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Numerical Simulation of a Three-dimensional Bingham Fluid Flow

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Abstract

Non-Newtonian fluids, especially Bingham fluids, are widely used in practice. The numerical simulation of such flows involves many mathematical problems. In this thesis we present two numerical methods for computing a Bingham fluid flow in a three-dimensional domain. Both methods are based on an existing algorithm for a two-dimensional problem.

Starting from a classical formulation for several problems, we will derive mixed as well as dual dual formulations for two problems. We will proof existence and uniqueness of a solution. The two-dimensional algorithm is extended in two different ways into three dimensions. The discretization of both problems is then done by finite elements. Existence, uniqueness as well as error estimates are then shown. Numerical results for both realizations confirm the mathematical theory. At the end an example with practical background is presented.

Contents

Chapter 1

Introduction

The topic of this thesis was motivated by a cooperation with the company dTech Steyr. I am thankful for the support given by Mr. Peter Fischer and his employees, especially by Mr. Walter Hinterberger and Mr. Simon Schneiderbauer.

We will study the flow of a medium containing two phases. In some regions the medium acts like a fluid, in others the medium can be interpreted as a rigid one. Such non-Newtonian fluids are of high interest since they appear in almost every environment. Typical examples are paints, ketchup, tooth paste, mud, lava, snow, blood and many more. The thesis is mainly based on a paper from He and Glowinski ([18]). Therein an Uzawa-type algorithm for the stationary flow of a Bingham fluid in a two-dimensional domain is developed. Our goal is to construct a similar algorithm for an extended three-dimensional problem. The thesis is structured in the following way:

In chapter 2 we derive equations for a non-Newtonian fluid with yield stress. If a certain function of the stress passes a limit τ_c, the medium starts to flow, what means that there exists relations between the strain rates and the stress. But if the yield limit is not reached, no strain-rates exist. Then the fluid acts like a rigid medium. A few constitutive laws, modeling such a medium are presented. For our study we will choose a Bingham fluid. The classical formulation of such a fluid is then given. Using special boundary conditions as well as a special domain, we will also consider a two-dimensional problem.

In chapter 3 suitable weak formulations for the vector-valued problem in 3D, as well as for the scalar problem in 2D are given. We will see, that the usual mixed formulation involving a variational inequality of the second kind, has a non-differentiable term to handle. In order to get rid of this term we will derive a dual formulation for both cases. At the end of the chapter existence and uniqueness for our problems are shown.

In chapter 4 we follow the ideas from He and Glowinski (see [18]). An approximation of the original problem by a time dependent one is suggested. It is shown that an additional but necessary stabilization leads to an algorithm which converges to the desired solution and which works well also for large values of τ_c. We then carry over the results to the vector-valued three-dimensional problem. For this we will use two different approaches. On the one hand we will interpret (u, p) as the primal variable and λ as the dual variable, on the other hand we will interpret u as the primal variable and (p, λ) as the dual variable. Both interpretations lead to two different algorithms for the vector-valued problem.

In chapter 5 we use Galerkin's principle to discretize our problems with finite elements. The two-dimensional problem is discretized in the same way as it was done in [18], in order to compare our results. For the three-dimensional problem we will use a continuous approximation of the pressure for the first approach and a discontinuous approximation for the second approach, respectively. Existence and uniqueness of a discrete solution is shown. The convergence of the discrete solution is then proven and error estimates are developed for the velocity.

In the last chapter we will present our numerical results. First we will test the convergence properties in the two-dimensional case for various values of τ_c and compare them with the results from [18]. Further we will test the three-dimensional algorithm by choosing special parameters and prescribing the pressure. The three-dimensional problem is then computed. Next, we will compare our developed methods for the three-dimensional problem. Finally we will use our algorithm for a problem with practical background. In cooperation with dTech Steyr, we will study the behavior of a snow cover.

Chapter 2

Mathematical Model

2.1 Bingham fluid flow in a three-dimensional domain

We want to study the stationary flow of an incompressible two-phase fluid with yield limit. In this section we will derive equations describing the desired flow. Our domain of interest should be an open bounded set $\Omega^{3D} \subset \mathbb{R}^3$. As in fluid dynamics common, we will describe our fluid in Eulerian coordinates. Using them, we can write down the equations of motion for the steady state case, the so called balance equations.

$$-\,\mathbf{div}\,\sigma = f(x) \tag{2.1}$$

with a volume force

$$f : \Omega^{3D} \to \mathbb{R}^3, \quad x \mapsto f(x)$$

and the stress tensor

$$\sigma : \Omega^{3D} \to \mathbb{R}^{3\times 3}, \quad u \mapsto \sigma(u).$$

In (2.1), the operator on the left hand side is defined by

$$\mathbf{div}\,\sigma := \begin{pmatrix} \operatorname{div}(\sigma_{11},\, \sigma_{12},\, \sigma_{13}) \\ \operatorname{div}(\sigma_{21},\, \sigma_{22},\, \sigma_{23}) \\ \operatorname{div}(\sigma_{31},\, \sigma_{32},\, \sigma_{33}) \end{pmatrix}.$$

Moreover for an incompressible fluid, it follows that the incompressibility condition

$$\operatorname{div} u = 0$$

has to hold. These equations are valid for all sort of continua. Further it can be shown (see [24]) that the stress tensor σ can be decomposed in a spherical part as well as a deviatoric part:

$$\sigma = -pI + \tau.$$

In the case of a fluid the variable p denotes the hydrostatic pressure. The tensor τ is called the deviator or the deviatoric stress.

Constitutive law

Up to now we have characterized an arbitrary fluid. In order to produce a closed model we have to classify the properties of the fluid. In many fluids this is done by a given relationship between the deviatoric stress τ and the tensor of strain rates.

Definition. The tensor of strain-rates is defined by

$$D_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for } i = 1, 2, 3.$$

Definition. The second invariant of the tensor of strain-rates is defined, using the Einstein summation convention, by

$$D_{\mathrm{II}} = \frac{1}{2} D_{ij} D_{ij}$$

and the second invariant of the deviatoric stress is defined by

$$\tau_{\mathrm{II}} = \frac{1}{2} \tau_{ij} \tau_{ij}$$

respectively.

Now a first model would be, if the deviatoric stress only depends linearly on the tensor of strain-rates. So

$$\tau = \mu D(u).$$

We then call the fluid *Newtonian*. The constant μ is called the viscosity. A more general class of fluids is the ones of *generalized (or non-) Newtonian models*, where τ is depending nonlinearly on u. For example the function μ can depend on the second invariant of the tensor of strain rates D_{II}:

$$\tau = \mu(D_{\mathrm{II}}) D(u).$$

A class of models doing so, is the class of *Herschel-Bulkley fluids*. This model is characterized by

$$\begin{aligned} &\tau = (2\mu D_{\mathrm{II}}^{n-1} + \tau_c/(D_{\mathrm{II}})^{1/2}) D(u) && \Leftrightarrow \tau_{\mathrm{II}}^{1/2} > \tau_c \\ &D(u) = 0 && \Leftrightarrow \tau_{\mathrm{II}}^{1/2} \leq \tau_c, \end{aligned} \tag{2.2}$$

where the variable τ_c is called the yield limit. So if a certain stress function (here $\tau_{\mathrm{II}}^{1/2}$) is under the given threshold τ_c, the medium acts like a rigid body since there are no strain rates. But if the stress increases and the yield limit is passed, the medium starts to flow following the constitutive law (2.2).

There is now the question which kind of Herschel-Bulkley model we choose. Here we are going to study the case $n = 1$. A fluid described by such a law is then called *Bingham fluid*. It is characterized by

$$\begin{aligned} &\tau = 2\mu D(u) + \tau_c D(u)/(D_{\mathrm{II}})^{1/2} && \Leftrightarrow \tau_{\mathrm{II}}^{1/2} > \tau_c \\ &D(u) = 0 && \Leftrightarrow \tau_{\mathrm{II}}^{1/2} \leq \tau_c. \end{aligned}$$

There are also variants of the Bingham model, for example a biviscous Bingham model in which two viscosities are considered (see [10]). But we do not want to go in detail here and refer to e.g. [27] for more modified models, to [3] for a general characterization of constitutive laws and to [30] for a comparison of some models for a practical problem. Further for more flow properties of Bingham fluids especially if the external loads vary, see [4].

Boundary Conditions

In order to complete our model, we need appropriate boundary conditions. For this we separate our boundary into three parts.

$$\partial\Omega^{3D} := \Gamma^{3D} = \Gamma_D^{3D} \cup \Gamma_N^{3D} \cup \Gamma_P^{3D}$$

with

$$\Gamma_D^{3D} \cap \Gamma_N^{3D} = \varnothing, \quad \Gamma_D^{3D} \cap \Gamma_P^{3D} = \varnothing, \quad \Gamma_P^{3D} \cap \Gamma_N^{3D} = \varnothing.$$

Now several boundary conditions are possible. For instance a part of the boundary, let us say Γ_D^{3D}, could defined by a wall. In this case we get a so called *no-slip* condition, i.e.

$$u = 0 \quad \text{on } \Gamma_D^{3D}.$$

Further on another part the traction $g(x)$, $x \in \Gamma_N^{3D}$ is known, so

$$\sigma \cdot n = g(x) \quad \text{on } \Gamma_N^{3D}.$$

Often the domain Ω^{3D} can be interpreted as a cylinder or prism with a large length let us say in z-direction. Simulating such a domain can be done by special boundary conditions for u and p. We assume that

$$\Omega^{3D} = \Omega \times (0, L),$$

where Ω denotes the cross section and L the length of the prism. Let Γ_P^{3D} be the part on which special boundary conditions are needed, then

$$\Gamma_P^{3D} := \Omega \times \{0\} \cup \Omega \times \{L\}.$$

The conditions on u and p then read as

$$u_{|_{z=0}} = u_{|_{z=L}} \quad \text{on } \Gamma_P^{3D}$$

and

$$p_{|z=0} = p_{|z=L} + h(x) \quad \text{on } \Gamma_P^{\mathrm{3D}},$$

respectively. The function $h(\cdot)$ is representing a jump on the interface. According to [29] we call these conditions periodic boundary conditions. Summarizing our results, a Bingham fluid flow (in a special domain Ω^{3D}) can be modeled by the following

Problem 2.1 (Classical Formulation)**.** *Find*

$$u \in C^2(\Omega^{3D}) \cap C(\partial\Omega^{3D}),\; p \in C^1(\Omega^{3D})$$

such that

$$\begin{aligned}
&\nabla p - \boldsymbol{div}\, \tau = f && \text{in } \Omega^{3D}\\
&\tau = 2\mu D(u) + \tau_c D(u)/(D_{II})^{1/2} && \Leftrightarrow \tau_{II}^{1/2} > \tau_c\\
&D(u) = 0 && \Leftrightarrow \tau_{II}^{1/2} \leq \tau_c\\
&\operatorname{div} u = 0 && \text{in } \Omega^{3D}\\
&u = 0 && \text{on } \Gamma_D^{3D}\\
&\sigma n = g(x) && \text{on } \Gamma_N^{3D}\\
&u_{|z=0} = u_{|z=L} && \text{on } \Gamma_P^{3D}\\
&p_{|z=0} = p_{|z=L} + i(x) && \text{on } \Gamma_P^{3D}.
\end{aligned}$$

2.2 Reduction on a 2D problem

Motivated by the work of He and Glowinski ([18]) we are going to set up boundary conditions and right hand side in such a way that we can reduce the three-dimensional problem to a two-dimensional one. For this our domain Ω^{3D} is assumed to be a prismatic domain with infinite length in z-direction. In order to simulate the flow along z-direction, we can take a domain with finite length and set appropriate periodic boundary conditions. So let

$$\Omega^{\mathrm{3D}} = \Omega \times (0, 1)$$

with quadratic cross section Ω in the xy-plane. The boundary is split into two parts

$$\Gamma^{\mathrm{3D}} = \Gamma_D^{\mathrm{3D}} \cup \Gamma_P^{\mathrm{3D}},$$

so $\Gamma_N^{\mathrm{3D}} = \varnothing$. On the one hand we consider a wall around the prism, so we have a no-slip condition on it:

$$u = 0 \quad \text{on } \Gamma_D^{\mathrm{3D}},$$

where

$$\Gamma_D^{3D} = \partial\Omega \times (0,1).$$

On the rest of the boundary

$$\Gamma_P^{3D} := \Omega \times \{0\} \cup \Omega \times \{1\}$$

we set homogenous periodic boundary conditions for the velocity u, i.e.

$$u(x,y,0) = u(x,y,1) \quad \text{on } \Gamma_P^{3D}.$$

For the pressure p we assume a constant drop along the z-axis. Let c be the drop in pressure per unit length, then we consider the following periodic boundary condition on p:

$$p(x,y,0) = p(x,y,1) + c \quad \text{on } \Gamma_P^{3D}.$$

We assume the flow to be laminar, which is automatically fulfilled if c is not too large (see [11]). Especially let $f = 0$. The model is then given by

Problem 2.2 (Classical Formulation with special boundary conditions)**.** *Find*

$$u \in C^2(\Omega^{3D}) \cap C(\partial\Omega^{3D}),\ p \in C^1(\Omega^{3D})$$

such that

$$\begin{aligned}
&\nabla p - \boldsymbol{div}\,\tau = 0 && \textit{in } \Omega^{3D}\\
&\tau = 2\mu D(u) + \tau_c D(u)/(D_{II})^{1/2} && \Leftrightarrow \tau_{II}^{1/2} > \tau_c\\
&D(u) = 0 && \Leftrightarrow \tau_{II}^{1/2} \le \tau_c\\
&\operatorname{div} u = 0 && \textit{in } \Omega^{3D}\\
&u = 0 && \textit{on } \Gamma_D^{3D}\\
&u(x,y,0) = u(x,y,1) && \textit{on } \Gamma_P^{3D}\\
&p(x,y,0) = p(x,y,1) + c && \textit{on } \Gamma_P^{3D}.
\end{aligned}$$

The following results can be found in [11].

Notation. For $u = (u_1, u_2)$ we denote the absolute value by $|\cdot|$, i.e.

$$|u| = \sqrt{u_1^2 + u_2^2}.$$

Under the assumptions made on the flow, it follows that our velocity field has the form $(0, 0, u_3(x,y))$. So our strain-rate tensor can be written as

$$D(u) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial y} \\ \frac{1}{2}\frac{\partial u_3}{\partial x} & \frac{1}{2}\frac{\partial u_3}{\partial y} & 0 \end{pmatrix} \tag{2.3}$$

and therefore the deviatoric stress tensor τ has the special form

$$\tau = \begin{pmatrix} 0 & 0 & \tau_{13} \\ 0 & 0 & \tau_{23} \\ \tau_{31} & \tau_{32} & 0 \end{pmatrix},$$

where the entries τ_{ij} only depend on x, y. If we consider now problem 2.2, the first three equations reduces to

$$\frac{\partial p}{\partial x} = 0 \tag{2.4}$$

$$\frac{\partial p}{\partial y} = 0 \tag{2.5}$$

$$\frac{\partial p}{\partial z} = \frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{32}}{\partial y}. \tag{2.6}$$

If we take a look at (2.6), the left hand side depends only on z because of (2.4) and (2.5), whereas the right hand side depends only on x, y. Therefore the pressure must be of the form

$$p = kz, \quad k \in \mathbb{R}$$

and as we assumed boundary conditions for p, this leads to

$$p = -cz.$$

In the case $\tau_{\text{II}}^{1/2} \leq \tau_c$ it follows from (2.3) that $D(u) = 0$ reduces to

$$\nabla u_3 = 0.$$

Because of the special form of the flow, the incompressibility condition is automatically fulfilled. Now together with

$$D_{\text{II}}^{1/2} = \frac{1}{2}|\nabla u_3|$$

and (2.3) it follows

$$\begin{aligned} \tau_{13} &= \mu \frac{\partial u_3}{\partial x} + \tau_c |\nabla u_3|^{-1} \frac{\partial u_3}{\partial x}, \\ \tau_{23} &= \mu \frac{\partial u_3}{\partial y} + \tau_c |\nabla u_3|^{-1} \frac{\partial u_3}{\partial y} \end{aligned}$$

for the case $\tau_{\text{II}}^{1/2} > \tau_c$. Now we can write down the classical formulation of our reduced problem:

Problem 2.3 (Classical formulation for the two-dimensional case)**.** *Find*

$$u \in C^2(\Omega) \cap C(\partial\Omega)$$

such that

$$
\begin{aligned}
&-\mu\Delta u - \tau_c \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = c && \Leftrightarrow \tau_{II}^{1/2} > \tau_c \\
&\nabla u = 0 && \Leftrightarrow \tau_{II}^{1/2} \leq \tau_c \\
&u = 0 && \textit{on } \partial\Omega = \Gamma
\end{aligned}
$$

Chapter 3

Variational formulations

We are now going to derive weak formulations for which we will have existence and uniqueness of a solution. We will formulate variational problems for the 3D problem 2.2 as well as for the reduced 2D problem 2.3. Further we will give dual formulations for both.

Throughout this thesis let V, V^{3D}, Q be Hilbert spaces with scalar products $(\cdot,\cdot)_V$,$(\cdot,\cdot)_{V^{3D}}$, $(\cdot,\cdot)_Q$ inducing norms, $\|\cdot\|_V$,$\|\cdot\|_{V^{3D}}$ and $\|\cdot\|_Q$ respectively. With V^*,V^{3D*}, Q^* we will denote their dual spaces, where $\langle\cdot,\cdot\rangle$ is the duality pairing. Moreover we will denote the L^2- scalar product by $(\cdot,\cdot)$.

Notation. We will denote the H^1-norm by $\|\cdot\|_1$ and the H^1-semi norm with $|\cdot|_1$ respectively. The L^2-norm will be denoted as $\|\cdot\|_0$.

3.1 Primal formulations

3.1.1 The three-dimensional problem

For the sake of simplicity we will denote the vector valued solution u^{3D} of the three-dimensional problem also by u. Note that in the next subsection u is the scalar solution of the reduced two-dimensional problem. For the derivation of the variational formulation we used [20],[21],[11].

We choose arbitrary but fixed functions $v \in V_0^{3D}$ and $q \in Q$ respectively, fulfilling

$$v = 0 \quad \text{on } \Gamma_D^{3D},$$

and

$$v_{|P_1} = v_{|P2} \quad \text{on } \Gamma_P^{3D}.$$

Then $v-u$ is a test function in V_0^{3D}. The test spaces V_0^{3D} and Q are specified later. Next we introduce Ω_1^{3D},Ω_0^{3D} in such a way that

$$\Omega^{3D} = \Omega_1^{3D} \cup \Omega_0^{3D}, \quad \Omega_1^{3D} \cap \Omega_0^{3D} = \varnothing, \tag{3.1}$$

where

$$\Omega_1^{\mathrm{3D}} := \{x \in \Omega^{\mathrm{3D}} \,\big|\, D(u) \neq 0\}$$

and

$$\Omega_0^{\mathrm{3D}} := \{x \in \Omega^{\mathrm{3D}} \,\big|\, D(u) = 0\}.$$

So Ω_0^{3D} is the region where $\tau_{\mathrm{II}}^{1/2} \leq \tau_c$, i.e. the fluid acts like a rigid medium. Ω_1^{3D} is then the set where τ_c is passed, so where it acts like a fluid. Note that either Ω_1^{3D} or Ω_0^{3D} can be empty. If we multiply now our equations by $v - u \in V_0^{\mathrm{3D}}, q \in Q$ and integrate over the whole domain, we get for the incompressibility condition

$$\int_{\Omega^{\mathrm{3D}}} q \operatorname{div} u \, dx = 0 \quad \forall q \in Q. \tag{3.2}$$

and for the balance equations

$$\int_{\Omega^{\mathrm{3D}}} - \mathbf{div}\, \sigma \cdot (v - u) \, dx = \int_{\Omega^{\mathrm{3D}}} f \cdot (v - u) \, dx \quad \forall v \in V_0^{\mathrm{3D}}. \tag{3.3}$$

By Gauss' theorem and the fact that σ is a symmetric tensor, the integral on the left hand side can be written as

$$\int_{\Omega^{\mathrm{3D}}} \sigma : D(v - u) \, dx - \int_{\partial\Omega^{\mathrm{3D}}} (\sigma n) \cdot (v - u) \, ds. \tag{3.4}$$

So (3.3) is equivalent to

$$\begin{aligned} - \int_{\Omega^{\mathrm{3D}}} p \operatorname{div}(v - u) \, dx + \int_{\Omega^{\mathrm{3D}}} \tau : D(v - u) \, dx - \int_{\partial\Omega^{\mathrm{3D}}} (\sigma n) \cdot (v - u) \, ds \\ = \int_{\Omega^{\mathrm{3D}}} f \cdot (v - u) \, dx \quad \forall\, v \in V_0^{\mathrm{3D}}, \end{aligned} \tag{3.5}$$

where we have used that the trace of D is just the divergence.

Now let us take a look on the second integral. According to the linearity of $D(\cdot)$ we have

$$\int_{\Omega^{\mathrm{3D}}} \tau : D(v - u) \, dx = \int_{\Omega^{\mathrm{3D}}} \tau : D(v) \, dx - \int_{\Omega^{\mathrm{3D}}} \tau : D(u) \, dx. \tag{3.6}$$

From (3.1) we get

$$\int_{\Omega^{\mathrm{3D}}} \tau : D(v) \, dx = \int_{\Omega_1^{\mathrm{3D}}} \tau : D(v) \, dx + \int_{\Omega_0^{\mathrm{3D}}} \tau : D(v) \, dx.$$

Now in Ω_1^{3D} we can use the definition of the tensor τ so that we have

$$\int_{\Omega_1^{\mathrm{3D}}} \tau : D(v) \, dx = 2\mu \int_{\Omega_1^{\mathrm{3D}}} D(u) : D(v) \, dx + \tau_c \int_{\Omega_1^{\mathrm{3D}}} D_{\mathrm{II}}^{-1/2}(u) D(u) : D(v) \, dx.$$

With the well known Cauchy-inequality it follows that

$$D(u) : D(v) \leq 2D_{\mathrm{II}}^{1/2}(u)D_{\mathrm{II}}^{1/2}(v), \tag{3.7}$$

we further have

$$\tau_c \int_{\Omega_1^{3\mathrm{D}}} D_{\mathrm{II}}^{-1/2}(u)D(u) : D(v)\, dx \leq \tau_c \int_{\Omega_1^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(v)\, dx$$

so that

$$\int_{\Omega_1^{3\mathrm{D}}} \tau : D(v)\, dx \leq 2\mu \int_{\Omega_1^{3\mathrm{D}}} D(u) : D(v)\, dx + \tau_c \int_{\Omega_1^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(v)\, dx.$$

In $\Omega_0^{3\mathrm{D}}$ we use the fact that $\tau_{\mathrm{II}}^{1/2} \leq \tau_c$. So applying inequality (3.7) once more we have

$$\int_{\Omega_0^{3\mathrm{D}}} \tau : D(v)\, dx \leq \int_{\Omega_0^{3\mathrm{D}}} 2\tau_{\mathrm{II}}^{1/2} D_{\mathrm{II}}^{1/2}(v)\, dx \leq \tau_c \int_{\Omega_0^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(v)\, dx.$$

Moreover

$$2\mu \int_{\Omega_1^{3\mathrm{D}}} D(u) : D(v)\, dx = 2\mu \int_{\Omega^{3\mathrm{D}}} D(u) : D(v)\, dx,$$

because of the definition of $\Omega_0^{3\mathrm{D}}$. So summarizing we get

$$\int_{\Omega^{3\mathrm{D}}} \tau : D(v)\, dx \leq 2\mu \int_{\Omega^{3\mathrm{D}}} D(u) : D(v)\, dx + \tau_c \int_{\Omega^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(v)\, dx.$$

Using the same technique for the second integral in (3.6), we obtain

$$\int_{\Omega^{3\mathrm{D}}} \tau : D(u)\, dx \leq 2\mu \int_{\Omega^{3\mathrm{D}}} D(u) : D(u)\, dx + \tau_c \int_{\Omega^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(u)\, dx.$$

Since Cauchy's inequality is an equation if its arguments are the same, we even have

$$\int_{\Omega^{3\mathrm{D}}} \tau : D(u)\, dx = 2\mu \int_{\Omega^{3\mathrm{D}}} D(u) : D(u)\, dx + \tau_c \int_{\Omega^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(u)\, dx.$$

In the end we get

$$\begin{aligned}\int_{\Omega^{3\mathrm{D}}} \tau : D(v-u)\, dx \leq 2\mu \int_{\Omega^{3\mathrm{D}}} D(u) : D(v-u)\, dx \\ + \tau_c \int_{\Omega^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(v)\, dx - \tau_c \int_{\Omega^{3\mathrm{D}}} 2D_{\mathrm{II}}^{1/2}(u)\, dx.\end{aligned} \tag{3.8}$$

Now, let us take a look on the boundary integral in (3.4). There holds

$$\int_{\Gamma^{3\mathrm{D}}} (\sigma n) \cdot (v-u)\, ds = \int_{\Gamma_D^{3\mathrm{D}}} (\sigma n) \cdot \underbrace{(v-u)}_{=0}\, ds + \int_{\Gamma_P^{3\mathrm{D}}} (\sigma n) \cdot (v-u)\, ds.$$

Especially

$$\int_{\Gamma_P^{3D}} (\sigma n) \cdot (v-u)\, ds = \int_{\Gamma_P^{3D}} (pI)n \cdot (v-u)\, ds + \int_{\Gamma_P^{3D}} \tau n \cdot (v-u)\, ds.$$

Using the identity $n(x,y,0) = -n(x,y,1)$ as well as boundary conditions, we have

$$\begin{aligned}\int_{\Gamma_P^{3D}} (pI)n \cdot (v-u)\, ds &= \int_\Omega pn_{|z=0} \cdot (v-u)_{|z=0}\, ds + \int_\Omega pn_{|z=1} \cdot (v-u)_{|z=1}\, ds \\ &= \int_\Omega \left(p_{|z=0} - p_{|z=1}\right) n_{|z=0} \cdot (v-u)_{|z=0}\, ds \\ &= \int_\Omega cn_{|z=0} (v-u)_{|z=0}\, ds \quad (3.9)\end{aligned}$$

and further

$$\int_{\Gamma_P^{3D}} \tau n \cdot (v-u)\, ds = \int_\Omega (\tau n)_{|z=0} \cdot (v-u)_{|z=0}\, ds + \int_\Omega (\tau n)_{|z=1} \cdot (v-u)_{|z=1}\, ds = 0.$$

Hence, together with (3.5),(3.8) we have

$$\begin{aligned}2\mu \int_{\Omega^{3D}} D(u) : D(v-u)\, dx - \int_{\Omega^{3D}} p \operatorname{div}(v-u)\, dx \\ + \tau_c \int_{\Omega^{3D}} 2D_{\mathrm{II}}^{1/2}(v)\, dx - \tau_c \int_{\Omega^{3D}} 2D_{\mathrm{II}}^{1/2}(u)\, dx \\ \geq \int_{\Omega^{3D}} f \cdot (v-u)\, dx + \int_\Omega cn_{|z=0}(v-u)_{|z=0}\, ds. \quad (3.10)\end{aligned}$$

We define bilinear forms

$$a(\cdot,\cdot) : V_0^{3D} \times V_0^{3D} \to \mathbb{R}, \qquad a(u,v) = \int_{\Omega^{3D}} 2D(u) : D(v)\, dx$$

$$b_1(\cdot,\cdot) : V_0^{3D} \times Q \to \mathbb{R}, \qquad b_1(v,p) = \int_{\Omega^{3D}} p \operatorname{div} v\, dx$$

as well as the continuous, convex but non-differentiable functional

$$j(\cdot) : V \to \mathbb{R}, \quad j(v) = \int_{\Omega^{3D}} 2D_{\mathrm{II}}^{1/2}(v)\, dx$$

and the linear, continuous operator

$$F \in V_0^{3D*}, \quad \langle F, v-u \rangle = \int_{\Omega^{3D}} f \cdot (v-u)\, dx + \int_\Omega cn_{|z=0}(v-u)_{|z=0}\, ds.$$

System (3.10),(3.2) can then be formulated as

$$\begin{aligned} &\mu a(u, v-u) + b_1(v-u, p) + \tau_c j(v) - \tau_c j(u) \geq \langle F, v-u\rangle && \forall v \in V_0^{3\mathrm{D}} \\ &b_1(u,q) = 0 && \forall q \in Q. \end{aligned} \tag{3.11}$$

We need that the first weak derivatives of u, v exist, and that they are in L^2. Moreover p, q have to be in L^2. Appropriate choices for our spaces are therefore

$$\begin{aligned} V^{3\mathrm{D}} &= [H^1(\Omega^{3\mathrm{D}})]^3 \\ V_0^{3\mathrm{D}} &= \{v \in [H^1(\Omega^{3\mathrm{D}})]^3 \,\big|\, v = 0 \text{ on } \Gamma_D^{3\mathrm{D}},\ v(x,y,0) = v(x,y,1)\} \\ Q &= L^2(\Omega^{3\mathrm{D}}) \end{aligned}$$

So we have showed, that the balance equations imply a so called *variational inequality of the second kind.*

Remark. Note that, although the constitutive law is dealing with two equivalences, the weak formulation (3.11) includes both cases: the one where the medium acts like a fluid and the case where $\tau_{\mathrm{II}}^{1/2}$ is under the given threshold τ_c.

Now we show that from (3.11), we obtain back the balance equations as well as the incompressibility condition. If $D(u) \neq 0$, then the functional $j(\cdot)$ is differentiable with directional derivative (using the chain rule)

$$\langle j'(u), v\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda} j(u+\lambda v)_{|\lambda=0} = \int_{\Omega^{3\mathrm{D}}} D_{\mathrm{II}}^{-1/2}(u) D(u) : D(v)\, dx. \tag{3.12}$$

Next, choosing test functions $u + \lambda v$ with $\lambda > 0$, the inequality in (3.11) equals

$$\mu a(u,v) + b_1(v,p) + \tau_c \frac{1}{\lambda}\left(j(u+\lambda v) - j(u)\right) \geq \langle F, v\rangle,$$

after dividing the whole inequality by λ. Now letting $\lambda \to 0$ we get

$$\mu a(u,v) + b_1(v,p) + \tau_c \langle j'(u), v\rangle \geq \langle F, v\rangle.$$

Taking now test functions $\pm v$, it follows

$$\mu a(u,v) + b_1(v,p) + \tau_c \langle j'(u), v\rangle = \langle F, v\rangle.$$

Using (3.12), the application of Gauss' theorem then yields

$$\begin{aligned} -\int_{\Omega^{3\mathrm{D}}} 2\mu\, \mathbf{div}\,(D(u)) \cdot v\, dx + \int_{\Omega^{3\mathrm{D}}} \nabla p \cdot v\, dx \\ - \tau_c \int_{\Omega^{3\mathrm{D}}} \mathbf{div}\left(D_{\mathrm{II}}^{-1/2}(u) D(u)\right) \cdot v\, dx + \int_{\partial\Omega^{3\mathrm{D}}} (\sigma n) \cdot v\, ds \\ = \int_{\Omega^{3\mathrm{D}}} f \cdot v\, dx + \int_{\Omega} c n_{|z=0} v_{|z=0}\, ds \quad \forall v \in V_0^{3\mathrm{D}}. \end{aligned} \tag{3.13}$$

Since (3.13) is valid for every test function v, we can choose v such that $v = 0$ on $\partial\Omega^{3D}$. So for $D(u) \neq 0$ this leads to

$$-\,\mathbf{div}\,(2\mu D(u)) - \mathbf{div}\left(\tau_c \frac{D(u)}{D_{\mathrm{II}}^{1/2}(u)}\right) + \nabla p = f. \tag{3.14}$$

And therefore the deviatoric stress tensor has the form

$$\tau = 2\mu D(u) + \tau_c D_{\mathrm{II}}^{-1/2}(u) D(u). \tag{3.15}$$

In this case the tensor of strain rates is given by

$$D(u) = \frac{1}{2\mu}(1 - \frac{\tau_c}{\tau_{\mathrm{II}}^{1/2}})\tau. \tag{3.16}$$

Now from (3.15) it follows

$$\tau_{\mathrm{II}}^{1/2} = \tau_c + 2\mu D_{\mathrm{II}}^{1/2}(u) > \tau_c.$$

Expression (3.15) makes no sense if $D(u) = 0$. In this case $\tau_{\mathrm{II}}^{1/2} \leq \tau_c$ has to hold since otherwise we get from (3.16) that $D_{\mathrm{II}}(u) > 0$ in contradiction to $D(u) = 0$.

Now we know that (3.14) is valid, we get from (3.13)

$$\int_{\partial\Omega^{3D}} (\sigma n) \cdot v\, ds - \int_{\Omega} c n_{|z=0} v_{|z=0}\, ds = 0.$$

So it follows with the same argumentation used in (3.9) that

$$p(x, y, 0) = p(x, y, 1) + c \quad \text{on } \Gamma_P^{3D}.$$

Finally, the equation in (3.11) holds for every test function q from which the incompressibility condition

$$\operatorname{div} u = 0 \tag{3.17}$$

follows. Now from (3.14),(3.17) we obtain our classical formulation.

Summarizing our results, under additional integrability as well as differentiability conditions, we have seen that problem 2.3 is equivalent to:

Problem 3.1 (Mixed formulation)**.** *Find $u \in V_0^{3D}$, $p \in Q$, such that*

$$\begin{aligned} &\mu a(u, v - u) + b_1(v - u, p) + \tau_c j(v) - \tau_c j(u) \geq \langle F, v - u\rangle \quad && \forall v \in V_0^{3D} \\ &b_1(u, q) = 0 && \forall q \in Q. \end{aligned}$$

with the bilinear forms

$$a(\cdot,\cdot) : V_0^{3D} \times V_0^{3D} \to \mathbb{R}, \qquad a(u,v) = \int_{\Omega^{3D}} 2D(u) : D(v)\, dx,$$

$$b_1(\cdot,\cdot) : V_0^{3D} \times Q \to \mathbb{R}, \qquad b_1(v,p) = \int_{\Omega^{3D}} p \operatorname{div} v\, dx,$$

the continuous, convex but non-differentiable functional

$$j(\cdot) : V_0^{3D} \to \mathbb{R}, \quad j(v) = \int_{\Omega^{3D}} 2D_{II}^{1/2}(v)\, dx$$

and the linear, continuous operator

$$F \in V_0^{3D*}, \quad \langle F, v-u \rangle = \int_{\Omega^{3D}} f \cdot (v-u)\, dx + \int_{\Omega} (cI) n_{|z=0} (v-u)_{|z=0}\, ds.$$

The solutions u, p are searched in

$$V_0^{3D} = \{ v \in [H^1(\Omega^{3D})]^3 \,\big|\, v = 0 \text{ on } \Gamma_D^{3D},\ v(x,y,0) = v(x,y,1) \}$$
$$Q = L^2(\Omega^{3D}).$$

3.1.2 The two-dimensional case

In the same way we can find an equivalent variational formulation for the problem reduced to the cross section Ω. As we mentioned in the previous subsection, the scalar solution of our problem will be also denoted by u. Analogously we get the following

Problem 3.2 (Primal formulation)**.** *Find $u \in V_0$, such that*

$$\mu a_2(u, v-u) + \tau_c j(v) - \tau_c j(u) \geq (c, v-u) \quad \forall\, v \in V_0$$

with the bilinear form

$$a_2(\cdot,\cdot) : V_0 \times V_0 \to \mathbb{R}, \quad a_2(u,v) = \int_{\Omega} \nabla u \cdot \nabla v\, dx$$

and the continuous, convex but non-differentiable functional

$$j(\cdot) : V_0 \to \mathbb{R}, \quad j(v) = \int_{\Omega} |\nabla v|\, dx.$$

The spaces V, V_0 are defined by

$$V = H^1(\Omega)$$

$$V_0 = \{ v \in V \,\big|\, v = 0 \text{ on } \Gamma \} = H_0^1(\Omega).$$

3.2 Dual formulations

To overcome the difficulties caused by the non-differentiable functional $j(\cdot)$, one usually does a regularization somewhere or, in our case, one derives another formulation in which the non-differentiability vanishes. The dual formulation given in the next subsections can be found in [11],[14].

3.2.1 The three-dimensional problem

Notation. The Frobenius norm of a tensor λ is given by

$$\|\lambda\|_F := (\lambda_{ij}\lambda_{ij})^{1/2}.$$

Theorem 3.2.1. *With the same notations as in the previous section problem 3.1 is equivalent to the following dual dual formulation:*

Problem 3.3 (Dual dual formulation)**.** *Find $u \in V_0^{3D}$, $p \in Q$, $\lambda \in \Lambda$ such that*

$$\begin{array}{ll} \mu a(u, v-u) + b_1(v-u, p) + \tau_c\sqrt{2} b_2(v-u, \lambda) = \langle F, v-u\rangle & \forall v \in V_0^{3D} \\ b_1(u, q) = 0 & \forall q \in Q \\ b_2(u, \eta - \lambda) \leq 0 & \forall \eta \in \Lambda \end{array}$$

where

$$b_2(\cdot,\cdot) : V_0^{3D} \times \Lambda \to \mathbb{R}, \quad b_2(v, \lambda) = \int_{\Omega^{3D}} \lambda : D(v)\, dx$$

and

$$\begin{aligned} V_0^{3D} &= \{v \in [H^1(\Omega^{3D})]^3 \,|\, v = 0 \text{ on } \Gamma_D^{3D},\ v(x,y,0) = v(x,y,1)\} \\ Q &= L^2(\Omega^{3D}) \\ \Lambda &= \{\lambda \in [L^2(\Omega^{3D})]^{3\times 3} \,|\, \lambda_{ij} = \lambda_{ji},\ \|\lambda\|_F \leq 1,\ i,j = 1,2,3\} \end{aligned}$$

Proof. The proof can be found in [11] for a more general case, or in [14] for the two-dimensional case. First we show that

$$b_2(u, \eta - \lambda) = \int_{\Omega^{3D}} (\eta - \lambda) : D(u)\, dx \leq 0 \tag{3.18}$$

is equivalent to

$$\int_{\Omega^{3D}} \lambda : D(u)\, dx = \int_{\Omega^{3D}} \|D(u)\|_F\, dx. \tag{3.19}$$

"$\Leftarrow$":
From (3.19) we get that

$$\int_{\Omega^{3D}} (\eta - \lambda) : D(u)\, dx = \int_{\Omega^{3D}} \eta : D(u)\, dx - \int_{\Omega^{3D}} \|D(u)\|_F\, dx$$
$$\leq \int_{\Omega^{3D}} \underbrace{\|\eta\|_F}_{\leq 1} : \|D(u)\|_F\, dx - \int_{\Omega^{3D}} \|D(u)\|_F\, dx$$
$$\leq 0 \qquad \forall \eta \in \Lambda.$$

"$\Rightarrow$":
On the other hand if we choose $\eta = D(u)/\|D(u)\|_F \in \Lambda$,

$$\int_{\Omega^{3D}} (\eta - \lambda) : D(u)\, dx \leq 0 \Rightarrow \int_{\Omega^{3D}} \frac{D(u) : D(u)}{\|D(u)\|_F}\, dx \leq \int_{\Omega^{3D}} \lambda : D(u)\, dx,$$

from which we get

$$\int_{\Omega^{3D}} (D(u) : D(u))^{1/2}\, dx \leq \int_{\Omega^{3D}} \lambda : D(u)\, dx.$$

Since we get from Cauchy's inequality that

$$\int_{\Omega^{3D}} \lambda : D(u)\, dx \leq \int_{\Omega^{3D}} \|\lambda\|_F \cdot \|D(u)\|_F\, dx \leq \int_{\Omega^{3D}} \|D(u)\|_F\, dx,$$

(3.19) follows. So the equivalence is shown.

Now let us take arbitrary but fixed test functions $v \in V_0^{3D}, q \in Q$. From problem 3.3 it follows that

$$\mu a(u, v-u) + b_1(v-u, p) + \tau_c j(v) - \tau_c j(u) - \langle F, v-u \rangle$$
$$= \tau_c \left(j(v) - j(u) - \sqrt{2}(\lambda, D(v) - D(u)) \right).$$

Since (3.19) holds, we get

$$j(u) = \int_{\Omega^{3D}} 2 \left(\frac{1}{2} D(u) : D(u) \right)^{1/2} dx = \sqrt{2}(\lambda, D(u))$$

and together with

$$\sqrt{2}(\lambda, D(v)) \leq \sqrt{2} \int_{\Omega^{3D}} \|\lambda\|_F \|D(v)\|_F\, dx \leq j(v)$$

we obtain

$$\tau_c \left(j(v) - j(u) - \sqrt{2}(\lambda, D(v) - D(u)) \right) \geq 0.$$

Next we assume problem 3.1. With the setting

$$(Y, v) = \mu a(u, v) + b_1(v, p) - \langle F, v\rangle$$

we can rewrite problem 3.1 as

$$(Y, v) + \tau_c j(v) - ((Y, u) + \tau_c j(u)) \geq 0 \quad \forall v \in V_0^{3\mathrm{D}}.$$

Now we choose test functions $v = \pm tw$ with $t > 0$ and $w \in V_0^{3\mathrm{D}}$. Then

$$t\,(\pm(Y, w) + \tau_c j(w)) - ((Y, u) + \tau_c j(u)) \geq 0 \quad \forall w \in V_0^{3\mathrm{D}}. \tag{3.20}$$

From this, tending t to zero, it follows that

$$(Y, u) + \tau_c j(u) \leq 0. \tag{3.21}$$

On the other hand, choosing test functions $v = \frac{1}{t}(u - w)$ we get from (3.20) that

$$\pm(Y, w) \geq \tau_c(j(u) - j(u - w)) \geq -\tau_c j(w),$$

since the inequality

$$j(w - u) \leq j(w) + j(u)$$

holds. In other words we get, that

$$|(Y, w)| \leq \tau_c j(w) \quad \forall\, w \in V_0^{3\mathrm{D}}. \tag{3.22}$$

From this automatically

$$(Y, u) + \tau_c j(u) \geq 0 \tag{3.23}$$

follows. Because of (3.21),(3.23)

$$(Y, u) + \tau_c j(u) = 0 \tag{3.24}$$

must be true. Next we introduce the space

$$\Phi = \{\phi \,|\, \phi \in [L^1(\Omega^{3\mathrm{D}})]^{3\times 3},\, \phi_{ij} = \phi_{ji}\}$$

equipped with the norm

$$\|\phi\|_\Phi = \int_{\Omega^{3\mathrm{D}}} \|\phi\|_F \, dx$$

as well as the mapping

$$\pi :\, V_0^{3\mathrm{D}} \to \Phi,\ w \mapsto \pi w = D(w).$$

With these notations we can rewrite (3.22) as

$$|(Y, w)| \leq \tau_c \sqrt{2} \|\pi w\|_\Phi.$$

Now, applying the theorem of Hahn-Banach yields that there exists

$$\lambda \in \Phi^* = \{\phi \,|\, \phi \in [L^\infty(\Omega^{3D})]^{3\times 3},\, \phi_{ij} = \phi_{ji}\}$$

such that

$$(Y, w) = -\tau_c\sqrt{2}(\lambda, D(w)) \tag{3.25}$$

and

$$|(\lambda, \pi w)| \leq \|\pi w\|_\Phi. \tag{3.26}$$

But (3.26) is equivalent to

$$\|\lambda\|_{\Phi^*} = \sup_{\pi w \in \Phi} \frac{|(\lambda, \pi w)|}{\|\pi w\|_\Phi} \leq 1$$

and (3.25) equals

$$\mu a(u, w-u) + b_1(w-u, p) + \tau_c\sqrt{2} b_2(w-u, \lambda) = \langle F, w-u\rangle \quad \forall w \in V_0^{3D}.$$

Finally (3.24) yields

$$\int_{\Omega^{3D}} \lambda : D(u)\, dx = \int_{\Omega^{3D}} (D(u) : D(u))^{1/2}\, dx.$$

and therefore (3.18) follows. □

Remark. As we have seen in the proof, it follows from the mixed problem not only that λ is an element of $[L^2(\Omega^{3D})]^{3\times 3}$. In fact it follows that λ is an $[L^\infty(\Omega^{3D})]^{3\times 3}$ function.

3.2.2 The two-dimensional case

It is clear that we can derive an equivalent dual formulation for the two-dimensional case (the proof can be found in [14]). Again u should be the same variable for the 2D case. Then for the problem on the cross section we have the following

Problem 3.4 (Dual formulation in the 2D case)**.** *Find $u \in V_0, w \in W$ such that*

$$\begin{array}{ll} \mu a_2(u, v-u) + \tau_c b(v-u, w) = (c, v-u) & \forall v \in V_0 \\ b(u, \phi - w) \leq 0 & \forall \phi \in W \end{array}$$

with the bilinear form

$$b(\cdot,\cdot) : V_0 \times W \to \mathbb{R}, \quad b(v, w) = \int_\Omega w \cdot \nabla v\, dx$$

and the convex set

$$W = \{w = (w_1, w_2) \in [L^2(\Omega)]^2 \,|\, |w| \leq 1\},$$

where $|w| = \sqrt{w_1^2 + w_2^2}$.

Equivalence to the three-dimensional problem

We will show that the dual formulations in two, as well as in three dimensions are equivalent if some extra settings hold. For this we will show that a solution (u, w) of the two-dimensional problem 3.4 can be extended to a solution of the three-dimensional problem 3.3.

Theorem 3.2.2. *Let $u_3 \in V_0$ and $w = (\tilde{\lambda}_{13}, \tilde{\lambda}_{23}) \in W$ be a solution of the two-dimensional problem 3.4, then*

$$u = (0, 0, u_3(x, y)), \quad \lambda = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \tilde{\lambda}_{13}(x,y) \\ 0 & 0 & \tilde{\lambda}_{23}(x,y) \\ \tilde{\lambda}_{31}(x,y) & \tilde{\lambda}_{32}(x,y) & 0 \end{pmatrix}$$

is a solution of the three-dimensional problem 3.3 in which $f = 0$ and $p = -cz$.

Proof. First it is easy to see, that every test function $v \in V_0^{\mathrm{3D}}$ can be written as

$$v(x,y,z) = \begin{pmatrix} v_1(x,y,z) \\ v_2(x,y,z) \\ v_3(x,y,z) - \bar{v}(x,y) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \bar{v}(x,y) \end{pmatrix} =: \hat{v} + \tilde{v}$$

where $\bar{v}$ is the mean value of v_3 concerning the z-component,i.e.

$$\bar{v}(x,y) = \int_{[0,1]} v_3(x,y,z)\, dz.$$

Such a decomposition is now chosen. We then have the following computations:

$$\begin{aligned}
&a(u, \hat{v}) = \\
&= \int_{\Omega \times [0,1]} 2 \begin{pmatrix} 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial x} \\ 0 & 0 & \frac{1}{2}\frac{\partial u_3}{\partial y} \\ \frac{1}{2}\frac{\partial u_3}{\partial x} & \frac{1}{2}\frac{\partial u_3}{\partial y} & 0 \end{pmatrix} : D(\hat{v})\, d\mathbf{x} \\
&= \int_{\Omega \times [0,1]} 2\frac{1}{2}(\frac{\partial u_3}{\partial x}(\frac{\partial \hat{v}_1}{\partial z} + \frac{\partial \hat{v}_3}{\partial x}) + \frac{\partial u_3}{\partial y}(\frac{\hat{v}_2}{\partial z} + \frac{\partial \hat{v}_3}{\partial y}))\, d\mathbf{x} \\
&= \int_\Omega \int_{[0,1]} \frac{\partial u_3}{\partial x}\frac{\partial v_1}{\partial z} + \frac{\partial u_3}{\partial x}\frac{\partial v_3}{\partial x} - \frac{\partial u_3}{\partial x}\frac{\partial \bar{v}}{\partial x} + \frac{\partial u_3}{\partial y}\frac{\partial v_2}{\partial z} + \frac{\partial u_3}{\partial y}\frac{\partial v_3}{\partial y} - \frac{\partial u_3}{\partial y}\frac{\partial \bar{v}}{\partial y}\, dz\, d\mathbf{x} \\
&\overset{u_3 = u_3(x,y)}{=} \int_\Omega \int_{[0,1]} \frac{\partial u_3}{\partial x}\frac{\partial v_1}{\partial z} + \frac{\partial u_3}{\partial y}\frac{\partial v_2}{\partial z}\, dz\, d\mathbf{x} \\
&= \int_\Omega \frac{\partial u_3}{\partial x} \int_{[0,1]} \frac{\partial v_1}{\partial z}\, dz\, d\mathbf{x} + \int_\Omega \frac{\partial u_3}{\partial y} \int_{[0,1]} \frac{\partial v_2}{\partial z}\, dz\, d\mathbf{x} \\
&= \int_\Omega \frac{\partial u_3}{\partial x}(v_1(x,y,1) - v_1(x,y,0))\, d\mathbf{x} + \int_\Omega \frac{\partial u_3}{\partial y}(v_2(x,y,1) - v_2(x,y,0))\, d\mathbf{x} \\
&\overset{v \in V_0^{\mathrm{3D}}}{=} 0,
\end{aligned}$$

and

$$\begin{aligned} a(u,\tilde{v}) &= \int_{\Omega\times[0,1]} 2D(u):D(\tilde{v})\,d\mathbf{x} \\ &= \int_\Omega \int_{[0,1]} \frac{\partial u_3}{\partial x}\frac{\partial \bar{v}}{\partial x} + \frac{\partial u_3}{\partial y}\frac{\partial \bar{v}}{\partial y}\,dz\,d\mathbf{x} \\ &= \int_\Omega \frac{\partial u_3}{\partial x}\frac{\partial \bar{v}}{\partial x} + \frac{\partial u_3}{\partial y}\frac{\partial \bar{v}}{\partial y}\,d\mathbf{x} \\ &= \int_\Omega \nabla u_3 \cdot \nabla \bar{v}\,d\mathbf{x} \\ &= a_2(u_3,\bar{v}). \end{aligned}$$

Moreover we have

$$\begin{aligned} b_1(\hat{v},p) &= -\int_{\Omega\times[0,1]} p \operatorname{div}\hat{v}\,dx \\ &= -\int_{\Omega\times[0,1]} p \operatorname{div} v\,dx \\ &= -\int_{\Omega\times[0,1]} c\nabla z\cdot v\,dx - \int_{\partial\Omega^{3\mathrm{D}}} pv\cdot n\,dx \\ &= -\int_{\Omega\times[0,1]} cv_3\,dx - \int_{\partial\Omega^{3\mathrm{D}}} pv\cdot n\,dx. \end{aligned}$$

Now, according to the periodic boundary conditions we have

$$b_1(\hat{v},p) = -\int_\Omega c\bar{v}\,dx - \int_\Omega cv_{|z=0}\cdot n_{|z=0}\,dx.$$

For the second component of v we get

$$b_1(\tilde{v},p) = -\int_{\Omega\times[0,1]} cz\frac{\partial \bar{v}}{\partial z}\,d\mathbf{x} = \mathbf{0}.$$

Clearly there holds

$$b_1(u,q) = 0.$$

Further

$$b_2(\hat{v},\lambda) = \frac{1}{\sqrt{2}}\int_{\Omega\times[0,1]} \tilde{\lambda}_{13}(\frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} - \frac{\partial \bar{v}}{\partial x}) + \tilde{\lambda}_{23}(\frac{\partial v_2}{\partial z} + \frac{\partial v_3}{\partial y} - \frac{\partial \bar{v}}{\partial y})\,dx.$$

Since the statements

$$\int_0^1 \tilde{\lambda}_{13}(x,y)\frac{\partial v_3}{\partial x}\,dz = \tilde{\lambda}_{13}\frac{\partial}{\partial x}\int_0^1 v_3\,dz = \tilde{\lambda}_{13}\frac{\partial \bar{v}}{\partial x}$$

and

$$\int_0^1 \tilde{\lambda}_{13}\frac{\partial v_1}{\partial z}\,dz = \tilde{\lambda}_{13}v_1(x,y,1) - \tilde{\lambda}_{13}v_1(x,y,0) = 0$$

are true (and also for the second component of v), we get that

$$\begin{aligned} b_2(\hat{v}, \lambda) &= \frac{1}{\sqrt{2}} \int_\Omega \int_{[0,1]} \tilde{\lambda}_{13} \frac{\partial v_1}{\partial z} + \tilde{\lambda}_{23} \frac{\partial v_2}{\partial z} \, dz \, d\mathbf{x} \\ &= 0. \end{aligned}$$

For the other component of the test function we have

$$\begin{aligned} b_2(\tilde{v}, \lambda) &= \frac{1}{\sqrt{2}} \int_\Omega \int_{[0,1]} \tilde{\lambda}_{13} \frac{\partial \bar{v}}{\partial x} + \tilde{\lambda}_{23} \frac{\partial \bar{v}}{\partial y} \, dz \, d\mathbf{x} \\ &= \frac{1}{\sqrt{2}} \int_\Omega \tilde{\lambda}_{13} \frac{\partial \bar{v}}{\partial x} + \tilde{\lambda}_{23} \frac{\partial \bar{v}}{\partial y} \, d\mathbf{x} \\ &= \frac{1}{\sqrt{2}} \int_\Omega w \cdot \nabla \bar{v} \, d\mathbf{x} \\ &= \frac{1}{\sqrt{2}} b(\bar{v}, w). \end{aligned}$$

In the same manner we get

$$b_2(u, \mu - \lambda) = \frac{1}{\sqrt{2}} b(u_3, \phi - w).$$

where $\phi = (\phi_{13}, \phi_{23}) \in W$ and

$$\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \phi_{13}(x, y) \\ 0 & 0 & \phi_{23}(x, y) \\ \phi_{31}(x, y) & \phi_{32}(x, y) & 0 \end{pmatrix}.$$

Now we use our assumption. The two-dimensional problem is satisfied and therefore the following must hold:

$$\mu a_2(u_3, \bar{v}) + \tau_c b(\bar{v}, w) = (c, \bar{v}).$$

Now plugging in our results from above we get (note that $f = 0$)

$$\mu a(u, v) + b_1(v, p) + \tau_c \sqrt{2} b_2(v, \lambda) = \langle F, v \rangle$$

and from the inequality

$$b(u_3, \phi - w) \leq 0,$$

we get

$$b_2(u, \mu - \lambda) \leq 0.$$

□

3.3 Existence and uniqueness of a solution

In this section we are going to list some results for mixed variational inequalities in an abstract way. Later we will apply these theorems to our problems. The following assumptions as well as the theorems can be found in [17]. Proofs and details are cited from [26],[13].

Let X, Y be Hilbert spaces equipped with scalar products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ respectively. Their norms, induced by the scalar products, are denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$. For the dual spaces we will use the notation $X^\star$ and $Y^\star$. Further on, let $\langle\cdot,\cdot\rangle$ be the duality pairing. We introduce

$$a(\cdot,\cdot) : X \times X \to \mathbb{R},\ b(\cdot,\cdot) : X \times Y \to \mathbb{R},\ j(\cdot) : X \to \mathbb{R}$$

$$f \in X^\star, \quad g \in Y^\star$$

then a variational inequality of the mixed type has the following abstract form:

Problem 3.5 (Abstract mixed formulation)**.** *Given $f \in X^\star$, $g \in Y^\star$, find $(u,p) \in X \times Y$ such that*

$$\begin{array}{ll} a(u, x-u) + b(x-u, p) + j(x) - j(u) \geq \langle f, x-u\rangle & \forall\, x \in X \\ b(u,y) = \langle g, y\rangle & \forall\, y \in Y. \end{array}$$

We will consider the linear case, so $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are bilinear forms. We assume that $a(\cdot,\cdot)$ as well as $b(\cdot,\cdot)$ are bounded, i.e. there exist constants $\alpha, \beta_1 > 0$, such that

$$a(x_1, x_2) \leq \alpha \|x_1\|_X \|x_2\|_X \quad \forall\, x_1, x_2 \in X$$

and

$$b(x,y) \leq \beta_1 \|x\|_X \|y\|_Y \quad \forall\, x \in X,\, \forall\, y \in Y.$$

By defining the linear operator

$$A : X \to X^\star, \quad \langle Ax, y\rangle = a(x,y) \quad \forall\, x \in X,\, \forall\, y \in X$$

as well as the linear operator

$$B : X \to Y^\star, \quad \langle Bx, y\rangle = b(x,y) \quad \forall\, x \in X,\, \forall\, y \in Y$$

and its adjoint operator

$$B^\star : Y \to X^\star, \quad \langle B^\star y, x\rangle = b(x,y) \quad \forall\, x \in X,\, \forall\, y \in Y$$

the abstract form of a mixed formulation looks like the following way

Problem 3.6 (Abstract mixed formulation in operator form)**.** *Given* $f \in X^\star, g \in Y^\star$*, search* $(u,p) \in X \times Y$ *:*

$$\begin{aligned} &\langle Au, x-u\rangle + \langle B^\star p, x-u\rangle + j(x) - j(u) \geq \langle f, x-u\rangle \quad \forall\, x \in X \\ &\langle Bu, y\rangle = \langle g, y\rangle \quad \forall\, y \in Y. \end{aligned}$$

Definition. Let B be an operator which maps from X onto Y. The kernel of B is defined via

$$\text{Ker } B := \{x \in X \,:\, Bx = 0\}.$$

The operator A is assumed to be coercive on Ker B, i.e. there exists $\alpha_0 > 0$ such that

$$\langle Ax, x\rangle \geq \alpha_0 \|x\|_X^2, \quad \forall\, x \in \text{Ker B}. \tag{3.27}$$

For the bilinear form $b(\cdot,\cdot)$ we assume that the inf-sup condition holds: there exists a constant $\beta > 0$, such that

$$\inf_{y\in Y} \sup_{x \in X} \frac{b(x,y)}{\|x\|_X \|y\|_Y} \geq \beta. \tag{3.28}$$

Definition. A function $f(\cdot)$ is called *lower semi continuous* if for all a the set

$$\{x \,|\, f(x) \leq a\}$$

is closed.

Definition. A function $f(\cdot)$ is called *proper* if it is not identically infinity,i.e.

$$f \not\equiv \infty.$$

Finally we assume $j(\cdot)$ to be a non-negative, continuous, convex, proper lower semi-continuous function.

Remark. It is important to know that we does not need differentiability for $j(\cdot)$.

3.3.1 Existence and uniqueness for the mixed problem

Definition. The polar set $X^0 \subset Y^\star$ of a subspace $X \subset Y$ is defined by

$$X^0 := \{y \in Y^\star \,:\, \langle y, x\rangle = 0 \quad \forall\, x \in X\}.$$

Definition. The orthogonal complement $K^\perp$ of a set $K \subset X$ is defined the following way

$$K^\perp := \{x \in X \,:\, (x,y)_X = 0 \quad \forall y \in K\}.$$

Notation. With

$$\text{Im } A$$

we will denote the range (the image) of the operator A.

Theorem 3.3.1 (Closed Range Theorem)**.** *Let X, Y be Hilbert spaces, $A : X \to Y^\star$ a linear, continuous operator and let $A^\star : Y \to X^\star$ be the adjoint operator, defined via $\langle A^\star y, x\rangle = \langle Ax, y\rangle$. Then the following statements are equivalent:*

(a) Im A is closed,

(b) Im $A^\star$ is closed,

(c) Im $A = (Ker\ A^\star)^0$,

(d) Im $A^\star = (Ker\ A)^0$.

Proof. The proof can be found in [33]. □

The following lemma gives us the possibility to use the inf-sup condition in another way. It can be found in [13],[34].

Lemma 3.3.2. *The three following properties are equivalent:*

1. *There exist a constant $\beta > 0$, such that*

$$\inf_{y\in Y} \sup_{x\in X} \frac{b(x,y)}{\|x\|_X \|y\|_Y} \geq \beta. \tag{3.29}$$

2. *The operator $B^\star$ is an isomorphism from Y onto $(Ker\ B)^0$ and*

$$\|B^\star y\|_{X^\star} \geq \beta \|y\|_Y \quad \forall\, y \in Y. \tag{3.30}$$

3. *The operator B is an isomorphism from $(Ker\ B)^\perp$ onto $Y^\star$ and*

$$\|Bx\|_{Y^\star} \geq \beta \|x\|_X \quad \forall\, x \in X. \tag{3.31}$$

Proof. We show first (1) ⇔ (2):

Due to the definition of the dual operator $B^\star$, (3.29) is equivalent to

$$\sup_{x\in X} \frac{\langle B^\star y, x\rangle}{\|x\|_X} \geq \beta \|y\|_Y$$

where the left hand side is the definition of the norm on $X^\star$, so that the inequality equals (3.30). From that, we get that $B^\star$ is injective. Furthermore it also implies that the inverse of $B^\star$ is continuous. Using the continuity of $B^\star$ as well as (3.30), it follows that Im $B^\star$ is a closed subspace of $X^\star$. Therefore it follows from the closed range theorem 3.3.1 that

$$\text{Im } B^\star = (\text{Ker } B)^0$$

what means that $B^\star$ is also surjective and therefore an isomorphism.

We prove (2) $\Leftrightarrow$ (3):
The equivalence of the isomorphisms is a direct consequence of theorem 3.3.1. There rests the equivalence of the inequalities:
We assume (3.30) and take an arbitrary $x \in (\text{Ker } B)^\perp$. Then $(x,.)_X \in (\text{Ker } B)^0$ and since we assumed (2), there exists $y \in Y$ with $B^\star y = (x,.)_X$. It follows then

$$\|Bx\|_{Y^\star} \geq \frac{b(x,y)}{\|y\|_Y} = \frac{\|x\|_X^2}{\|y\|_Y} = \frac{\|(x,.)_X\|_{X^\star}}{\|y\|_Y}\|x\|_X = \frac{\|B^\star y\|_X}{\|y\|_Y}\|x\|_X \geq \beta\|x\|_X.$$

We assume (3.31) and take an arbitrary $y \in Y$, then it follows that $(y,.)_Y \in Y^\star$. So there exists $x \in (\text{Ker } B)^\perp$ such that $Bx = (y,.)_Y$. It follows then

$$\|B^\star y\|_{X^\star} \geq \frac{b(x,y)}{\|x\|_X} = \frac{\|y\|_Y^2}{\|x\|_X} = \frac{\|(y,.)_X\|_{X^\star}}{\|x\|_X}\|y\|_Y = \frac{\|Bx\|_Y}{\|x\|_X}\|y\|_Y \geq \beta\|y\|_Y.$$

□

According to Lemma 3.3.2, there exists a unique solution $u_0 \in (\text{Ker } B)^\perp$ of

$$Bu_0 = g \quad \text{in } Y^\star. \tag{3.32}$$

We are now going to prove existence and uniqueness for the mixed problem 3.6. For this we introduce a variational inequality of the second kind on the kernel Ker B, which provides a unique solution for the original problem. Let u_0 being the solution of (3.32). Then we introduce the following auxiliary problem:

Problem 3.7 (Auxiliary problem on Ker B)**.** *Find $w \in$ Ker B such that*

$$\langle Aw, x-w\rangle + j(x+u_0) - j(w+u_0) \geq \langle \tilde{f}, x-w\rangle \quad \forall\, x \in \textit{Ker } B.$$

with

$$\langle \tilde{f}, x-w\rangle = \langle f, x-w\rangle - \langle Au_0, x-w\rangle.$$

Remark. If we can find a unique solution $w \in$ Ker B of problem 3.7, then $u = u_0 + w \in X$ is a unique solution of problem 3.6.

Since the functional $x \mapsto j(x+u_0)$ is convex and continuous from Ker B to $\mathbb{R}$, f is bounded and we can apply the following theorem:

Theorem 3.3.3. *Let A be a bounded, linear and coercive operator. Let f be a bounded functional. Further let j be a non-negative, continuous, convex, proper and lower-semi continuous functional. Then the following variational inequality of the second kind has a unique solution:*
Find $u \in X$ such that

$$a(u, x-u) + j(x) - j(u) \geq \langle f, x-u \rangle \quad \forall\, x \in X. \tag{3.33}$$

The proof requires the next lemma.

Lemma 3.3.4. *Let A be a bounded, linear and coercive operator. Let f be a bounded functional. Further let j be a non-negative, continuous, convex, proper and lower-semi continuous functional. Let $\rho > 0$ then for an arbitrary but fixed $u \in X$, the problem of finding $y \in X$ such that*

$$(y, x-y) + \rho j(x) - \rho j(y) \geq (u, x-y) + \rho \langle f, x-y \rangle - \rho a(u, x-y) \quad \forall\, x \in X$$

has a unique solution.

Proof. For the detailed proof see [14] and the references therein. The main idea is the following: the problem is a variational inequality of the second kind with the special bilinear form $(\cdot,\cdot)$. Since the inner product is clearly symmetric and positive definite, it follows the equivalence to a minimization problem: find $y \in X$ such that

$$J_\rho(u, y) \leq J_\rho(u, x) \quad \forall\, x \in X$$

with

$$J_\rho(u, x) = \frac{1}{2}\|x\|_X^2 + \rho j(x) - (u, x) + \rho a(u, x) - \rho \langle f, x \rangle.$$

With the assumptions on j it follows then from results out of convex analysis that the optimization problem has a unique solution. □

We are now able to proof theorem 3.3.3. It can be found in [14].

Proof of theorem 3.3.3. First we will prove uniqueness. Let u_1, u_2 be two solutions of the variational problem,i.e.

$$a(u_1, x-u_1) + j(x) - j(u_1) \geq \langle f, x-u_1 \rangle \quad \forall\, x \in X \tag{3.34}$$

$$a(u_2, x-u_2) + j(x) - j(u_2) \geq \langle f, x-u_2 \rangle \quad \forall\, x \in X \tag{3.35}$$

From the above inequalities it follows

$$j(u_i) \leq a(u_i, x-u_i) + j(x) - \langle f, x-u_i \rangle \quad i = 1,2\ \forall\, x \in X \tag{3.36}$$

Since $j(\cdot)$ is proper, what implies that $j \not\equiv +\infty$, there exists $x_0 \in X$ such that $j(x_0) < +\infty$, so we take $x = x_0$ in (3.36) and get that $j(u_i)$ is finite for $i = 1, 2$. Therefore we can choose special test functions $x = u_2$ in (3.34) as well as $x = u_1$ in (3.35). Adding these two inequalities yields

$$a(u_1, u_2 - u_1) + a(u_2, u_1 - u_2) \geq 0 \Leftrightarrow a(u_2 - u_1, u_2 - u_1) \leq 0.$$

Since $a(\cdot, \cdot)$ is bounded, we get

$$\alpha_0 \|u_2 - u_1\|^2 \leq 0$$

what is equivalent to

$$u_2 = u_1.$$

Now for proving existence of a solution we formulate an auxiliary problem. For the meanwhile let $u \in X$ arbitrary, then we define z as the solution of finding $z \in X$ such that

$$(z, x - z) + \rho j(x) - \rho j(z) \geq (u, x - z) + \rho\langle f, x - z\rangle - \rho a(u, x - z) \quad \forall x \in X \tag{3.37}$$

with $\rho > 0$. Now lemma 3.3.4 provides a unique solution.

Defining the mapping

$$S_\rho : X \to X,\ u \mapsto S_\rho(u) = z,$$

one can see that (3.33) is equivalent to finding a solution u of the fixed point problem

$$u = S_\rho(u). \tag{3.38}$$

So for showing existence it is sufficient to prove that S_ρ is a contraction mapping. With the notation $S_\rho(u_1) = z_1$, $S_\rho(u_2) = z_2$ and the choice $x = z_2$ and $x = z_1$ respectively we get from (3.37)

$$(z_1, z_2 - z_1) + \rho j(z_2) - \rho j(z_1) \geq (u_1, z_2 - z_1) + \rho\langle f, z_2 - z_1\rangle - \rho a(u_1, z_2 - z_1)$$

and

$$(z_2, z_1 - z_2) + \rho j(z_1) - \rho j(z_2) \geq (u_2, z_1 - z_2) + \rho\langle f, z_1 - z_2\rangle - \rho a(u_2, z_1 - z_2).$$

Adding these two inequalities leads to

$$\|z_2 - z_1\|^2 \leq (u_2 - u_1, z_2 - z_1) - \rho a(u_2 - u_1, z_2 - z_1). \tag{3.39}$$

Now we use the operator $A \in L(X, X)$ to rewrite (3.39).

$$\|z_2 - z_1\|^2 \leq ((I - \rho A)(u_2 - u_1), z_2 - z_1) \leq \|I - \rho A\|_{L(X,X)} \|u_2 - u_1\| \|z_2 - z_1\|$$

or

$$\|z_2 - z_1\| \le \|I - \rho A\|_{L(X,X)} \|u_2 - u_1\|.$$

We assume that ρ satisfies

$$0 < \rho < \frac{2\alpha_0}{\|A\|^2} \tag{3.40}$$

then

$$\begin{aligned} \|(I-\rho A)x\|^2 &= \|x\|^2 - 2\rho(x, Ax) + \rho^2\|Ax\|^2 \\ &\le \|x\|^2 - 2\alpha_0\rho\|x\|^2 + \rho^2\|A\|^2\|x\|^2 \\ &= (1 - 2\alpha_0\rho + \rho^2\|A\|^2)\|x\|^2 \end{aligned}$$

Since (3.40) holds, it follows that $(1 - 2\alpha_0\rho + \rho^2\|A\|^2) < 1$ and therefore

$$\|I - \rho A\| < 1.$$

So it follows

$$\|S_\rho(u_2) - S_\rho(u_1)\| = \|z_2 - z_1\| \le \|I - \rho A\| \|u_2 - u_1\|.$$

As the contraction property of S_ρ is shown, it follows from Banach's fixed point theorem that (3.38) and thus (3.33) has a unique solution. □

So problem 3.7 has a unique solution $w \in \text{Ker } B$. To find a solution of the mixed problem, we are going to approximate problem 3.6 as well as 3.7 by regularizing the nondifferentiable functional j. It can be shown ([26]) that we can introduce a family of functionals $j_\varepsilon : X \to \mathbb{R}$ parameterized by $\varepsilon \in (0, 1]$ with the following properties:

a) j_ε is convex and differentiable with Gateaux derivative

$$j'_\varepsilon : X \to X^\star. \tag{3.41}$$

b)

$$j_\varepsilon(x) \to j(x) \text{ for } \varepsilon \to 0, \tag{3.42}$$

uniformly with respect to $x \in X$.

c)

$$\|j'_\varepsilon(x)\|_{X^\star} \le c, \quad \forall x \in X, \tag{3.43}$$

where the constant c should be independent of ε and x.

Our new problem then reads as

Problem 3.8 (Regularized abstract mixed formulation)**.** *Given* $f \in X^\star, g \in Y^\star$*, search* $(u_\varepsilon, p_\varepsilon) \in X \times Y$ *:*

$$\begin{array}{ll}\langle Au_\varepsilon, x - u_\varepsilon\rangle + \langle B^\star p_\varepsilon, x - u_\varepsilon\rangle + j_\varepsilon(x) - j_\varepsilon(u_\varepsilon) \geq \langle f, x - u_\varepsilon\rangle & \forall\, x \in X \\ \langle Bu_\varepsilon, y\rangle = \langle g, y\rangle & \forall\, y \in Y.\end{array}$$

The auxiliary problem is replaced by

Problem 3.9 (Regularized auxiliary problem)**.** *Find* $w_\varepsilon \in Ker\ B$ *such that*

$$\langle Aw_\varepsilon, x - w_\varepsilon\rangle + j_\varepsilon(x + u_0) - j_\varepsilon(w_\varepsilon + u_0) \geq \langle \tilde{f}, x - w_\varepsilon\rangle \quad \forall\, x \in Ker\ B.$$

with

$$\langle \tilde{f}, x - w_\varepsilon\rangle = \langle f, x - w_\varepsilon\rangle - \langle Au_0, x - w_\varepsilon\rangle.$$

Applying theorem 3.3.3 leads to the existence of a unique solution $w_\varepsilon \in \text{Ker } B$ for the auxiliary problem 3.9. Now, as j_ε is differentiable, it follows with the same technique used in section 3.1, that problem 3.9 is equivalent to the problem of finding $u_\varepsilon = w_\varepsilon + u_0 \in X$ such that

$$\langle Au_\varepsilon, x\rangle + \langle j'_\varepsilon(u_\varepsilon), x\rangle = \langle f, x\rangle \quad \forall\, x \in \text{Ker } B. \tag{3.44}$$

In the same manner the inequality of problem 3.8 is equivalent to

$$\langle Au_\varepsilon, x\rangle + \langle j'_\varepsilon(u_\varepsilon), x\rangle + \langle B^\star p_\varepsilon, x\rangle = \langle f, x\rangle \quad \forall\, x \in X.$$

We observe that $f - Au_\varepsilon - j'_\varepsilon(u_\varepsilon) \in (\text{Ker } B)^0$, since (3.44) holds. Therefore we can apply lemma 3.3.2 and get a unique $p_\varepsilon \in Y$ such that

$$B^\star p_\varepsilon = f - Au_\varepsilon - j'_\varepsilon(u_\varepsilon). \tag{3.45}$$

Thus we have shown

Lemma 3.3.5. *Problem 3.8 has a unique solution* $(u_\varepsilon, p_\varepsilon) \in X \times Y$*.*

So regularizing problems 3.6, 3.7 leads to a unique solution. Now the only thing to do is, to show that for $\varepsilon \to 0$ the limits of our solutions solve the original problems. We will see that u_ε converges strongly to u. In the case of the multiplier p_ε this result is unfortunately not given. We will have only weak convergence of a subsequence.

First let us choose special test functions $x = w_\varepsilon$ in problem 3.7 as well as $x = w$ in problem 3.9. Then adding this two inequalities leads to

$$\langle \tilde{A}w_\varepsilon - \tilde{A}w, w - w_\varepsilon\rangle + j_\varepsilon(u) - j(u) + j(u_\varepsilon) - j_\varepsilon(u_\varepsilon) \geq 0.$$

With $u_\varepsilon - u = w_\varepsilon - w$, (3.42) and the boundedness of A we get

$$\begin{aligned}\alpha_0\|w - w_\varepsilon\|^2 &\leq j_\varepsilon(u) - j(u) \\ &\leq C_2\varepsilon\end{aligned}$$

or equivalently

$$\|w - w_\varepsilon\| \le \sqrt{C_2\varepsilon/\alpha_0}.$$

Thus we end up with the next lemma.

Lemma 3.3.6. *The solution w_ε of problem 3.9 converges strongly to the solution w of problem 3.7.*

Next we will to show that our sequences u_ε, p_ε are bounded independently from ε. The result above sure gives us the desired attribute for u_ε, since $\|u_\varepsilon\| \le \sqrt{C/\alpha_0} + \|u\|$. For p_ε we use the inf-sup condition and get

$$\begin{aligned}
\beta\|p_\varepsilon\|_Y &\le \sup_{0\ne x\in X} \frac{|\langle B^\star p_\varepsilon, x\rangle|}{\|x\|_X} \\
&\overset{(3.45)}{\le} \sup_{0\ne x\in X} \frac{|\langle f - Au_\varepsilon - j'_\varepsilon(u_\varepsilon), x\rangle|}{\|x\|_X} \\
&\le \|f\|_{X^\star} + \|A\|_{L(X,X^\star)}\|u_\varepsilon\|_X + \|j'_\varepsilon(u_\varepsilon)\|_{X^\star} \\
&\le \|f\|_{X^\star} + \|A\|_{L(X,X^\star)}(\sqrt{C/\alpha_0} + \|u\|_X) + C_3
\end{aligned}$$

where we used the boundedness of u_ε as well as the assumption on $\|j'_\varepsilon(u_\varepsilon)\|_{X^\star}$. So it follows (see [19]) that there exists a subsequence $p_{\varepsilon'}$ of p_ε which converges weakly to a limit $p \in Y$. Since we have $u_\varepsilon \to u$ for every subsequence $u_{\varepsilon'}$, there clearly holds $u_{\varepsilon'} \to u$.

Finally we have to show that the limits of our subsequences really solve the original problem. Starting from our approximate problem 3.8 and using the second assumption on $j(\cdot)$ we get

$$\begin{aligned}
\langle B^\star p_{\varepsilon'}, x\rangle &+ \langle Au_{\varepsilon'}, x\rangle - \langle f, x - u_{\varepsilon'}\rangle + j_\varepsilon(x) \\
&\ge \langle B^\star p_{\varepsilon'}, u_{\varepsilon'}\rangle + \langle Au_{\varepsilon'}, u_{\varepsilon'}\rangle + j_\varepsilon(u_{\varepsilon'}) \\
&\qquad \ge \langle B^\star p_{\varepsilon'}, u_{\varepsilon'}\rangle + \langle Au_{\varepsilon'}, u_{\varepsilon'}\rangle + j(u_\varepsilon).
\end{aligned}$$

With the assumptions on A,$B^\star$, the convergence of the subsequences, the continuity of $j(\cdot)$ we get, after taking the limit $\varepsilon' \to 0$ on both sides,

$$\langle Au, x - u\rangle + j(x) - j(u) + \langle B^\star p, x - u\rangle \ge \langle f, x - u\rangle.$$

Together with (3.32) we have

Theorem 3.3.7. *Let $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ be bounded bilinear forms. For $b(\cdot,\cdot)$ we assume that the inf-sup condition (3.28) holds. Let the operator A be coercive. Further let $j(\cdot)$ be a non-negative, continuous, convex, proper and lower semi-continuous functional. Its regularization family $j_\varepsilon(\cdot)$ is assumed to fulfill (3.41)-(3.43). Then there exists a solution $(u, p) \in X \times Y$ for problem 3.6. Furthermore, u is unique.*

By regularizing the original problems 3.6 and 3.7, we have seen that the auxiliary problem reduces to an equation what was essential for the existence and uniqueness of the regularized mixed problem. So showing existence and uniqueness of the mixed formulation requires finding a suitable regularizing family j_ε with the wright assumptions.

In [17] one can see that the theorem also holds for more general assumptions.

3.3.2 Application to a fluid flow in a quadratic pipe

We will now prove that our mixed problem fulfills all assumptions, such that we can apply the theorems from the previous subsection. From now on we consider the case

$$\Gamma^{3D} = \Gamma^{3D}_D.$$

So our mixed problem, derived in the previous section has the form

$$\begin{aligned} &\mu a(u, v-u) + b_1(v-u, p) + \tau_c j(v) - \tau_c j(u) \geq (f, v-u) \quad && \forall v \in V_0^{3D} \\ &b_1(u, q) = 0 && \forall q \in Q. \end{aligned}$$

So in the notation of the previous subsection, we choose

$$X = V_0^{3D}, \quad Y = Q, \quad b(\cdot,\cdot) = b_1(\cdot,\cdot)$$

and A, j stay the same.

We know that if we can show all assumptions, then there exists a unique solution u for the two-dimensional case and a solution u, p for the three-dimensional case where u is unique but p not. The non-uniqueness of p comes from the fact that the kernel of the operator B_1 is non-trivial. It contains all constant functions. So if we demand another assumption on the space Q we get a uniquely defined p too. In the literature usually one demands that

$$\int_{\Omega^{3D}} p \, dx = 0. \tag{3.46}$$

So we seek $p \in L_0^2(\Omega^{3D})$, where

$$L_0^2(\Omega^{3D}) := \{ v \in L^2(\Omega^{3D}) \mid \int_{\Omega^{3D}} v \, dx = 0 \}.$$

From now on we will define $Q := L_0^2(\Omega^{3D})$.

We define the operator

$$A : V_0^{3D} \to V_0^{3D\star}, \ \langle Au, v \rangle = a(u,v) = \int_{\Omega^{3D}} 2D(u) : D(v) \, dx, \quad \forall\, v \in V_0^{3D}$$

then A is linear. The boundedness of the bilinear form $a(\cdot,\cdot)$ is simple. For the coercitivity of the operator A on V_0^{3D}, we can use the following lemma:

Lemma 3.3.8 (Korn's inequality)**.** *Let* $\Omega^{3D} \subset \mathbb{R}^3$ *an open bounded set with a Lipschitz continuous boundary. Then there exists a constant* $c_K = c_K(\Omega^{3D}) > 0$ *such that*

$$\int_{\Omega^{3D}} D(v) : D(v)\, dx \geq c_K^2 \|v\|_1^2 \quad \forall\, v \in [H_0^1(\Omega^{3D})]^3.$$

Proof. Can be found in [11]. □

So the operator A is elliptic. By Cauchy's inequality the bilinear form $b_1(\cdot,\cdot)$ is bounded, so we define the bounded linear operator

$$B_1 : V_0^{\mathrm{3D}} \to Q^\star,\ \langle B_1 v, p\rangle = b_1(v,p) = -\int_{\Omega^{\mathrm{3D}}} p \operatorname{div} v\, dx, \quad \forall\, p \in Q$$

as well as the adjoint operator

$$B_1^\star : Q \to V_0^{\mathrm{3D}\star},\ \langle B_1^\star p, v\rangle = b_1(v,p) = -\int_{\Omega^{\mathrm{3D}}} p \operatorname{div} v\, dx, \quad \forall\, v \in V_0^{\mathrm{3D}}.$$

Now we will show that the inf-sup condition holds. This condition for the operator $B_1 = -\operatorname{div}$ is proved in several papers. First of all we have the following

Definition. For the dual space of H_0^1, H^{-1} its norm is defined by

$$\|p\|_{-1} := \sup_{0 \neq q \in H_0^1(\Omega^{\mathrm{3D}})} \frac{(p,q)}{\|q\|_1}.$$

The next lemma is our starting point for proving the inf sup condition.

Lemma 3.3.9 (Necas)**.** *Let* $\Omega^{3D} \subset \mathbb{R}^3$ *be an open bounded set with a Lipschitz continuous boundary. Then there exists a constant* $c > 0$ *such that for all* $p \in L^2(\Omega^{3D})$

$$\|p\|_0 \leq c(\|p\|_{-1} + \|\nabla p\|_{-1}).$$

Proof. The proof is rather technical and can be found in [11]. For an alternative proof see [5]. □

With it we have the following theorem.

Theorem 3.3.10. *Let* $\Omega^{3D} \subset \mathbb{R}^3$ *an open bounded set with a Lipschitz continuous boundary. Then there exists a constant* $c > 0$ *such that for all* $p \in L_0^2(\Omega^{3D})$

$$\|p\|_0 \leq c\|\nabla p\|_{-1} \tag{3.47}$$

holds.

The proof is cited from [34].

Proof. As the embedding $[H_0^1(\Omega^{3D})]^3 \to L^2(\Omega^{3D})$ is compact, its adjoint embedding $L^2(\Omega^{3D}) \to H^{-1}(\Omega^{3D})$ is also compact. Now we assume that (3.47) doesn't hold. Then there exists a sequence $(p_k) \in L_0^2(\Omega^{3D})$ such that $\|p_k\|_0 = 1$ and $\|\nabla p_k\|_{-1} \to 0$. According to the compact embedding $L^2(\Omega^{3D}) \to H^{-1}(\Omega^{3D})$ there exists a convergent subsequence $(p'_k) \in H^{-1}(\Omega^{3D})$. From the last lemma we then have, that (p'_k) is Cauchy in $L^2(\Omega^{3D})$ and therefore $p'_k \to p$ in $L^2(\Omega^{3D})$ where $p \in L_0^2(\Omega^{3D})$.
Moreover the following must hold

$$\nabla p = \lim_{k\to\infty} \nabla p'_k = 0$$

and therefore p has to be constant. As $p \in L_0^2(\Omega^{3D})$ it follows that $p = 0$. On the other hand this is inconsistent with $\|p_k\|_0 = 1$. □

Finally we need a regularization family $j_\varepsilon(\cdot)$ fulfilling (3.41)-(3.43). According to [26] we define

$$j_\varepsilon(u) = \tau_c \int_{\Omega^{3D}} \sqrt{D(u) : D(u) + \varepsilon^2}\, dx.$$

With it, conditions (3.41) and (3.42) are clearly satisfied. To show (3.43), we need the derivative of $j_\varepsilon(\cdot)$. Similarly to (3.12) the derivative in direction v is given by

$$\langle j'_\varepsilon(u), v\rangle = \tau_c \int_{\Omega^{3D}} \frac{D(u) : D(v)}{\sqrt{D(u) : D(u) + \varepsilon^2}}\, dx.$$

So we can estimate

$$\begin{aligned} |\langle j'_\varepsilon(u), v\rangle| &\leq \tau_c \int_{\Omega^{3D}} \frac{2D_{\mathrm{II}}(u)^{1/2} D_{\mathrm{II}}(v)^{1/2}}{\sqrt{D(u) : D(u) + \varepsilon^2}}\, dx \\ &\leq \tau_c \int_{\Omega^{3D}} \sqrt{2} D_{\mathrm{II}}(v)^{1/2}\, dx \\ &\leq 2\tau_c \|v\|_{V_0^{3D}} \end{aligned}$$

which implies

$$\|j'_\varepsilon(u)\|_{V_0^{3D\star}} \leq c$$

with a constant c. In the same way we can prove the assumptions for the two-dimensional case. Since we have already showed the equivalence between the mixed problem and the dual formulation, we end up with

Theorem 3.3.11. *The dual dual formulation 3.3 and the dual formulation 3.4 have a unique solution $(u, p) \in V_0^{3D} \times Q$ and $u \in V_0$ respectively. Moreover there exists a solution $\lambda \in \Lambda$ and $w \in W$ for problem 3.3 and problem 3.4 respectively.*

For the dual variables λ for the three-dimensional problem and w for the two-dimensional problem respectively, the kernel of the operators

$$B_2 : V_0^{3\mathrm{D}} \to \Lambda^\star, \quad \langle B_2 u, \lambda \rangle = b_2(u, \lambda)$$

and for the two-dimensional case

$$B : V_0 \to W^\star, \quad \langle Bu, w \rangle = b(u, w)$$

respectively, are not so easy to handle as it was for B_1 (see also [16],[18]). For example looking at the two-dimensional problem, suppose that u is a unique solution and let w, w' the dual solutions. Plugging in and subtracting yields

$$\operatorname{div}(w' - w) = 0. \tag{3.48}$$

From functional analysis we have the decomposition

$$[L^2(\Omega)]^{3\times 3} = \nabla[H_0^1(\Omega)]^3 \oplus S_0$$

with

$$S_0 := \{w \in [L^2(\Omega)]^{3\times 3} \mid \mathbf{div}\, w = 0\}$$

so that we can decompose uniquely

$$w = w_1 + w_2, \quad w_1 \in \nabla[H_0^1(\Omega)]^3,\ w_2 \in S_0.$$

Now we see from (3.48), that if u is a unique solution, then we have a uniquely component of w in $\nabla H_0^1(\Omega)$ but w_2 and w_2' do not have to be the same. The space S_0 is much larger than the space of constant functions, so we need more than a simple condition like (3.46) in order to get uniqueness. We will do this in the next chapter.

Chapter 4

A method for solving the weak systems

Now we want to find a suitable method solving the variational inequalities. There are at least two possibilities to do that:

1. Regularizing the non-differentiable functional leads to an variational equation which can then be solved with more common methods. The answers of how to regularize the functional and which methods are used, can be found in [14] and [16]. Similarly one can regularize the constitutive law (viscosity regularization), see for example [12],[1],[2].

2. Using a dual formulation has the advantage not to handle a non-differentiable functional but has an extra variable to be solved. Methods based an such formulations can be found in [15],[18], [25],[16] and [28].

In this chapter we are going to use dual formulations, developed in chapter 3, combined with an Uzawa type method.

4.1 An Uzawa type method

4.1.1 A time dependent approximation

We use the ideas from [18]. Therein the developed method is applied to the problem reduced onto the cross section of the prism which we already derived in chapter 2 and chapter 3. Here we will list the results for the $2D$-problem and then we will carry them over to the three-dimensional case. Starting point is the dual formulation 3.4. First we will show that we can replace the inequality therein by an equation.

We introduce the orthogonal projection onto the set W,

$$P_{\mathrm{W}} : L^2(\Omega) \times L^2(\Omega) \to W,$$

for the norm

$$\|\mu\|_{L^2(\Omega)\times L^2(\Omega)} := \left(\int_\Omega |\mu|^2\,dx\right)^{1/2}.$$

It is given by

$$P_{\mathrm{W}}(\mu(x)) = \frac{\mu(x)}{\sup\,(1, |\mu(x)|)} \quad \forall\,\mu \in L^2(\Omega)\times L^2(\Omega).$$

where

$$W = \{w = (w_1, w_2)\,|\, w \in L^2(\Omega)\times L^2(\Omega),\ |w| \le 1\}.$$

Lemma 4.1.1. *The orthogonal projection P_W fulfills the following property:*

$$\|P_W(\mu) - P_W(\eta)\|_{L^2\times L^2} \le \|\mu - \eta\|_{L^2\times L^2}. \tag{4.1}$$

Proof. Since P_{W} is the orthogonal projection, there holds for all $w \in L^2(\Omega)\times L^2(\Omega)$:

$$(w - P_{\mathrm{W}}(w), v - P_{\mathrm{W}}(w))_{L^2\times L^2} \le 0 \tag{4.2}$$

Let $\mu, \eta \in L^2(\Omega)\times L^2(\Omega)$ arbitrary but fixed, then it follows from (4.2), with the choice $w = \mu$ and $v = P_{\mathrm{W}}(\eta)$, that

$$(P_{\mathrm{W}}(\mu), P_{\mathrm{W}}(\eta) - P_{\mathrm{W}}(\mu))_{L^2\times L^2} \ge (\mu, P_{\mathrm{W}}(\eta) - P_{\mathrm{W}}(\mu))_{L^2\times L^2}. \tag{4.3}$$

On the other hand, choosing $w = \eta$ and $v = P_{\mathrm{W}}(\mu)$ leads to

$$(P_{\mathrm{W}}(\eta), P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta))_{L^2\times L^2} \ge (\eta, P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta))_{L^2\times L^2}. \tag{4.4}$$

Adding (4.3) and (4.4) now yields

$$(P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta), P_{\mathrm{W}}(\eta) - P_{\mathrm{W}}(\mu))_{L^2\times L^2} \ge (\mu - \eta, P_{\mathrm{W}}(\eta) - P_{\mathrm{W}}(\mu))_{L^2\times L^2},$$

what is equivalent to

$$\begin{aligned}\|P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta)\|^2_{L^2\times L^2} &\le (\mu - \eta, P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta))_{L^2\times L^2}\\ &\le \|\mu - \eta\|_{L^2\times L^2}\cdot\|P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta)\|_{L^2\times L^2}.\end{aligned}$$

So we get

$$\|P_{\mathrm{W}}(\mu) - P_{\mathrm{W}}(\eta)\|_{L^2\times L^2} \le \|\mu - \eta\|_{L^2\times L^2}.$$

□

Notation. We will denote the norm $\|\cdot\|_{L^2\times L^2}$ also by $\|\cdot\|_0$.

Now the inequality from problem 3.4,

$$\langle Bu, \phi - w \rangle \leq 0 \quad \forall \phi \in W,$$

is clearly equivalent to

$$\langle w + rBu - w, \phi - w \rangle \leq 0 \quad \forall \phi \in W \tag{4.5}$$

for $r > 0$. And this is equivalent to

$$w = P_{\mathrm{W}}(w + rBu).$$

Therefore the dual formulation 3.4 is equivalent to the following problem:

Problem 4.1 (Dual formulation for the steady state case)**.** *Find* $u \in V_0$, $w \in W$ *such that*

$$\begin{aligned} &\mu a_2(u, v) + \tau_c b(v, w) = (c, v) \quad \forall\, v \in V_0 \\ &w = P_W(w + rBu) \end{aligned} \tag{4.6}$$

with $r > 0$.

Now we could compute a solution (u, w) by the following natural algorithm:

Algorithm 4.1.2. *If*

$$w_0 \in W$$

is given, then, for $m \geq 0$, w_m *being known, we compute* u_m *and* w_{m+1} *by*

$$\begin{aligned} \mu a_2(u_m, v) &= (c, v) - \tau_c b(v, w_m) \quad \forall v \in V_0 \\ w_{m+1} &= P_W(w_m + rBu_m) \end{aligned}$$

Concerning the convergence we have the following result:

Theorem 4.1.3. *Under the assumption*

$$0 < r < \frac{2\mu}{\tau_c} \tag{4.7}$$

we have for all $w_0 \in W$

$$\lim_{m \to \infty} u_m = u \quad \textit{in } V_0 = H_0^1(\Omega)$$

and

$$\lim_{m \to \infty} w_m = w \quad \textit{weakly in } L^2(\Omega) \times L^2(\Omega)$$

Proof. We denote $u_m - u$ and $w_m - w$ by $\bar{u}_m$ and $\bar{w}_m$ respectively. Since (u, w) solves problem 4.1 we have

$$\mu a_2(\bar{u}_m, v) = -\tau_c b(v, \bar{w}_m) \quad \forall v \in V_0 \tag{4.8}$$

and

$$\|\bar{w}_{m+1}\|_0 = \|P_{\mathrm{w}}(w_m + rBu_m) - P_{\mathrm{W}}(w - rBu)\|_0 \overset{(4.1)}{\leq} \|\bar{w}_m + rB\bar{u}_m\|_0. \tag{4.9}$$

Using the identity

$$\|\bar{w}_m + rB\bar{u}_m\|_0^2 = \|\bar{w}_m\|_0^2 + 2r(B\bar{u}_m, \bar{w}_m) + r^2\|B\bar{u}_m\|_0^2,$$

(4.9) is equivalent to

$$\|\bar{w}_m\|_0^2 - \|\bar{w}_{m+1}\|_0^2 \geq -2rb(\bar{u}_m, \bar{w}_m) - r^2\|B\bar{u}_m\|_0^2. \tag{4.10}$$

Using test functions $v = \bar{u}_m$ in (4.8) leads to

$$b(\bar{u}_m, \bar{w}_m) = -\frac{\mu}{\tau_c} a_2(\bar{u}_m, \bar{u}_m) = -\frac{\mu}{\tau_c}\|B\bar{u}_m\|_0^2,$$

so that (4.10) is equal to

$$\|\bar{w}_m\|_0^2 - \|\bar{w}_{m+1}\|_0^2 \geq r\left(\frac{2\mu}{\tau_c} - r\right)\|B\bar{u}_m\|_0^2.$$

Now using (4.7), we get that the sequence $\|\bar{w}_m\|_0^2$ is decreasing. As the sequence is bounded from below by zero, we get

$$\lim_{m\to\infty}(\|\bar{w}_m\|_0^2 - \|\bar{w}_{m+1}\|_0^2) = 0$$

and therefore

$$\lim_{m\to\infty}\|B\bar{u}_m\|_0^2 = \lim_{m\to\infty}\|\nabla\bar{u}_m\|_0^2 = 0.$$

Since the H_0^1-semi norm is equivalent to the H^1-norm in V_0, this is equivalent to

$$\lim_{m\to\infty}\|u_m - u\|_1 = 0.$$

For the convergence of the dual solution w_m we refer to [18]. □

Remark. In each step of the algorithm we have to solve an equation of the form

$$\mu a_2(u_m, v) = f,$$

what is the weak formulation of the Dirichlet-problem

$$\begin{aligned} -\mu\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma. \end{aligned}$$

So the initial problem of solving a variational inequality of the second kind, changed to the problem of solving a Poisson equation several times.

As we can see from the assumption, the algorithm deteriorates as the yield limit τ_c is getting larger and larger (under the assumption that every thing else stays the same). So we are going to develop another method with better properties. The main idea in [18] is to approximate the solution u_∞ of

$$\mu a_2(u_\infty, v - u_\infty) + \tau_c j(v) - \tau_c j(u_\infty) \geq (c, v - u_\infty) \quad \forall\, v \in V_0 \tag{4.11}$$

by the time dependent problem

Problem 4.2 (Time dependent problem)**.** *Find $u(t) \in L^2_{loc}(0,\infty; H^1_0(\Omega))$ such that*

$$\begin{aligned} \int_\Omega \frac{\partial}{\partial t}\nabla u(t) \cdot \nabla(v - u(t))\, dx + \mu a_2(u(t), v - u(t)) \\ + \tau_c j(v) - \tau_c j(u(t)) \geq (c, v - u(t)) \quad \forall\, v \in V_0 \end{aligned} \tag{4.12}$$

$$u(0) = u_0 \tag{4.13}$$

where

$$L^2_{loc}(0,\infty; H^1_0(\Omega)) := \{v \,:\, [0,\infty) \to H^1_0(\Omega) \,|\, \|v\|_{L^2([0,t],H^1_0(\Omega))} < \infty, \forall t \in \mathbb{R}^+\}$$

and

$$\|v\|_{L^2([0,t],H^1_0(\Omega))} := \left(\int_0^t \|v(s)\|^2_{H^1_0(\Omega)}\, ds\right)^{1/2}.$$

Remark. We denoted the solution of (4.11) now by (u_∞, w_∞) in order to confirm that it is a steady state solution.

The next theorem guarantees that the time-dependent solution converges to the steady state solution.

Theorem 4.1.4. *Let $u(t)$ be the solution of problem 4.2 and let u_∞ be the solution of (4.11). Then there exists $\tilde{c} \in \mathbb{R}$ such that the estimate*

$$\|u(t) - u_\infty\|_1 \leq \tilde{c}\, e^{-\mu t}\|u_0 - u_\infty\|_1, \quad \forall\, t \geq 0 \tag{4.14}$$

holds.

Proof. We set $u(t) - u_\infty = \bar{u}(t)$. If we choose test functions $v = u_\infty$ in problem 4.2, $v = u(t)$ in problem 4.1, respectively, we get, after adding the two inequalities,

$$\langle \frac{\partial}{\partial t}\nabla u(t), \nabla(u(t) - u_\infty)\rangle + \mu a_2(\bar{u}(t), \bar{u}(t)) \leq 0. \tag{4.15}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t), w(t)\rangle = \langle \frac{\partial v(t)}{\partial t}, w(t)\rangle + \langle v(t), \frac{\partial w(t)}{\partial t}\rangle$$

and

$$\frac{\partial u_\infty}{\partial t} = 0,$$

(4.15) is equivalent to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \bar{u}(t)|^2\,dx \le -\mu \int_\Omega |\nabla \bar{u}(t)|^2\,dx.$$

After integrating this ordinary differential equation with respect to t we get

$$\int_\Omega |\nabla \bar{u}(t)|^2\,dx \le e^{-2\mu t}\int_\Omega |\nabla \bar{u}(0)|^2\,dx$$

or in another notation, using the initial value (4.13),

$$|u(t) - u_\infty|_1^2 \le e^{-2\mu t}|u_0 - u_\infty|_1^2.$$

Thanks to Friedrich's inequality we then have

$$|u(t) - u_\infty|_1 \ge \frac{1}{\tilde{c}}\|u(t) - u_\infty\|_1.$$

Since

$$|u_0 - u_\infty|_1^2 \le \|u_0 - u_\infty\|_0^2$$

holds, we have showed (4.14). □

For the time discretization of problem 4.2, the authors use a backward Euler scheme as the following: Given

$$u^0 = u_0$$

a solution u^{n+1} can be computed (for given u^n) by

$$\frac{1}{\Delta t}\int_\Omega \nabla(u^{n+1} - u^n)\cdot\nabla(v - u^{n+1})\,dx + \mu a_2(u^{n+1}, v - u^{n+1}) \\ + \tau_c j(v) - \tau_c j(u^{n+1}) \ge (c, v - u^{n+1}) \quad \forall\, v \in V_0$$

or equivalently

$$(1 + \mu\Delta t)a_2(u^{n+1}, v - u^{n+1}) + \Delta t\tau_c j(v) - \Delta t\tau_c j(u^{n+1}) \\ \ge a_2(u^n, v) + \Delta t(c, v - u^{n+1}) \quad \forall\, v \in V_0, \quad (4.16)$$

where Δt denotes the time discretization step. In the same manner as we did it for (4.11), one can show that (4.16) is equivalent to the time discretized mixed dual system:

Problem 4.3 (Time discretized dual dual system). *Find $u \in V_0, w \in W$ such that for all $v \in V_0$*

$$(1 + \mu\Delta t)a_2(u^{n+1}, v) + \tau_c\Delta t b(v, w^{n+1}) = a_2(u^n, v) + \Delta t(c, v)$$
$$w^{n+1} = P_W(w^{n+1} + rBu^{n+1})$$

for given $u^0 = u_0, w^0 \in W$.

4.1.2 A stabilized scheme

Computing a solution of problem 4.3 now seems to be straight forward by using a fixed point algorithm. Unfortunately there rests the problem of bad convergence properties for large values of τ_c, everything else staying the same. So we will use a special regularization for problem 4.3 which resolves the problem of bad convergence property and makes the dual variable unique. A so called (pseudo-) time disretized regularization term is added to the projection such that our problem to be solved (after time discretizing) looks like the following:

Problem 4.4 (Regularized time discretized dual dual formulation)**.** *Given* $u^0 = u_0 \in V_0, w^0 \in W$*. For* u^n, w^n *being known, find* $u^{n+1} \in V_0, w^{n+1} \in W$ *such that for all* $v \in V_0$

$$
\begin{aligned}
&(1 + \mu\Delta t)a_2(u^{n+1}, v) + \tau_c \Delta t\, b(v, w^{n+1}) = a_2(u^n, v) + \Delta t(c, v) \\
&\varepsilon \frac{w^{n+1} - w^n}{\Delta t} + w^{n+1} = P_W(w^{n+1} + rBu^{n+1})
\end{aligned}
\tag{4.17}
$$

with $\varepsilon > 0$.

Remark. The regularization term guarantees that $w^{n+1} \in W$ since (4.17) can be written as the convex combination

$$
w^{n+1} = \frac{\varepsilon}{\varepsilon + \Delta t} w^n + \frac{\Delta t}{\varepsilon + \Delta t} P_{\mathrm{W}}(w^{n+1} + rBu^{n+1}).
$$

As we mentioned, the additional term on the left hand side causes that the dual variable is unique.

Theorem 4.1.5. *Under the assumption*

$$
0 < r < \frac{2}{\tau_c}(\mu + \frac{1}{\Delta t}) \tag{4.18}
$$

problem 4.4 has a unique solution $(u^{n+1}, w^{n+1}) \in V_0 \times W$ *for all* $(u^n, w^n) \in V_0 \times W$*.*

Proof. We define the map

$$
T^n : W \to W, \quad T^n(\phi) = \frac{\varepsilon}{\varepsilon + \Delta t} w^n + \frac{\Delta t}{\varepsilon + \Delta t} P_{\mathrm{W}}(\phi + rBu_\phi^{n+1}),
$$

where u_ϕ^{n+1} is denoted as the solution of the Dirichlet-problem

$$
(1 + \mu\Delta t)a_2(u_\phi^{n+1}, v) = a_2(u^n, v) - \Delta t\tau_c b(v, \phi) + \Delta t(c, v) \quad \forall\, v \in V_0. \tag{4.19}
$$

We remark that for a fixed ϕ, u_ϕ is a unique solution of our problem since we can apply the Lax-Milgram theorem to (4.19). Now we take arbitrary

$\phi_1, \phi_2 \in W$ and denote $\phi_1 - \phi_2$ by $\bar{\phi}$. In the same manner we will denote $u_{\phi_1}^{n+1} - u_{\phi_2}^{n+1}$ by $\bar{u}$. Since P_{W} fulfills (4.1), it follows

$$\|T^n(\phi_1) - T^n(\phi_2)\|_0 \leq \frac{\Delta t}{\varepsilon + \Delta t} \|\bar{\phi} + rB\bar{u}\|_0. \tag{4.20}$$

From the definition of the norm it follows

$$\|\bar{\phi} + rB\bar{u}\|_0^2 = \|\bar{\phi}\|_0^2 + 2r(B\bar{u}, \bar{\phi}) + r^2 \|B\bar{u}\|_0^2. \tag{4.21}$$

Furthermore, from (4.19) we get (by subtracting the equations for $u_{\phi_1}^{n+1}$ and $u_{\phi_2}^{n+1}$, using test functions $v = \bar{u}$)

$$(1 + \mu\Delta t) a_2(\bar{u}, \bar{u}) = -\Delta t \tau_c b(\bar{u}, \bar{\phi}) = -\Delta t \tau_c (B\bar{u}, \bar{\phi}), \tag{4.22}$$

so that we can combine (4.21),(4.22) and get out of (4.20)

$$\|T^n(\phi_1) - T^n(\phi_2)\|_0^2 \leq \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^2 \left(\|\bar{\phi}\|_0^2 - 2r \frac{1 + \mu\Delta t}{\tau_c \Delta t} a_2(\bar{u}, \bar{u}) + r^2 \|B\bar{u}\|_0^2\right),$$

or equivalently

$$\|T^n(\phi_1) - T^n(\phi_2)\|_0^2 + r\left[\frac{2}{\tau_c}\left(\mu + \frac{1}{\Delta t}\right) - r\right]\left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^2 \|B\bar{u}\|_0^2 \leq \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^2 \|\bar{\phi}\|_0^2,$$

since

$$a_2(\bar{u}, \bar{u}) = \int_\Omega |\nabla \bar{u}|^2 \, dx = \|B\bar{u}\|_0^2.$$

Now under the assumption (4.18) it follows that

$$\|T^n(\phi_1) - T^n(\phi_2)\|_0 \leq \frac{\Delta t}{\varepsilon + \Delta t} \|\phi_1 - \phi_2\|_0 \quad \forall \phi_1, \phi_2 \in W.$$

Since $\frac{\Delta t}{\varepsilon + \Delta t} < 1$, T^n is a contraction and therefore has a unique fixed point ϕ together with a unique solution u_ϕ. □

As we have shown that there exists a solution for the regularized problem, we can write down a natural fixed point iteration for solving the system. Remember that n denotes the time discretization step whereas m is now introduced to denote the inner iteration index.

Algorithm 4.1.6. *If*

$$w_0^{n+1} = w^n$$

then for $m \geq 0, w_m^{n+1}$ *being known, we can compute* u_m^{n+1}, w_{m+1}^{n+1} *by*

$$(1 + \mu\Delta t) a_2(u_m^{n+1}, v) = a_2(u^n, v) - \tau_c \Delta t \, b(v, w_m^{n+1}) - \Delta t (c, v) \quad \forall v \in V_0$$

$$w_{m+1}^{n+1} = \frac{\varepsilon}{\varepsilon + \Delta t} w^n + \frac{\Delta t}{\varepsilon + \Delta t} P_W(w_m^{n+1} + rBu_m^{n+1}).$$

4.2 Convergence results

In this section we will proof that (u_m^n, w_{m+1}^n) - computed by algorithm 4.1.6 - converges to the desired solution (u_∞, w_∞), as well as that the regularization resolves the problem of bad convergence properties for large values of τ_c.

Notation. We denote the measure of a set Ω by meas(Ω).

Theorem 4.2.1. *Under the assumption*

$$0 < r < \frac{2}{\tau_c}\left(\mu + \frac{1}{\Delta t}\right) \tag{4.23}$$

there holds for all $m \geq 0$

$$\begin{aligned} \|w_m^{n+1} - w^{n+1}\|_0 &\leq \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^m \|w^{n+1} - w^n\|_0 \\ &\leq \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^m 2(meas(\Omega))^{1/2}. \end{aligned} \tag{4.24}$$

Further we obtain

$$\begin{aligned} \|u_m^{n+1} - u^{n+1}\|_1 &\leq \frac{\tau_c}{(\mu + 1/\Delta t)}\|w^{n+1} - w^n\|_0 \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^m \\ &\leq 2(meas(\Omega))^{1/2}\frac{\tau_c}{(\mu + 1/\Delta t)}\left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^m. \end{aligned} \tag{4.25}$$

Proof. Again we use the mapping

$$T^n : W \to W, \quad T^n(\phi) = \frac{\varepsilon}{\varepsilon + \Delta t} w^n + \frac{\Delta t}{\varepsilon + \Delta t} P_{\mathrm{W}}(\phi + rBu_\phi^{n+1}).$$

Then we have from the last theorem that T^n is a contraction with

$$\|T^n(\phi_1) - T^n(\phi_2)\|_0 \leq \frac{\Delta t}{\varepsilon + \Delta t}\|\phi_1 - \phi_2\|_0 \quad \forall\, \phi_1, \phi_2 \in W. \tag{4.26}$$

From algorithm 4.1.6 we get that

$$w_{m+1}^{n+1} = T^n(w_m^{n+1})$$

and

$$w^{n+1} = T^n(w^{n+1}).$$

The contraction property (4.26) then yields

$$\|w_{m+1}^{n+1} - w^{n+1}\|_0 \leq \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)\|w_m^{n+1} - w^{n+1}\|_0 \quad \forall\, m \geq 0.$$

As we assumed $w_0^{n+1} = w^n$ we get, applying (4.26) further m times,

$$\|w_m^{n+1} - w^{n+1}\|_0 \le \left(\frac{\Delta t}{\varepsilon + \Delta t}\right)^m \|w^{n+1} - w^n\|_0 \quad \forall\, m \ge 0. \tag{4.27}$$

By definition of the convex set W, we get for an arbitrary $w \in W$

$$\|w\|_0 = \left(\int_\Omega |w|^2\, dx\right)^{1/2} \le \left(\int_\Omega 1\, dx\right)^{1/2} = (meas(\Omega))^{1/2}. \tag{4.28}$$

Combining (4.27) and (4.28) results in (4.24).

To prove (4.25) we use the fact that u_m^{n+1}, u^{n+1} are solutions of the Dirichlet problem in algorithm 4.1.6. Therefore $u_m^{n+1} - u^{n+1}$ is a solution of

$$(1+\mu\Delta t)a_2(u_m^{n+1} - u^{n+1}, v) = -\tau_c \Delta t\, b(v, w_m^{n+1} - w^{n+1}) \quad \forall\, v \in V_0,\ \forall\, m \ge 0.$$

Choosing test functions $v = u_m^{n+1} - u^{n+1}$ leads to

$$(1+\mu\Delta t)\|\nabla(u_m^{n+1} - u^{n+1})\|_0 \le \tau_c \Delta t \|w_m^{n+1} - w^{n+1}\|_0 \quad \forall\, m \ge 0.$$

From (4.24) and Friedrich's inequality we get (4.25). □

So we have seen that the solution (u_m^n, w_m^n) of an inner iteration converges to the desired limit (u^{n+1}, w^{n+1}). Next we have to show the convergence of (u^n, w^n) to the steady state solution (u_∞, w_∞). The next theorem guarantees the strong convergence of u^n as well as the weak convergence of w^n.

Theorem 4.2.2. *Under the assumption*

$$0 < r < \frac{2\mu}{\tau_c} \tag{4.29}$$

we have

$$\lim_{n\to\infty} u^n = u_\infty \ \textit{in}\ V_0 = H_0^1(\Omega) \tag{4.30}$$

as well as

$$\lim_{n\to\infty} w^n = w_\infty \ \textit{weakly in}\ L^2(\Omega) \times L^2(\Omega) \tag{4.31}$$

for all $(u^0, w^0) \in V_0 \times W$.

Proof. We only give a detailed proof for (4.30). For (4.31) we refer to [18] and the references therein. Again we will denote $u^{n+1} - u_\infty$ and $w^{n+1} - w_\infty$ by $\bar{u}^{n+1}$ and $\bar{w}^{n+1}$, respectively. As (u_∞, w_∞) is a solution of (4.1) and (u^{n+1}, w^{n+1}) is a solution of problem 4.4 we then have

$$(1+\mu\Delta t)a_2(\bar{u}^{n+1}, v) = a_2(\bar{u}^n, v) - \tau_c \Delta t b(v, \bar{w}^{n+1}) \quad \forall\, v \in V_0, \tag{4.32}$$

as well as (from (4.1))

$$\left\| \varepsilon \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} + \bar{w}^{n+1} \right\|_0 \leq \|\bar{w}^{n+1} + rB\bar{u}^{n+1}\|_0. \tag{4.33}$$

In the same manner as in the proofs before, we now choose test functions $v = \bar{u}^{n+1}$ in (4.32) leading to

$$(1 + \mu\Delta t)\|\nabla\bar{u}^{n+1}\|_0^2 = a_2(\bar{u}^n, \bar{u}^{n+1}) - \tau_c \Delta t b(\bar{u}^{n+1}, \bar{w}^{n+1}). \tag{4.34}$$

Further we have (remembering that $\|Bu\|_0^2 = \|\nabla u\|_0^2$)

$$\|\bar{w}^{n+1} + rB\bar{u}^{n+1}\|_0^2 = \|\bar{w}^{n+1}\|_0^2 + 2rb(\bar{u}^{n+1}, \bar{w}^{n+1}) + r^2\|\nabla\bar{u}^{n+1}\|_0^2. \tag{4.35}$$

Together with (4.34),(4.35) we then get from (4.33)

$$\begin{aligned} \left\| \varepsilon \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} + \bar{w}^{n+1} \right\|_0^2 \quad \leq \quad & \|\bar{w}^{n+1}\|_0^2 + 2rb(\bar{u}^{n+1}, \bar{w}^{n+1}) + r^2\|\nabla\bar{u}^{n+1}\|_0^2 \\ & = \|\bar{w}^{n+1}\|_0^2 - 2r\frac{1 + \mu\Delta t}{\tau_c \Delta t}\|\nabla\bar{u}^{n+1}\|_0^2 \\ & + \frac{2r}{\tau_c \Delta t} a_2(\bar{u}^n, \bar{u}^{n+1}) + r^2\|\nabla\bar{u}^{n+1}\|_0^2. \end{aligned} \tag{4.36}$$

Another formulation for the left hand side is

$$\varepsilon^2 \left\| \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} \right\|_0^2 + 2\frac{\varepsilon}{\Delta t} \int_\Omega (\bar{w}^{n+1} - \bar{w}^n) \cdot \bar{w}^{n+1}\, dx + \|\bar{w}^{n+1}\|_0^2,$$

so that we get from (4.36)

$$\begin{aligned} \varepsilon^2 \left\| \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} \right\|_0^2 + 2\frac{\varepsilon}{\Delta t} \int_\Omega (\bar{w}^{n+1} - \bar{w}^n) \cdot \bar{w}^{n+1}\, dx \\ \leq \\ - 2r\frac{1 + \mu\Delta t}{\tau_c \Delta t}\|\nabla\bar{u}^{n+1}\|_0^2 + \frac{2r}{\tau_c \Delta t} a_2(\bar{u}^n, \bar{u}^{n+1}) + r^2\|\nabla\bar{u}^{n+1}\|_0^2. \end{aligned} \tag{4.37}$$

Moreover, using the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned} 2\int_\Omega (\bar{w}^{n+1} - \bar{w}^n) \cdot \bar{w}^{n+1}\, dx &= 2\|\bar{w}^{n+1}\|_0^2 - 2\int_\Omega \bar{w}^n \cdot \bar{w}^{n+1}\, dx \\ &\leq 2\|\bar{w}^{n+1}\|_0^2 - \|\bar{w}^n\|_0^2 - \|\bar{w}^{n+1}\|_0^2 \\ &= \|\bar{w}^{n+1}\|_0^2 - \|\bar{w}^n\|_0^2 \end{aligned}$$

as well as

$$2a_2(\bar{u}^n, \bar{u}^{n+1}) = 2\int_\Omega \nabla\bar{u}^n \cdot \nabla\bar{u}^{n+1}\, dx \leq \|\nabla\bar{u}^n\|_0^2 + \|\nabla\bar{u}^{n+1}\|_0^2.$$

Summing up these two results we get from (4.37)

$$\varepsilon^2 \left\| \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} \right\|_0^2 + r\left(\frac{2\mu}{\tau_c} - r\right) \|\nabla \bar{u}^{n+1}\|_0^2 \\ \leq \frac{\varepsilon}{\Delta t}\left(\|\bar{w}^n\|_0^2 - \|\bar{w}^{n+1}\|_0^2\right) + \frac{r}{\tau_c \Delta t}\left(\|\nabla \bar{u}^n\|_0^2 - \|\nabla \bar{u}^{n+1}\|_0^2\right). \quad (4.38)$$

From our assumption on r, (4.29), we get

$$0 < \varepsilon^2 \left\| \frac{\bar{w}^{n+1} - \bar{w}^n}{\Delta t} \right\|_0^2 + r\left(\frac{2\mu}{\tau_c} - r\right) \|\nabla \bar{u}^{n+1}\|_0^2.$$

So it follows that the sequence $(\varepsilon\|\bar{w}^n\|_0^2 + (r/\tau_c)\|\nabla \bar{u}^n\|_0^2)_n$ is decreasing. Therefore the left hand side of (4.38) converges to zero, so it follows

$$\lim_{n\to\infty} \nabla \bar{u}^{n+1} = 0 \text{ in } L^2(\Omega) \times L^2(\Omega),$$

which is equivalent to (4.30). □

4.3 Application to the problem in 3D

Our task is now to derive an analogous algorithm for the three-dimensional problem. For this we use two different ways treating the variables u, p, λ. A first choice would be, if we interpret (u, p) as the solution of a Stokes problem, instead of u being the solution of a Dirichlet problem. This step sounds like a natural one. Another possibility is that we interpret (p, λ) as one dual variable. This is also a logical generalization because in many applications p is already a dual variable.

4.3.1 The first approach

First we will look at problem 3.3:

Find $u \in V_0^{3\text{D}}$, $p \in Q$, $\lambda \in \Lambda$ such that

$$\begin{aligned} &\mu a(u, v) + b_1(v, p) + \tau_c \sqrt{2} b_2(v, \lambda) = (f, v) && \forall v \in V_0^{3\text{D}} \\ &b_1(u, q) = 0 && \forall q \in Q \\ &b_2(u, \eta - \lambda) \leq 0 && \forall \eta \in \Lambda. \end{aligned}$$

Now with the same technique as in the $2D$ model, the system is equivalent to

Problem 4.5 (Dual dual formulation)**.** *Find $u \in V_0^{3D}$, $p \in Q$, $\lambda \in \Lambda$ such that*

$$\begin{aligned} &\mu a(u, v) + b_1(v, p) + \tau_c \sqrt{2} b_2(v, \lambda) = (f, v) && \forall v \in V_0^{3D} \\ &b_1(u, q) = 0 && \forall q \in Q \\ &\lambda = P_\Lambda(\lambda + rD(u)) \end{aligned}$$

with $r > 0$. The operator occurring on the right hand side is now the orthogonal projection onto Λ given by:

$$P_\Lambda : [L^2(\Omega^{3D})]^{3\times 3} \to \Lambda, \quad P_\Lambda(\mu)(x) = \frac{\mu(x)}{\sup(1, \|\mu(x)\|_F)} \quad \forall \mu \in [L^2(\Omega^{3D})]^{3\times 3}.$$

A natural generalization of problem 4.4 looks then like

Problem 4.6 (Regularized time discretized dual dual formulation for 3D)**.** *Given $u^0 = u_0 \in V_0^{3D}, p^0 \in Q, \lambda^0 \in \Lambda$. For u^n, p^n, λ^n being known, find $u^{n+1} \in V_0^{3D}, p^{n+1} \in Q, \lambda^{n+1} \in \Lambda$ such that for all $v \in V_0^{3D}, q \in Q$*

$$(1 + \mu\Delta t)a(u^{n+1}, v) + \tau_c\sqrt{2}\Delta t\, b_2(v, w^{n+1}) + \Delta t\, b_1(v, p^{n+1}) = a(u^n, v) + \Delta t(f, v)$$

$$(p^{n+1}, q) = (p^n, q) + \bar{r}(B_1 u^{n+1}, q)$$

$$\varepsilon\frac{\lambda^{n+1} - \lambda^n}{\Delta t} + \lambda^{n+1} = P_\Lambda(\lambda^{n+1} + rB_2 u^{n+1})$$

with $\varepsilon > 0$.

Then the generalization of algorithm 4.1.6 can be written as

Algorithm 4.3.1. *If*

$$(u^0, p^0, \lambda^0) \in V_0^{3D} \times Q \times \Lambda$$

is given, and if

$$p_0^{n+1} = p^n,\ \lambda_0^{n+1} = \lambda^n$$

then, for $m \geq 0$, $p_m^{n+1}, \lambda_m^{n+1}$ being known, we compute u_m^{n+1},p_{m+1}^{n+1} and λ_{m+1}^{n+1} as follows:

$$(1 + \mu\Delta t)a(u_m^{n+1}, v) = a(u^n, v) - \Delta t b_1(v, p_m^{n+1}) - \tau_c\sqrt{2}\Delta t b_2(v, \lambda_m^{n+1}) + (f, v)$$

$$p_{m+1}^{n+1} = p_m^{n+1} + \bar{r}\,\mathrm{div}\, u_m^{n+1}$$

$$\lambda_{m+1}^{n+1} = \frac{\varepsilon}{\varepsilon + \Delta t}\lambda^n + \frac{\Delta t}{\varepsilon + \Delta t}P_\Lambda(\lambda_m^{n+1} + rD(u_m^{n+1}))$$

where $r, \bar{r} > 0$.

4.3.2 The second approach

In this case we introduce the following settings:

$$W = Q \times \Lambda,$$

$$b :\ V_0^{3\mathrm{D}} \times W \to \mathbb{R},\ b(v, w) = b(v, (w_1, w_2)) = \bar{b}_1(v, w_1) + b_2(v, w_2)$$

with

$$\bar{b}_1 : V_0^{3\mathrm{D}} \times Q \to \mathbb{R}, \quad \bar{b_1}(v,p) = \frac{1}{\tau_c\sqrt{2}} b_1(v,p),$$

as well as

$$P_{\mathrm{W}} : L^2(\Omega^{3\mathrm{D}}) \times [L^2(\Omega^{3\mathrm{D}})]^{3\times 3}, \quad (p,\mu) \mapsto (p, P_\Lambda(\mu))$$

and

$$B : V_0^{3\mathrm{D}} \to [L^2(\Omega^{3\mathrm{D}})]^{3\times 3} \times [L^2(\Omega^{3\mathrm{D}})]^{3\times 3}, \quad u \mapsto Bu = (\bar{B}_1 u, B_2 u)$$

with the linear operators $\bar{B_1}, B_2$ defined by the bilinear forms $\bar{b_1}(\cdot,\cdot), b_2(\cdot,\cdot)$. Finally we introduce the norm for the product space W by

$$\|w\|_W := \max\{\|w_1\|_0, \|w_2\|_0\}.$$

Then our problem looks like the following:

Problem 4.7 (Regularized time discretized dual dual formulation 3D). *Find* $u_\infty \in V_0^{3D}, w_\infty \in W = \{w = (w_1, w_2) \,|\, w_1 \in Q,\ w_2 \in \Lambda,\ \|w_2\|_F \leq 1\}$ *such that*

$$\begin{aligned} &a(u_\infty, v) + \tau_c\sqrt{2} b(v, w_\infty) = (f, v) \quad \forall\, v \in V_0^{3D} \\ &w_\infty = P_W(w_\infty + rBu_\infty) \end{aligned}$$

what is just the form of problem 4.1 with an extra factor $\sqrt{2}$ in the first line. So we can exactly apply the results from the previous section which means that a solution of (4.39) can be computed the following way (again n denotes the time discretization step whereas m denotes the inner iteration step)

Algorithm 4.3.2. *If*

$$(u^0, p^0, \lambda^0) \in V_0^{3D} \times Q \times \Lambda$$

is given, and if

$$p_0^{n+1} = p^n,\ \lambda_0^{n+1} = \lambda^n$$

then, for $m \geq 0$, $p_m^{n+1}, \lambda_m^{n+1}$ *being known, we compute* u_m^{n+1},p_{m+1}^{n+1} *and* λ_{m+1}^{n+1} *as follows:*

$$(1+\mu\Delta t)a(u_m^{n+1}, v) = a(u^n, v) - \Delta t b_1(v, p_m^{n+1}) - \tau_c\sqrt{2}\Delta t b_2(v, \lambda_m^{n+1}) + (f, v)$$

$$p_{m+1}^{n+1} = \frac{\varepsilon}{\varepsilon+\Delta t} p^n + \frac{\Delta t}{\varepsilon+\Delta t}(p_m^{n+1} + \bar{r}\,\mathrm{div}\, u_m^{n+1})$$

$$\lambda_{m+1}^{n+1} = \frac{\varepsilon}{\varepsilon+\Delta t}\lambda^n + \frac{\Delta t}{\varepsilon+\Delta t} P_\Lambda(\lambda_m^{n+1} + rD(u_m^{n+1}))$$

where

$$\bar{r} = r/\tau_c\sqrt{2}.$$

Remark. Applying the analysis leads to the same results except an extra constant factor because of the factor $\sqrt{2}$ in the dual formulation.

Remark. We want to mention that applying algorithms 4.3.1,4.3.2 leads to the computation of the exact solution of a linear system in each step of the fixed-point iteration. In order to avoid this fact one can use an algorithm dealing with an approximate solution. For more details for such an inexact Uzawa-type method, we refer to [7].

Chapter 5

Approximation with Finite Elements

In this chapter we will use the finite element method for discretizing our developed dual systems in the three-dimensional case as well as in the two-dimensional one. We are going to discretize the generalizations for the three-dimensional problems in two different ways and then we will compare the convergence properties in chapter 6. Definitions from the first section are cited from [9],[32],[31].

5.1 Discretization of the domain

For the three-dimensional case we consider a mesh $\mathcal{T}_h^{3D}$ of Ω^{3D}, where

$$\mathcal{T}_h^{3D} := \{T \,\big|\, T \text{ is a tetrahedron in } \Omega^{3D}\}.$$

The index h denotes the maximum of all diameters h_T of all tetrahedrons T. The set of nodes is denoted by $\mathcal{N}_h^{3D}$, the set of edges is denoted by $\mathcal{E}_h^{3D}$ and finally we denote the set of faces by $\mathcal{F}_h^{3D}$. We further distinguish the inner nodes from the ones lying on the boundary of the mesh,i.e.

$$\mathcal{N}_h^{3D} = \mathcal{N}_{h,\Omega^{3D}}^{3D} \cup \mathcal{N}_{h,\Gamma^{3D}}^{3D}$$

analogously for the edges

$$\mathcal{E}_h^{3D} = \mathcal{E}_{h,\Omega^{3D}}^{3D} \cup \mathcal{E}_{h,\Gamma^{3D}}^{3D}$$

as well, for the faces

$$\mathcal{F}_h^{3D} = \mathcal{F}_{h,\Omega^{3D}}^{3D} \cup \mathcal{F}_{h,\Gamma^{3D}}^{3D}.$$

Analogously we can discretize the two-dimensional domain Ω by a mesh $\mathcal{T}_h$ with

$$\mathcal{T}_h := \{T \,\big|\, T \text{ is a triangle in } \Omega\}.$$

The set of all nodes is here denoted by $\mathcal{N}_h$, whereas the set of edges is denoted by $\mathcal{E}_h$. Moreover

$$\mathcal{N}_h = \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,\Gamma}$$

and

$$\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\Gamma}.$$

We introduce the following definitions:

Definition. A mesh $\mathcal{T}_h^{\text{3D}}$ of a domain Ω is called an admissible triangulation, if

(R1) $\bar{\Omega} = \bigcup\limits_{T \in \mathcal{T}_h^{\text{3D}}} T$

(R2) $T_1 \cap T_2 = \begin{cases} \varnothing \\ \text{a node} \\ \text{an edge} \\ \text{a face} \end{cases}$ for all $T_1 \neq T_2 \in \mathcal{T}_h^{\text{3D}}$,

holds.

Definition. Let $\mathcal{T}_h^{\text{3D}}$ be a triangulation. For an element $T \in \mathcal{T}_h^{\text{3D}}$ let ρ_T be the radius of the ball (circle in the two-dimensional case) with maximal diameter B_T. We call $\mathcal{T}_h^{\text{3D}}$ shape regular if the ratio of h_T and ρ_T is bounded from above, independently from T and h, i.e. there exists $\kappa > 0$ such that

$$\frac{h_T}{\rho_T} \leq \kappa \quad \forall T \in \mathcal{T}_h.$$

Definition. A shape regular mesh $\mathcal{T}_h$ fulfilling

$$\exists \kappa > 0 : \frac{h}{h_T} \leq \kappa, \quad \forall T \in \mathcal{T}_h$$

is called quasi-uniform.

Our meshes are now assumed to be admissible and quasi-uniform. The neighborhoods of a node x and a tetrahedron K are defined by

$$\omega_x := \bigcup_{T \in \mathcal{T}_h^{\text{3D}}, x \in T} T$$

and

$$\tilde{\omega}_K := \bigcup_{T \in \mathcal{T}_h^{\text{3D}}, T \cap K \neq 0} T$$

respectively. So ω_x contains all tetrahedrons which contain the node x, whereas $\tilde{\omega}_K$ denotes the set which contains all tetrahedrons T which have a common node, a common edge or a common face with the tetrahedron K.

5.2 The discrete models

We use Galerkin's principle for the discretization with finite elements. So the infinite dimensional spaces $V_0^{\mathrm{3D}}, Q, \Lambda$ and V_0, W are replaced by finite dimensional subspaces $V_{0,h}^{\mathrm{3D}}, Q_h, \Lambda_h$ and $V_{0,h}, W_h$, respectively. Defining the subspaces later, we first write down the discrete models for the mixed problem 3.1

Problem 5.1 (Discrete mixed formulation). *Find $u_h \in V_{0,h}^{3D}, p_h \in Q_h$ such that $\forall\, v_h \in V_{0,h}^{3D}, \forall\, q_h \in Q_h$*

$$\begin{aligned}
&\mu a(u_h, v_h - u_h) + b_1(v_h - u_h, p_h) + \tau_c j(v_h) - \tau_c j(u_h) \geq (f, v_h - u_h) \\
&b_1(u_h, q_h) = 0
\end{aligned}$$

and for the problem reduced to the cross section 3.2

Problem 5.2 (Discrete primal formulation). *Find $u_h \in V_{0,h}$ such that $\forall\, v_h \in V_{0,h}$*

$$\mu a_2(u_h, v_h - u_h) + \tau_c j(v_h) - \tau_c j(u_h) \geq (c, v_h - u_h)$$

respectively. The dual formulations can then be written as

Problem 5.3 (Discrete dual dual formulation). *Find $u_h \in V_{0,h}^{3D}$, $p_h \in Q_h$, $\lambda_h \in \Lambda$ such that $\forall\, v_h \in V_{0,h}^{3D}, \forall\, q_h \in Q_h$, $\forall\, \eta_h \in \Lambda_h$*

$$\begin{aligned}
&\mu a(u_h, v_h) + b_1(v_h, p_h) + \tau_c \sqrt{2}\, b_2(v_h, \lambda_h) = (f, v_h) \\
&b_1(u_h, q_h) = 0 \\
&b_2(u_h, \eta_h - \lambda_h) \leq 0
\end{aligned}$$

and in the 2D case

Problem 5.4 (Discrete dual formulation). *Find $u_h \in V_{0,h}$, $w_h \in W_h$ such that $\forall\, v_h \in V_{0,h}$, $\forall\, \phi_h \in W_h$*

$$\begin{aligned}
&\mu a_2(u_h, v_h) + \tau_c \sqrt{2} b(v_h, w_h) = (c, v_h) \\
&b_2(u_h, \phi_h - w_h) \leq 0
\end{aligned}$$

respectively.

Lemma 5.2.1. *The discrete problems 5.1 and 5.2 are equivalent to their discrete dual formulations 5.3 and 5.4 respectively.*

Proof. Can be done in the same way as we did it for the continuous case. □

5.2.1 The two-dimensional case

In [18] the following finite dimensional subspaces are used for the approximation of $V, V_0, L^2(\Omega) \times L^2(\Omega), W$:

$$V_h = \{v_h \in C^0(\overline{\Omega_h}) \,\big|\, v_h|_T \in P_1,\, \forall T \in \mathcal{T}_h\} \subset V,$$

$$V_{0,h} = \{v_h \in V_h \,\big|\, v_h|_{\Gamma_h} = 0\} \subset V_0,$$

$$L_h = \{\lambda_h \,\big|\, \lambda_h \in L^2(\Omega_h) \times L^2(\Omega_h),\, \lambda_h|_T \in P_0,\, \forall T \in \mathcal{T}_h\} \subset L^2(\Omega_h) \times L^2(\Omega_h),$$

$$W_h = \{w_h \,\big|\, w_h \in L_h,\, |w_{h|T}| \le 1 \,\forall T \in \mathcal{T}_h\} \subset W.$$

Therein P_0, P_1 denote the spaces of polynomials in two variables of degree 0 and 1 respectively. The finite element discretization of algorithm 4.1.6 looks then as follows:

Algorithm 5.2.2 (Discrete 2D algorithm)**.** *If*

$$w_{h,0}^{n+1} = w_h^n$$

then for $m \ge 0, w_{h,m}^{n+1}$ *being known, we can compute* $u_{h,m}^{n+1}, w_{h,m+1}^{n+1}$ *by*

$$(1 + \mu\Delta t) a_2(u_{h,m}^{n+1}, v_h) = a_2(u_h^n, v_h) - \tau_c \Delta t\, b(v_h, w_{h,m}^{n+1}) - \Delta t(c, v_h) \quad \forall\, v_h \in V_{0,h}$$

$$w_{h,m+1}^{n+1} = \frac{\varepsilon}{\varepsilon + \Delta t} w_h^n + \frac{\Delta t}{\varepsilon + \Delta t} P_{W_h}(w_{h,m}^{n+1} + rBu_{h,m}^{n+1}).$$

5.2.2 The three-dimensional case

Continuous approximation of the pressure

Here we are going to discretize algorithm 4.3.1. We introduce the following finite dimensional subspaces for $V^{3D}, V_0^{3D}, Q, [L^2(\Omega^{3D})]^{3\times 3}, \Lambda$:

$$V_h^{3D} = \{v_h \in [C^0(\overline{\Omega_h^{3D}})]^3 \,\big|\, v_h|_T \in [P_1]^3,\, \forall T \in \mathcal{T}_h^{3D}\} \subset [H^1(\Omega^{3D})]^3,$$

$$V_{0,h}^{3D} = \{v_h \in V_h \,\big|\, v_h|_{\Gamma_h^{3D}} = 0\} \subset V_0^{3D},$$

$$Q_h = \{q_h \in C^0(\overline{\Omega_h^{3D}}) \,\big|\, q_h|_T \in P_1,\, \forall T \in \mathcal{T}_h\} \subset Q,$$

$$L_h = \{\lambda_h \,\big|\, \lambda_h \in [L^2(\Omega_h^{3D})]^{3\times 3},\, \lambda_h|_T \in [P_0]^{3\times 3},\, \forall T \in \mathcal{T}_h^{3D}\} \subset [L^2(\Omega_h^{3D})]^{3\times 3},$$

$$\Lambda_h = \{\lambda_h \,\big|\, \lambda_h \in L_h,\, \|\lambda_{h|T}\|_F \le 1 \,\forall T \in \mathcal{T}_h^{3D}\} \subset \Lambda.$$

The usage of linear ansatz functions for u_h as well as for p_h leads to an unstable system, i.e. the discrete inf-sup condition for $b_1(\cdot,\cdot)$ is not fulfilled. By adding so called bubble functions the system becomes stable.
A bubble function is defined by

$$b_T(x) = \lambda_1\lambda_2\lambda_3\lambda_4$$

where the λ_i are the barycentric coordinates of x on T. With this definition we see that $b_T \in P_4$ vanishes on the boundary ∂T. We replace $V_{0,h}^{3\mathrm{D}}$ by

$$\overline{V}_{0,h}^{3\mathrm{D}} = \{v_h \in [C^0(\Omega^{3\mathrm{D}})]^3 \,\big|\, v_h|_T = p_T + \beta_T b_T,\, p_T \in [P_1]^3,\, \beta_T \in \mathbb{R}^3 \;\forall T \in \mathcal{T}_h^{3\mathrm{D}}\}.$$

By adding bubble functions, we have to deal with more degrees of freedom. Luckily one can show that the bubble functions can be locally eliminated because of their definition. As we use problem 5.3 for our implementation, we will show the elimination for this system. For $u_h \in \bar{V}_{0,h}^{3\mathrm{D}}$ we make the ansatz

$$u_h = u_h^1 + u_h^b, \quad u_h^1 \in V_{0,h}^{3\mathrm{D}},\; u_h^b = \sum_{T\in\mathcal{T}_h^{3\mathrm{D}}} \beta_T b_T \in [\mathrm{span}\{b_T : T \in \mathcal{T}_h^{3\mathrm{D}}\}]^3.$$

We get, by using special test functions $b_T e_i$ for $i = 1, 2, 3$ (where the e_i should be the unit vectors in $\mathbb{R}^3$) with a fixed T

$$\mu \sum_{j=1}^{3} a(b_T e_j, b_T e_i)\beta_{T,j} + \mu a(u_h^1, b_T e_i) \\ + b_1(b_T e_i, p_h) + \sqrt{2}\tau_c b_2(b_T e_i, \lambda_h) = (f, b_T e_i). \quad (5.1)$$

Now we make some observations. As the bubble function b_T is zero outside of a tetrahedron T, it follows

$$\begin{aligned} a(u_h^1, b_T e_i) &= \int_T 2D(u_h^1) : D(b_T e_i)\, dx \\ &= \underbrace{-\int_T 2\,\mathbf{div}\, D(u_h^1)\cdot b_T e_i\, dx}_{=0,\, u_h^1 \text{ linear on } T} + \underbrace{\int_{\partial T} 2D(u_h^1)n \cdot b_T e_i\, ds}_{=0,\, b_T=0 \text{ on } \partial T} \\ &= 0 \end{aligned}$$

as well as

$$\begin{aligned} b_2(b_T e_i, \lambda_h) &= \int_T \lambda_h : D(b_T e_i)\, dx \\ &= \underbrace{-\int_T \mathbf{div}\, \lambda_h \cdot b_T e_i\, dx}_{=0,\, \lambda_h \text{ const. on } T} + \underbrace{\int_{\partial T} \lambda_h n \cdot b_T e_i\, ds}_{=0} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} b_1(b_T e_i, p_h) &= -\int_T p_h \operatorname{div} b_T e_i \, dx \\ &= \int_T \nabla p_h \cdot b_T e_i \, dx - \underbrace{\int_{\partial T} p_h b_T e_i \cdot n \, ds}_{=0} \\ &= \int_T \nabla p_h \cdot b_T e_i \, dx. \end{aligned}$$

So that we get from (5.1)

$$\mu \sum_{j=1}^{3} a(b_T e_j, b_T e_i) \beta_{T,j} = \int_T (f - \nabla p_h) \cdot b_T e_i \, dx. \tag{5.2}$$

With the notations

$$\gamma_T = \int_T b_T \, dx, \quad \overline{f}^T = \frac{1}{\gamma_T} \int_T b_T f \, dx, \quad \delta_T = \mu(a(b_T e_j, b_T e_i))_{i,j \in \{1,2,3\}}$$

we can write down the formula for the unknowns β_T. From (5.2) we have

$$\beta_T = \delta_T^{-1} \left(\int_T b_T (f - \nabla p_h) \, dx \right) = \gamma_T \delta_T^{-1} (\overline{f}^T - \nabla p_h). \tag{5.3}$$

One can easily check that δ_T is positive definite, so the inverse exists. Now we can plug in our result (5.3) into the second as well as the third equation (inequality resp.) of problem 5.3 and get

$$\begin{aligned} b_1(u_h, q_h) &= b_1(u_h^1, q_h) + b_1(u_h^b, q_h) \\ &= -\int_{\Omega^{3D}} q_h \operatorname{div} u_h^1 \, dx - \sum_{T \in \mathcal{T}_h} \int_T q_h \operatorname{div} b_T \beta_T \, dx \\ &= -\int_{\Omega^{3D}} q_h \operatorname{div} u_h^1 \, dx + \sum_{T \in \mathcal{T}_h} \int_T \nabla q_h \cdot b_T \beta_T \, dx - \underbrace{\int_{\partial T} q_h n \cdot b_T \beta_T \, ds}_{=0} \\ &\overset{(5.3)}{=} -\int_{\Omega^{3D}} q_h \operatorname{div} u_h^1 \, dx + \sum_{T \in \mathcal{T}_h} \alpha(T) \int_T (\delta_T^{-1} (\overline{f}^T - \nabla p_h)) \cdot \nabla q_h \, dx \\ &= 0 \end{aligned}$$

with

$$\alpha(T) = \frac{\gamma_T^2}{\operatorname{meas}(T)}.$$

Moreover we have

$$
\begin{aligned}
b_2(u_h, \eta_h - \lambda_h) &= b_2(u_h^1, \eta_h - \lambda_h) + b_2(u_h^b, \eta_h - \lambda_h) \\
&= b_2(u_h^1, \eta_h - \lambda_h) + \sum_{T \in \mathcal{T}_h} \int_T (\eta_h - \lambda_h) : D(b_T \beta_T)\, dx \\
&= b_2(u_h^1, \eta_h - \lambda_h) - \sum_{T \in \mathcal{T}_h} \underbrace{\int_T \mathbf{div}\,(\eta_h - \lambda_h) \cdot b_T \beta_T\, dx}_{=0} \\
&\quad - \underbrace{\int_{\partial T} (\eta_h - \lambda_h) n \cdot b_T \beta_T\, ds}_{=0} \\
&= b_2(u_h^1, \eta_h - \lambda_h) \leq 0 \quad \forall\, \eta_h \in \Lambda_h.
\end{aligned}
$$

Summarizing our results, the additional degrees of freedom coming from the bubble functions, can be eliminated and the new system to be solved has the following form:

Problem 5.5 (Discrete dual dual system with continuous pressure)**.** *Search* $u_h \in V_{0,h}^{3D}, p_h \in Q_h, \lambda_h \in \Lambda_h$

$$
\begin{aligned}
&\mu a(u_h, v_h) + b_1(v_h, p_h) + \sqrt{2}\tau_c\, b_2(v_h, \lambda_h) = (f, v_h) && \forall\, v_h \in V_{0,h}^{3D} \\
&b_1(u_h, q_h) - c_h(p_h, q_h) = \langle G_h, q_h \rangle && \forall\, q_h \in Q_h \\
&b_2(u_h, \eta_h - \lambda_h) \leq 0
\end{aligned}
$$

with the mesh-dependent bilinear form

$$
c_h(p_h, q_h) = \sum_{T \in \mathcal{T}_h^{3D}} \alpha(T) \int_T (\delta_T^{-1} \nabla p_h) \cdot \nabla q_h\, dx
$$

and the mesh-dependent linear functional

$$
\langle G_h, q_h \rangle = \sum_{T \in \mathcal{T}_h^{3D}} \alpha(T) \int_T (\delta_T^{-1} \overline{f}^T) \cdot \nabla q_h\, dx.
$$

So our algorithm can be written as

Algorithm 5.2.3. *If*

$$
(u_h^0, p_h^0, \lambda_h^0) \in V_{0,h}^{3D} \times Q_h \times \Lambda_h
$$

is given, and if

$$
p_{0,h}^{n+1} = p_h^n,\ \lambda_{0,h}^{n+1} = \lambda_h^n
$$

then, for $m \geq 0$, $p^{n+1}_{m,h}, \lambda^{n+1}_{m,h}$ *being known, we compute* $u^{n+1}_{m,h}$,$p^{n+1}_{m+1,h}$ *and* $\lambda^{n+1}_{m+1,h}$ *as follows:*

$$(1 + \mu\Delta t)a(u^{n+1}_{m,h}, v_h) = a(u^n_h, v_h) - \Delta t b_1(v_h, p^{n+1}_{m,h}) - \tau_c\sqrt{2}\Delta t b_2(v_h, \lambda^{n+1}_{m,h}) + (f, v_h)$$

$$(p^{n+1}_{m+1,h}, q_h) = (p^{n+1}_{m,h}, q_h) + \bar{r}(\langle G_h, q_h\rangle - (B_1 u^{n+1}_{m,h}, q_h) + c_h(p_h, q_h)$$

$$\lambda^{n+1}_{m+1,h} = \frac{\varepsilon}{\varepsilon + \Delta t}\lambda^n_h + \frac{\Delta t}{\varepsilon + \Delta t} P_{\Lambda_h}(\lambda^{n+1}_{m,h} + rD(u^{n+1}_{m,h}))$$

where $r, \bar{r} > 0$.

Discontinuous approximation of the pressure

An alternative method would be if we treat p, λ as one variable. In this case we use for p the same ansatz functions as for λ. We choose the finite dimensional subspaces of $V_0^{3D}, Q, [L^2(\Omega^{3D})]^{3\times 3}, \Lambda$ in the following way:

$$V_h^{3D} = \{v_h \in [C^0(\overline{\Omega_h^{3D}})]^3 \,|\, v_h|_T \in [P_1]^3, \forall T \in \mathcal{T}_h^{3D}\} \subset [H^1(\Omega^{3D})]^3,$$

$$V_{0,h}^{3D} = \{v_h \in V_h^{3D} \,|\, v_h|_{\Gamma_h^{3D}} = 0\} \subset V_0^{3D},$$

$$Q_h = \{q_h \,|\, q_h|_T \in P_0, \forall T \in \mathcal{T}_h\} \subset Q,$$

$$L_h = \{\lambda_h \,|\, \lambda_h \in [L^2(\Omega_h^{3D})]^{3\times 3}, \lambda_h|_T \in [P_0]^{3\times 3}, \forall T \in \mathcal{T}_h^{3D}\} \subset [L^2(\Omega_h^{3D})]^{3\times 3},$$

$$\Lambda_h = \{\lambda_h \,|\, \lambda_h \in L_h, \|\lambda_{h|T}\|_F \leq 1 \,\forall T \in \mathcal{T}_h^{3D}\} \subset \Lambda.$$

So the discrete version of algorithm 4.3.2 looks like

Algorithm 5.2.4. *If*

$$(u^0_h, p^0_h, \lambda^0_h) \in V^{3D}_{0,h} \times Q_h \times \Lambda_h$$

is given, and if

$$p^{n+1}_{0,h} = p^n_h,\; \lambda^{n+1}_{0,h} = \lambda^n_h$$

then, for $m \geq 0$, $p^{n+1}_{m,h}, \lambda^{n+1}_{m,h}$ *being known, we compute* $u^{n+1}_{m,h}$,$p^{n+1}_{m+1,h}$ *and* $\lambda^{n+1}_{m+1,h}$ *as follows:*

$$(1 + \mu\Delta t)a(u^{n+1}_{m,h}, v) = a(u^n_h, v) - \Delta t b_1(v, p^{n+1}_{m,h}) - \tau_c\sqrt{2}\Delta t b_2(v, \lambda^{n+1}_{m,h}) + (f, v)$$

$$p^{n+1}_{m+1,h} = \frac{\varepsilon}{\varepsilon + \Delta t} p^n_h + \frac{\Delta t}{\varepsilon + \Delta t}(p^{n+1}_{m,h} + \bar{r}\,\mathrm{div}\, u^{n+1}_{m,h})$$

$$\lambda^{n+1}_{m+1,h} = \frac{\varepsilon}{\varepsilon + \Delta t}\lambda^n_h + \frac{\Delta t}{\varepsilon + \Delta t} P_{\Lambda}(\lambda^{n+1}_{m,h} + rD(u^{n+1}_{m,h}))$$

where

$$\bar{r} = r/\tau_c\sqrt{2}.$$

5.3 Existence and uniqueness for the discrete mixed problems

Here we want to show that our discretized problems also provide unique solutions.

5.3.1 The two-dimensional problem

For the two-dimensional problem we can apply the theorem cited from [14].

Theorem 5.3.1. *The discrete problem 5.2 admits one and only one solution.*

Proof. The statement follows from the equivalence of the primal problem 5.2 to minimizing

$$J(v_h) := \frac{1}{2}a_2(v_h, v_h) - (c, v_h) + j(v_h)$$

which is shown in [14]. The functional J is continuous and strictly convex on V_h too and we have

$$\lim_{\|v_h\|\to\infty} J(v_h) = +\infty$$

since the discrete assumption

$$a_2(v_h, v_h) \geq \alpha_1 \|v_h\|_1^2 \quad \forall\, v_h \in V_h$$

is fulfilled. The uniqueness of the solution u_h is then clear. □

5.3.2 The three-dimensional problem

Similarly to chapter 2, we can guarantee the existence of a discrete solution, if the discrete assumptions are satisfied. So we have the following theorem which can be found in [17]

Theorem 5.3.2. *Under the assumptions from theorem 3.3.7 with the exception that the inf sup condition (3.28) is replaced by*

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_{0,h}^{3D}} \frac{b_1(v_h, q_h)}{\|v_h\|_{V_{0,h}^{3D}} \|q_h\|_{Q_h}} \geq \beta \tag{5.4}$$

and the condition on coercivity (3.27) is replaced by the condition

$$\langle Av, v\rangle \geq \alpha_0 \|v\|_{V_h^{3D}}^2, \quad \forall\, v \in Ker\ B_{1,h}, \tag{5.5}$$

the discrete problem 5.1 has a unique solution.

Since

$$\text{Ker } B_{1,h} = \{v_h \mid \int_{\Omega^{3\mathrm{D}}} q_h \operatorname{div} v_h \, dx = 0 \text{ for all } q_h \in Q_h\} \subset V_{0,h}^{3\mathrm{D}},$$

condition (5.5) is fulfilled. For the discrete inf sup condition (5.4) we first need the following lemma, which can be found in [6],[34].

Lemma 5.3.3 (Fortin). *We assume that there exists a surjective mapping* $\Pi_h : V^{3D} \to V_h^{3D}$ *with*

1. $b_1(\Pi_h v, q_h) = b_1(v, q_h)$ *for all* $q_h \in Q_h$ *as well as for all* $v \in V^{3D}$ *and*
2. $\|\Pi_h v\|_{V^{3D}} \leq c\|v\|_{V^{3D}}$ *for all* $v \in V^{3D}$ *with* $c > 0$*, independent of* h*.*

If further the inf-sup condition for the spaces V^{3D}, Q *holds, then the discrete counterpart follows.*

Proof. There holds

$$\beta_1 \|q_h\|_Q \leq \sup_{0 \neq v \in V^{3\mathrm{D}}} \frac{b_1(v, q_h)}{\|v\|_{V^{3\mathrm{D}}}} \leq c \sup_{0 \neq v \in V^{3\mathrm{D}}} \frac{b_1(\Pi_h v, q_h)}{\|\Pi_h v\|_{V^{3\mathrm{D}}}} \leq c \sup_{0 \neq v_h \in V_h^{3\mathrm{D}}} \frac{b_1(v_h, q_h)}{\|v_h\|_{V^{3\mathrm{D}}}}$$

and therefore

$$\inf_{0 \neq q_h \in Q_h} \sup_{0 \neq v_h \in V_h^{3\mathrm{D}}} \frac{b_1(v_h, q_h)}{\|v_h\|_{V^{3\mathrm{D}}} \|q_h\|_Q} \geq \frac{1}{c} \beta_1.$$

□

Definition. The modified Clement interpolation operator is defined by

$$R_h : L^1(\Omega^{3\mathrm{D}}) \to V_{0,h}^{3\mathrm{D}}, \quad R_h u \mapsto \sum_{x \in \mathcal{N}_{0,\Omega^{3\mathrm{D}}}^{3\mathrm{D}}} (Pu)\phi_x$$

with

$$P : L^1(\omega_x^{3\mathrm{D}}) \to \mathbb{R}, \quad u \mapsto Pu = \frac{1}{\operatorname{meas}(\omega_x)} \int_{\omega_x} u \, dx$$

and ϕ_x denotes the nodal basis function.

Remark. By standard theory ([24]) it follows the following error estimate for the modified Clement interpolation operator

$$\|u - R_h u\|_{L^2(T)} \leq c h_T \|u\|_{[H^1(\tilde{\omega_T})]^3}. \tag{5.6}$$

Fortin's lemma now provides a method for proving the discrete inf-sup condition. With it we have (see [31])

Theorem 5.3.4. *Assume that we have a shape regular triangularization then for the spaces* $\bar{V}_h^{3D}$ *and* Q_h *the discrete inf-sup condition is fulfilled.*

Proof. We do the proof with the previous lemma. So let us define

$$\Pi_h u := R_h u + \sum_{T \in \mathcal{T}_h^{3\mathrm{D}}} \psi_T \int_T (u - R_h u) \left(\int_T \psi_T \, dx \right)^{-1} dx$$

where

$$\psi_T = 256 b_T \in \operatorname{span}\{b_T : T \in \mathcal{T}_h^{3\mathrm{D}}\}, \quad \Rightarrow \max_{x \in T} \psi_T(x) = 1.$$

With the previous definitions it is clear that $\Pi_h u$ is a linear operator. Further we have

$$\int_T \Pi_h u \, dx = \int_T u \, dx \tag{5.7}$$

and therefore

$$\begin{aligned}
\int_{\Omega^{3\mathrm{D}}} p_h \operatorname{div} \Pi_h u \, dx &= -\int_{\Omega^{3\mathrm{D}}} \nabla p_h \cdot \Pi_h u \, dx \\
&= -\sum_{T \in \mathcal{T}_h^{3\mathrm{D}}} \nabla p_h \int_T \Pi_h u \, dx \\
&\stackrel{(5.7)}{=} -\sum_{T \in \mathcal{T}_h^{3\mathrm{D}}} \nabla p_h \int_T u \, dx \\
&= -\int_{\Omega^{3\mathrm{D}}} \nabla p_h \cdot u \, dx \\
&= \int_{{}^{3\mathrm{D}}\Omega} p_h \operatorname{div} u \, dx.
\end{aligned}$$

So the first condition from lemma 5.3.3 is fulfilled. Proving the second assumption, we first need the following standard results (see [24]):

$$|\psi_T|_{[H^1(T)]^3} \leq c h_T^{n/2-1} \tag{5.8}$$

and

$$\int_T \psi_T \, dx \geq c h_T^n. \tag{5.9}$$

So it follows with (5.6),(5.8),(5.9)

$$\begin{aligned}
|\Pi_h u|_{[H^1(T)]^3} &\leq |R_h u|_{[H^1(T)]^3} + |\psi_T|_{[H^1(T)]^3} \left| \int_T (u - R_h u) \, dx \right| \left| \int_T \psi_T \, dx \right|^{-1} \\
&\leq c_1 |u|_{[H^1(\tilde{\omega_T})]^3} + c_2 h_T^{-n/2-1} \text{meas } (T)^{1/2} \|u - R_h u\|_{L^2(T)} \\
&\leq c_3 |u|_{[H^1(\tilde{\omega_T})]^3}.
\end{aligned}$$

Summation over all elements T concludes the proof. □

Remark. For the three-dimensional problem with piecewise constant ansatz functions, theorem 5.3.1 can be applied.

5.4 Convergence and discretization errors

Up to now we do not know if the unique solution derived from our discretized systems, is an approximation for the infinite-dimensional one. So we have to prove that for $h \to 0$, the discrete solution converges. The next lemmata concerning the convergence of the finite element method, as well as the theorem for the two-dimensional case can be found in [14].

Further in this section we will give error estimates for the velocity u for the two-dimensional problem as well as for the three-dimensional problem with continuous pressure. The result for the two-dimensional problem can be found in [14]. For the three-dimensional problem see [25].

5.4.1 The two-dimensional problem

Proving the convergence of the discrete solution requires the next lemma.

Lemma 5.4.1. *$\forall\, v \in V_0 = H_0^1(\Omega)$ there exists $v_h \in V_{0,h}$ such that $v_h \to v$ strongly in $H_0^1(\Omega)$*

Proof. We take an arbitrary $v \in \mathcal{D}(\Omega)$, since the space of test functions

$$\mathcal{D}(\Omega) := \{v \in C^\infty(\Omega) \,\big|\, \exists K \subset \Omega \textit{ compact}\} = C_0^\infty(\Omega)$$

is dense in $H_0^1(\Omega)$. Let a triangle $T \in \mathcal{T}_h$ be defined by its nodes M_1, M_2, M_3, we then introduce the affine linear function Φ_T on T (the nodal basis) such that

$$\Phi_T(M_i) = v(M_i), \quad \text{for } i = 1, 2, 3$$

We now define v_h by

$$v_h = \Phi_T \quad \text{in } T. \tag{5.10}$$

It is shown in [8], using Taylor series expansion, that

$$|\Phi_T(x) - v(x)| \leq c_1(v)h \quad \forall\, x \in T$$

and

$$|\operatorname{grad} \Phi_T(x) - \operatorname{grad} v(x)| \leq c_2(v)\sqrt{h} \quad \forall\, x \in T$$

where the constants $c_1(v), c_2(v)$ depend on v. Together with (5.10) we get that

$$v_h \to v \quad \text{in } H_0^1(\Omega).$$

□

Theorem 5.4.2. *Let $\mathcal{T}_h$ be an admissible and shape regular mesh, then the solution u_h of problem 5.2 converges strongly to the solution u of problem 3.2 in $H_0^1(\Omega)$.*

Proof. The soluion u_h of problem 5.2 is given by

$$\mu a_2(u_h, v_h - u_h) + \tau_c j(v_h) - \tau_c j(u_h) \geq (c, v_h - u_h) \quad \forall\, v_h \in V_{0,h}.$$

By choosing test functions $v_h = 0$, we get

$$\mu a_2(u_h, u_h) + \tau_c j(u_h) \leq (c, u_h).$$

Since $j(u_h) \geq 0$, it follows with Friedrich's inequality

$$\|u_h\|_1 \leq c_F \|c\|_0 \quad \forall\, h.$$

Now, from lemma 5.4.1 we get, for r_h being the nodal interpolation operator on $V_{0,h}$, that

$$r_h v \in V_{0,h} \quad \forall\, h, \text{ and } r_h v \to v \text{ strongly in } H_0^1(\Omega).$$

As u is the solution of problem 3.2 and $u_h \in V_{0,h} \subset V_0$, we can choose $v = u_h$ and get

$$\mu a_2(u, u_h - u) + \tau_c j(u_h) - \tau_c j(u) \geq \langle c, u_h - u \rangle \quad \forall\, h. \tag{5.11}$$

In the same way with $v = r_h u \in V_{0,h} \subset V_0$ in problem 5.2, we get

$$\mu a_2(u_h, r_h u - u) + \tau_c j(r_h u) - \tau_c j(u_h) \geq (c, r_h u - u_h). \tag{5.12}$$

Adding (5.11) and (5.12) yields

$$\mu a_2(u_h - u, u_h - u) \leq a_2(u_h, r_h u - u) + \tau_c(j(r_h u) - j(u)) + (c, u - r_h u).$$

From this we get

$$\|u_h - u\|_1^2 \leq \left(c_F \|c\|_0 + \tau_c \sqrt{\text{meas}(\Omega)}\right) \|r_h u - u\|_1 + \|c\|_0 \|r_h u - u\|_0.$$

From the proof of lemma 5.4.1 we have for the nodal interpolation operator the estimates

$$\|r_h v - v\|_1 \leq c_1(v)\sqrt{h}$$

and

$$\|r_h v - v\|_0 \leq c_2(v) h.$$

Using the Galerkin orthogonality

$$a_2(v_h, r_h v - v) = 0 \quad \forall\, v \in V, \forall\, v_h \in V_h,$$

we get that

$$\|u_h - u\|_1^2 \leq \alpha \tau_c \sqrt{h} + \beta h$$

with α, β being independent of τ_c, h. So we have shown that $(u_h)_h$ is strongly convergent in $H_0^1(\Omega)$. $\square$

Remark. Theorem 5.4.2 also provides an error estimate for the approximate solution u_h:

$$\|u_h - u\|_1 = \mathcal{O}(\sqrt[4]{h}).$$

Under more restrictive assumptions, one can show that

$$\|u_h - u\|_1 = \mathcal{O}(h|\log h|^{1/2}).$$

For this we refer to [14] and the references therein.

5.4.2 The three-dimensional case using piecewise linear functions for p

The following results are from [25]. Looking at the three-dimensional problem 5.5, we need the following standard assumptions:

$$\begin{aligned} a(u,v) &\le \alpha_2 \|u\|_1 \|v\|_1 \\ a(u,u) &\ge \alpha_1 \|u\|_1^2 \\ b(v,q) &\le \beta_2 \|v\|_1 \|q\|_0 \\ j(u) - j(v) &\le \beta_1 \|u - v\|_1 \end{aligned}$$

for all $u, v \in V_0^{3D}$ and for all $p, q \in Q$. These assumptions are all fulfilled as we have already shown in chapter 2. Further we need the following assumptions on the mesh-dependent bilinear form $c_h(\cdot,\cdot)$:

(C1) $c_h(p,q)$ is defined for any couple of functions $p, q \in H^1(\Omega^{3D})$.

(C2) $[\cdot]_h$, defined by $[q_h]_h^2 = c_h(q_h, q_h)$ is a norm.

(C3) $\forall\, p_h, q_h \in Q_h,\ c_h(p_h, q_h) \le [p_h]_h [q_h]_h$

(C4) There exists a positive constant independent of h, γ, and $k > 0$ such that

$$\forall\, v_h \in V_h^{3D}, \forall\, q_h \in Q_h,\ b(v_h, q_h) \le \gamma \frac{1}{h^k} \|v_h\|_0 [q_h]_h$$

(C5) There exists a positive constant independent of h, c, such that

$$\forall\, q \in H^1(\Omega^{3D}),\ [q]_h \le c h^k \|q\|_1.$$

Conditions (C1),(C2),(C3),(C5) are easily checked. Showing (C4) requires the following lemma:

Lemma 5.4.3. *Let the family of triangulations be quasi-uniform. Then there exists a constant $c > 0$, independent of h such that, for any $u_h \in V_{0,h}^{3D}$ and $q_h \in Q_h$, we have*

$$|b_1(v_h, q_h)| \leq c\frac{1}{h}\|v_h\|_0[q_h]_h.$$

Proof. First we deduce that $\alpha(T) = \mathcal{O}(h_T^2)$ and therefore $\alpha(T) \geq c_\alpha h_T^2$. We then have

$$\begin{aligned}
[q_h]_h^2 = \sum_{T\in\mathcal{T}_h^{3D}} \alpha(T)\int_T \nabla q_h \dot{\nabla} q_h\,dx &\geq \min_{T\in\mathcal{T}_h^{3D}} \alpha(T) \sum_{T\in\mathcal{T}_h^{3D}} \int_T \nabla q_h \cdot \nabla q_h\,dx \\
&\geq c_\alpha(\frac{\min h_T}{h^2})^2 h^2 |q_h|_1^2 \\
&\geq c_\alpha c_b h^2 |q_h|_1^2,
\end{aligned}$$

since the ratio h_T/h is bounded away from zero by c_b (quasi-uniform). So on the one hand we have

$$|q_h|_1 \leq c\frac{1}{h}[q_h]_h \tag{5.13}$$

with $c = \sqrt{1/c_\alpha c_b}$. On the other hand we get

$$b_1(v_h, q_h) = -\int_{\Omega^{3D}} q_h \operatorname{div} v_h\,dx = \int_{\Omega^{3D}} \nabla q_h \cdot v_h\,dx$$

and, therefore, using Cauchy-Schwarz inequality together with (5.13),

$$b_1(v_h, q_h) = \int_{\Omega^{3D}} \nabla q_h \cdot v_h\,dx \leq \|v_h\|_0 |q_h|_1 \leq c\frac{1}{h}\|v_h\|_0[q_h]_h$$

follows. □

With it we have an appropriate error estimate

Theorem 5.4.4. *Let u and p a solution of problem 3.1 and u_h,p_h a solution of*

$$\begin{aligned}
&\mu a(u_h, v_h - u_h) + \tau_c j(v_h) - \tau_c j(u_h) + b_1(v_h - u_h, p_h) \geq (f, v_h - u_h) \\
&b_1(u_h, q_h) - c_h(p_h, q_h) = \langle G_h, q_h\rangle
\end{aligned}$$

for all $v_h \in V_{0,h}^{3D}$ and for all $q_h \in Q_h$, what is the equivalent mixed formulation to problem 5.5. Further we assume that $u \in [H^2(\Omega^{3D})]^3$ and $p \in H^1(\Omega^{3D})$. Then the following error bound holds:

$$\|u - u_h\|_1 \leq ch^{1/2}(|u|_2 + |p|_1)$$

Proof. The technical proof can be found in [25]. □

Remark. The convergence of the discrete solution for the approximation with constant functions follows from theorem 5.4.2.

Remark. The regularity condition $u \in [H^2(\Omega^{3D})]^3$ is a result which can be found in [14].

Chapter 6

Numerical Results

In this chapter we want to present our numerical results. For the scalar two-dimensional problem, we tested our algorithm by comparing with the results from [18]. Then we will test the three-dimensional algorithm using special settings which lead to the results of the two-dimensional problem. At the end we will show results for the three-dimensional problem as well as for a three-dimensional problem related to a practical application.

6.1 Scalar problem in 2D

6.1.1 Realization

For the two-dimensional problem we implemented algorithm 4.1.6 in C++. In each step of the fixed-point iteration we solved the arising linear system for $u_{h,m}^{n+1}$ with a preconditioned CG-Solver. For preconditioning we used a simple Jacobi-preconditioner. The Dirichlet boundary conditions are implemented by a penalty term. For a better comparison, we used the same parameters as it was done in [18]:

$$\varepsilon = \Delta t = \frac{1}{\mu}, \quad \mu = 1.0, \quad r = \frac{\mu}{\tau_c}.$$

For the drop in pressure per length, we choose $c = 10$. Our yield limit τ_c is chosen to be $0.5, 1.5$ and 2.5. To initialize algorithm 4.1.6 we have used

$$u_h^0 = 0, \quad w_h^0 = 0.$$

Concerning the stopping criterion, we used for the fixed-point iteration

$$\|w_{h,m+1}^{n+1} - w_{h,m}^{n+1}\|_0 \leq 10^{-4} \quad \text{or } m \geq 5.$$

So the inner iteration is bounded by 5 steps. In the experiments it turned out that this bound is enough. For the uzawa-like iteration (the outer one) we used, analogously to the related paper, that

$$\|\nabla u_h^{n+1} - \nabla u_h^n\|_0 \leq 10^{-6}.$$

6.1.2 Results

In figure 6.1 we see the discretized quadratic cross section of our domain Ω using triangles. We then used various values for the yield limit τ_c in order to see properties of the solution vector u. If τ_c increases, then the rigid behavior grows. One can see that for τ_c being large enough, no flow exists, since the domain of rigid behavior then captures the whole domain and due to boundary conditions $u = 0$ follows. In figure 6.2 the rigid phase is very small, whereas in figure 6.3 we see that in the center a large region with $D(u) = 0$ is given.

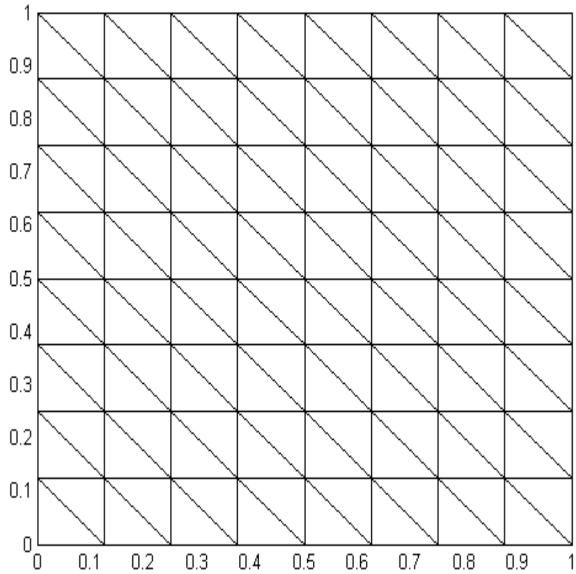

Figure 6.1: The discretization of the domain Ω with triangles.

Concerning the convergence rate we see in figure 6.7 and figure 6.8 that the convergence rates for u and w slightly improve if we use finer meshes. Further it can be seen, that for a fixed mesh size the convergence gets worse if we increase the yield limit. This was already mentioned. Moreover we see that adding the stabilization to the projection has the desired effect that the algorithm does not deteriorate (see figures 6.5 and 6.6). Finally in table 6.4 uzawa-iteration numbers for various values of h and τ_c are given.

Our results are in agreement with them from [18].

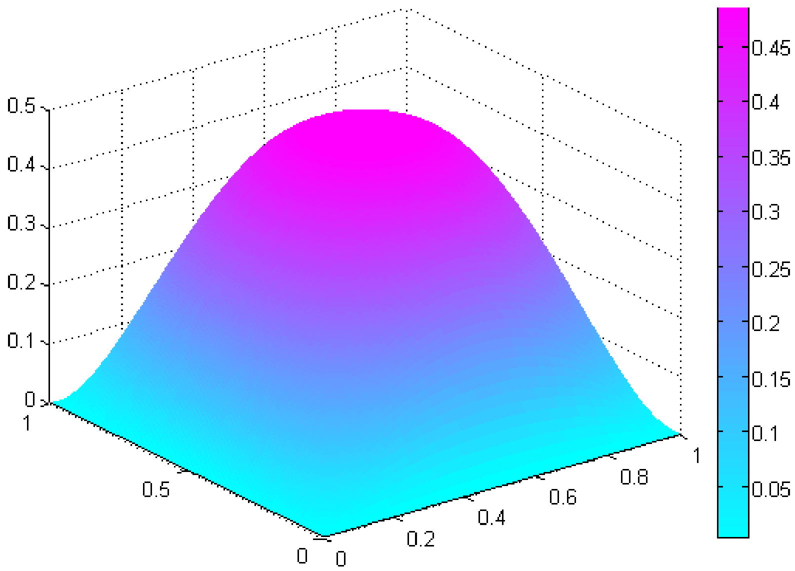

Figure 6.2: Plot for the scalar velocity u with $\tau_c = 0.5$. Discretization with 16641 nodes. The rigid behavior of the flow can be seen in the center of the domain (constant velocity).

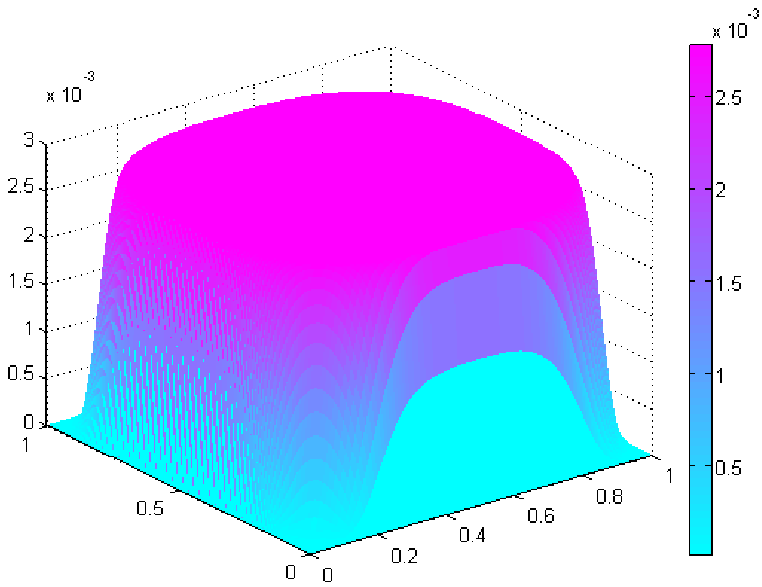

Figure 6.3: Plot for the scalar velocity u with $\tau_c = 2.5$. Discretization with 16641 nodes. The rigid behavior of the flow can be seen on nearly the whole domain.

Number of Uzawa-iterations

	$\tau_c = 0.5$	$\tau_c = 1.5$	$\tau_c = 2.5$	$\tau_c = 3.5$
$h = 1/32$: 1089 nodes	31	80	196	26
$h = 1/64$: 4225 nodes	29	86	158	26
$h = 1/128$: 16641 nodes	26	67	179	26

Figure 6.4: Number of outer iterations for various discretizations and various values of τ_c. We see that the iteration numbers vary if τ_c increases, but stays nearly the same if h varies.

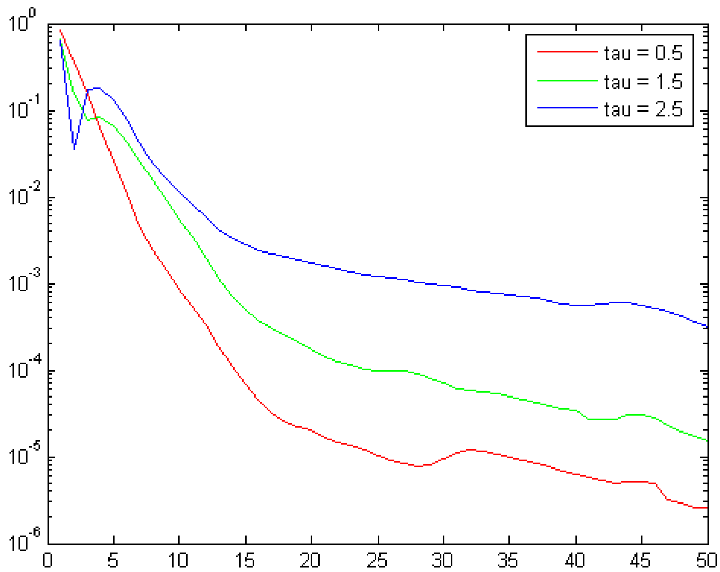

Figure 6.5: Visualization of $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ (y-axis) versus iteration numbers (x-axis). Discretization with $h = 1/128$. The convergence rate gets worse if τ_c increases.

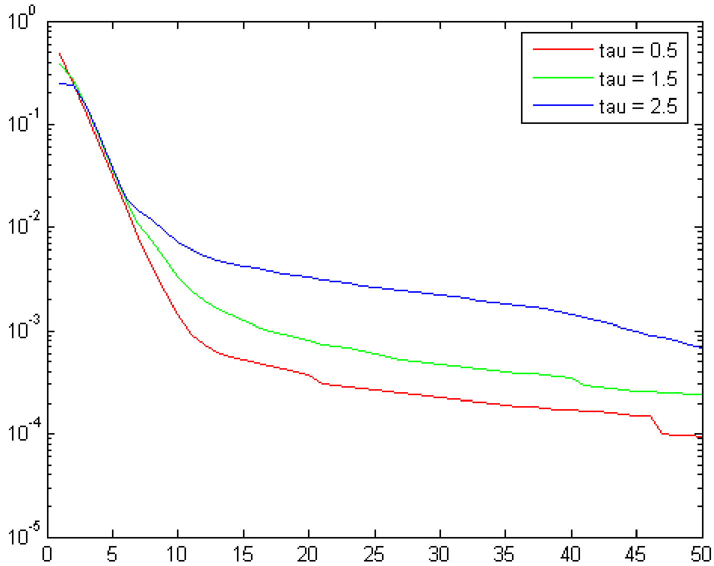

Figure 6.6: Visualization of $\|w_h^{n+1} - w_h^n\|_0$ (y-axis) versus iteration numbers (x-axis). Discretization with $h = 1/128$. The convergence rate gets worse if τ_c increases.

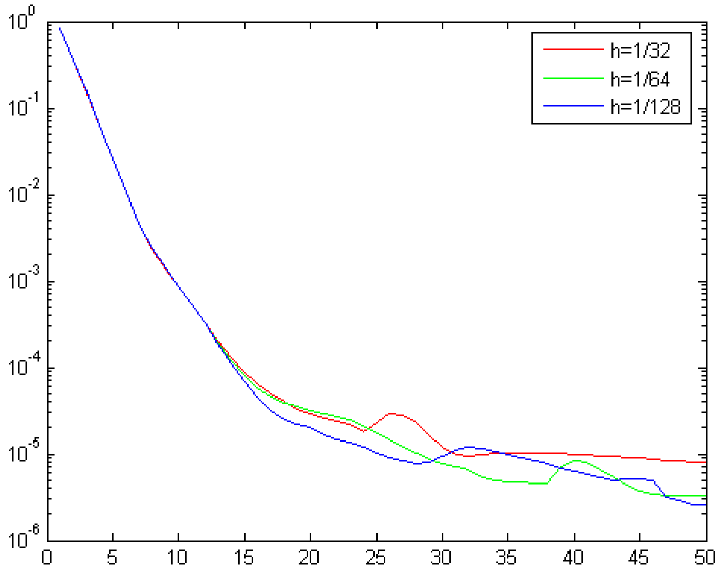

Figure 6.7: Visualization of $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ (y-axis) versus iteration numbers for $\tau = 0.5$. Discretization with various meshes. The convergence rate improves for finer meshes.

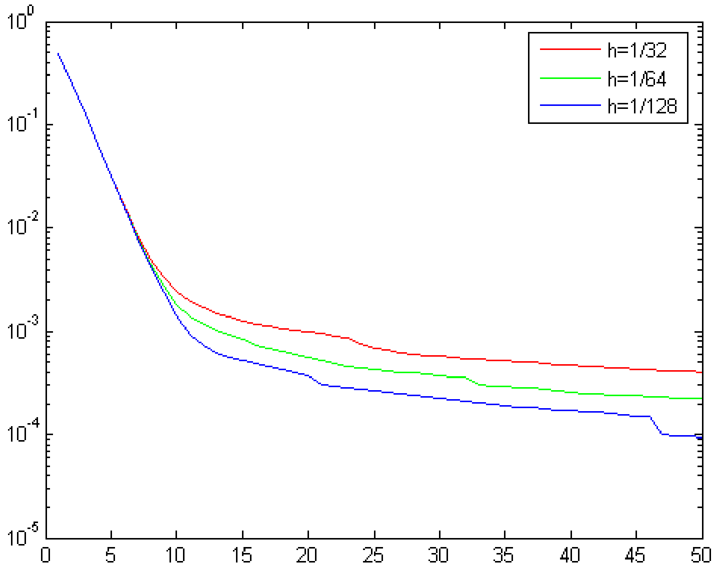

Figure 6.8: Visualization of $\|w_h^{n+1} - w_h^n\|_0$ (y-axis) versus iteration numbers for $\tau = 0.5$. Discretization with various meshes. The convergence rate improves for finer meshes.

6.2 The three-dimensional problem

6.2.1 Test

Similar to the previous section we will test our algorithm. Using the first approach, we choose our parameters in the same way as we did it in theorem 3.2.2. So, under the assumptions

$$f = 0, \quad c = 10.0, \quad p = -cz, \quad \tau_c = 0.5,$$

and the choice

$$\varepsilon = \Delta t = \frac{1}{\mu}, \quad \mu = 1.0, \quad r = \frac{\sqrt{2}\mu}{\tau_c}, \quad \bar{r} = \frac{r}{\sqrt{2}\tau_c} = \frac{\mu}{\tau_c^2},$$

we will compare the results from the three-dimensional algorithm with the results from the two-dimensional one. The error of u is again computed by $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ and the error for λ is $\|w_h^{n+1} - w_h^n\|_0$. The stopping criterion is the same as in the 2D case:

$$\|w_{h,m+1}^{n+1} - w_{h,m}^{n+1}\|_0 \leq 10^{-4} \quad \text{or } m \geq 5$$

for the inner iteration, and

$$\|\nabla u_h^{n+1} - \nabla u_h^n\|_0 \leq 10^{-6}$$

for the outer iteration. The initializations for u and λ are also the same as in the 2D case. For the implementation of the periodic boundary conditions we refer to [29]. The discretization of Ω^{3D} is done with tetrahedrons (see figure 6.14). In figures 6.9 and 6.10 the test of the algorithm is presented. It can be seen that in each step of the algorithm, the error for u and λ in the three-dimensional case coincides with the error in the two-dimensional case.

Next we are going to apply the same algorithm without prescribing p. All other parameters stay the same. To initialize the pressure we use

$$p_h^0 = -4z^2 - 6z.$$

The error for p is computed by

$$\|p_h^{n+1} - p_h^n\|_0.$$

In figures 6.11-6.13 the error between the solution u from algorithm 5.2.3 and the two-dimensional solution, computed by algorithm 5.2.2, in the euclidean norm is plotted in the plane $z = 0$. One can see that there are minimal variations in some nodes. These variations makes sense, since the

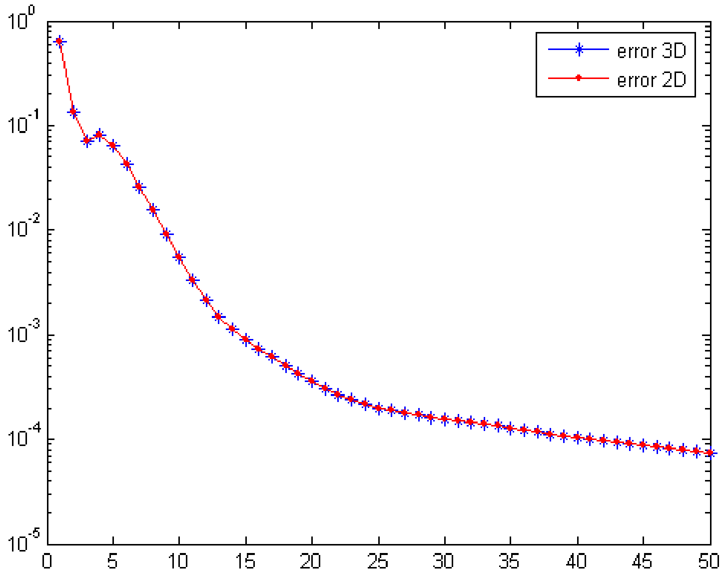

Figure 6.9: Verification of the 3D algorithm for u in the case $h = 1/8$, $\tau_c = 1.5$. Visualization of $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ (y-axis) versus outer iteration number (x-axis).

finite element space can not exactly approximate the pressure $p = -cz$.

The absolute value of the solution u as well as the pressure p can then be seen in figures 6.15 and 6.16 respectively. In figure 6.15 the rigid behavior of the flow is located in the center of the prism. This part is very small since the yield limit is $\tau_c = 0.5$. In figure 6.16 the linear behavior of the pressure can be seen.

We have seen that our algorithm in the three-dimensional case furnishes results which, on the one hand coincide with the results in the two-dimensional case if we prescribe the pressure, and on the other hand are a good approximation of the two-dimensional results if we does not prescribe the pressure. Further we have seen that the part of the domain Ω^{3D} where the medium acts like a rigid body is located in the center. For the choice $\tau_c = 0.5$ the rigid behavior is not very dominant.

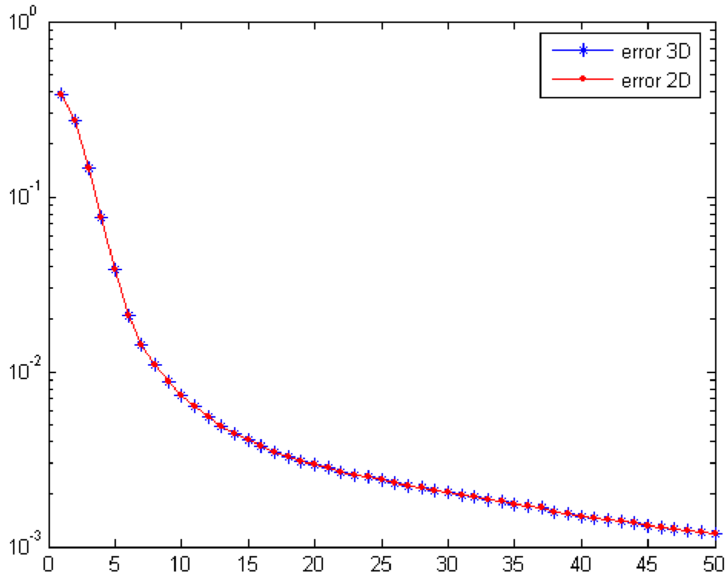

Figure 6.10: Verification of the 3D algorithm for λ in the case $h = 1/8$, $\tau_c = 1.5$. Visualization of $\|w_h^{n+1} - w_h^n\|_0$ (y-axis) versus outer iteration number.

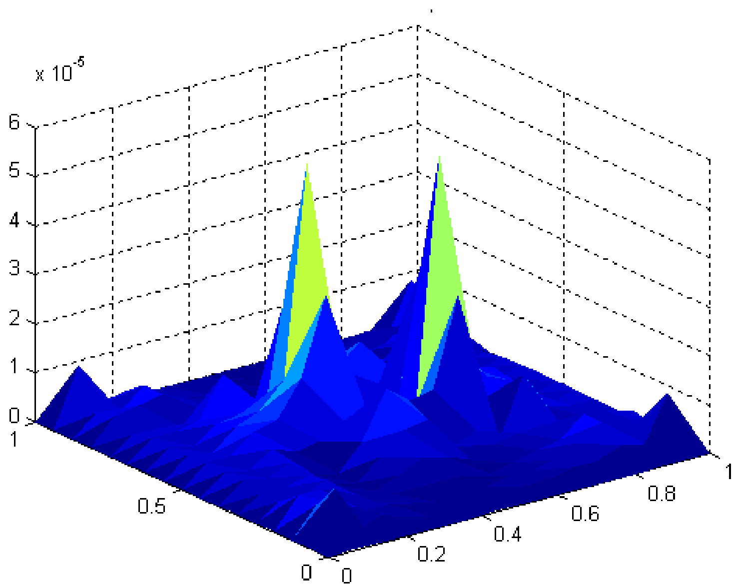

Figure 6.11: Error in the first component u_1 between the three-dimensional solution, using linear ansatz functions for p, and the two-dimensional solution. Mesh size: $h = 1/16$, yield limit: $\tau_c = 0.5$. The error is computed in the euclidean norm.

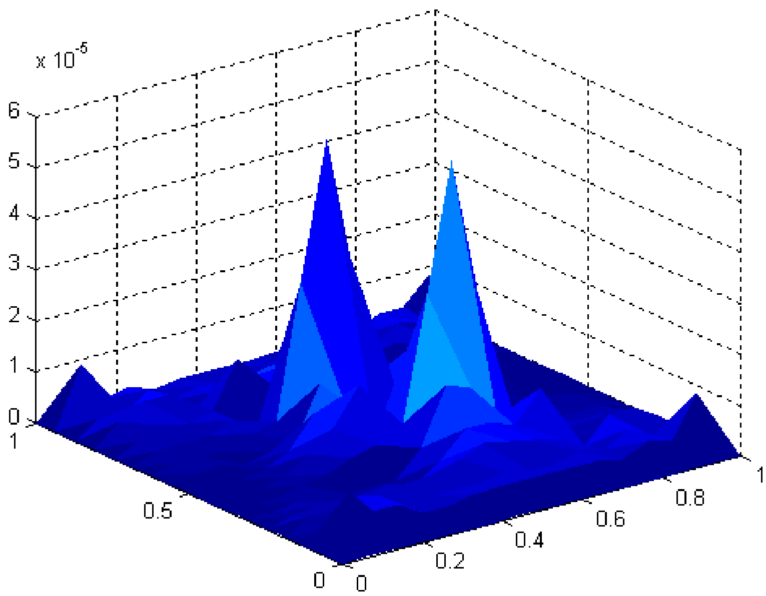

Figure 6.12: Error in the second component u_2 between the three-dimensional solution, using linear ansatz functions for p, and the two-dimensional solution. Mesh size: $h = 1/16$, yield limit: $\tau_c = 0.5$. The error is computed in the euclidean norm.

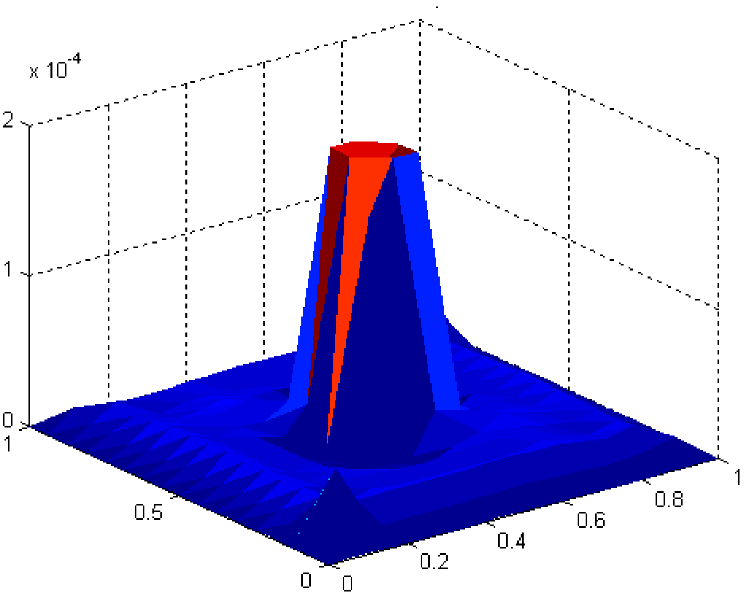

Figure 6.13: Error in the third component u_3 between the three-dimensional solution, using linear ansatz functions for p, and the two-dimensional solution u. Mesh size: $h = 1/16$, yield limit: $\tau_c = 0.5$. The error is computed in the euclidean norm.

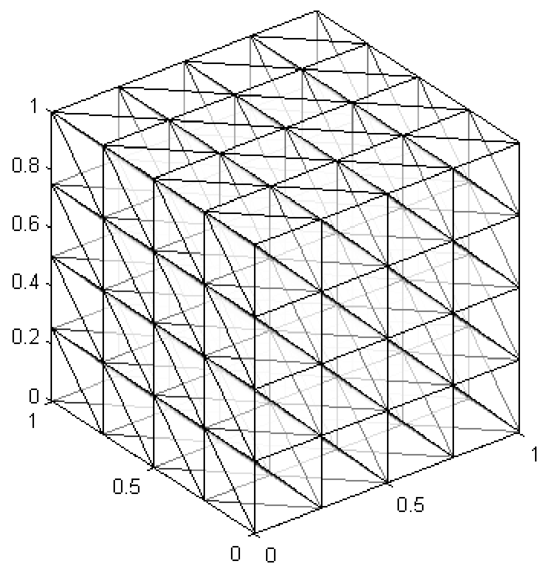

Figure 6.14: Discretization of the three-dimensional domain Ω^{3D} with tetrahedrons.

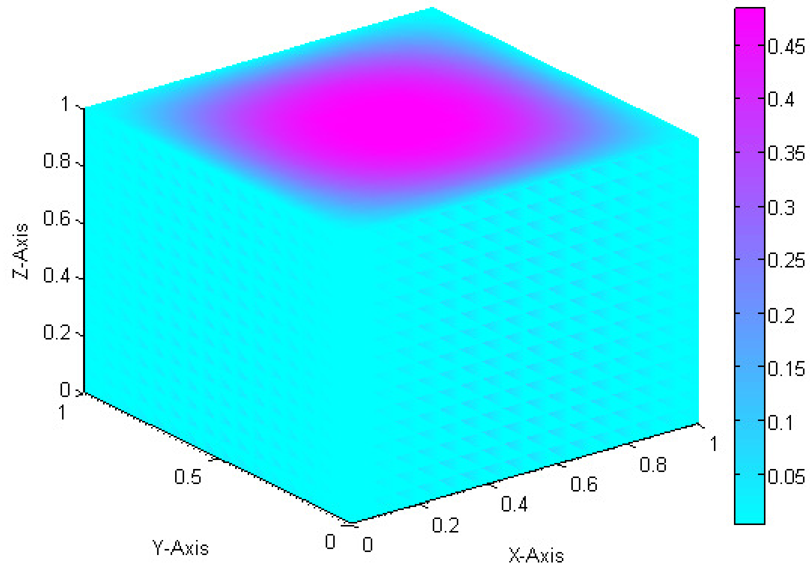

Figure 6.15: Visualization of the absolute value $|u|$ of the three-dimensional solution, using linear ansatzfunctions for p. The discretization is done with 4096 nodes. The part of the domain where the fluid acts like a rigid medium can be seen in the center of the prism.

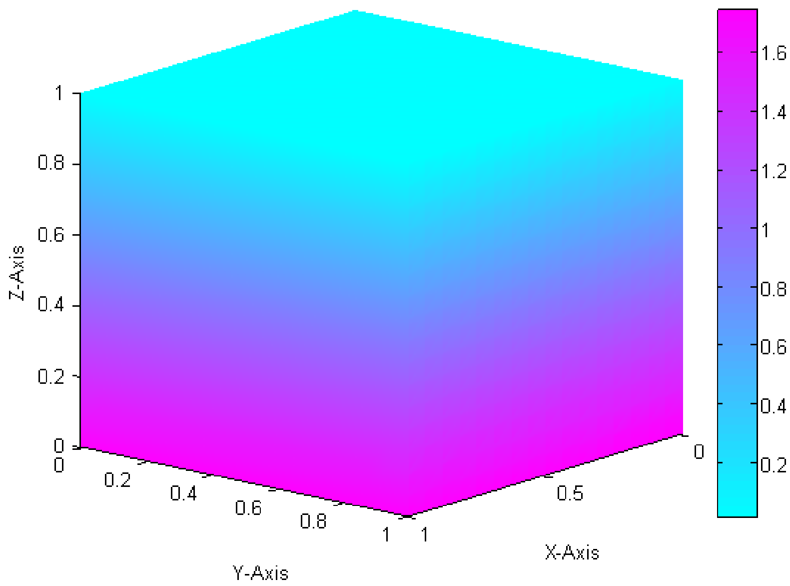

Figure 6.16: Visualization of the hydrostatic pressure p of the three-dimensional problem using linear ansatzfunctions for p. The discretization is done with 4096 nodes. The linear behavior of the pressure can be seen.

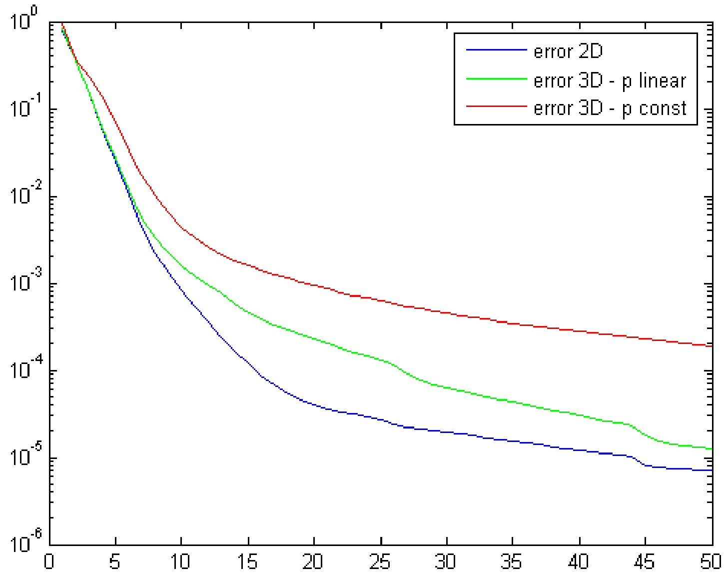

Figure 6.17: Comparison of the different algorithms. Visualization of $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ (y-axis) for $h = 1/16$ and $\tau_c = 0.5$ versus iteration numbers.

6.2.2 Comparison of the different algorithms

In chapter 4 we developed two different algorithms for the three-dimensional problem. There rests the question which one provides better convergence properties. In figures 6.17-6.20 the residual for the two algorithms as well as for the two-dimensional algorithm is plotted. We see that first the step from two into three dimensions is costly and second that the continuous approximation of the pressure is leading to a better convergence rate. Table 6.18 confirms this statement with iteration numbers for the two-dimensional algorithm, the three-dimensional algorithm using linear ansatz functions for p and the three-dimensional algorithm using constant ansatz functions.

Number of Uzawa-iterations for $\tau_c = 0.5$

	2D-algorithm 5.2.2	3D algorithm 5.2.3	3D algorithm 5.2.4
64 nodes	19	125	100
512 nodes	21	81	188
4096 nodes	45	56	207

Figure 6.18: Number of outer iterations for various discretizations. The yield limit is given by $\tau_c = 0.5$.

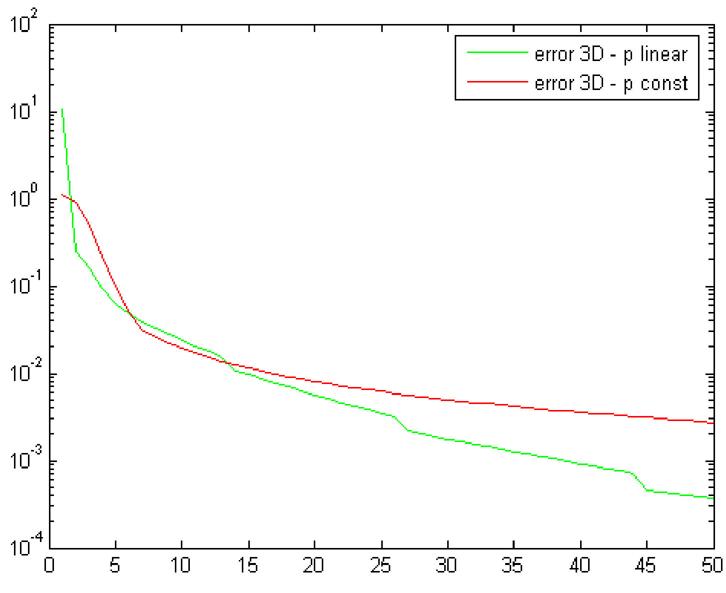

Figure 6.19: Comparison of the different algorithms. Visualization of $\|p_h^{n+1} - p_h^n\|_0$ (y-axis) for $h = 1/16$ and $\tau_c = 0.5$ versus iteration numbers.

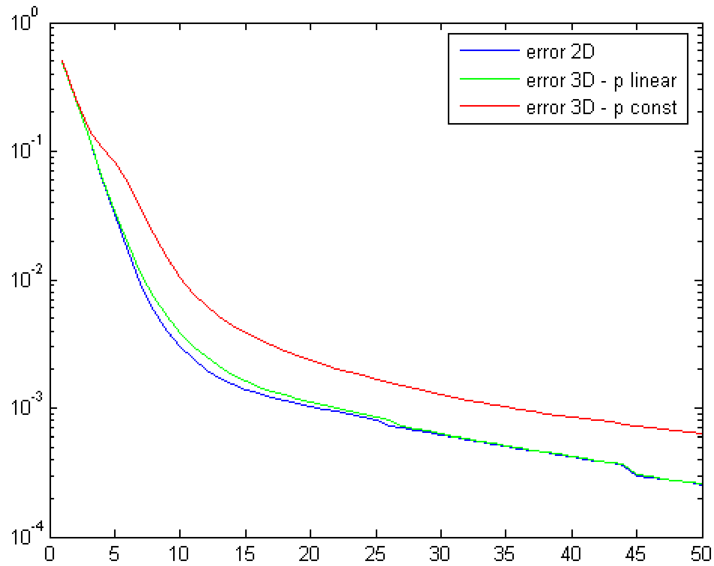

Figure 6.20: Comparison of the different algorithms. Visualization of $\|\lambda_h^{n+1} - \lambda_h^n\|_0$ (y-axis) for $h = 1/16$ and $\tau_c = 0.5$ versus iteration numbers.

6.3 An example related to a practical problem

At the end we wantto apply the three-dimensional algorithm to a problem with practical background. In collaboration with dTech Steyr, we studied the behavior of a snow cover on a mountainside. Snow can be seen as a material with yield stress, so there are regions where the medium acts like a fluid as well as regions where the yield limit is not reached yet, i.e. where the medium acts like a rigid body. The latter are the more interesting, since these are the regions which eventually can slide down. Due to [22],[30] we decided to use a Bingham model for simulating the flow on a snow covered mountainside.

Our domain Ω is chosen to be a rectangular prism $([0, x_{max}] \times [0, y_{max}] \times [0, z_{max}])$ with boundary Γ, under an angle of inclination Θ. We split the

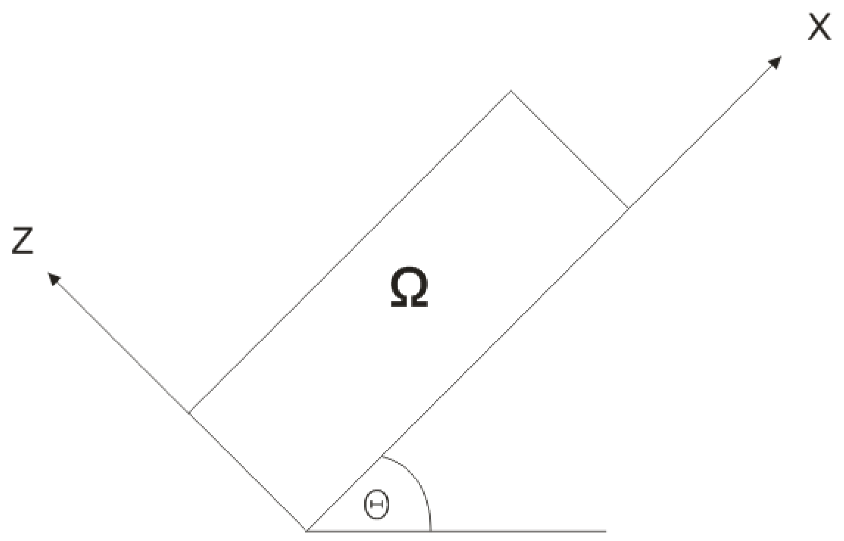

Figure 6.21: Cross section in the xz-plane of the domain Ω rotated by Θ with rotation axis y.

boundary into four parts,

$$\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_{P_x} \cup \Gamma_{P_y}$$

where

$$\Gamma_D = [0, x_{max}] \times [0, y_{max}] \times \{0\}$$
$$\Gamma_N = [0, x_{max}] \times [0, y_{max}] \times \{z_{max}\}$$
$$\Gamma_{P_x} = \{0\} \times [0, y_{max}] \times [0, z_{max}] \cup \{x_{max}\} \times [0, y_{max}] \times [0, z_{max}]$$

$$\Gamma_{P_y} = [0, x_{max}] \times \{0\} \times [0, z_{max}] \cup [0, x_{max}] \times \{y_{max}\} \times [0, z_{max}].$$

On the bottom of the snow cover,Γ_D, the snow adheres to the mountainside, so we set a no-slip condition

$$u = 0 \quad \text{on } \Gamma_D.$$

On the top,Γ_N, we demand that there are zero-tractions, i.e. this boundary should be strainless, so

$$\sigma n = 0 \quad \text{on } \Gamma_N.$$

On the rest of the boundary, we want to simulate a very long (on Γ_{P_x}) as well as very broad (on Γ_{P_y}) mountainside, so we set homogenous periodic boundary conditions:

$$u(0, y, z) = u(x_{max}, y, z) \quad \text{on } \Gamma_{P_x},$$
$$u(x, 0, z) = u(x, y_{max}, z) \quad \text{on } \Gamma_{P_y}.$$

Along x-direction the pressure decreases. The drop in pressure is directly proportional to the difference in altitude Δh of the snow cover. The proportionality factor is $m * g$, where m denotes the mass of the snow cover and g the acceleration of gravity. So

$$p(0, y, z) = p(x_{max}, y, z) + mg\Delta h \quad \text{on } \Gamma_{P_x},$$
$$p(x, 0, z) = p(x, y_{max}, z) \quad \text{on } \Gamma_{P_y}.$$

Our classical formulation then reads as follows:

Problem 6.1 (Classical Formulation for the snow cover)**.** *Find*

$$u \in C^2(\Omega) \cap C(\partial\Omega),\ p \in C^1(\Omega)$$

such that

$$\begin{aligned}
&\nabla p - \boldsymbol{div}\,\tau = f && \textit{in } \Omega \\
&\tau = 2\mu D(u) + \tau_c D(u)/(D_{II})^{1/2} && \Leftrightarrow \tau_{II}^{1/2} > \tau_c \\
&D(u) = 0 && \Leftrightarrow \tau_{II}^{1/2} \leq \tau_c \\
&\operatorname{div} u = 0 && \textit{in } \Omega \\
&u = 0 && \textit{on } \Gamma_D \\
&\sigma n = 0 && \textit{on } \Gamma_N \\
&u(0, y, z) = u(x_{max}, y, z) && \textit{on } \Gamma_{P_x} \\
&p(0, y, z) = p(x_{max}, y, z) + mg\Delta h && \textit{on } \Gamma_{P_x} \\
&u(x, 0, z) = u(x, y_{max}, z) && \textit{on } \Gamma_{P_y} \\
&p(x, 0, z) = p(x, y_{max}, z) && \textit{on } \Gamma_{P_y}.
\end{aligned}$$

The volume force f is given by

$$f = (0, 0, -\rho * g)$$

where ρ denotes the density.

For the numerical simulation we used the first approach developed in chapter 5. The regions where the snow acts like a rigid medium can be seen if the snow cover is high enough and the mountainside has a strong rise. Therefore we choose

$$z_{max} = 3m,\ \Theta = 45°.$$

The physical parameters for snow are given by

$$\rho = 400kg/m^3,\ \mu = 0.03\frac{kg * s}{m},\ \tau_c = 750Pa$$

The mathematical parameters are chosen to be the same as there were for the previous problem:

$$\varepsilon = \Delta t = \frac{1}{\mu},\ r = \frac{\sqrt{2}\mu}{\tau_c},\ \bar{r} = \frac{\mu}{\tau_c^2}.$$

Further we used the following stopping criterions for the outer iteration:

$$\|\nabla u_h^{n+1} - \nabla u_h^n\|_0 \leq 10^{-5}.$$

The stopping criterion for the inner iteration stays the same:

$$\|\lambda_{h,m+1}^{n+1} - \lambda_{h,m}^{n+1}\|_0 \leq 10^{-4} \text{ or } m \geq 5.$$

In figure 6.22 the discretized domain is shown. Figure 6.23 then shows the plot for $|u|$. One can clearly see the regions where $|u| = const.$, i.e. where the snow cover can slide down. Figures 6.24-6.26 show the error for u, p and λ. Using a discretization with 125 nodes, the algorithm converges after 249 iterations.

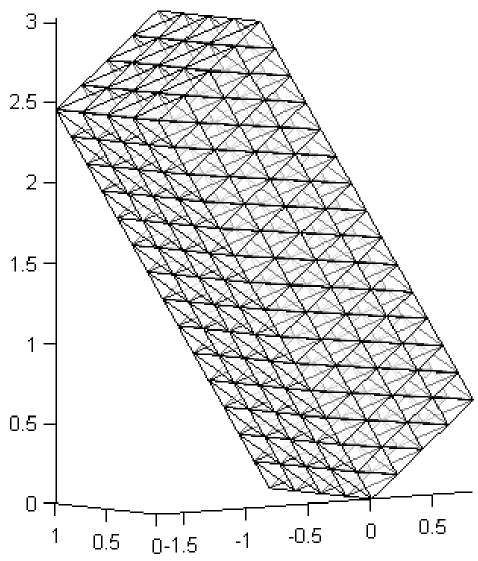

Figure 6.22: Discretization of the domain with tetrahedrons.

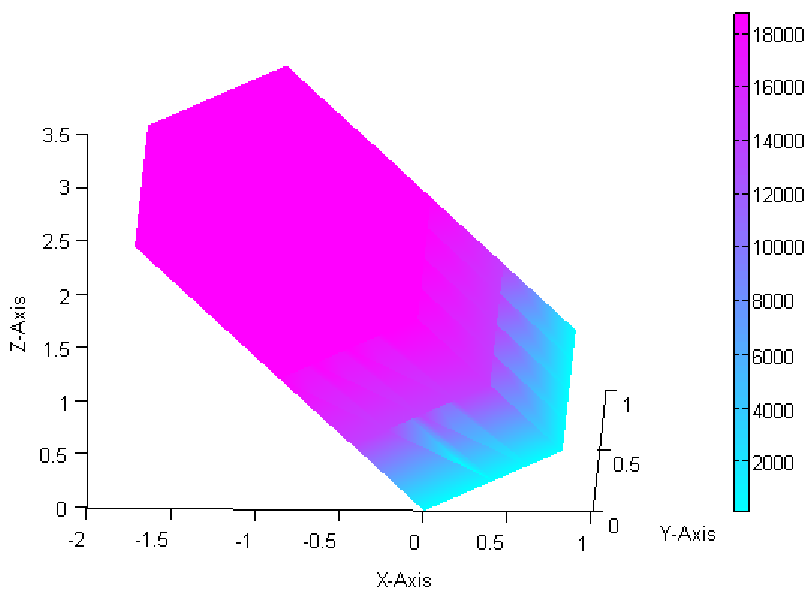

Figure 6.23: Absolute Value of the velocity u. Discretization with 125 nodes. A large part of the domain involves the rigid behavior (constant velocity).

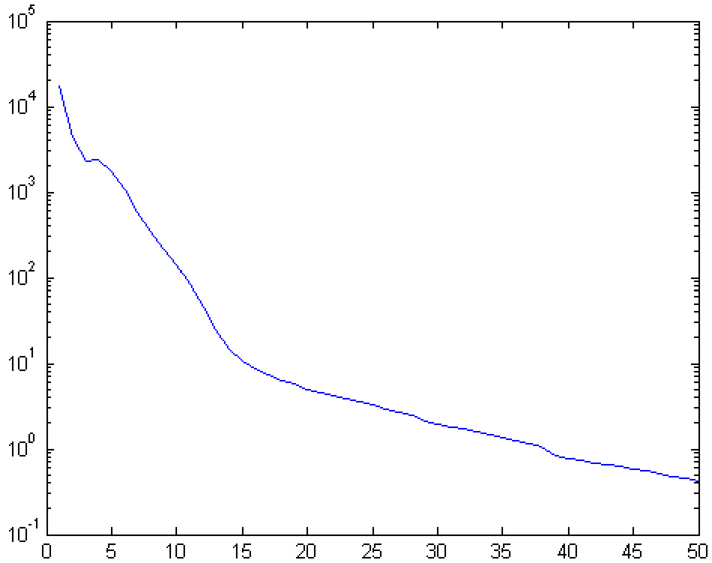

Figure 6.24: Visualization of $\|\nabla u_h^{n+1} - \nabla u_h^n\|_0$ versus iteration number. The algorithm converges after 249 iterations.

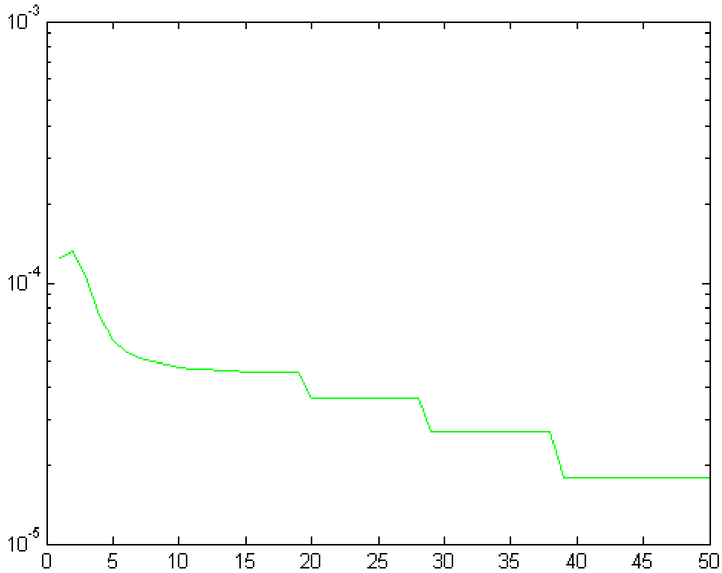

Figure 6.25: Visualization of $\|p_h^{n+1} - p_h^n\|_0$ versus iteration number.

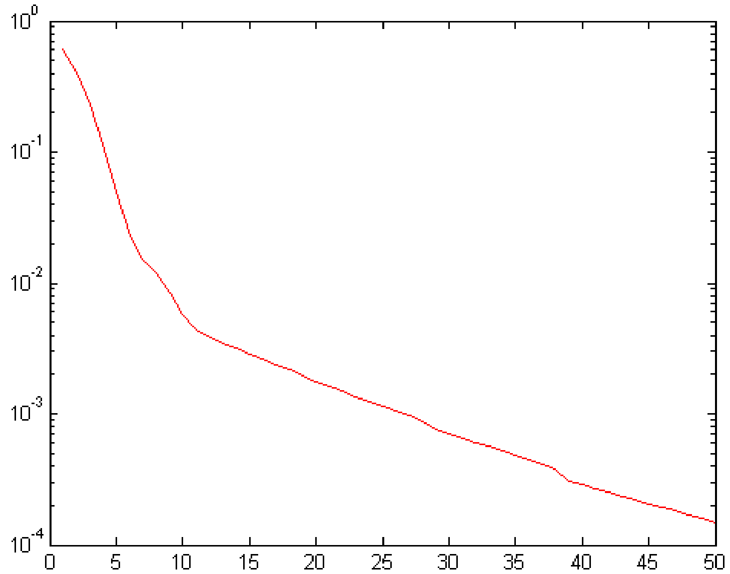

Figure 6.26: Visualiztation of $\|\lambda_h^{n+1} - \lambda_h^n\|_0$ versus iteration number.

Summary

The simulation of a Bingham fluid led to a mixed system involving a variational inequality of the second kind. We have seen that the arising non-differentiable functional characterizes the difficulty of the problem. In order to get rid of this we developed a dual dual formulation consisting of a system of equations and a projection. Although we did not have to treat the problem of non-differentiability, there occurred the problem with the large kernel of the convex set Λ. For this reason the uniqueness of a dual variable is not given.

We presented two methods for solving the three-dimensional problem. The idea of using a time-dependent problem in order to approximate the steady state problem, led to an uzawa-type method coupled with a fixed-point iteration together with a stabilization for the projection. The stabilization is necessary due to the non-uniqueness of the dual variable. Discretization with finite elements was done for both three-dimensional approaches. On the one hand we used a continuous approximation of the pressure stabilized with bubble functions on the other we chose a discontinuous approximation of the pressure without additional stabilization. We saw that we can eliminate the additional degrees of freedom coming from the bubbles. In return we get an additional mesh dependent bilinear form. For both approaches we derived existence and convergence results.

In the numerical results we have seen that the chosen approaches lead to an algorithm which works correctly. We have seen that the parameter representing the yield limit causes higher iteration numbers if it increases. On the other hand it seemed that the iteration number decreases for finer meshes. Further we compared our results for the three-dimensional problem. It could be seen that the approach using the continuous pressure approximation provides better convergence properties. Finally we have seen that we can apply the algorithm to a problem with practical background. The behavior of a snow cover on a mountainside was computed. If the angle of inclination and the height of the snow cover are big enough, a domain in which the fluid acts like a rigid medium can be seen.

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Ich, Philipp Laaber, erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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