Symbolic local Fourier analysis for determining an approximation error estimate

This *Mathematica* notebook accompanies the paper

"Using cylindrical algebraic decomposition and local Fourier analysis to analyze numerical methods: two examples" by S. Takacs

A preprint version is available at

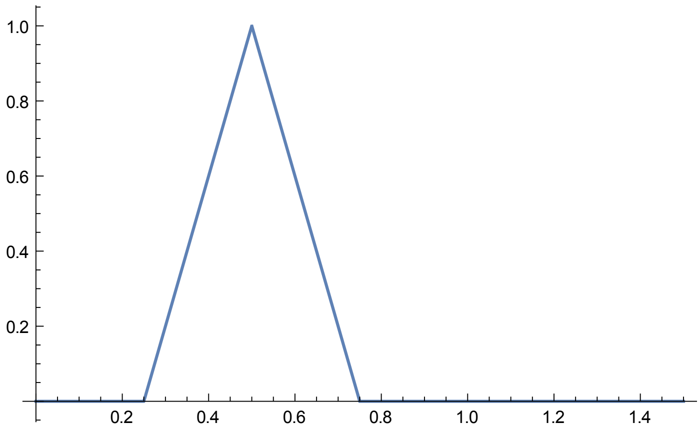
http://www.numa.uni-linz.ac.at/~stefant/J3362/slfa/

Case 1: The Courant element

First, we define the basis functions:

```
φk_,i_[x_] :=
 If[hk (i - 1) < x ≤ hk i, x / hk - (i - 1), 0] + If[hk i < x ≤ hk (i + 1), -x / hk + (i + 1), 0]
```

```
Plot[φ2,2[x] /. hk_ :→ 2^-k, {x, 0, 1.5}]
```



The next step is to compute the integrals that determine the mass matrix. As the support of the basis functions consists of two elements, we obtain a tri-diagonal matrix, where the diagonal entries and the off-diagonal entries have the following values:

```
Integrate[φk,i[x]^2, {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{2\,h_k}{3}$$

```
Integrate[φk,i[x] * φk,i+1[x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{h_k}{6}$$

So, we obtain that the mass matrix M_k has the following tri-diagonal form:

$$M_k = \frac{h_k}{6}\begin{pmatrix} 4 & 1 & & & & & & & & \\ 1 & 4 & 1 & & & & & & & \\ & 1 & 4 & 1 & & & & & & \\ & & 1 & 4 & 1 & & & & & \\ & & & 1 & 4 & 1 & & & & \\ & & & & 1 & 4 & 1 & & & \\ & & & & & 1 & 4 & 1 & & \\ & & & & & & 1 & 4 & 1 & \\ & & & & & & & 1 & 4 & 1 \\ & & & & & & & & 1 & 4 \end{pmatrix}$$

By multiplying M_k with the complex exponentials $\underline{\phi}_k(\theta) = \left(e^{i\,\theta\, j}\right)_j$ obtain in the j-th row:

$$\frac{h_k}{6}\left(1 * e^{i\,\theta\,(j-1)} + 4 * e^{i\,\theta\, j} + 1 * e^{i\,\theta\,(j+1)}\right) = \frac{h_k}{6}\left(4 + e^{-i\,\theta} + e^{i\,\theta}\right) e^{i\,\theta\, j}.$$

So, $M_k\, \underline{\phi}_k(\theta) = \frac{h_k}{6}\left(4 + e^{-i\,\theta} + e^{i\,\theta}\right) \underline{\phi}_k(\theta)$.

So, the symbol reads as follows:

```
Mk_[θ_] := hk/6 (4 + e^(-i θ) + e^(i θ))
```

Now we compute the integrals that determine the stiffness matrix. As the support of the basis functions consists of two elements, we obtain a tri-diagonal matrix, where the diagonal entries and the off-diagonal entries have the following values:

```
Integrate[D[φk,i[x], x]^2, {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{2}{h_k}$$

```
Integrate[D[φk,i[x], x] D[φk,i+1[x], x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$-\frac{1}{h_k}$$

So, we obtain that the stiffness matrix K_k has the following tri-diagonal form:

$$K_k = \frac{1}{h_k}\begin{pmatrix} 2 & -1 & & & & & & & & \\ -1 & 2 & -1 & & & & & & & \\ & -1 & 2 & -1 & & & & & & \\ & & -1 & 2 & -1 & & & & & \\ & & & -1 & 2 & -1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & -1 & 2 & -1 & \\ & & & & & & & -1 & 2 & -1 \\ & & & & & & & & -1 & 2 \end{pmatrix}$$

The symbol reads as follows:

```
Kk_[θ_] := 1/hk (2 - e^(-i θ) - e^(i θ))
```

The next step is to determine the intergrid-taransfer matrix, as outlined in the paper.

```
ϕk_[θ_, x_] := Sum[φk,i[x] Exp[i θ I], {i, -∞, ∞}]
```

$\phi_{k\_}[\theta\_, x\_] := \sum_{i=-\infty}^{\infty} \varphi_{k,i}[x]\ \mathrm{Exp}[i\ \theta\ I]$

Here, we have to solve the equations (11) and (13) from the paper:

$\mathrm{Simplify}[\phi_{k-1}[2\,\theta, 0] == A\,\phi_k[\theta, 0] + B\,\phi_k[\theta + \pi, 0], \mathrm{Assumptions} \to \{h_k > 0, h_{k-1} == 2\,h_k\}]$

$A + B == 1$

$\mathrm{Simplify}[\phi_{k-1}[2\,\theta, h_k] == A\,\phi_k[\theta, h_k] + B\,\phi_k[\theta + \pi, h_k],$
$\mathrm{Assumptions} \to \{h_k > 0, h_{k-1} == 2\,h_k\}]$

$\frac{1}{2}\left(1 + e^{2\,i\,\theta}\right) == (A - B)\, e^{i\,\theta}$

```
Solve[{%, %%}]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$\left\{\left\{A \to \frac{1}{4} e^{-i\,\theta}\left(1 + e^{i\,\theta}\right)^2, B \to -\frac{1}{4} e^{-i\,\theta}\left(-1 + e^{i\,\theta}\right)^2\right\}\right\}$

```
{A, B} /. (%[[1]]) // Expand
```

$\left\{\frac{1}{2} + \frac{e^{-i\,\theta}}{4} + \frac{e^{i\,\theta}}{4}, \frac{1}{2} - \frac{e^{-i\,\theta}}{4} - \frac{e^{i\,\theta}}{4}\right\}$

$P[\theta\_] := \left\{\left\{\frac{1}{2} + \frac{e^{-i\,\theta}}{4} + \frac{e^{i\,\theta}}{4}, \frac{1}{2} - \frac{e^{-i\,\theta}}{4} - \frac{e^{i\,\theta}}{4}\right\}\right\}$

As outlined in the paper, we have to set up also the symbols for the mass matrix and the stiffnes matrix for the two-dimensional basis:

$M2_{k\_}[\theta\_] := \mathrm{DiagonalMatrix}[\{M_k[\theta], M_k[\theta + \pi]\}]$

$K2_{k\_}[\theta\_] := \mathrm{DiagonalMatrix}[\{K_k[\theta], K_k[\theta + \pi]\}]$

Now, we check if the Galerkin identity is satisfied (up to scaling):

$\mathrm{FullSimplify}\left[P[\theta].M2_k[\theta].\mathrm{Transpose}[P[\theta]] == \frac{1}{2} M_{k-1}[2\,\theta],\right.$
$\left.\mathrm{Assumptions} \to \{h_k > 0, h_{k-1} == 2\,h_k\}\right]$

```
True
```

$\mathrm{FullSimplify}\left[P[\theta].K2_k[\theta].\mathrm{Transpose}[P[\theta]] == \frac{1}{2} K_{k-1}[2\,\theta],\right.$
$\left.\mathrm{Assumptions} \to \{h_k > 0, h_{k-1} == 2\,h_k\}\right]$

```
True
```

The symbol of the whole operator $G_k = \frac{1}{h_k^2}(I - \Pi_k)\,K_k^{-1}\,M_k$ reads as follows:

```
G_k_[θ_] = FullSimplify[
  (IdentityMatrix[2] - Transpose[P[θ]].Inverse[1/2 {{K_(k-1)[2 θ]}}].P[θ].K2_k[θ]).
    Inverse[K2_k[θ]].M2_k[θ] / h_k^2, Assumptions → {h_(k-1) == 2 h_k}]
```

$$\left\{\left\{\frac{1}{12}(2+\text{Cos}[\theta]),\ \frac{1}{12}(-2+\text{Cos}[\theta])\right\},\ \left\{\frac{1}{12}(-2-\text{Cos}[\theta]),\ \frac{1}{12}(2-\text{Cos}[\theta])\right\}\right\}$$

As this symbol has rank 1, the spectral radius is equal to to the sum of the eigenvalues, which coincides with the trace of the matrix:

```
Simplify[G_k[θ][[1, 1]] + G_k[θ][[2, 2]]]
```

$$\frac{1}{3}$$

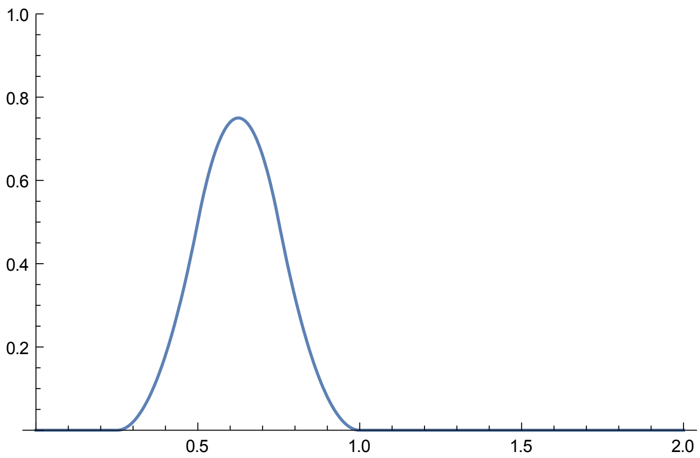
In this case, we do not have to determine the supremum anymore, because the spectral radius takes the value $\frac{1}{3}$ for all frequencies θ.

Case 2: A P^2-spline discretization

First, we define the basis functions:

```
φ_k_,i_[x_] := If[h_k (i - 1) < x ≤ h_k i, 1/(2 h_k^2) (x - h_k (i - 1))^2, 0] +
  If[h_k i < x ≤ h_k (i + 1), 3/4 - 1/(4 h_k^2) (2 x - h_k i - h_k (i + 1))^2, 0] +
  If[h_k (i + 1) < x ≤ h_k (i + 2), 1/(2 h_k^2) (x - h_k (i + 2))^2, 0]
```

```
Plot[φ_2,2[x] /. h_k_ :> 2^-k, {x, 0, 2}, PlotRange → {0, 1}]
```



The next step is to compute the integrals that determine the mass matrix. As the support of the basis functions consists of three elements, we obtain a multi-diagonal matrix, where the diagonal entries and the off-diagonal entries have the following values:

```
Integrate[φ_k,i[x]^2, {x, -∞, ∞}, Assumptions → h_k > 0]
```

$$\frac{11\,h_k}{20}$$

```
Integrate[φk,i[x] * φk,i+1[x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{13\, h_k}{60}$$

```
Integrate[φk,i[x] * φk,i+2[x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{h_k}{120}$$

So, we obtain that the mass matrix M_k has the following form:

$$M_k = \frac{h_k}{120}\begin{pmatrix} 66 & 26 & 1 & & & & & & & \\ 26 & 66 & 26 & 1 & & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & & \\ & & 1 & 26 & 66 & 26 & 1 & & & \\ & & & 1 & 26 & 66 & 26 & 1 & & \\ & & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & & 1 & 26 & 66 & 26 \\ & & & & & & & 1 & 26 & 66 \end{pmatrix}$$

The symbol reads as follows:

$$M_k\_[\theta\_] := \frac{h_k}{120}\left(66 + 26\, e^{-i\theta} + 26\, e^{i\theta} + e^{-2 i\theta} + e^{2 i\theta}\right)$$

Now we compute the integrals that determine the stiffness matrix. As the support of the basis functions consists of three elements, we obtain a multi-diagonal matrix, where the diagonal entries and the off-diagonal entries have the following values:

```
Integrate[D[φk,i[x], x]^2, {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\frac{1}{h_k}$$

```
Integrate[D[φk,i[x], x] D[φk,i+1[x], x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$-\frac{1}{3\, h_k}$$

```
Integrate[D[φk,i[x], x] D[φk,i+2[x], x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$$-\frac{1}{6\, h_k}$$

So, we obtain that the stiffness matrix K_k has the following form:

$$K_k = \frac{1}{6 h_k} \begin{pmatrix} 6 & -2 & -1 & & & & & & & \\ -2 & 6 & -2 & -1 & & & & & & \\ -1 & -2 & 6 & -2 & -1 & & & & & \\ & -1 & -2 & 6 & -2 & -1 & & & & \\ & & -1 & -2 & 6 & -2 & -1 & & & \\ & & & -1 & -2 & 6 & -2 & -1 & & \\ & & & & -1 & -2 & 6 & -2 & -1 & \\ & & & & & -1 & -2 & 6 & -2 & -1 \\ & & & & & & -1 & -2 & 6 & -2 \\ & & & & & & & -1 & -2 & 6 \end{pmatrix}$$

The symbol reads as follows:

$$K_{k\_}[\theta\_] := \frac{1}{6\, h_k} \left(6 - 2\, e^{-i\theta} - 2\, e^{i\theta} - e^{-2 i\theta} - e^{2 i\theta}\right)$$

The next step is to determine the intergrid-taransfer matrix, as outlined in the paper:

```
L = 10;
```

$$\phi_{k\_}[\theta\_, x\_] := \sum_{i=-L}^{L} \varphi_{k,i}[x]\ \mathrm{Exp}[i\,\theta\, I]$$

Here, we have to solve the equations (11) and (13) from the paper:

$$\mathrm{Simplify}[\phi_{k-1}[2\theta, 0] == A\,\phi_k[\theta, 0] + B\,\phi_k[\theta+\pi, 0],\ \mathrm{Assumptions} \to \{h_k > 0,\ h_{k-1} == 2\, h_k\}]$$

$$\frac{1}{2}\left(1 + e^{-2 i\theta}\right) == \frac{1}{2}\left(A + B + A\, e^{-i\theta} - B\, e^{-i\theta}\right)$$

$$\mathrm{Simplify}[\phi_{k-1}[2\theta, h_k] == A\ \phi_k[\theta, h_k] + B\,\phi_k[\theta+\pi, h_k],$$
$$\mathrm{Assumptions} \to \{h_k > 0,\ h_{k-1} == 2\, h_k\}]$$

$$1 + (6 - 4 A - 4 B)\, e^{2 i\theta} + e^{4 i\theta} == 4\ (A - B)\ e^{3 i\theta}$$

```
Solve[{%, %%}, {A, B}]
```

$$\left\{\left\{A \to \frac{1}{8}\, e^{-2 i\theta}\left(1 + e^{i\theta}\right)^3,\ B \to -\frac{1}{8}\, e^{-2 i\theta}\left(-1 + e^{i\theta}\right)^3\right\}\right\}$$

```
{A, B} /. (%[[1]]) // Expand
```

$$\left\{\frac{3}{8} + \frac{3\, e^{-i\theta}}{8} + \frac{e^{i\theta}}{8} + \frac{1}{8}\, e^{-2 i\theta},\ \frac{3}{8} - \frac{3\, e^{-i\theta}}{8} - \frac{e^{i\theta}}{8} + \frac{1}{8}\, e^{-2 i\theta}\right\}$$

$$P[\theta\_] := \left\{\left\{\frac{3}{8} + \frac{3\, e^{-i\theta}}{8} + \frac{e^{i\theta}}{8} + \frac{1}{8}\, e^{-2 i\theta},\ \frac{3}{8} - \frac{3\, e^{-i\theta}}{8} - \frac{e^{i\theta}}{8} + \frac{1}{8}\, e^{-2 i\theta}\right\}\right\}$$

As outlined in the paper, we have to set up also the symbols for the mass matrix and the stiffnes matrix for the two-dimensional basis:

$$M2_{k\_}[\theta\_] := \mathrm{DiagonalMatrix}[\{M_k[\theta],\ M_k[\theta+\pi]\}]$$

$$K2_{k\_}[\theta\_] := \mathrm{DiagonalMatrix}[\{K_k[\theta],\ K_k[\theta+\pi]\}]$$

Now we check if the Galerkin identity is satisfied (up to scaling):

```
FullSimplify[P[θ].M2k[θ].Conjugate[Transpose[P[θ]]] == {{1/2 Mk-1[2 θ]}},
  Assumptions → {hk > 0, hk-1 == 2 hk, θ ∈ Reals}]
```

True

```
FullSimplify[P[θ].K2k[θ].Conjugate[Transpose[P[θ]]] == {{1/2 Kk-1[2 θ]}},
  Assumptions → {hk > 0, hk-1 == 2 hk, θ ∈ Reals}]
```

True

The symbol of the whole operator $G_k = \frac{1}{h_k^2}(I - \Pi_k) K_k^{-1} M_k$ reads as follows:

```
Gk_[θ_] = FullSimplify[ (IdentityMatrix[2] -
     Conjugate[Transpose[P[θ]]].Inverse[1/2 {{Kk-1[2 θ]}}].P[θ].K2k[θ]).
    Inverse[K2k[θ]].M2k[θ] / hk^2, Assumptions → {hk > 0, hk-1 == 2 hk, θ ∈ Reals}]
```

$$\left\{\left\{-\frac{(-2+\mathrm{Cos}[\theta])\,(33+26\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}\left[\frac{\theta}{2}\right]^2}{80\,(2+\mathrm{Cos}[\theta])\,(2+\mathrm{Cos}[2\,\theta])},\ -\frac{i\,(65\,\mathrm{Sin}[\theta]-26\,\mathrm{Sin}[2\,\theta]+\mathrm{Sin}[3\,\theta])}{320\,(2+\mathrm{Cos}[2\,\theta])}\right\},\left\{\frac{i\,(65\,\mathrm{Sin}[\theta]+26\,\mathrm{Sin}[2\,\theta]+\mathrm{Sin}[3\,\theta])}{320\,(2+\mathrm{Cos}[2\,\theta])},\ -\frac{\mathrm{Cos}\left[\frac{\theta}{2}\right]^2(2+\mathrm{Cos}[\theta])\,(33-26\,\mathrm{Cos}[\theta]+\mathrm{Cos}[2\,\theta])}{80\,(-2+\mathrm{Cos}[\theta])\,(2+\mathrm{Cos}[2\,\theta])}\right\}\right\}$$

As this symbol has rank 1, the spectral radius is equal to to the sum of the eigenvalues, which coincides with the trace of the matrix:

```
spectralradius = Simplify[Gk[θ][[1, 1]] + Gk[θ][[2, 2]]]
```

$$\frac{-51+14\,\mathrm{Cos}[2\,\theta]+\mathrm{Cos}[4\,\theta]}{40\,(-2+\mathrm{Cos}[\theta])\,(2+\mathrm{Cos}[\theta])\,(2+\mathrm{Cos}[2\,\theta])}$$

Now, we rewrite this term as rational function by replacing cos(θ) by c:

```
spectralradius = spectralradius /. Cos[x_] :→ ChebyshevT[x / θ, c]
```

$$\frac{-50-8\,c^2+8\,c^4+14\,(-1+2\,c^2)}{40\,(-2+c)\,(2+c)\,(1+2\,c^2)}$$

Here, we can use CAD to determine the supremum:

```
Resolve[ForAll[c, -1 ≤ c ≤ 1, -λ ≤ spectralradius ≤ λ]]
```

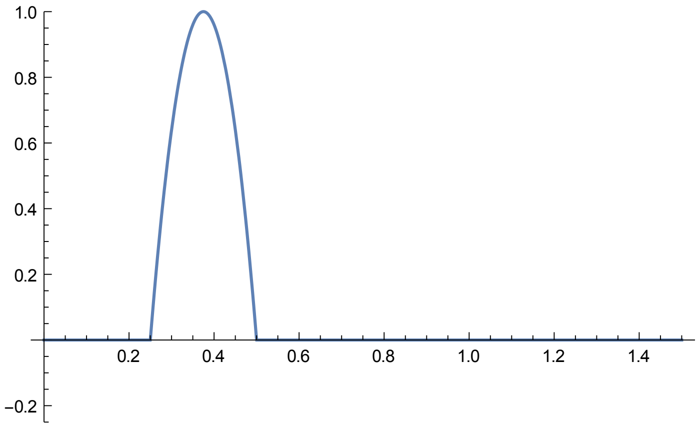
$$\lambda \geq \frac{2}{5}$$

So, we obtain that the supremum is $\frac{2}{5}$.

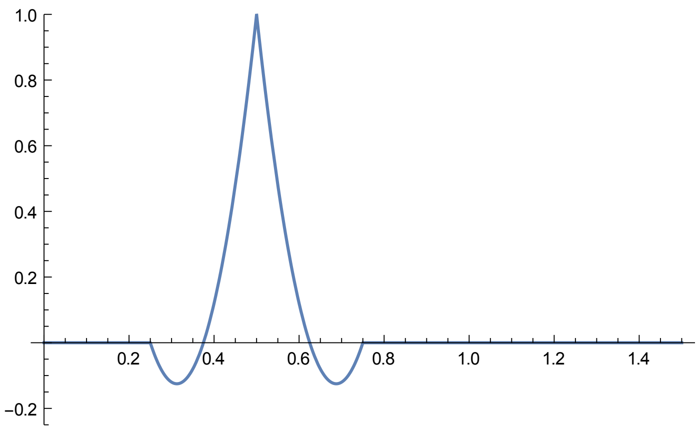
Case 3: A stanadard P^2 discretization

First, we define the basis functions:

```
φ[k_,i_,1][x_] := If[h[k] (i - 1) < x ≤ h[k] i, 2/h[k]^2 (x - h[k] (i - 1)) (x - h[k] (i - 1/2)), 0] +
  If[h[k] i < x ≤ h[k] (i + 1), 2/h[k]^2 (x - h[k] (i + 1)) (x - h[k] (i + 1/2)), 0]

φ[k_,i_,2][x_] := If[h[k] i < x ≤ h[k] (i + 1), -4/h[k]^2 (x - h[k] i) (x - h[k] (i + 1)), 0]

φ[k_,i_][x_] := If[Mod[i, 2] == 0, φ[k,i/2,1][x], φ[k,(i-1)/2,2][x]]

Plot[φ[2,3][x] /. h[k_] :> 2^-k, {x, 0, 1.5}, PlotRange → {-.25, 1}]
```




```
Plot[φ[2,4][x] /. h[k_] :> 2^-k, {x, 0, 1.5}, PlotRange → {-.25, 1}]
```



The next step is to compute the integrals that determine the mass matrix. Also here, we obtain a multi-diagonal matrix. However, as there are two kinds of basis functions, the coefficients are alternating:

```
Integrate[φ[k,i][x]^2, {x, -∞, ∞}, Assumptions → h[k] > 0]
```

$$\begin{cases} \frac{4 h_k}{15} & \text{Mod}[i, 2] == 0 \,\&\&\, h_k > 0 \\ \frac{8 h_k}{15} & 0 < \text{Mod}[i, 2] < 2 \,\&\&\, h_k > 0 \\ 0 & \text{True} \end{cases}$$

```
Integrate[φ[k,i][x] * φ[k,i+1][x], {x, -∞, ∞}, Assumptions → h[k] > 0]
```

$$\text{ConditionalExpression}\left[\frac{h_k}{15}, \text{Mod}[i, 2] == 0\right]$$

```
Integrate[φk,i[x] * φk,i+2[x], {x, -∞, ∞}, Assumptions → hk > 0]
```

```
ConditionalExpression[-hk/30, Mod[i, 2] == 0]
```

We obtain that the mass matrix M_k has the following form:

$$M_k = \frac{h_k}{30}\begin{pmatrix} 8 & 2 & -1 & & & & & & & \\ 2 & 16 & 2 & 0 & & & & & & \\ -1 & 2 & 8 & 2 & -1 & & & & & \\ & 0 & 2 & 16 & 2 & 0 & & & & \\ & & -1 & 2 & 8 & 2 & -1 & & & \\ & & & 0 & 2 & 16 & 2 & 0 & & \\ & & & & -1 & 2 & 8 & 2 & -1 & \\ & & & & & 0 & 2 & 16 & 2 & 0 \\ & & & & & & -1 & 2 & 8 & 2 \\ & & & & & & & 0 & 2 & 16 \end{pmatrix} =$$

$$\frac{h_k}{60}\begin{pmatrix} 24 & 4 & -1 & & & & & & & \\ 4 & 24 & 4 & -1 & & & & & & \\ -1 & 4 & 24 & 4 & -1 & & & & & \\ & -1 & 4 & 24 & 4 & -1 & & & & \\ & & -1 & 4 & 24 & 4 & -1 & & & \\ & & & -1 & 4 & 24 & 4 & -1 & & \\ & & & & -1 & 4 & 24 & 4 & -1 & \\ & & & & & -1 & 4 & 24 & 4 & -1 \\ & & & & & & -1 & 4 & 24 & 4 \\ & & & & & & & -1 & 4 & 24 \end{pmatrix} +$$

$$\frac{h_k}{60}\begin{pmatrix} -8 & 0 & -1 & & & & & & & \\ 0 & 8 & 0 & 1 & & & & & & \\ -1 & 0 & -8 & 0 & -1 & & & & & \\ & 1 & 0 & 8 & 0 & 1 & & & & \\ & & -1 & 0 & -8 & 0 & -1 & & & \\ & & & 1 & 0 & 8 & 0 & 1 & & \\ & & & & -1 & 0 & -8 & 0 & -1 & \\ & & & & & 1 & 0 & 8 & 0 & 1 \\ & & & & & & -1 & 0 & -8 & 0 \\ & & & & & & & 1 & 0 & 8 \end{pmatrix} =: A_k + B_k$$

By multiplying A_k with the complex exponentials $\underline{\phi}_k(\theta) = \left(e^{i\,\theta\, j}\right)_j$, we obtain in the j-th row:

$$\frac{h_k}{60}\left(-1 * e^{i\,\theta\,(j-2)} + 4 * e^{i\,\theta\,(j-1)} + 24 * e^{i\,\theta\, j} + 4 * e^{i\,\theta\,(j+1)} - 1 * e^{i\,\theta\,(j+2)}\right) =.$$

$$\frac{h_k}{60}\left(24 + 4\,e^{-i\,\theta} + 4\,e^{i\,\theta} - e^{-2\,i\,\theta} - 4\,e^{2\,i\,\theta}\right) e^{i\,\theta\, j}$$

So, $A_k\,\underline{\phi}_k(\theta) = \underbrace{\frac{h_k}{60}\left(24 + 4\,e^{-i\,\theta} + 4\,e^{i\,\theta} - e^{-2\,i\,\theta} - 4\,e^{2\,i\,\theta}\right)}_{=:\hat{A}_k(\theta)}\,\underline{\phi}_k(\theta)$

By multiplying B_k with the complex exponentials $\underline{\phi}_k(\theta) = \left(e^{i\theta j}\right)_j$, we obtain in the j-th row:

$$\frac{h}{60}\left((-1)^{j+1} * e^{i\theta(j-2)} + (-1)^{j+1} * 8 * e^{i\theta j} + (-1)^{j+1} * e^{i\theta(j+2)}\right) = \frac{h}{60}\left(-e^{-2i\theta} - 8 - e^{2i\theta}\right) e^{i(\theta+\pi)j}.$$

So, $B_k\,\underline{\phi}_k(\theta) = \underbrace{\frac{h_k}{60}\left(-e^{-2i\theta} - 8 - e^{2i\theta}\right)}_{=:\hat{B}_k(\theta)}\underline{\phi}_k(\theta+\pi)$

Concluding, we obtain

$$M_k\,\underline{\phi}_k(\theta) = \hat{A}_k(\theta)\,\phi_k(\theta) + \hat{B}_k(\theta)\,\underline{\phi}_k(\theta+\pi)$$

So, ob observe that $M_k\,\underline{\phi}_k(\theta)$ is not an element of span $\{\underline{\phi}_k(\theta)\}$, but of span $\{\underline{\phi}_k(\theta), \underline{\phi}_k(\theta+\pi)\}$.

So, the symbol has to be formulated in the basis $(\underline{\phi}_k(\theta), \underline{\phi}_k(\theta+\pi))$.

As we obviously also have

$$M_k\,\underline{\phi}_k(\theta+\pi) = \hat{B}_k(\theta+\pi)\,\underline{\phi}_k(\theta) + \hat{A}_k(\theta+\pi)\,\underline{\phi}_k(\theta+\pi),$$

we see immediately that

$$\hat{M}_k(\theta) = \begin{pmatrix} \hat{A}_k(\theta) & \hat{B}_k(\theta) \\ \hat{B}_k(\theta+\pi) & \hat{A}_k(\theta+\pi) \end{pmatrix}.$$

in the basis $(\underline{\phi}_k(\theta), \underline{\phi}_k(\theta+\pi))$.

So, we define as follows:

```
M_k_[θ_] :=
```

$$\frac{h_k}{60}\begin{pmatrix} 24 + 4\,e^{-i\theta} + 4\,e^{i\theta} - e^{-2i\theta} - e^{2i\theta} & -8 - e^{-2i\theta} - e^{2i\theta} \\ -8 - e^{-2i(\theta+\pi)} - e^{2i(\theta+\pi)} & 24 + 4\,e^{-i(\theta+\pi)} + 4\,e^{i(\theta+\pi)} - e^{-2i(\theta+\pi)} - e^{2i(\theta+\pi)} \end{pmatrix}$$

Now we compute the integrals that determine the stiffness matrix. As the support of the basis functions consists of three elements, we obtain a multi-diagonal matrix, where the diagonal entries and the off-diagonal entries have the following values:

```
Integrate[D[φk,i[x], x]^2, {x, -∞, ∞}, Assumptions → hk > 0]
```

$$\begin{cases} \frac{14}{3 h_k} & \text{Mod}[i, 2] == 0 \;\&\&\; h_k > 0 \\ \frac{16}{3 h_k} & 0 < \text{Mod}[i, 2] < 2 \;\&\&\; h_k > 0 \\ 0 & \text{True} \end{cases}$$

```
Integrate[D[φk,i[x], x] D[φk,i+1[x], x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$\text{ConditionalExpression}\left[-\frac{8}{3 h_k}, \text{Mod}[i, 2] == 0\right]$

```
Integrate[D[φk,i[x], x] D[φk,i+2[x], x], {x, -∞, ∞}, Assumptions → hk > 0]
```

$\text{ConditionalExpression}\left[\frac{1}{3 h_k}, \text{Mod}[i, 2] == 0\right]$

So, we obtain that the stiffness matrix K_k has the following form:

$$K_k = \frac{1}{3\,h_k}\begin{pmatrix} 14 & -8 & 1 & & & & & & & \\ -8 & 16 & -8 & 0 & & & & & & \\ 1 & -8 & 14 & -8 & 1 & & & & & \\ & 0 & -8 & 16 & -8 & 0 & & & & \\ & & 1 & -8 & 14 & -8 & 1 & & & \\ & & & 0 & -8 & 16 & -8 & 0 & & \\ & & & & 1 & -8 & 14 & -8 & 1 & \\ & & & & & 0 & -8 & 16 & -8 & 0 \\ & & & & & & 1 & -8 & 14 & -8 \\ & & & & & & & 0 & -8 & 16 \end{pmatrix} =$$

$$\frac{1}{6\,h_k}\begin{pmatrix} 30 & -16 & 1 & & & & & & & \\ -16 & 30 & -16 & 1 & & & & & & \\ 1 & -16 & 30 & -16 & 1 & & & & & \\ & 1 & -16 & 30 & -16 & 1 & & & & \\ & & 1 & -16 & 30 & -16 & 1 & & & \\ & & & 1 & -16 & 30 & -16 & 1 & & \\ & & & & 1 & -16 & 30 & -16 & 1 & \\ & & & & & 1 & -16 & 30 & -16 & 1 \\ & & & & & & 1 & -16 & 30 & -16 \\ & & & & & & & 1 & -16 & 30 \end{pmatrix} +$$

$$\frac{1}{6\,h_k}\begin{pmatrix} -2 & 0 & 1 & & & & & & & \\ 0 & 2 & 0 & -1 & & & & & & \\ 1 & 0 & -2 & 0 & 1 & & & & & \\ & -1 & 0 & 2 & 0 & -1 & & & & \\ & & 1 & 0 & -2 & 0 & 1 & & & \\ & & & -1 & 0 & 2 & 0 & -1 & & \\ & & & & 1 & 0 & -2 & 0 & 1 & \\ & & & & & -1 & 0 & 2 & 0 & -1 \\ & & & & & & 1 & 0 & -2 & 0 \\ & & & & & & & -1 & 0 & 2 \end{pmatrix} =: A + B$$

So, the symbol reads as follows:

$$\mathrm{K}_{k\_}[\theta\_] := \frac{1}{6\,h_k}\begin{pmatrix} 30 - 16\,e^{-i\theta} - 16\,e^{i\theta} + e^{-2i\theta} + e^{2i\theta} & -2 + e^{-2i\theta} + e^{2i\theta} \\ -2 + e^{-2i(\theta+\pi)} + e^{2i(\theta+\pi)} & 30 - 16\,e^{-i(\theta+\pi)} - 16\,e^{i(\theta+\pi)} + e^{-2i(\theta+\pi)} + e^{2i(\theta+\pi)} \end{pmatrix}$$

The next step is to determine the intergrid-taransfer matrix:

```
L = 10;
```

$$\phi_{k\_}[\theta\_, x\_] := \sum_{i=-L}^{L} \varphi_{k,i}[x]\ \mathrm{Exp}[i\,\theta\,I]$$

Here, we need 4 frequencies to be able to reconstruct a coarse-grid function on the fine grid:

```
Simplify[
 ϕk-1[2 θ, 0] == A0 ϕk[θ, 0] + A1 ϕk[θ + π / 2, 0] + A2 ϕk[θ + 2 π / 2, 0] + A3 ϕk[θ + 3 π / 2, 0],
 Assumptions → {hk > 0, hk-1 == 2 hk}]
```

$$A0 + A1 + A2 + A3 == 1$$

```
Simplify[ϕk-1[2 θ, hk / 2] == A0 ϕk[θ, hk / 2] + A1 ϕk[θ + π / 2, hk / 2] +
   A2 ϕk[θ + 2 π / 2, hk / 2] + A3 ϕk[θ + 3 π / 2, hk / 2], Assumptions → {hk > 0, hk-1 == 2 hk}]
```

$$\frac{1}{8}\left(3 + 6\, e^{2 i \theta} - e^{4 i \theta}\right) == (A0 + i\,(A1 + i\,A2 - A3))\, e^{i \theta}$$

```
Simplify[ϕk-1[2 θ, hk] ==
  A0 ϕk[θ, hk] + A1 ϕk[θ + π / 2, hk] + A2 ϕk[θ + 2 π / 2, hk] + A3 ϕk[θ + 3 π / 2, hk],
 Assumptions → {hk > 0, hk-1 == 2 hk}]
```

$$A0 + A2 == 1 + A1 + A3$$

```
Simplify[ϕk-1[2 θ, 3 hk / 2] ==
  A0 ϕk[θ, 3 hk / 2] + A1 ϕk[θ + π / 2, 3 hk / 2] + A2 ϕk[θ + 2 π / 2, 3 hk / 2] +
   A3 ϕk[θ + 3 π / 2, 3 hk / 2], Assumptions → {hk > 0, hk-1 == 2 hk}]
```

$$3\, e^{2 i \theta}\left(2 + e^{2 i \theta}\right) == 1 + 8\,(A0 - i\,A1 - A2 + i\,A3)\, e^{3 i \theta}$$

```
Solve[{%, %%, %%%, %%%%}, {A0, A1, A2, A3}]
```

$$\left\{\left\{A0 \to -\frac{1}{32} e^{-3 i \theta}\left(1 + e^{i \theta}\right)^4\left(1 - 4\, e^{i \theta} + e^{2 i \theta}\right),\ A1 \to \frac{1}{32}\, i\, e^{-3 i \theta}\left(-1 + e^{2 i \theta}\right)^3,\ A2 \to \frac{1}{32} e^{-3 i \theta}\left(-1 + e^{i \theta}\right)^4\left(1 + 4\, e^{i \theta} + e^{2 i \theta}\right),\ A3 \to -\frac{1}{32}\, i\, e^{-3 i \theta}\left(-1 + e^{2 i \theta}\right)^3\right\}\right\}$$

```
{a0, a1, a2, a3} = {A0, A1, A2, A3} /. (%[[1]]) // Expand
```

$$\left\{\frac{1}{2} + \frac{9\, e^{-i \theta}}{32} + \frac{9\, e^{i \theta}}{32} - \frac{1}{32} e^{-3 i \theta} - \frac{1}{32} e^{3 i \theta},\ \frac{3}{32}\, i\, e^{-i \theta} - \frac{3}{32}\, i\, e^{i \theta} - \frac{1}{32}\, i\, e^{-3 i \theta} + \frac{1}{32}\, i\, e^{3 i \theta},\ \frac{1}{2} - \frac{9\, e^{-i \theta}}{32} - \frac{9\, e^{i \theta}}{32} + \frac{1}{32} e^{-3 i \theta} + \frac{1}{32} e^{3 i \theta},\ -\frac{3}{32}\, i\, e^{-i \theta} + \frac{3}{32}\, i\, e^{i \theta} + \frac{1}{32}\, i\, e^{-3 i \theta} - \frac{1}{32}\, i\, e^{3 i \theta}\right\}$$

Here, the symbol of the intergrid transfer is not only a 4-dimensional vector. Becuase we already have 2x2-symbols on the coarse grid, the intergrid transfer is a 2x4-matrix, mapping between

$$\operatorname{span}\left\{\underline{\phi}_{k-1}(2\,\theta),\ \underline{\phi}_{k-1}(2\,\theta + \pi)\right\} \quad \text{and}$$
$$\operatorname{span}\left\{\underline{\phi}_{k}(\theta),\ \underline{\phi}_{k}(\theta + \pi/2),\ \underline{\phi}_{k}(\theta + \pi),\ \underline{\phi}_{k}(\theta + 3\,\pi/2)\right\}$$

The symbol reads as follows:

```
{{a0, a1, a2, a3}, {a3, a0, a1, a2} /. θ → θ + π / 2} // FullSimplify // MatrixForm
```

$$\begin{pmatrix} -\cos\left[\frac{\theta}{2}\right]^4 (-2 + \cos[\theta]) & \frac{\sin[\theta]^3}{4} & (2 + \cos[\theta])\sin\left[\frac{\theta}{2}\right]^4 & \ldots \\ -\frac{1}{4}\cos[\theta]^3 & \frac{1}{4}\left(\cos\left[\frac{\theta}{2}\right] - \sin\left[\frac{\theta}{2}\right]\right)^4 (2 + \sin[\theta]) & \frac{\cos[\theta]^3}{4} & -\frac{1}{4}\left(\text{Co}\ldots\right. \end{pmatrix}$$

$$P[\theta\_] := \begin{pmatrix} -\cos\left[\frac{\theta}{2}\right]^4 (-2 + \cos[\theta]) & \frac{\sin[\theta]^3}{4} & (2 + \cos[\theta])\sin\left[\frac{\theta}{2}\right]\ldots \\ -\frac{1}{4}\cos[\theta]^3 & \frac{1}{4}\left(\cos\left[\frac{\theta}{2}\right] - \sin\left[\frac{\theta}{2}\right]\right)^4 (2 + \sin[\theta]) & \frac{\cos[\theta]^3}{4} \end{pmatrix}$$

As outlined in the paper, we have to set up also the symbols for the mass matrix and the stiffnes matrix for the four-dimensional basis

$\left(\underline{\phi}_k(\theta), \underline{\phi}_k(\theta+\pi/2), \underline{\phi}_k(\theta+\pi), \underline{\phi}_k(\theta+3\pi/2)\right)$

Here, we define the symbol based on the symbol for the two-dimensional basis $\left(\underline{\phi}_k(\theta), \underline{\phi}_k(\theta+\pi)\right)$ as follows:

```
M2k_[θ_] := ( Mk[θ][[1, 1]]        0                    Mk[θ][[1, 2]]        0
              0                    Mk[θ + π / 2][[1, 1]]  0                  Mk[θ + π / 2][[1, 2]]
              Mk[θ][[2, 1]]        0                    Mk[θ][[2, 2]]        0
              0                    Mk[θ + π / 2][[2, 1]]  0                  Mk[θ + π / 2][[2, 2]] )

K2k_[θ_] := ( Kk[θ][[1, 1]]        0                    Kk[θ][[1, 2]]        0
              0                    Kk[θ + π / 2][[1, 1]]  0                  Kk[θ + π / 2][[1, 2]]
              Kk[θ][[2, 1]]        0                    Kk[θ][[2, 2]]        0
              0                    Kk[θ + π / 2][[2, 1]]  0                  Kk[θ + π / 2][[2, 2]] )
```

Now, we check if the Galerkin identity is satisfied (up to scaling):

```
FullSimplify[P[θ].M2k[θ].Transpose[P[θ]] == 1/2 Mk-1[2 θ],
 Assumptions → {hk > 0, hk-1 == 2 hk}]
```

```
True
```

```
FullSimplify[P[θ].K2k[θ].Transpose[P[θ]] == 1/2 Kk-1[2 θ],
 Assumptions → {hk > 0, hk-1 == 2 hk}]
```

```
True
```

The symbol of the whole operator $G_k = \frac{1}{h_k^2}(I - \Pi_k) K_k^{-1} M_k$ reads as follows:

```
G_k_[θ_] = FullSimplify[(IdentityMatrix[4] - FullSimplify[
    Transpose[P[θ]].FullSimplify[Inverse[1/2 K_{k-1}[2 θ]]].P[θ].K2_k[θ],
    Assumptions → {h_k > 0, h_{k-1} == 2 h_k}]).Inverse[K2_k[θ]].M2_k[θ] / h_k^2]
```

$$\Big\{\Big\{\frac{1}{960}(125+122\,\mathrm{Cos}[\theta]+9\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]^2,\ \frac{1}{3840}(29\,\mathrm{Cos}[\theta]-12\,\mathrm{Cos}[2\,\theta]-9\,\mathrm{Cos}[3\,\theta]-4\,(7+4\,\mathrm{Sin}[\theta]+23\,\mathrm{Sin}[2\,\theta])),$$
$$\frac{1}{960}\mathrm{Cos}\Big[\frac{\theta}{2}\Big]^2(-85+98\,\mathrm{Cos}[\theta]-9\,\mathrm{Cos}[2\,\theta]),\ \frac{1}{3840}(-28+29\,\mathrm{Cos}[\theta]-12\,\mathrm{Cos}[2\,\theta]-9\,\mathrm{Cos}[3\,\theta]+16\,\mathrm{Sin}[\theta]+92\,\mathrm{Sin}[2\,\theta])\Big\},$$
$$\Big\{\frac{1}{3840}(-28+16\,\mathrm{Cos}[\theta]+12\,\mathrm{Cos}[2\,\theta]-29\,\mathrm{Sin}[\theta]-92\,\mathrm{Sin}[2\,\theta]-9\,\mathrm{Sin}[3\,\theta]),$$
$$-\frac{(1+\mathrm{Sin}[\theta])\,(-125+9\,\mathrm{Cos}[2\,\theta]+122\,\mathrm{Sin}[\theta])}{1920},\ \frac{1}{3840}(-28-16\,\mathrm{Cos}[\theta]+12\,\mathrm{Cos}[2\,\theta]-29\,\mathrm{Sin}[\theta]+92\,\mathrm{Sin}[2\,\theta]-9\,\mathrm{Sin}[3\,\theta]),$$
$$-\frac{(-85+9\,\mathrm{Cos}[2\,\theta]-98\,\mathrm{Sin}[\theta])\,(-1+\mathrm{Sin}[\theta])}{1920}\Big\},$$
$$\Big\{-\frac{1}{960}(85+98\,\mathrm{Cos}[\theta]+9\,\mathrm{Cos}[2\,\theta])\,\mathrm{Sin}\Big[\frac{\theta}{2}\Big]^2,\ \frac{1}{3840}(-28-29\,\mathrm{Cos}[\theta]-12\,\mathrm{Cos}[2\,\theta]+9\,\mathrm{Cos}[3\,\theta]-16\,\mathrm{Sin}[\theta]+92\,\mathrm{Sin}[2\,\theta]),$$
$$\frac{1}{960}\mathrm{Cos}\Big[\frac{\theta}{2}\Big]^2(125-122\,\mathrm{Cos}[\theta]+9\,\mathrm{Cos}[2\,\theta]),\ \frac{1}{3840}(-28-29\,\mathrm{Cos}[\theta]-12\,\mathrm{Cos}[2\,\theta]+9\,\mathrm{Cos}[3\,\theta]+16\,\mathrm{Sin}[\theta]-92\,\mathrm{Sin}[2\,\theta])\Big\},$$
$$\Big\{\frac{1}{3840}(-28+16\,\mathrm{Cos}[\theta]+12\,\mathrm{Cos}[2\,\theta]+29\,\mathrm{Sin}[\theta]+92\,\mathrm{Sin}[2\,\theta]+9\,\mathrm{Sin}[3\,\theta]),$$
$$\frac{(1+\mathrm{Sin}[\theta])\,(-85+9\,\mathrm{Cos}[2\,\theta]+98\,\mathrm{Sin}[\theta])}{1920},\ \frac{1}{3840}(-28-16\,\mathrm{Cos}[\theta]+12\,\mathrm{Cos}[2\,\theta]+29\,\mathrm{Sin}[\theta]-92\,\mathrm{Sin}[2\,\theta]+9\,\mathrm{Sin}[3\,\theta]),$$
$$\frac{(-125+9\,\mathrm{Cos}[2\,\theta]-122\,\mathrm{Sin}[\theta])\,(-1+\mathrm{Sin}[\theta])}{1920}\Big\}\Big\}$$

```
Eigenvalues[%]
```

$$\Big\{\frac{1}{10},\ \frac{1}{30},\ 0,\ 0\Big\}$$

In this case, we do not have to determine the supremum anymore, because the spectral radius takes the value $\frac{1}{10}$ for all frequencies θ.