Monotone operators in nonlinear PDEs
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Exercise sheet 1

1. (Convergence principles) Let X be a Banach space. Then it holds:
   (a) If $x_n \rightharpoonup x$ weakly in X $(n \to \infty)$, then there is a constant c with $\|x_n\|_X \leq c$ for any $n \in \mathbb{N}$.
   (b) If $x_n \rightharpoonup x$ weakly in X $(n \to \infty)$ and $f_n \to f$ weakly in X^* $(n \to \infty)$, then it yields $\langle f_n, x_n \rangle_X \to \langle f, x \rangle_X$ $(n \to \infty)$.
   (c) If $x_n \to x$ weakly in X $(n \to \infty)$ and $f_n \rightharpoonup f$ weakly in X^* $(n \to \infty)$, then it yields $\langle f_n, x_n \rangle_X \to \langle f, x \rangle_X$ $(n \to \infty)$.
   (d) Let X be additionally reflexive and let $(x_n) \subset X$ be a bounded sequence. If all weakly converging subsequences of (x_n) converge to the same limit x, then the whole sequence (x_n) converges weakly to x.

2. (Inverse operator) Let X be a separable, reflexive Banach space and let $A : X \to X^*$ be strictly monotone, coercive, hemicontinuous. Then there exists the operator $A^{-1} : X^* \to X$ and it is strictly monotone and demicontinuous. Here, an operator $B : X^* \to X$ is called strictly monotone if it holds
$$\langle f - g, Bf - Bg \rangle_X > 0$$
for any $f \neq g \in X^*$.

3. (Radial continuous operator) Let X be a separable and reflexive Banach space. An operator $A : X \to X^*$ is called radially continuous, if $t \mapsto \langle A(u + tv), v \rangle_X$ is continuous for any $u, v \in V$.
   (a) (Minty's trick) If A is radially continuous and it holds $\langle f - Av, u - v \rangle_X \geq 0$ for any $v \in V$, then it yields $f = Au$.
   (b) If A is radially continuous and monotone, then it is demicontinuous. In particular, hemicontinuous, demicontinuous and radially continuous are equivalent for a monotone operator.

4. Consider the function $g : \mathbb{R} \to \mathbb{R}$ with
$$g(u) = \begin{cases} |u|^{p-2}u & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$
Show that:
   (a) For $p > 1$ the function g is strictly monotone.
   (b) For $p \geq 2$ it holds $\langle g(u) - g(v), u - v \rangle \geq c|u - v|^p$ for any $u, v \in \mathbb{R}$.
   (c) For $p = 2$ the function g is strongly monotone.

Hint: You may use that $c \sum_i |a_i|^p \leq \sum_i |a_i| \leq C \sum_i |a_i|^p$ *for any* $p > 0$. *Moreover, by Jensen's inequality* ($x \mapsto |x|^p$ *is convex) it holds* $|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$.