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Some Benchmark Problems in Electromagnetics

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Chapter 1

Introduction

An important part of the modern life of most people is affected by electronic devices like radios, computers, smartphones and many more. Do these objects have something in common? Yes, they do, namely they all obey special laws of electromagnetism that can be described in a very mathematical manner which is great if someone wants to analyze the electromagnetic properties. The bad news are, that in most cases the solutions of the individual problems cannot be computed analytically and so improved numerical solvers are needed.

1.1 Motivation

If one wants to describe states of electric and/or magnetic kind, the main approach for this task will be writing down the corresponding *Maxwell Equations*. These equations deal with electric and magnetic quantities and describe, how these fields and scalars are generated and altered by each other. They are applicable in microscopic behaviors like atomic models as well as in macroscopic cases like electric motors and transformers. They are named after *James Clerk Maxwell*, who published them in the years 1861 - 1862.

The purpose of this thesis is to give an overview of the electromagnetic laws, derive Maxwell's equations and solve them in simplified settings, where the complexity is reduced to a level, in which an analytic solution can be computed.

1.2 Physical Quantities

Throughout this theses there are frequently used physical quantities that are now described for better understanding. We will make excessive use of some letters consistently corresponding to them, so whenever they appear in this thesis, the physical meaning is refered to Table 1.1.

The total electric charge Q of any region Ω and the electric charge density are related by $Q(\Omega) = \int_\Omega \rho(x)\,\mathrm{d}x$, similarly the total current I passing a surface S and

Notation	Unit	Description
$E = (E_1, E_2, E_3)^T$	$[V/m]$	electric field intensity
$D = (D_1, D_2, D_3)^T$	$[As/m^2]$	electric flux density (electric induction)
$H = (H_1, H_2, H_3)^T$	$[A/m]$	magnetic field intensity
$B = (B_1, B_2, B_3)^T$	$[Vs/m^2]$	magnetic flux density (magnetic induction)
$J = (J_1, J_2, J_3)^T$	$[A/m^2]$	electric current density
$\rho = \rho(x, t)$	$[As/m^3]$	electric charge density
$M = (M_1, M_2, M_3)^T$	$[Vs/m^2]$	magnetization
$P = (P_1, P_2, P_3)^T$	$[As/m^2]$	electric polarization

Table 1.1: Table of physical quantities.

the current density are related by $I(S) = \int_S J \cdot n \, \mathrm{d}S$, where n denotes the normal vector of S. The electric current density J can be split up into the sum of the conduct current density J_c and the impressed current density J_i, thus $J = J_c + J_i$. A conduct current only occurs in conductive media like wires (i.e., $\sigma \neq 0$ in (2.15c)) and can be imagined as the movement of charge in a conductor. Now the difference between these two densities is that the impressed current density does not represent an actual current, at least not in conventional sense. Changing electric fields produce changing magnetic fields even when no charges are present. This is the reason for introducing this somehow unexpected type of current.

We also define the following material parameters:

- μ $[Vs/Am]$ the magnetic permeability of a special matter
- ε $[As/Vm]$ the electric permittivity of a special matter

Magnetic permeability is the degree of magnetization that a material obtains in response to an applied magnetic field. The electric permittivity is a measure of the resistance that is encountered when forming an electric field in a medium. In vacuum (or air) these variables are equal to their reference values $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am and $\varepsilon_0 = 8.8542 \cdot 10^{-12}$ As/Vm (permeability and permittivity of free space). In an arbitrary medium, the relations $\mu = \mu_r \mu_0$ and $\varepsilon = \varepsilon_r \varepsilon_0$ relate the general values to the fixed ones via μ_r and ε_r, the relative permeability and relative permittivity, respectively. With $c \approx 3 \cdot 10^8$ m/s the speed of light, the relation $\varepsilon_0 \, \mu_0 \, c^2 = 1$ holds. In ferromagnetic materials μ_r is a nonlinear function that depends on H, the magnetic field intensity. For further details determining μ_r see [8].

1.3 Notation

We will frequently use the well-known mathematical Operators:

Definition 1.1 (Divergence)**.** *Let* $U : \mathbb{R}^3 \to \mathbb{R}^3$ *be a sufficiently smooth vector field.*

Then the divergence *of the vector field U is defined as*

$$\nabla \cdot U := \operatorname{div}(U) := \sum_{i=1}^{3} \frac{\partial U}{\partial x_i}$$

Definition 1.2 (Curl). *Let $U : \mathbb{R}^3 \to \mathbb{R}^3$ be a sufficiently smooth vector field. Then the* curl *of a vector field U is defined as*

$$\nabla \times U := \operatorname{curl}(U) := \begin{pmatrix} \partial_2 U_3 - \partial_3 U_2 \\ \partial_3 U_1 - \partial_1 U_3 \\ \partial_1 U_2 - \partial_2 U_1 \end{pmatrix}$$

1.4 Preliminaries and Integral Identities

Theorem 1.3 (Stokes' Theorem). *Let S be a surface in $\mathbb{R}^3$ parametrized by $\varphi : M \to \mathbb{R}^3$, where M is a subset of a sufficiently smooth set $K \subset \mathbb{R}^3$ and $\varphi \in C^2(M)$ with boundary ∂S. Let $f \in (C^1(\varphi(M)))^3$, then it holds that*

$$\int_S \operatorname{curl} f \cdot n \, \mathrm{d}S = \int_{\partial S} f \cdot \tau \, \mathrm{d}\tau, \tag{1.1}$$

where n and τ denote the unit normal of S and the tangent of ∂S, respectively, in each point.

Proof. See Theorem 8.50 in [1]. □

Theorem 1.4 (Gauss's Theorem). *Let V be a sufficiently smooth subset of $\mathbb{R}^3$ and ∂V its boundary. Let $M \supset V$ be open and $f \in (C^1(M))^3$. Then it holds that*

$$\int_V \operatorname{div} f \, \mathrm{d}x = \int_{\partial V} f \cdot n \, \mathrm{d}S, \tag{1.2}$$

where n denotes the unit normal vector of ∂V in each point.

Proof. See Theorem 8.58 and Remark 8.59 in [1]. □

Lemma 1.5. *For a twice continuously differentable scalar field $u : \mathbb{R}^3 \to \mathbb{R}$ and a twice continuously differentable vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ the following identities hold:*

$$\Delta u = \operatorname{div}(\nabla u) \tag{1.3}$$
$$\operatorname{div}(\operatorname{curl} F) = 0 \tag{1.4}$$
$$\operatorname{curl}(\nabla u) = 0 \tag{1.5}$$
$$\operatorname{curl}(\operatorname{curl} F) = \nabla(\operatorname{div} F) - \Delta F \tag{1.6}$$

Proof. By definition we obtain for (1.3)

$$\operatorname{div}(\nabla u) = \operatorname{div}\begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{pmatrix} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = \Delta u.$$

For the (1.4) we again plug in the definitions of div and curl and use Schwarz's theorem:

$$\begin{aligned} \operatorname{div}(\operatorname{curl} F) &= \operatorname{div}\begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} \\ &= \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) \\ &= 0. \end{aligned}$$

Identity (1.5) is shown by

$$\operatorname{curl}(\nabla u) = \operatorname{curl}\begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \partial_2\partial_3 u - \partial_3\partial_2 u \\ \partial_3\partial_1 u - \partial_1\partial_3 u \\ \partial_1\partial_2 u - \partial_2\partial_1 u \end{pmatrix} = 0,$$

where we again used Schwarz's theorem. For (1.6) we rewrite its left side as

$$\operatorname{curl}(\operatorname{curl} F) = \operatorname{curl}\begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix} = \begin{pmatrix} \partial_2(\partial_1 F_2 - \partial_2 F_1) - \partial_3(\partial_3 F_1 - \partial_1 F_3) \\ \partial_3(\partial_2 F_3 - \partial_3 F_2) - \partial_1(\partial_1 F_2 - \partial_2 F_1) \\ \partial_1(\partial_3 F_1 - \partial_1 F_3) - \partial_2(\partial_2 F_3 - \partial_3 F_2) \end{pmatrix} \tag{1.7}$$

and its right hand side

$$\nabla(\operatorname{div} F) - \Delta F = \begin{pmatrix} \partial_1(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \\ \partial_2(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \\ \partial_3(\partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3) \end{pmatrix} - \begin{pmatrix} \partial_1^2 F_1 + \partial_2^2 F_1 + \partial_3^2 F_1 \\ \partial_1^2 F_2 + \partial_2^2 F_2 + \partial_3^2 F_2 \\ \partial_1^2 F_3 + \partial_2^2 F_3 + \partial_3^2 F_3 \end{pmatrix}, \tag{1.8}$$

which proves the statement, since (1.7) equals (1.8) again by Schwarz's theorem. □

In the derivation of the vector potential formulation we will need the following two lemmas:

Lemma 1.6. *Let $\Omega \subset \mathbb{R}^3$ be simply connected and $B \in (L^2(\Omega))^3$ be a vector field fulfilling $\operatorname{div} B = 0$. Then there exists a vector field $A \in (H^1(\Omega))^3$ such that*

$$B = \operatorname{curl} A$$

Proof. See Theorem 3.4 in [3]. □

One can also show the statement of Lemma 1.6 for $B \in (C^1(\Omega))^3$ and $A \in (C^2(\Omega))^3$.

Lemma 1.7. *Let $\Omega \subset \mathbb{R}^3$ be simply connected and $F \in (C^1(\Omega))^3$ be a vector field fulfilling $\operatorname{curl} F = 0$. Then there exists a scalar field $\phi \in C^2(\Omega)$ such that*

$$F = \nabla\phi$$

Proof. This is a direct consequence of Corollary 8.24 in [1]. □

Chapter 2

Derivation of Maxwell's Equations

In the next four subsections Maxwell's equations are derived. These are a system of four equations that relates the main physical quantities which describe electromagnetic behavior and illustrates which types of physical phenomena can give rise to electric and magnetic fields. Two of these equations are 3 dimensional, thus in fact we obtain a system of in total 8 partial differential equations,

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad \text{(Ampère's Law)}$$
$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Faraday's Law)}$$
$$\nabla \cdot D = \rho \qquad \text{(Gauss's Law - electric)}$$
$$\nabla \cdot B = 0, \qquad \text{(Gauss's Law - magnetic)}$$

called *Maxwell's equations.*

For the rest of this thesis we assume the involved quantities to be sufficiently smooth in the sense that the conditions of Theorem 1.4 (Gauss's theorem) and Theorem 1.3 (Stokes' theorem) are fulfilled. In general these assumptions on the smoothness are satisfied in nature, i.e., no limitation on the field of observation is made.

In the following we will figure out that an electric current induces a magnetic field, which is a directed quantity. To determine its orientation the *Right-hand rule* can be used.

Remark 2.1 (Right-hand Rule)**.** *If an electric current passes through a straight wire, let the thumb of your right hand point in the direction of the current. Then the remaining fingers of your right hand show the orientation of the induced magnetic field. This rule can analogously be applied backwards, i.e., if the magnetic field is given and one wants the know the orientation of the induced electric current.*

The relation between electric forces and charges is described by *Coulomb's Law* which is an experimental postulate. It states that similarly poled charges attract each other while oppositely poled charges repel.

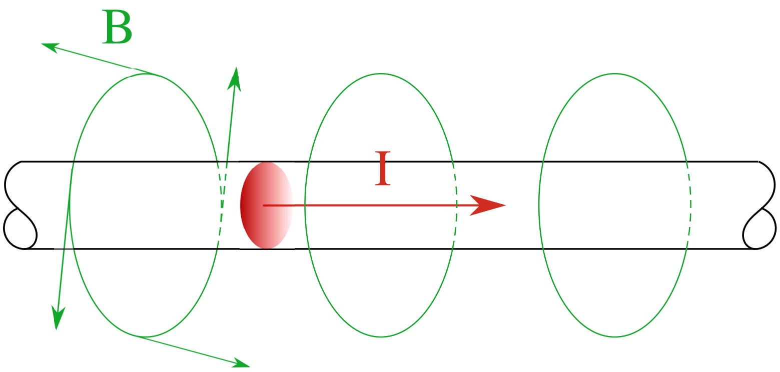

Figure 2.1: Wire with current and arising magnetic field

Postulate 2.2 (Coulomb's Law)**.** *The electrostatic force K experienced by a charge Q_1 at position $a \in \mathbb{R}^3$ in the vicinity of another charge Q_2 at position $b \in \mathbb{R}^3$ in vacuum is equal to*

$$K = \frac{Q_1 Q_2}{4\varepsilon_0 \pi \|a-b\|^3}(a-b) = \frac{Q_1 Q_2}{4\varepsilon_0 \pi r^2}\mathbf{e}_r, \tag{2.1}$$

where $r = \|a-b\|$ and $\mathbf{e}_r = \frac{a-b}{r}$, the unit vector pointing from b to a.

It can be seen easily that if Q_1, Q_2 are unequally charged in (2.1), e.g. $Q_1 > 0$ and $Q_2 < 0$, then

$$K = \underbrace{\frac{Q_1 Q_2}{4\varepsilon_0 \pi r^2}}_{<0}\mathbf{e}_r$$

is directed from a to b, i.e., the two charges attract each other. An analogous result follows with equal charges. Observing a point charge Q, the electrostatic force K is related to the electric field intensity E by $K = EQ$. Thus, the electric field in $a \in \mathbb{R}^3$ of a point charge placed in $b \in \mathbb{R}^3$ is

$$E(a) = \frac{Q}{4\varepsilon_0 \pi}\frac{a-b}{\|a-b\|^3}. \tag{2.2}$$

If we have a continuous charge density q inside a volume V, (2.2) changes to

$$E(a) = \frac{1}{4\varepsilon_0 \pi}\int_V q(x)\frac{a-x}{\|a-x\|^3}\,\mathrm{d}x. \tag{2.3}$$

2.1 Ampère's Law

If a conductor(e.g. a wire) is flooded by an electric current, in its surrounding a magnetic field is generated in the sense of Remark 2.1 (right-hand rule). This is illustrated in Figure 2.1. In *Ampère's law* the integrated magnetic field around a closed loop is related to the electric current passing through the loop by

$$\oint_{\partial S} H \cdot \tau \,\mathrm{d}s = \int_{S} J \cdot n \,\mathrm{d}S, \tag{2.4}$$

where τ is the tangential vector of the curve ∂S and n the normal vector of the surface S. We want to transform the integral form into a differential one, so we use Theorem 1.3 (Stokes' theorem) and get

$$\int_{S} (\nabla \times H) \cdot n \,\mathrm{d}S = \oint_{\partial S} H \cdot \tau \,\mathrm{d}s = \int_{S} J \cdot n \,\mathrm{d}S.$$

Since the surface S is arbitrary, it follows that the integrands must be equal, i.e.,

$$\nabla \times H = J. \tag{2.5}$$

It was Maxwell's amendment, what made law (2.5) complete: He found out that a magnetic field is not only induced by a conductive current but also by a so-called displacement current. This current is proportional to the variation of the electric flux density D, thus we have to add the term $\partial D/\partial t$ to (2.5), so we get

$$\nabla \times H = J + \frac{\partial D}{\partial t}, \tag{2.6}$$

the first of four Maxwell equations[1].

Summarizing, this law states that there are two ways of generating a magnetic field: by the presence of an electric current or by changing an electric field.

2.2 Faraday's Law

Let us assume, that we have a conductive loop and on its ends we are able to measure the voltage. *Faraday's law* describes how a time varying magnetic field passing the loop in normal direction induces an electric field inside the loop.

If Φ denotes the magnetic flux through the cross section S of the loop, it holds that $\Phi = \int_S B \cdot n \,\mathrm{d}S$, where B is the magnetic induction. Faraday found out that the relation between the induced voltage u_i (also called electromotive force) and the magnetic flux is

$$u_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{S} B \cdot n \,\mathrm{d}S = -\int_{S} \frac{\partial}{\partial t} B \cdot n \,\mathrm{d}S, \tag{2.7}$$

where the last equality follows by the time independence of the surface S. A change of the magnetic flux in time can be achieved by a time-dependent magnetic field itself or by a motion of the conductive loop.

[1] In fact (2.6) represents an equation in 3 dimensions.

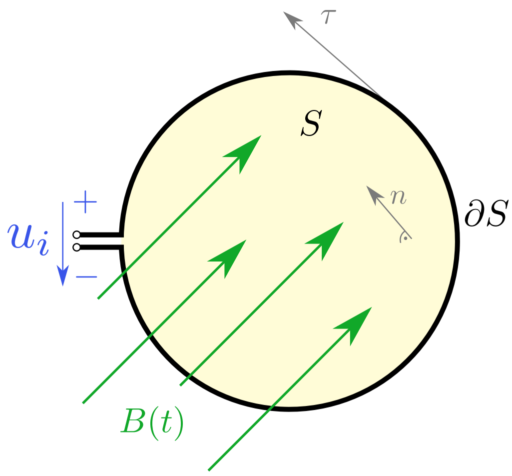

Figure 2.2: Conductive loop exposed to a time-dependent magnetic field.

The electric field intensity E is a measure of how fast the voltage is changing along a path, so the voltage u along a path Γ is equal to the line integral $\int_\Gamma E \cdot \tau \, \mathrm{d}\Gamma$. With this equality and Theorem 1.3 (Stokes' theorem) it follows for our example of the conductive loop that

$$u_i = \int_{\partial S} E \cdot \tau \, \mathrm{d}s = \int_S \operatorname{curl} E \cdot n \, \mathrm{d}S. \tag{2.8}$$

Now we can use that the electromotive force u_i is basically the same in (2.7) and (2.8), i.e.,

$$\int_S \operatorname{curl} E \cdot n \, \mathrm{d}S = -\int_S \frac{\partial}{\partial t} B \cdot n \, \mathrm{d}S.$$

Again the surface S was chosen arbitrarily, so the quantities inside the integral have to be equal and the second of four Maxwell equations[2] follows:

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.9}$$

2.3 Gauss's Law - electric

This law dictates how the electric field behaves around electric charges. That is, if there exists electric charge then the divergence of the electric flux density D at that point is non-zero, otherwise it is equal to zero. Another way of describing the connection of charge and electric flux is the following: The amount of total charge in a volume V is equal to the electric flux exiting its surface $S = \partial V$, i.e.,

$$\int_S D \cdot n \, \mathrm{d}S = \int_V \rho \, \mathrm{d}x, \tag{2.10}$$

[2] Again this is an equation in 3 dimensions.

where ρ is the electric charge density. Applying Theorem 1.4 (Gauss's theorem) to the left hand side of (2.10) we get $\int_S D \cdot n \, \mathrm{d}S = \int_V \operatorname{div} D \, \mathrm{d}x$. Using that V was arbitrarily chosen, this leads to the third Maxwell equation[3]:

$$\nabla \cdot D = \rho \tag{2.11}$$

If there is a positive total charge within the volume V, the electric flux exits the surface. Otherwise, if the total charge inside V is negative, there is an electric flux entering the surface.

2.4 Gauss's Law - magnetic

The electric law of Gauss states that the divergence of the electric flux density is equal to the electric charge density. Gauss's law in the magnetic case is the analogous version: The divergence of the magnetic flux density is equal to the "magnetic charge" density. Since magnetism is always caused by the presence of a positive and a negative magnetic pole, known as the north and south pole, no magnetic charge density exists, because that would mean, that there is a magnetic monopole generating the magnetic charge. But no one has ever found magnetic monopoles, i.e., the divergence of the magnetic flux density is equal to zero. Thus, the integral form of Gauss's law in the magnetic case is

$$\int\limits_{\partial V} B \cdot n \, \mathrm{d}S = 0, \tag{2.12}$$

i.e., the amount of magnetic field lines entering the volume V through its surface ∂V is equal to the amount of magnetic field lines exiting V. Using Theorem 1.4 (Gauss's theorem), we get $\int_{\partial V} B \cdot n \, \mathrm{d}S = \int_V \operatorname{div} B \, \mathrm{d}x$. Again we have chosen the volume V arbitrarily, so the last of four Maxwell equations[4] follows:

$$\nabla \cdot B = 0. \tag{2.13}$$

2.5 Constitutive Equations - Material Laws

A constitutive equation in physics is a relation between two physical quantities that is specific to a material or substance. It describes the response of that material which is exposed to external stimuli, e.g. the change of the magnetic induction subject to the magnetic field. In our case, the magnetic and electric quantities of Table 1.1 are involved and are related as follows:

$$\begin{aligned} B &= \mu H + \mu_0 M \\ D &= \varepsilon E + P \\ J &= J_c + J_i = \sigma \left(E + v \times B \right) + J_i. \end{aligned}$$

[3]Note that this equation is 1-dimensional.
[4]Note that this equation is 1-dimensional.

Here J_c is the conduct current density, J_i is the impressed current density, σ is the electric conductivity (only conductive matter has $\sigma \neq 0$) and $v = (v_1, v_2, v_3)^T$ is the velocity with which the observed region or body is moved in space. In most cases the regions exposed to magnetic and electric influence do not move and are not deformed, i.e., have a velocity equal to 0. So for the rest of this thesis we assume $v = 0$.

2.6 Summary

In the previous sections we derived 4 partial differential equations (2 of them are 3-dimensional, so in fact 8 PDEs) and material laws. Here is a compact overview of the derived system.

Maxwell's equations are

$$\nabla \times H = J + \frac{\partial D}{\partial t} \tag{2.14a}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.14b}$$

$$\nabla \cdot D = \rho \tag{2.14c}$$

$$\nabla \cdot B = 0 \tag{2.14d}$$

and the corresponding constitutive equations are

$$B = \mu H + \mu_0 M \tag{2.15a}$$

$$D = \varepsilon E + P \tag{2.15b}$$

$$J = J_c + J_i = \sigma E + J_i. \tag{2.15c}$$

Chapter 3

Alternative Formulations

Using vector analysis one can derive other formulations of the Maxwell equations (2.14). Based on [6] the vector potential formulation and the E-field based formulation are discussed.

3.1 Vector Potential Formulation

First we need the following tool:

Lemma 3.1. *Let $\phi, \psi \in C^2(\Omega)$ be scalar fields and $A \in \left(C^1(\Omega)\right)^3$ a vector field. The substitutions $\tilde{A} = A + \nabla\psi$ and $\tilde{\phi} = \phi - \partial\psi/\partial t$ for A and ϕ, respectively, lead to the same magnetic and electric fields, i.e.,*

$$B = \operatorname{curl} A = \operatorname{curl} \tilde{A} \qquad \textit{and} \qquad E = -\frac{\partial A}{\partial t} - \nabla\phi = -\frac{\partial \tilde{A}}{\partial t} - \nabla\tilde{\phi}.$$

Proof. Due to (1.5) in Lemma 1.5 we know that $\operatorname{curl} \nabla F = 0$ for all $F \in C^2$ and it follows that

$$\operatorname{curl} \tilde{A} = \operatorname{curl}(A + \nabla\psi) = \operatorname{curl} A + \operatorname{curl} \nabla\psi = \operatorname{curl} A = B.$$

Because we can swap the order of the time derivative and the gradient of ψ the second identity follows with

$$\begin{aligned} -\frac{\partial \tilde{A}}{\partial t} - \nabla\tilde{\phi} &= -\frac{\partial(A + \nabla\psi)}{\partial t} - \nabla\left(\phi - \frac{\partial\psi}{\partial t}\right) \\ &= -\frac{\partial A}{\partial t} - \nabla\phi \underbrace{- \frac{\partial(\nabla\psi)}{\partial t} + \nabla\frac{\partial\psi}{\partial t}}_{=0} = E \end{aligned}$$

□

Gauss's law in the magnetic case (2.14d) states that the divergence of the magnetic flux density is zero, so Lemma 1.6 is applicable, hence we know that there exists an A such that $B = \operatorname{curl} A$. Substituting this into Faraday's law (2.14b), we get

$$\operatorname{curl} E = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\operatorname{curl} A) \tag{3.1}$$

or, equivalently,

$$\operatorname{curl} E + \frac{\partial}{\partial t}(\operatorname{curl} A) = \operatorname{curl}\left(E + \frac{\partial A}{\partial t}\right) = 0.$$

Now, using Lemma 1.7, there is a scalar field ϕ such that $E + \frac{\partial A}{\partial t} = -\nabla\phi$ or, equivalently,

$$E = -\frac{\partial A}{\partial t} - \nabla\phi. \tag{3.2}$$

With $\nu = 1/\mu$ and material law (2.15a) Ampère's law (2.14a) can be rewritten as

$$\operatorname{curl}\left(\nu B - \frac{\mu_0}{\mu}M\right) = \operatorname{curl} H = J + \frac{\partial D}{\partial t}.$$

Reordering the terms and using the previous assumption $B = \operatorname{curl} A$ and the material laws (2.15b), (2.15c) leads to

$$\operatorname{curl}\nu\operatorname{curl} A = \sigma E + J_i + \varepsilon\frac{\partial E}{\partial t} + \frac{\partial P}{\partial t} + \operatorname{curl}\frac{\mu_0}{\mu}M. \tag{3.3}$$

Substituting E in (3.3) by equality (3.2) we get

$$\operatorname{curl}\nu\operatorname{curl} A + \sigma\frac{\partial A}{\partial t} + \varepsilon\frac{\partial^2 A}{\partial t^2} = J_i + \operatorname{curl}\frac{\mu_0}{\mu}M + \frac{\partial P}{\partial t} - \sigma\nabla\phi - \varepsilon\frac{\partial}{\partial t}\nabla\phi. \tag{3.4}$$

Lemma 3.1 allows us to subsitute A and ϕ by $\tilde{A} = A + \nabla\psi$ and $\tilde{\phi} = \phi - \frac{\partial}{\partial t}\psi$, respectively, without changing the electric and magnetic field. We set $\psi = \int_0^t \phi$, then $\tilde{\phi} = \phi - \frac{\partial}{\partial t}\int_0^t \phi = 0$ and we obtain the following vector potential formulation from (3.4) with vector identity (1.5):

$$\varepsilon\frac{\partial^2 \tilde{A}}{\partial t^2} + \sigma\frac{\partial \tilde{A}}{\partial t} + \operatorname{curl}\nu\operatorname{curl}\tilde{A} = J_i + \operatorname{curl}\frac{\mu_0}{\mu}M + \frac{\partial P}{\partial t}.$$

Once $\tilde{A}$ is determined, the magnetic and electric fields can be computed as follows:

- $B = \operatorname{curl}\tilde{A}$
- $E = -\frac{\partial\tilde{A}}{\partial t} - \underbrace{\nabla\tilde{\phi}}_{=0} = -\frac{\partial\tilde{A}}{\partial t}$
- $H = \nu B - \frac{\mu_0}{\mu}M$
- $D = \varepsilon E + P$

3.2 E-field based Formulation

By the law of Faraday (2.14b) and material law (2.15a) we get

$$\operatorname{curl} E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} - \mu_0 \frac{\partial M}{\partial t}. \tag{3.5}$$

Multiplying (3.5) with $\nu = 1/\mu$ and applying the curl-Operator on both sides we obtain

$$\operatorname{curl} \nu \operatorname{curl} E = -\frac{\partial}{\partial t} \operatorname{curl} H - \frac{\mu_0}{\mu} \frac{\partial}{\partial t} \operatorname{curl} M, \tag{3.6}$$

where we used that the order of the time derivative and the curl can be exchanged. Considering Ampère's law (2.14a) and substituting the current density J and the electric flux density D with the material laws (2.15b) and (2.15c), respectively, we obtain

$$\operatorname{curl} H = J + \frac{\partial D}{\partial t} = \sigma E + J_i + \varepsilon \frac{\partial E}{\partial t} + \frac{\partial P}{\partial t}. \tag{3.7}$$

If we assume that the polarization P in (3.7) can be neglected, we get

$$\operatorname{curl} H = \sigma E + J_i + \varepsilon \frac{\partial E}{\partial t}. \tag{3.8}$$

Substituting $\operatorname{curl} H$ in equation (3.6) by identity (3.8), the so-called E-field based formulation follows:

$$\varepsilon \frac{\partial^2 E}{\partial t^2} + \sigma \frac{\partial E}{\partial t} + \operatorname{curl} \nu \operatorname{curl} E = -\frac{\partial J_i}{\partial t} - \frac{\mu_0}{\mu} \frac{\partial}{\partial t} \operatorname{curl} M \tag{3.9}$$

Chapter 4

Special Regimes

There are many different special cases of Maxwell's equations (2.14), for example fields which are constant in time or only weakly time dependent. Each regime allows simplifications of the equations and under special simplifying assumptions we are able to solve them in an analytic way. Time independent cases are split up into electrostatic and magnetostatic ones.

The entity of Electromagnetism is arranged like Figure 4.1 shows.

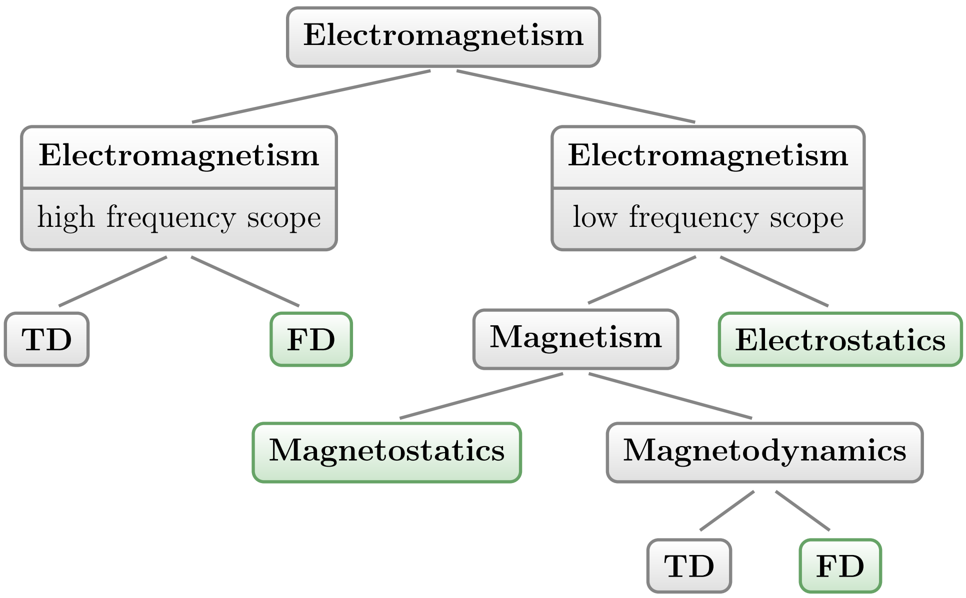

Figure 4.1: Split-up of electromagnetism (TD = Time Domain, FD = Frequency Domain).

In the following four subsections we will take a closer look at the highlighted branches of Figure 4.1. See [6] for further details.

4.1 The Magnetostatic Case

In this case we assume that all involved quantities are time-independent. Then the equations for electric and magnetic fields decouple and the Maxwell equations (2.14) reduce to

$$\int\limits_{\partial S} H \cdot \tau \, \mathrm{d}s = \int\limits_{S} J \cdot n \, \mathrm{d}S \qquad \longrightarrow \qquad \nabla \times H = J \tag{4.1}$$

$$\int\limits_{S} B \cdot n \, \mathrm{d}S = 0 \qquad \longrightarrow \qquad \nabla \cdot B = 0. \tag{4.2}$$

The magnetic induction is solenoidal and the magnetic field intensity possesses curls at positions where a current appears.

4.1.1 Biot-Savart Formulation

This formulation serves for calculating the magnetic field H by a given electric current density J. For that purpose the starting point is the magnetostatic case, in which the dependency of the magnetic quantities of time is neglected. If the electric flux density D is constant in time, the time derivative $\partial D/\partial t$ is zero and Ampère's law (2.14a) reduces to

$$\operatorname{curl} H = J. \tag{4.3}$$

From (4.2) we know that B is solenoid. According to Lemma 1.6, for the magnetic induction B, a vector field A can b introduced, which fulfills the conditions $B = \operatorname{curl} A$ and $\operatorname{div} A = 0$. The second condition is necessary for uniqueness of the chosen A and is called *Coulomb gauging*. Assuming that we have a linear, homogenous and isotropic material, i.e., the permeability μ is constant, we get from (4.3) and material law (2.15a)

$$\operatorname{curl} H = \frac{1}{\mu} \operatorname{curl} (\operatorname{curl} A) = \frac{1}{\mu} (\nabla \operatorname{div} A - \Delta A) = J \tag{4.4}$$

with the vector identity (1.6) of Lemma 1.5. By definition A fulfills $\operatorname{div} A = 0$, so $\nabla \operatorname{div} A = 0$ as well and it follows that

$$-\Delta A = \mu J.$$

With the Green function of the Laplace operator Δ we obtain

$$A(y) = \frac{\mu}{4\pi} \int\limits_{\mathbb{R}^3} \frac{J(x)}{|y - x|} \, \mathrm{d}x. \tag{4.5}$$

Substituting (4.5) back in the relation $B = \operatorname{curl} A$, the result is called the *Biot-Savart formula*

$$
\begin{aligned}
B(y) = \operatorname{curl}_y A(y) &= \frac{\mu}{4\pi} \int\limits_{\mathbb{R}^3} \nabla_y \times \frac{J(x)}{|y-x|} \, \mathrm{d}x \\
&\overset{(4.7)}{=} \frac{\mu}{4\pi} \int\limits_{\mathbb{R}^3} \frac{y-x}{|y-x|^3} \times J(x) \, \mathrm{d}x,
\end{aligned} \tag{4.6}
$$

where the index y in curl_y and ∇_y indicates that the differential operator only acts on the y variable. The needed equality for (4.6) follows with simple differentiation and the obvious fact that $\nabla_y J(x) = 0$

$$
\begin{aligned}
\nabla_y \times \left(J(x) \cdot \frac{1}{|y-x|} \right) &= J(x) \times \left(\nabla_y \frac{1}{|y-x|} \right) \\
&= J(x) \times \left(-\frac{1}{|y-x|^2} \cdot \frac{1}{2} \frac{1}{|y-x|} \cdot 2(y-x) \right) \\
&= -J(x) \times \frac{y-x}{|y-x|^3} \\
&= \frac{y-x}{|y-x|^3} \times J(x)
\end{aligned} \tag{4.7}
$$

4.2 The Electrostatic Case

In this case we assume that all involved quantities are time-independent. Then the equations for electric and magnetic fields decouple. The electric field is irrotational and its sources are charges. The Maxwell equations (2.14) reduce to

$$
\int\limits_{\partial S} E \cdot \tau \, \mathrm{d}s = 0 \qquad \longrightarrow \qquad \nabla \times E = 0 \tag{4.8}
$$

$$
\int\limits_{S} D \cdot n \, \mathrm{d}S = \int\limits_{V} \rho \, \mathrm{d}x \qquad \longrightarrow \qquad \nabla \cdot D = \rho. \tag{4.9}
$$

The electric field intensity is irrotational due to (4.8), so with Lemma 1.7 there exists a sufficiently smooth potential field ϕ with $E = \nabla\phi$. This potential is very convenient for practical applications because it is a scalar quantity. Further, by setting $E = \nabla\phi$ Faraday's law in the electrostatic case (4.8) is automatically satisfied.

In a more physical way one could define ϕ as the energy that is needed to move a point charge from S_0 to S, i.e.,

$$
\phi(S) := -\int\limits_{S_0}^{S} E \cdot \tau \, \mathrm{d}\tau.
$$

It is easy to see that the principle of superposition holds if we consider an electric field consisting of different charges:

$$\phi(S) = -\int_{S_0}^{S} \left(\sum_i E_i \right) \cdot \tau \, \mathrm{d}\tau = -\sum_i \int_{S_0}^{S} E_i \cdot \tau \, \mathrm{d}\tau = \sum_i \phi_i(S)$$

4.3 Time-harmonic Regime

In this section we assume linear material laws and time-harmonic excitation with the frequency ω. This leads to the following ansatz for the variable $U(x,t)$, where U stands for any of the quantities H, B, E, D, J, ρ and $\Re$ is the projection onto the real part:

$$U(x,t) = \Re\left(\hat{U}(x)e^{i\omega t}\right). \tag{4.10}$$

This separation ansatz yields

$$\frac{\partial U}{\partial t} = i\omega U \tag{4.11}$$

$$\frac{\partial^2 U}{\partial t^2} = -\omega^2 U. \tag{4.12}$$

With the assumptions (4.10) - (4.12) and setting $M = 0$ and $P = 0$ one can derive a time-harmonic variant of the Maxwell equations (2.14)

$$\operatorname{curl} H(x) - (i\omega\varepsilon + \sigma)E(x) = J_i(x) \tag{4.13a}$$

$$\operatorname{curl} E(x) + i\omega\mu H(x) = 0 \tag{4.13b}$$

$$\operatorname{div} \varepsilon E(x) = \rho(x) \tag{4.13c}$$

$$\operatorname{div} \mu H(x) = 0 \tag{4.13d}$$

and the continuity equation

$$i\omega\rho(x) + \operatorname{div} J(x) = 0, \tag{4.14}$$

where (4.14) follows by taking the divergence on both sides of (4.13a), material law (2.15c), vector identity (1.4) and (4.13c). In the vector potential formulation we have $E = -i\omega A$ and $B = \operatorname{curl} A$, so system (4.13) simplifies to

$$\operatorname{curl} \mu^{-1} \operatorname{curl} A - (\omega^2\varepsilon - i\omega\sigma)A = J_i.$$

In the E-field based formulation we get

$$\operatorname{curl} \mu^{-1} \operatorname{curl} E - (\omega^2\varepsilon - i\omega\sigma)E = -i\omega J_i.$$

4.4 Quasistatic Case – Eddy Current Problem

In this regime we assume that the frequency is low, i.e., we will neglect the displacement currents. In other words $|\partial D/\partial t| \ll |J|$. Further, we obtain the so-called *eddy-current approximation* to the Maxwell equations (2.14)

$$\begin{aligned}
\operatorname{curl} H &= J \\
\operatorname{div} B &= 0 \\
\operatorname{curl} E &= -\frac{\partial B}{\partial t} \\
\operatorname{div} D &= \rho
\end{aligned}$$

and to the material laws

$$\begin{aligned}
B &= \mu H + \mu_0 M \\
D &= \varepsilon E \\
J &= \sigma E + J_i.
\end{aligned}$$

Then, the vector potential formulation from Section 3.1 in the time domain reads as

$$\sigma \frac{\partial \tilde{A}}{\partial t} + \operatorname{curl} \nu \operatorname{curl} \tilde{A} = J_i + \operatorname{curl} \frac{\mu_0}{\mu} M$$

and in the frequency domain

$$i\omega\sigma\tilde{A} + \operatorname{curl} \nu \operatorname{curl} \tilde{A} = J_i + \operatorname{curl} \frac{\mu_0}{\mu} M.$$

The E-field based formulation from Section 3.2 in the time domain is

$$\sigma \frac{\partial E}{\partial t} + \operatorname{curl} \nu \operatorname{curl} E = -\frac{\partial J_i}{\partial t}$$

and in the frequency domain

$$i\omega\sigma E + \operatorname{curl} \nu \operatorname{curl} E = -i\omega J_i,$$

where we set $\frac{\mu_0}{\mu}\frac{\partial}{\partial t} M = 0$ due to the assumption at the beginning that the frequency is low.

Chapter 5

Benchmark Problems

The purpose of a benchmark is to assess the relative performance of an object. In our case we especially look at numerical solvers for the Maxwell equations (2.14), where we want to validate whether the solution of the numerical solver is feasible. Normally, the numerical approach is applicable in very general situations and configurations of the problem, whereas the exact solution can only be computed in simplified cases. In these simplified situations both solutions can be compared and, of course, should be equal (in some sense). Otherwise, this indicates that the numerical solver is not working properly. In the following (sub-)sections we will derive analytic solutions of some simple magnetostatic, electrostatic and eddy current problems. These benchmark problems are based on exercises in [2, 4, 7].

5.1 Benchmarks in Magnetostatics

5.1.1 Straight Wire

In this benchmark we consider a straight and thin conductor (e.g. a wire) with infinite length and a constant current. Let the material surrounding the wire be air, so the relative permeability $\mu_r = 1$ and thus $\mu = \mu_0$. The wire is aligned in the x_3-direction passing the origin in the ordinary Euclidian vector space $\mathbb{R}^3$, see Figure 5.1. As we already know from Ampère's law in Section 2.1, the current in the conductor causes a magnetic field and a magnetic induction B in the periphery. Under these special assumptions to symmetry and simplicity of the problem, we are able to calculate the caused induction in an arbitrary point $P = (P_1, P_2, P_3)^T \in \mathbb{R}^3$.

From the Biot-Savart formula (4.6) we get

$$B(P) = \frac{\mu_0}{4\pi} \int\limits_{\mathbb{R}^3} \frac{P - x}{|P - x|^3} \times J(x)\,\mathrm{d}x, \tag{5.1}$$

where

$$J(x) = \begin{cases} \hat{J}(x_3) & x \in \{(y_1, y_2, y_3)^T \in \mathbb{R}^3 : y_1 = y_2 = 0\} \\ 0 & \text{otherwise.} \end{cases}$$

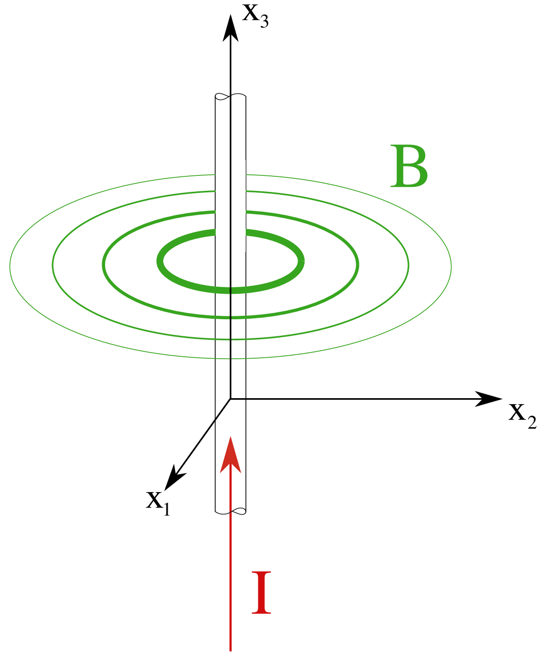

Figure 5.1: Alignment of the straight wire problem.

This follows due to the chosen geometry of the wire. $\hat{J}(x_3)$ is the given current density flowing through the cross-section in x_3-direction. In our case, we can simply set $\hat{J}(x_3) = I$ with a constant current I, so that the cross product in (5.1) reduces to

$$\begin{aligned} \frac{P-x}{|P-x|^3} \times J(x) &= \frac{1}{|P-x|^3} \cdot \begin{pmatrix} P_1 \\ P_2 \\ P_3 - x_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \\ &= \frac{1}{|P-x|^3} \cdot \begin{pmatrix} P_2 I \\ -P_1 I \\ 0 \end{pmatrix}. \end{aligned}$$

The vector $(P_2 I, -P_1 I, 0)^T$ is independent of x and x itself may only be integrated over the x_3-part, so (5.1) becomes

$$\begin{aligned} B(P) &= \frac{\mu_0}{4\pi} \begin{pmatrix} P_2 I \\ -P_1 I \\ 0 \end{pmatrix} \int\limits_{\{0\}\times\{0\}\times\mathbb{R}} \frac{1}{|P-x|^3}\,\mathrm{d}x \\ &= \frac{\mu_0}{4\pi} \begin{pmatrix} P_2 I \\ -P_1 I \\ 0 \end{pmatrix} \int\limits_{-\infty}^{\infty} \left| \begin{pmatrix} P_1 \\ P_2 \\ P_3 - z \end{pmatrix} \right|^{-3} \mathrm{d}z. \end{aligned} \tag{5.2}$$

The integral in (5.2) is computed via

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\begin{pmatrix}P_1\\P_2\\P_3-z\end{pmatrix}\right|^{-3}\mathrm{d}z &= \int_{-\infty}^{\infty}(P_1^2+P_2^2+(P_3-z)^2)^{-3/2}\,\mathrm{d}z\\ &= \left.\frac{z-P_3}{(P_1^2+P_2^2)(P_1^2+P_2^2+(P_3-z)^2)^{1/2}}\right|_{-\infty}^{\infty}\\ &= \frac{2}{P_1^2+P_2^2},\end{aligned}$$

where the last equality follows with $C := P_1^2+P_2^2$, l'Hôpital's rule and

$$\begin{aligned}\lim_{z\to\pm\infty}\frac{z-P_3}{C(C+(P_3-z)^2)^{1/2}} &= \lim_{z\to\infty}\pm\left(\frac{(z-P_3)^2}{C^3+C^2(P_3-z)^2}\right)^{1/2}\\ &= \pm\left(\lim_{z\to\infty}\frac{(z-P_3)^2}{C^3+C^2(P_3-z)^2}\right)^{1/2}\\ \text{(l'Hôpital)}\qquad &= \pm\left(\lim_{z\to\infty}\frac{2(z-P_3)}{-2C^2(P_3-z)}\right)^{1/2} = \pm\left(\frac{1}{C^2}\right)^{1/2} = \pm\frac{1}{C}.\end{aligned}$$

Back in (5.2) this leads to

$$B(\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}) = \frac{\mu_0}{2\pi}\frac{I}{P_1^2+P_2^2}\begin{pmatrix}P_2\\-P_1\\0\end{pmatrix}. \tag{5.3}$$

Because of symmetry, the absolute value of B is constant for a fixed radius $r = \sqrt{P_1^2+P_2^2}$. So without loss of generality we can neglect P_2. The graph for $P_2 = 0$ is shown in Figure 5.2, where (5.3) simplifies to

$$B(\begin{pmatrix}P_1\\0\\P_3\end{pmatrix}) = -\frac{\mu_0}{2\pi}\frac{I}{P_1}\begin{pmatrix}0\\1\\0\end{pmatrix},$$

a hyperbola.

5.1.2 Circular Conductor Loop

In this scenario we are considering a circular loop made of a conductive material with infinitesimally small cross-section flooded by a constant current density J and radius r. Its center is located in the origin of the 3-dimensional Euclidian coordinate system, see Figure 5.3 for the details. Now we are interested in the magnetic field B arising due to the current.

We assume that $P = (P_1, 0, 0)^T$, i.e., lies on the x_1-axis. Then, because of symmetry reasons, the x_2 and x_3-components of $B(P)$ vanish. This follows directly from

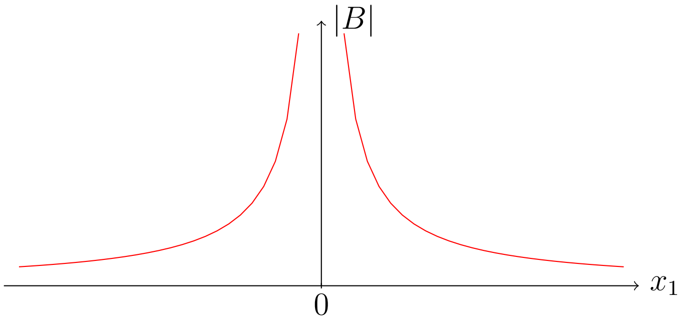

Figure 5.2: Magnitude of the magnetic Induction B of the straight wire problem of points lying on the x_1-x_3-plane.

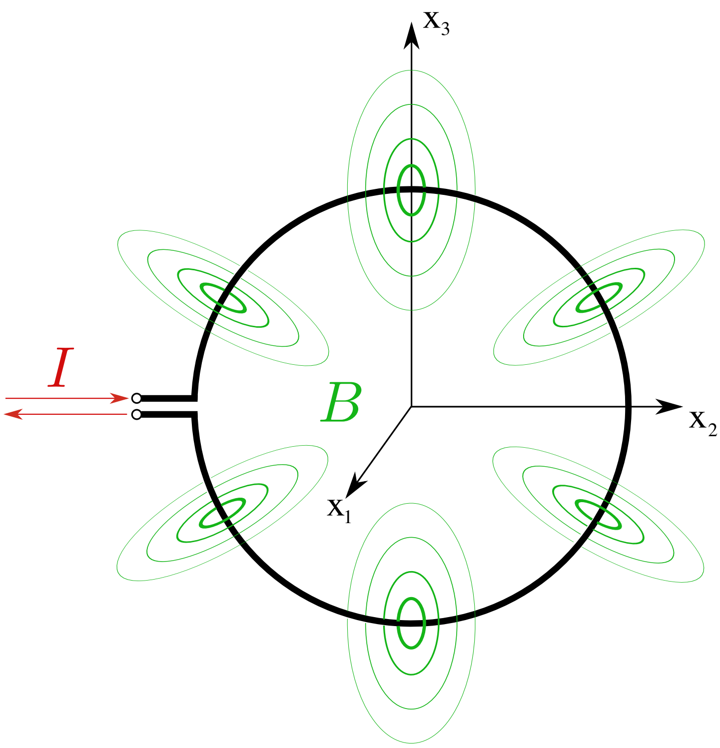

Figure 5.3: Alignment of the circular conductor loop with constant current I.

the awareness that each magnetic field line in x_2 direction is compensated by a corresponding one pointing in $-x_2$-direction (and analogously x_3). Only the x_1-component remains and is in general not equal to 0. We will see this mathematically during the calculation. We use the Biot-Savart formulation (4.6) to compute B as follows:

$$B(P) = \frac{\mu_0}{4\pi} \int\limits_{\mathbb{R}^3} \frac{P - x}{|P - x|^3} \times J(x)\, \mathrm{d}x \tag{5.4}$$

Since the current only flows in $C := \{(x_1, x_2, x_3)^T : x_2^2 + x_3^2 = r, x_1 = 0\}$ and $J = I\tau$ in C and 0 otherwise (τ denotes the unit tangent of C), the 3-dimensional integral in (5.4) reduces to a line integral with $x \in C \;\Rightarrow\; x = (0, r\cos\varphi, r\sin\varphi)^T$ for some $\varphi \in (0, 2\pi)$, i.e.,

$$\begin{aligned} B(P) &= \frac{\mu_0}{4\pi} \int\limits_0^{2\pi} \left| \begin{pmatrix} P_1 \\ 0 \\ 0 \end{pmatrix} - r \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} \right|^{-3} \left[\begin{pmatrix} P_1 \\ 0 \\ 0 \end{pmatrix} - r \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} \right] \times I \begin{pmatrix} 0 \\ \sin\varphi \\ -\cos\varphi \end{pmatrix} \mathrm{d}\varphi \\ &= \frac{\mu_0}{4\pi} I \int\limits_0^{2\pi} \frac{1}{(P_1^2 + r^2)^{3/2}} \begin{pmatrix} r \\ P_1\cos\varphi \\ P_1\sin\varphi \end{pmatrix} \mathrm{d}\varphi. \end{aligned}$$

Integrating $\cos\varphi$ and $\sin\varphi$ over the whole interval $(0, 2\pi)$ makes them vanish, i.e., the second and third component of $B(P)$ are 0 (as forecasted above). Denoting the first component of $B(P)$ with $B_1(P)$ we get

$$B_1(P) = \frac{\mu_0}{4\pi} I \int\limits_0^{2\pi} \frac{r}{(P_1^2 + r^2)^{3/2}}\, \mathrm{d}\varphi, \tag{5.5}$$

where the integrand is a constant and the integral is equal to the circumference of the circle with radius r

$$B_1(P) = \frac{\mu_0}{4\pi} I \frac{r}{(P_1^2 + r^2)^{3/2}} 2r\pi = \frac{\mu_0 I r^2}{2(P_1^2 + r^2)^{3/2}}. \tag{5.6}$$

This function is plotted in Figure 5.4.

5.1.3 Helmholtz Coil

See Example 8.4 in [4] for further details. A Helmholtz coil is used to produce a region of nearly uniform magnetic field. For this purpose two identical short coils with radius a and distance d to the plane $\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_3 = 0\}$ are aligned as shown in Figure 5.5. The magnetic flux of the first and second coil is doneted by B_1 and B_2, respectively, and we again restrict our calculations to points placed on the x_3-axis. Recall that the x_1 and x_2-components of the magnetic field vanish for these

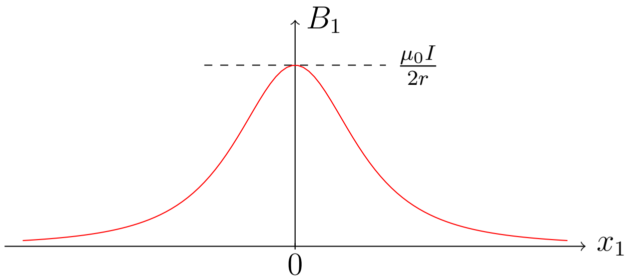

Figure 5.4: Magnitude of the magnetic Induction B of the circular conductor loop for points lying on the x_1-axis.

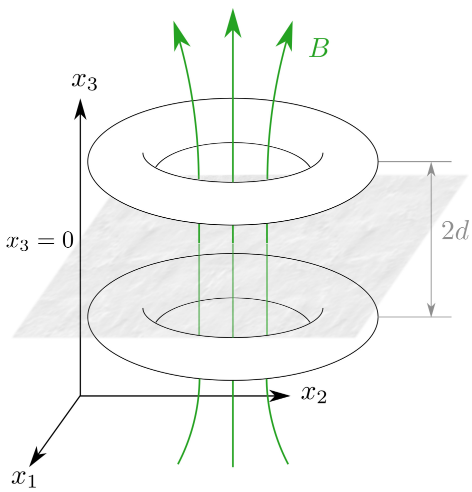

Figure 5.5: Geometry of a Helmholtz coil built up by two identical coils to produce a region of nearly uniform magnetic field.

points because of symmetry. We compute B_1 the same way we did in Subsection 5.1.2. There we derived formula (5.6) which we adapt to

$$B_1(x_3) = \frac{1}{2}\frac{\mu_0 I a^2}{(a^2 + (x_3 - d)^2)^{3/2}}.$$

We get the field B_2 from B_1 by substituting d with $-d$, i.e.,

$$B_2(x_3) = \frac{1}{2}\frac{\mu_0 I a^2}{(a^2 + (x_3 + d)^2)^{3/2}}.$$

The resulting magnetic field B is just the sum $B = B_1 + B_2$. If we assume that the radius a is already chosen and fixed, the only adjustable parameter is d. We want to choose d, such that the arising magnetic field is nearly homogenous in some region. This can be achieved, if the derivatives $\mathrm{d}^n B/\mathrm{d}x_3^n$ vanish in $x_3 = 0$. The first derivative is

$$\frac{\mathrm{d}B}{\mathrm{d}x_3} = \frac{\mathrm{d}(B_1 + B_2)}{\mathrm{d}x_3} = -\frac{3}{2}\mu_0 a^2 I \left[\frac{x_3 - d}{(a^2 + (x_3 - d)^2)^{5/2}} + \frac{x_3 + d}{(a^2 + (x_3 + d)^2)^{5/2}}\right]$$

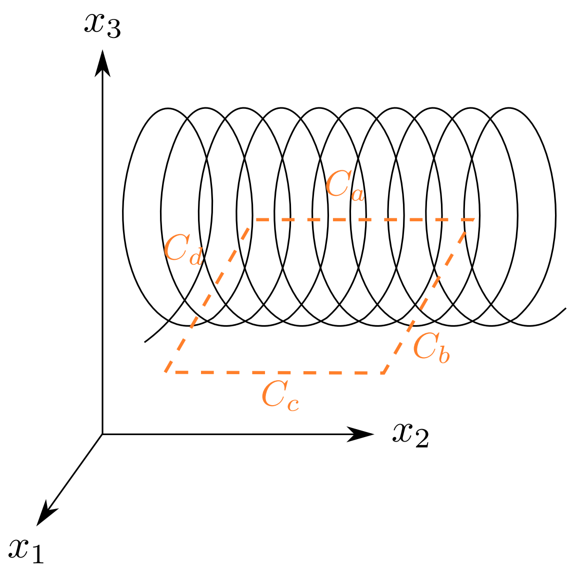

Figure 5.6: Solenoid with special Ampèrian loop, where $C = C_a \cup C_b \cup C_c \cup C_d$.

and vanishes in $x_3 = 0$ independent of the choice of d. The second derivative is

$$\frac{\mathrm{d}^2 B}{\mathrm{d}x_3^2} = -\frac{3}{2}\mu_0 a^2 I \left[\frac{a^2 - 4(x_3 - d)^2}{(a^2 + (x_3 - d)^2)^{7/2}} + \frac{a^2 - 4(x_3 + d)^2}{(a^2 + (x_3 + d)^2)^{7/2}} \right]$$

and vanishes in $x_3 = 0$, if we set $a = 2d$. The Taylor expansion of $B(x_3)$ around 0 is

$$B(x_3) = B(0) + x_3 \left.\frac{\mathrm{d}B}{\mathrm{d}x_3}\right|_{x_3=0} + \frac{x_3^2}{2} \left.\frac{\mathrm{d}^2 B}{\mathrm{d}x_3^2}\right|_{x_3=0} + \mathcal{O}(x_3^3) \tag{5.7}$$

and if we restrict to $-\frac{a}{10} \leq x_3 \leq \frac{a}{10}$ we get an estimate for the difference

$$\forall x_3 : |x_3| \leq \frac{a}{10} : \quad |B(x_3) - B(0)| \leq 1.2 \cdot 10^{-6},$$

which means that the magnetic field in this region is nearly homogenous and this is exactly what we wanted to produce.

5.1.4 Solenoid

This benchmark follows Problem 4.2 in [2]. There we consider a long solenoid like in Figure 5.6 and a corresponding so-called *Ampèrian loop* C. We use Ampère's law in integral form (2.4) to compute the magnetic field inside the solenoid. The term $\int_C B \cdot \tau \, \mathrm{d}s$ can be split up into the four edges (denoted by C_a, C_b, C_c, C_d, as in Figure 5.6) of the Ampèrian loop, i.e.,

$$\int_C B \cdot \tau \, \mathrm{d}s = \int_{C_a} B_a \cdot \tau_a \, \mathrm{d}s + \int_{C_b} B_b \cdot \tau_b \, \mathrm{d}s + \int_{C_c} B_c \cdot \tau_c \, \mathrm{d}s + \int_{C_d} B_d \cdot \tau_d \, \mathrm{d}s. \tag{5.8}$$

We can assume that the magnetic field inside the solenoid is oriented in x_2-direction which means $B = |B|\hat{i}$, where $\hat{i}$ is the x_2 unit vector. Let the distance from the solenoid to the edge C_c be very large, then we can assume $B_c = 0$. Perambulating the loop clockwise, we can take a closer look on the integrals of (5.8) which now simplify to

$$\begin{aligned}
\int\limits_{C_a} B_a \cdot \tau_a \, \mathrm{d}s &= |B| \cdot \int\limits_{C_a} \mathrm{d}s &&= |B| \cdot |C_a| \\
\int\limits_{C_b} B_b \cdot \tau_b \, \mathrm{d}s &= |B| \cos\frac{\pi}{2} \int\limits_{C_b} \mathrm{d}s = 0 \\
\int\limits_{C_c} B_c \cdot \tau_c \, \mathrm{d}s &= 0 \cdot \int\limits_{C_c} \mathrm{d}s &&= 0 \\
\int\limits_{C_d} B_d \cdot \tau_d \, \mathrm{d}s &= |B| \cos\frac{\pi}{2} \int\limits_{C_d} \mathrm{d}s = 0.
\end{aligned}$$

Let S denote the area spanned by C, then using Ampère's law in integral form (2.4) gives

$$|B| \cdot |C_a| = \int\limits_{C} B \cdot \tau \, \mathrm{d}s = \mu_0 \int\limits_{S} J \cdot n \, \mathrm{d}S = \mu_0 I_{enc}, \tag{5.9}$$

where I_{enc} is the total electric current enclosed by the loop. Equivalently to (5.9) $|B| = \mu_0 I_{enc}/a$. To find the current enclosed by the Ampèrian loop, just multiply the current in each turn of the solenoid by the number of turns within the loop. If n is the relative number of turns per unit, then $I_{enc} = aI \cdot n$ and

$$|B| = \frac{\mu_0 a I n}{a} = \mu_0 I n = \mu_0 I \frac{N}{L}, \tag{5.10}$$

where $n = N/L$ with N is the total number of turns and L is the length of the solenoid. Summarized, the magnetic field inside the solenoid is constant and oriented in x_2-direction. With (5.10) it follows that

$$B = \mu_0 I \frac{N}{L} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{5.11}$$

5.1.5 Toroidal

This benchmark refers to Example 8.1 in [4]. We are looking for the magnetic field of a tightly wound and closed toroidal like it is shown in Figure 5.8. The magnitude of the current is I and the number of turns is N. Three different Ampèrian loops C_1, C_2, C_3, corresponding to the areas S_1, S_2, S_3 in Figure 5.8, are used for the calculation. Recall Ampère's law in integral form (2.4):

$$\int\limits_{\partial S} H \cdot \tau \, \mathrm{d}s = \int\limits_{S} J \cdot n \, \mathrm{d}S.$$

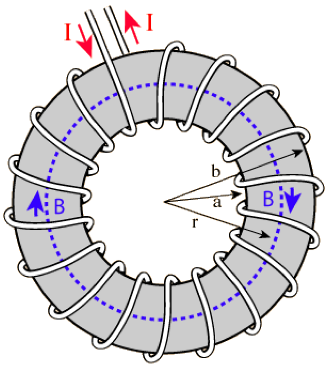

Figure 5.7: Example of a toroidal.

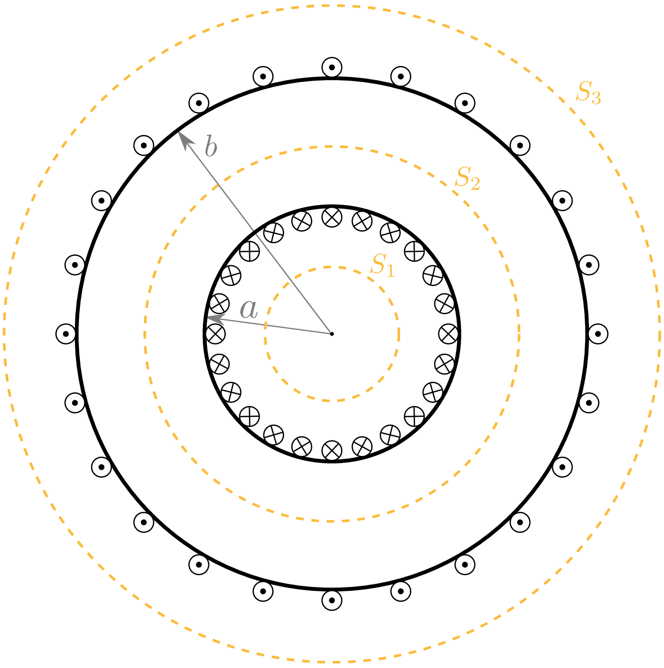

Figure 5.8: Geometry of a toroidal.

For C_1 and hence $r < a$ it is clear that

$$\int_{C_1} B \cdot \tau \, \mathrm{d}s = 0 \qquad \Rightarrow \qquad B_\varphi = 0$$

because there is no current flowing inside S_1. For C_2 and therefore $a < r < b$ we get

$$\begin{aligned} \int_{C_2} B \cdot \tau \, \mathrm{d}s &= \mu_0 \int_{S_2} J \cdot n \, \mathrm{d}S = -\mu_0 N I \\ &= B_\varphi 2r\pi \\ \Rightarrow \qquad B_\varphi &= -\frac{\mu_0 N I}{2r\pi}, \end{aligned} \tag{5.12}$$

where we used that the orientation of the electric current and the normal of S_2 are parallel. The right-hand rule described in Remark 2.1 gives us the negative sign and B is oriented in tangential direction in (5.12) and constant according to its magnitude around a loop with fixed radius. For C_3 and $r > b$ it is again clear that

$$\int_{C_3} B \cdot \tau \, \mathrm{d}s = 0 \qquad \Rightarrow \qquad B_\varphi = 0,$$

because each current entering S_3 in one direction leaves it in the opposite direction, i.e., the total current is 0. Summarized, the magnetic induction vanishes outside the toroidal and inside its absolute value is proportional to NI.

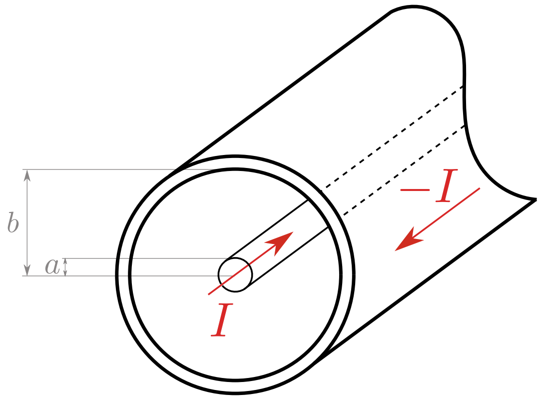

Figure 5.9: Geometry of a coaxial cable.

5.1.6 Coaxial Cable

This subsection is based on Example 4.4 in [4]. The geometry of a standard coaxial cable is shown in Figure 5.9. It has an inner conductor surrounded by a tubular insulating layer. The radius of the inner wire is a and the distance from the center to the layer is b. Through the inner conductor an electric current flows and returns through the layer. Now, we are interested in the magnetic flux caused by the current. As we already know, there will be no flux outside the cable, since the total current passing a surface containing the cross-section of the cable is zero. This follows directly by applying Ampère's law in its integral form (2.4) over an area S_0 enclosing the whole cross-section of the cable:

$$\int\limits_{\partial S_0} H \cdot \tau \, \mathrm{d}s = \int\limits_{S_0} J \cdot n \, \mathrm{d}S = 0.$$

So we only calculate the inside flux.

Let S be the area of an Ampèrian loop in between the two conductors. With Ampère's law in its integral form (2.4) and material law (2.15a) we get

$$\int\limits_{\partial S} B \cdot \tau \, \mathrm{d}s = \mu_0 \int\limits_{S} J \cdot n \, \mathrm{d}S = \mu_0 I. \tag{5.13}$$

On the whole loop B and τ point in the same direction, so we can pull out B from the integral, i.e.,

$$\int\limits_{\partial S} B \cdot \tau \, \mathrm{d}s = |B| \int\limits_{\partial S} \mathrm{d}s = |B| 2r\pi \tag{5.14}$$

and therefore, combining (5.13) and (5.14), we get

$$|B| = \frac{\mu_0 I}{2r\pi}.$$

5.2 Benchmarks in Electrostatics

5.2.1 Cylindric Charge

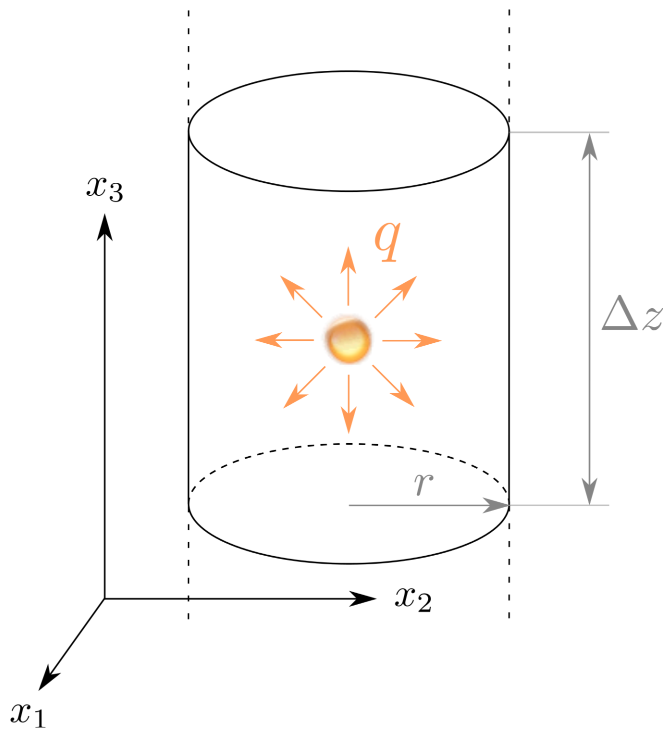

Figure 5.10: Infintely long cylinder with homogenous charge.

This benchmark follows Example 3.3 in [4]. We are looking for the electric field of an infinitely long cylinder with homogenous space charge. Let r be its radius. Because of the symmetry, E can only depend on the radial distance a to the center, so $E = E(a)$ and is constant for a fixed a. Particularly, E is constant on the surface of the cylinder. In the following, we chop a new zylinder V with length Δz out of the initial one, as depicted in Figure 5.10. Since the electric field points radially outwards from the axis of V, E is perpendicular to the normal vector of the bottom and the top surfaces of V, i.e., with Gauss's law for electrostatics (4.9) and $D = \varepsilon_0 E$ (neglecting P in material law (2.15b) and assuming that $\varepsilon = \varepsilon_0$ like in air) we observe

$$\int_{S_{bottom}} (\varepsilon_0 E) \cdot n \, \mathrm{d}S = 0 = \int_{S_{top}} (\varepsilon_0 E) \cdot n \, \mathrm{d}S.$$

E is parallel to the normal vector of the curved side $S = \partial V \backslash (S_{bottom} \cup S_{top})$ of the cylinder. Thus, considering V, one gets for $a \geq r$:

$$\begin{aligned}\int_{\partial V} D \cdot n \, \mathrm{d}S = \int_{S} D \cdot n \, \mathrm{d}S = \int_{S} (\varepsilon_0 E) \cdot n \, \mathrm{d}S = \varepsilon_0 E(a) \int_{S} \mathrm{d}S = \varepsilon_0 E(a) 2\pi a \Delta z \\ = \int_{V} \rho \, \mathrm{d}x = \rho r^2 \pi \Delta z,\end{aligned}$$

since ρ is constant inside V and zero outside. Equivalently, we write

$$E(a) = \frac{\rho r^2}{2\varepsilon_0 a}. \tag{5.15}$$

For $a < r$ and the corresponding S we get analogously

$$\begin{aligned}\int_{S} D \cdot n \, \mathrm{d}S = \int_{S} (\varepsilon_0 E) \cdot n \, \mathrm{d}S = \varepsilon_0 E(a) \int_{S} \mathrm{d}S = \varepsilon_0 E(a) 2\pi a \Delta z \\ = \int_{V} \rho \, \mathrm{d}x = \rho a^2 \pi \Delta z.\end{aligned}$$

Thus, for the electric field

$$E(a) = \frac{\rho a}{2\varepsilon_0} \tag{5.16}$$

follows. Merging (5.15) and (5.16) we obtain

$$E(a) = \begin{cases} \dfrac{\rho}{2\varepsilon_0} \, a & \text{if } a < r \\ \dfrac{\rho r^2}{2\varepsilon_0} \, \dfrac{1}{a} & \text{if } a \geq r \end{cases} \tag{5.17}$$

Figure 5.11 shows the graph of this function. E is continuous in r because

$$\begin{aligned}\lim_{a \to r^-} E(a) = \lim_{a \to r} \tfrac{\rho}{2\varepsilon_0} \, a = \tfrac{\rho}{2\varepsilon_0} \, r \\ = \lim_{a \to r^+} E(a) = \lim_{a \to r} \tfrac{\rho r^2}{2\varepsilon_0} \, \tfrac{1}{a} = \tfrac{\rho}{2\varepsilon_0} \, r\end{aligned}$$

5.2.2 Spherical Charge

The following is based on Example 1.5 in [2]. We want to find the electric field E at distance r to the center of a sphere with uniform volume charge density ρ and radius a. It is reasonable to assume that we have a entirely radial field because the charge distribution is spherically symmetric and so no preferred direction exists. Now we consider a Gaussian surface S centered on the charged sphere, as shown in Figure

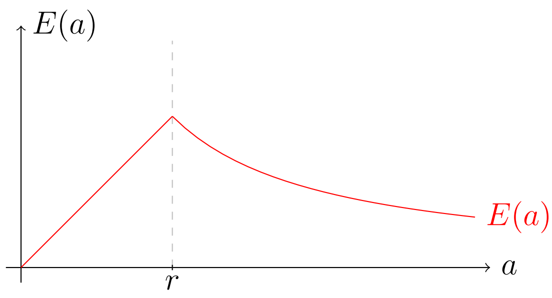

Figure 5.11: Electric field intensity of the cylindric charge problem.

5.12. We denote the sphere spanned by this Gaussian surface by V. For S it is clear that its outer normal and the electric field are parallel, so

$$\int_S E \cdot n \,\mathrm{d}S = |E| \int_S \mathrm{d}S = |E| 4r^2\pi \tag{5.18}$$

follows, where r is the radius of V. Using Gauss's law in integral form (2.10) and material law (2.15b) we get with (5.18)

$$\int_S E \cdot n\mathrm{d}S = |E|4r^2\pi = \frac{1}{\varepsilon_0} \int_V \rho \,\mathrm{d}x$$

or, equivalently,

$$|E| = \frac{1}{4r^2\pi\varepsilon_0} \int_V \rho \,\mathrm{d}x \tag{5.19}$$

which still depends on V, i.e., the choice of the Gaussian surface. Let us first consider the field outside the charged sphere, thus $r > a$. In this case the integral in (5.19) is independent of r and therefore constant:

$$|E| = \frac{1}{4r^2\pi\varepsilon_0} \int_V \rho \,\mathrm{d}x = \frac{1}{4r^2\pi\varepsilon_0}\, \rho\, \frac{4a^3\pi}{3} = \frac{\rho a^3}{3\varepsilon_0 r^2}, \qquad \text{for } r > a. \tag{5.20}$$

Inside the charged sphere we have $r \leq a$ and the amount of enclosed charge decreases, so the value of the integral gets smaller, in particular:

$$|E| = \frac{1}{4r^2\pi\varepsilon_0} \int_V \rho \,\mathrm{d}x = \frac{1}{4r^2\pi\varepsilon_0}\, \rho\, \frac{4r^3\pi}{3} = \frac{\rho r}{3\varepsilon_0}, \qquad \text{for } r \leq a. \tag{5.21}$$

Combining (5.20) and (5.21) gives us the radial electric field

$$|E(r)| = \begin{cases} \dfrac{\rho r}{3\varepsilon_0} & r \leq a \\[2ex] \dfrac{\rho a^3}{3\varepsilon_0 r^2} & r > a \end{cases}$$

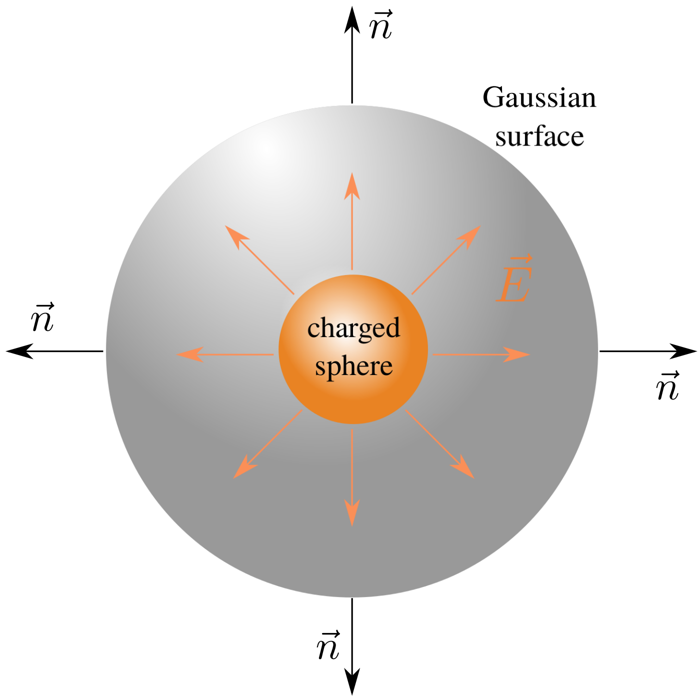

Figure 5.12: Charged sphere with surrounding Gaussian surface.

that is plotted in Figure 5.13, where we recognize a linear increase of electric field intensity inside the charged sphere. Outside E decreases faster than in the zylindric case in Subsection 5.2.1. Continuity of E can be seen with

$$\begin{aligned} &\lim_{r\to a^-} |E(r)| = \lim_{r\to a} \tfrac{\rho r}{3\varepsilon_0} = \tfrac{\rho a}{3\varepsilon_0} \\ = &\lim_{r\to a^+} |E(r)| = \lim_{r\to a} \tfrac{\rho a^3}{3\varepsilon_0 r^2} = \tfrac{\rho a}{3\varepsilon_0}. \end{aligned}$$

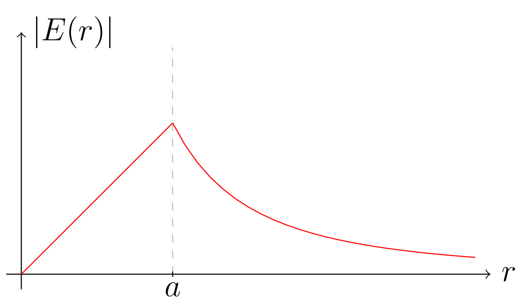

Figure 5.13: Electric field intensity of the spherical charge problem.

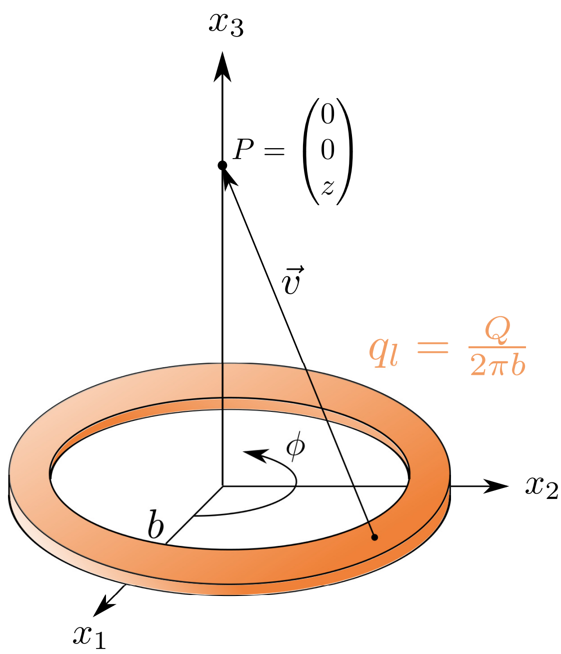

Figure 5.14: Geometry of the circular charged annulus.

5.2.3 Circular Charged Annulus

This example was adopted from "On-Axis E-Field of a Circular Charged Annulus" in [7]. Given is a homogenous line charge arranged in shape of an annulus placed in the x_1-x_2-plane with its center in the origin, see Figure 5.14. The electric field intensity E has to be determined for all points on the x_3-axis, i.e., for all points in $\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 = x_2 = 0\}$. Consider a radius b of the ring and a line charge density $q_l = Q/2\pi b$. Let $A := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1^2 + x_2^2 = b^2, x_3 = 0\}$ represent the annulus.

Because of symmetry, we already know that the electric field only has a component oriented in x_3-direction. We want to determine the electric field intensity in an arbitrary point $(0, 0, x_3)^T$. Recall that the electric field in a point $y \in \mathbb{R}^3$ caused by a continuous charge density inside a volume V is given by law (2.3):

$$E(y) = \frac{1}{4\varepsilon_0 \pi} \int_V q(x) \frac{y - x}{\|y - x\|^3} \,\mathrm{d}x. \tag{5.22}$$

In this case $V = A$, which means that we only have to integrate over a 1-dimensional curve. Changing to cylindrical coordinates, i.e., $x_1 = r\cos\phi, x_2 = r\sin\phi$ and x_3 stays unchanged, we observe that the integration over r and x_3 is only a pointwise evaluation at b and 0, respectively. Thus, the integration over ϕ remains, i.e., we can

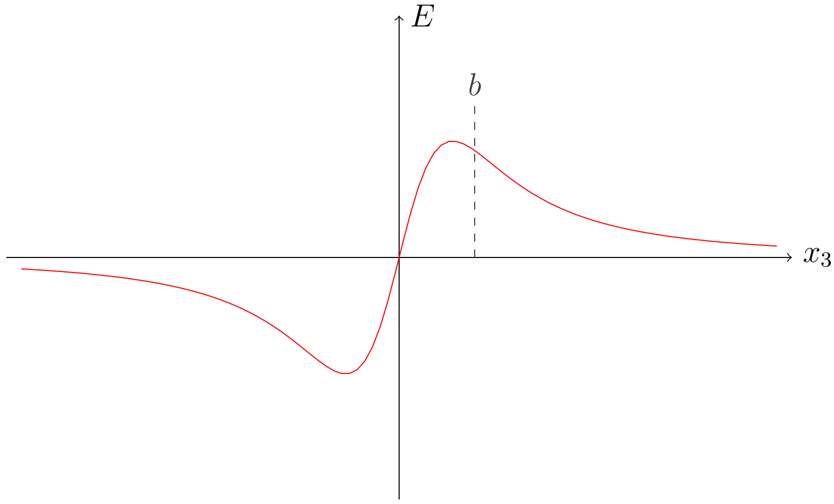

Figure 5.15: Electric field intensity of the circular charged annulus.

rewrite (5.22) in the third component (the x_1 and x_2-components are zero) as

$$\begin{aligned} E(x_3) &= \frac{1}{4\varepsilon_0\pi}\int\limits_0^{2\pi} \frac{x_3 q_l}{(b^2\cos^2\phi + b^2\sin^2\phi + x_3^2)^{3/2}}\, b\,\mathrm{d}\phi \\ &= \frac{1}{4\varepsilon_0\pi}\,\frac{x_3 q_l}{(b^2 + x_3^2)^{3/2}}\, b \int\limits_0^{2\pi} \mathrm{d}\phi \\ &= \frac{q_l b x_3}{2\varepsilon_0(b^2 + x_3^2)^{3/2}}. \end{aligned}$$

If we plug in the previous definition $q_l = Q/2\pi b$ we get

$$E(x_3) = \frac{q_l b x_3}{2\varepsilon_0(b^2 + x_3^2)^{3/2}} = \frac{Q x_3}{4\varepsilon_0\pi(b^2 + x_3^2)^{3/2}}. \tag{5.23}$$

$E(x_3)$ given by (5.23) is plotted in Figure 5.15. Mathematically it is clear that the maximum value of the electric field on the x_3-axis only depends on the radius b of the annulus. It is interesting that $|E|$ reaches its maximum if $x_3 \approx \pm 0.7b$, i.e., x_3 is about 70% of the radius.

5.3 A Benchmark in the Quasistatic Case

We are now looking at an eddy current problem which is based on Problem 3.7 in [2]. Here, a long solenoid is given carrying an electric current $I(t) = I_0 \sin(\omega t)$, i.e., a time harmonic current with frequency ω. Our goal is to find the induced electric field E inside and outside the solenoid as a function of r, the distance from the solenoid axis.

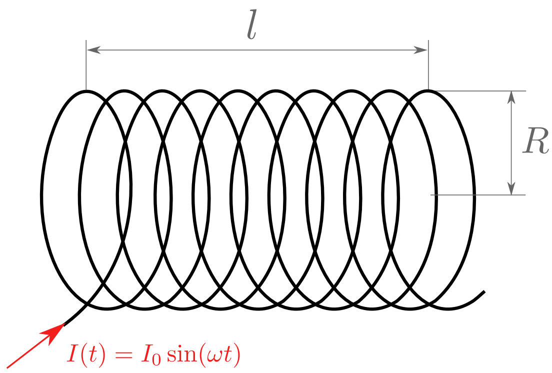

Figure 5.16: Solenoid with time harmonic current.

Let R be the radius, l the length and N the number of turns of the solenoid. Faraday's law (2.14b) in its integral form gives

$$\int_{\partial S} E \cdot \tau \,\mathrm{d}s = -\frac{\mathrm{d}}{\mathrm{d}t} \int_{S} B \cdot n \,\mathrm{d}S. \tag{5.24}$$

Let S be a circle with radius $r \leq R$ with its center on the axis of the solenoid, then the first integral of (5.24) simplifies to

$$\int_{\partial S} E \cdot \tau \,\mathrm{d}s = E(r) \int_{\partial S} \mathrm{d}s = E(r) 2r\pi,$$

where we used that E and τ are parallel. Since the magnetic field B is perpendicular to S, the right hand side of (5.24) simplifies to

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{S} B \cdot n \,\mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} |B| \int_{S} \mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mu_0 N I(t)}{l} r^2 \pi \right),$$

where $|B| = \mu_0 N I(t)/l$, as derived in (5.10) in Subsection 5.1.4. So equality (5.24) reads as

$$E(r) 2r\pi = -\frac{r^2 \pi \mu_0 N}{l} \frac{\mathrm{d}I(t)}{\mathrm{d}t}$$

which can be rewritten with the previous definition $I(t) = I_0 \sin(\omega t)$ to

$$E(r) = -\frac{\mu_0 r N}{2l} \omega I_0 \cos(\omega t). \tag{5.25}$$

Never mind the negative sign in (5.25), it only indicates that the electric field opposes the change in magnetic flux. For $r > R$ the calculation is nearly the same, but B is negligible outside the solenoid, so the right hand side of (5.24) gets

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_{S} B \cdot n \,\mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} |B| \int_{S} \mathrm{d}S = -\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mu_0 N I(t)}{l} R^2 \pi \right).$$

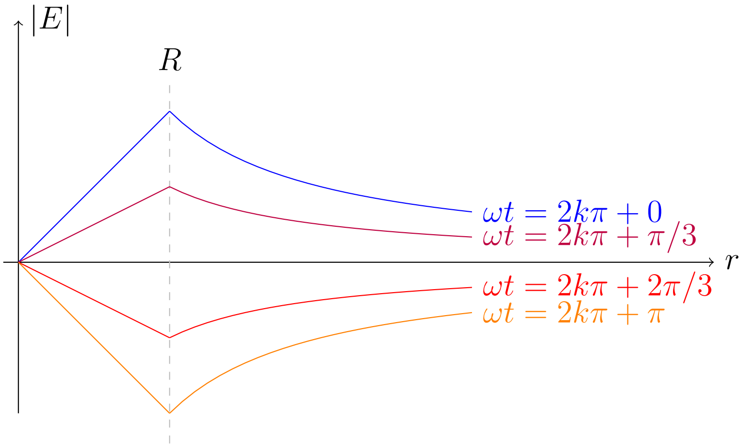

Figure 5.17: Electric field of a solenoid with time harmonic current $I(t) = I_0 \sin(\omega t)$, where $k \in \mathbb{Z}$.

Proceeding the way we did above, (5.25) changes to

$$E(r) = -\frac{\mu_0 R^2 N}{2rl}\omega I_0 \cos(\omega t). \tag{5.26}$$

Finally, we get

$$E(r) = \begin{cases} -\dfrac{\mu_0 r N}{2l}\omega I_0 \cos(\omega t) & r \leq R \\ -\dfrac{\mu_0 R^2 N}{2rl}\omega I_0 \cos(\omega t) & r > R \end{cases} \tag{5.27}$$

which is continuous in $r = R$. We see in Figure 5.17 that the electric field intensity increases linearly in r inside the solenoid but ouside E decreases with $1/r$.

Chapter 6

Conclusion

As we have seen, Maxwell's equations (2.14) in the very general setting are an intractable topic if one wants to solve them analytically. We have derived the full system of equations in combination with material laws. In some special cases we were able to compute a solution in an analytic way. It is somehow clear that we were forced to make excessive simplifications in the geometry and symmetry, so that the complicated system of equations decoupled and became more dealable. Otherwise, we have to use more powerful tools: numerical solvers. Likewise the analytic way, numerical solving starts from a mathematical model of the main problem, i.e., the differential equations we are faced to, some conditions on the region and conditions depending on time. On the one hand, the suitability of the numerical solution, of course, depends on the correctness of this model. On the other hand, it also depends on the correctness of the algorithm itself. To verify the solver's propriety, the analytic solutions derived in Chapter 5 can be used in the corresponding scenarios to compare them to the numerical results.

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