JOHANNES KEPLER UNIVERSITY LINZ

Exercise sheet 1

1. (a) For all $f \in X^*$, the sequence $(\langle f, x_n \rangle_X)$ is bounded because of the weak convergence of (x_n) we know that the sequence of real numbers $\langle f, x_n \rangle_X$ converges to $\langle f, x \rangle_X$. Therefore, we have $\sup_n |\langle f, x_n \rangle_X| \leq c(f)$. Using the canonical isometry $\iota : X \to X^{**}$, given by $\langle \iota x, f \rangle_{X^*} = \langle f, x \rangle_X$, it follows that the sequence $(\iota x_n) \subseteq X^{**}$ is pointwise bounded. The principle of uniform boundedness yields $\sup_n \|\iota x_n\|_{X^{**}} \leq c$. Using $\|\iota x_n\|_{X^{**}} = \|x_n\|_X$, the claim follows.

   (b) We have:

$$\begin{aligned} |\langle f_n, x_n \rangle_X - \langle f, x \rangle_X| &\leq |\langle f_n, x_n \rangle_X - \langle f, x_n \rangle_X| + |\langle f, x_n - x \rangle_X| \\ &\leq \| \|f_n - f\|_{X^*} \| \|x_n\|_X + |\langle f, x_n - x \rangle_X|. \end{aligned}$$

   Now, due to the given conditions: $\|f_n - f\|_{X^*} \to 0$ as $n \to \infty$, $|\langle f, x_n - x \rangle_X| \to 0$ as $n \to \infty$, and $\|x_n\|_X \leq c$ according to (i). Consequently, we have $|\langle f_n, x_n \rangle_X - \langle f, x \rangle_X| \to 0$ as $n \to \infty$.

   (c) The proof is analogous to the proof of claim (ii).

   (d) Proof by contradiction. If (x_n) does not weakly converge to x, i.e., there exist $f \in X^*$, $\epsilon > 0$, and a subsequence (x_{n_k}) such that $|\langle f, x_{n_k} \rangle_X - \langle f, x \rangle_X| \geq \epsilon$ for all $k \in \mathbb{N}$. According to the given assumption, the subsequence (x_{n_k}) is bounded. Therefore, by the Eberlein–Smulian theorem, there exists a sub-subsequence $(x_{n_{k_l}})$ that weakly converges, and, as per the assumption, it converges weakly to x. This leads to a contradiction. Hence, the claim holds.

2. If A is strictly monotone, we have

$$\langle A(u_1) - A(u_2), u_1 - u_2 \rangle_X = \langle f - f, u_1 - u_2 \rangle_X = 0, \tag{1}$$

which is possible only if $u_1 = u_2$. In other words, the equation $Au = f$ has a unique solution, so the inverse A^{-1} does exist. The mapping A^{-1} is strictly monotone: For $f_1, f_2 \in X^*$, where $f_1 \neq f_2$, put $u_1 = A^{-1}(f_1)$. Then $u_1 \neq u_2$. As A is strictly monotone, one has

$$\langle f_1 - f_2, A^{-1}(f_1) - A^{-1}(f_2) \rangle_X = \langle A(u_1) - A(u_2), u_1 - u_2 \rangle_X > 0.$$

The mapping A^{-1} is bounded: by the coercivity of A, there is $\zeta : \mathbb{R}^+ \to \mathbb{R}$ such that $\lim_{\xi \to \infty} \zeta(\xi) = +\infty$, and $\langle A(u), u \rangle_X \geq \|u\|_X \zeta(\|u\|_X)$. Therefore,

$$\zeta(\|u\|_X) \leq \langle A(u), u \rangle_X = \langle f, u \rangle_X \leq \|f\|_{X^*} \|u\|_X,$$

so that $\zeta(\|A^{-1}(f)\|_X) = \zeta(\|u\|_X) \leq \|f\|_{X^*}$. Thus, A^{-1} maps bounded sets in X^* into bounded sets in X. The mapping A^{-1} is demicontinuous: take $f_k \to f$ in X^*. As A^{-1} was shown to be bounded, the sequence $(A^{-1}(f_k))$ is bounded and (possibly up to a subsequence) $u_k = A^{-1}(f_k) \rightharpoonup u$ in X by the Eberlein–Smulian theorem. It remains to show $A(u) = f$. By the monotonicity of A, for any $v \in V$:

$$0 \leq \langle A(u_k) - A(v), u_k - v\rangle_X = \langle f_k - A(v), u_k - v\rangle_X. \tag{2}$$

Therefore, by the continuity of the duality pairing, passing to the limit with $k \to \infty$ yields

$$0 \leq \lim_{k\to\infty} \langle f_k - A(v), u_k - v\rangle_X = \langle f - A(v), u - v\rangle_X. \tag{3}$$

Then we apply the Minty's trick again, which gives $A(u) = f$. Thus, even the whole sequence (u_k) converges weakly.

3. (a) Replace v with $u + \epsilon w$ with $w \in X$ arbitrary. This gives

$$\langle f - A(u + \epsilon w), -\epsilon w\rangle_X \geq 0.$$

Divide it by $\epsilon > 0$ and pass to the limit with ϵ by using the radial continuity of A:

$$0 \geq \langle f - A(u + \epsilon w), w\rangle \to \langle f - A(u), w\rangle_X.$$

As w is arbitrary, one gets $A(u) = f$.

(b) Take a sequence (u_k) convergent to some $u \in X$. Then $(A(u_k))$ is bounded in X^*, and by the Eberlein-Smulian theorem, we can select a subsequence $(A(u_{k_l}))$ converging weakly to some $f \in X^*$. Then, by the monotonicity of A, we have

$$0 \leq \lim_{l\to\infty} \langle A(u_{k_l}) - A(v), u_{k_l} - v\rangle_X = \langle f - A(v), u - v\rangle_X.$$

As v is arbitrary, and we assume the radial continuity of A, Minty's trick (i) yields $f = A(u)$. Since f is thus determined uniquely, even the whole sequence $(A(u_k))$ must converge to it weakly.

4. (a) Consider the case of $p > 1$. It always holds $u \neq v$ in the following. First, we consider wlog $u \neq 0$ and $v = 0$, which gives

$$(g(u) - g(0))u = |u|^{p-2}u^2 = |u|^p > 0$$

for $u \neq 0$. Next, we consider $u \neq 0$ and $v \neq 0$. By a direct computation,

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle = |u|^p + |v|^p - |u|^{p-2}u \cdot v - |v|^{p-2}v \cdot u.$$

By Young's inequality, it holds

$$||u|^{p-2}u \cdot v| \leq |u|^{p-1}|v| \leq \frac{|u|^p}{p'} + \frac{|v|^p}{p},$$

where $p' = \frac{p}{p-1}$. Similarly, $||v|^{p-2}v \cdot u| \leq \frac{|v|^p}{p'} + \frac{|u|^p}{p}$. Hence,

$$-|u|^{p-2}u \cdot v - |v|^{p-2}v \cdot u \geq -|u|^p - |v|^p,$$

from which we conclude

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq \frac{1}{p}|u|^p + \frac{1}{p'}|v|^p \geq 0.$$

(b) Consider $p \geq 2$. If either $u = 0$ or $v = 0$, the result follows as shown in the proof of (i). Thus, we assume $u \neq 0$ and $v \neq 0$. For $p = 2$ it yields

$$(g(u) - g(v))(u - v) = (u - v)(u - v) = |u - v|^2.$$

In the case of $p > 2$, we make use of (i) to get

$$\langle |u|^{p-2}u - |v|^{p-2}v, u - v\rangle \geq \frac{1}{p}|u|^p + \frac{1}{p'}|v|^p,$$

and at this point we use Jensen's inequality $2^{p-1}(|u|^p + |v|^p) \geq |u - v|^p$ to conclude the statement.

(c) Already shown in the proof of (ii).