}_\ell = (I \otimes M_\ell)(K_\ell \otimes I) + (M_\ell \otimes I)(I \otimes K_\ell) \quad (33)$$

with $\mathcal{O}(m_\ell^2 p)$ operations, while a naive approach would require $\mathcal{O}(m_\ell^2 p^2)$ operations, i.e., the order of complexity would be as large as the number of non-zero entries of the matrix.

In summary, the preparatory steps require $\mathcal{O}(m_\ell p^2 + p^6)$ operations and the smoother itself requires $\mathcal{O}(m_\ell^2 p)$ operations. The overall costs are therefore $\mathcal{O}(m_\ell^2 p + p^6)$ or, assuming $p^5 \leq m_\ell^2$, $\mathcal{O}(m_\ell^2 p)$ operations. Note that the overall costs are therefore on the same order as the costs for multiplication with the matrix \mathcal{K}_ℓ . Note that also the coarse-grid correction can be completed with $\mathcal{O}(m_\ell^2 p)$ operations if the tensor product structure of the intergrid transfer matrices $\mathcal{I}_\ell^{\ell-1}$ and $\mathcal{I}_{\ell-1}^\ell$ is used. The method can be called asymptotically optimal since the overall multigrid solver requires $\mathcal{O}(m_\ell^2 p)$ operations, which is the same effort as for the matrix-vector product with \mathcal{K}_ℓ .

6.2 Experimental results

As a numerical example, we solve the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

on the domain $\Omega = (0, 1)^d$, $d = 1, 2$, where the right-hand side and boundary conditions are chosen in accordance with the exact solution

$$u(\mathbf{x}) = \prod_{j=1}^d \sin(\pi x_j).$$

We perform a (tensor product) B-spline discretization using uniformly sized knot spans and maximum-continuity splines for varying spline degrees p . We start from a coarse discretization with only a single interval and perform ℓ uniform, dyadic refinement steps to obtain a finer discretization.

We then set up a two-grid method according to the framework established in Section 3 and using the proposed smoother (14) for the one-dimensional domain and (24) for the two dimensional domain. We always use one pre- and one post-smoothing step. The damping parameter τ was chosen experimentally and has the value $\tau = 0.14$ for the one-dimensional domain and $\tau = 0.11$ for the two-dimensional domain, independently of p . For the two-dimensional domain, τ had to be slightly decreased to 0.10 for the cases $p \leq 3$.

We perform two-grid iteration until the Euclidean norm of the initial residual is reduced by a factor of 10^{-8} . The iteration numbers using different spline degrees p as well as different refinement levels ℓ for the one-dimensional domain are given in Table 1, and those for the two-dimensional domain in Table 2. As predicted by the theory, the iteration numbers remain uniformly bounded with respect to the spline degree p as well as the refinement level.

p	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20
$\ell = 10$	22	20	20	21	21	20	20	20	20	20	20	18	18	17	17
$\ell = 11$	23	20	20	21	20	20	20	20	20	19	19	18	19	18	17
$\ell = 12$	23	20	20	20	20	20	20	20	20	20	19	18	18	18	18

Table 1 Two-grid iteration numbers in 1D.

p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\ell = 5$	82	80	75	76	76	73	72	72	70	71	70	68	69	69	66
$\ell = 6$	83	87	76	74	75	73	72	72	70	70	69	68	68	67	65

Table 2 Two-grid iteration numbers in 2D.

7 Conclusions and outlook

We have analyzed in detail the convergence properties of a geometric multigrid method for a simple isogeometric model problem using B-splines. In a first step, we discussed a one-dimensional domain. Based on the obtained insights, in particular on the importance of boundary effects on the convergence rate, we have proposed a boundary-corrected mass-Richardson smoother. We then proved that this new smoother yields convergence rates which are robust with respect to the spline degree, a result which was not obtainable with the use of classical [13] or purely mass-based smoothers [11].

By exploiting the tensor product structure of spline spaces commonly used in IGA, we extended the construction of the smoother to the two-dimensional setting and again proved robust convergence for this case. We have shown how the proposed smoother can be efficiently realized such that the overall multigrid method has quasi-optimal complexity. Although we have restricted ourselves to simple tensor product domains, our technique is easily extended to problems with non-singular geometry mappings as discussed in Remark 1.

The extension of this approach to three or more dimensions remains open. In particular, the representation of the smoother as the Kronecker product of one-dimensional smoothers plus some low-rank correction as in (24) encounters some difficulties in this case. Therefore, the construction of robust smoothers for the three- and higher-dimensional cases is left as future work.

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