Hierarchical Extension Operators plus Smoothing in Domain Decomposition Preconditioners

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Abstract

The paper presents a cheap technique for the approximation of the harmonic extension from the boundary into the interior of a domain with respect to a given differential operator. The new extension operator is based on the hierarchical splitting of the given f.e. space together with smoothing sweeps and an exact discrete harmonic extension on the lowest level and will be used as a component in a domain decomposition (DD) preconditioner. In combination with an additional algorithmical improvement of this DD-preconditioner solution times faster then the previously studied were achieved for the preconditioned parallelized cg-method. The analysis of the new extension operator gives the result that in the 2D-case $\mathcal{O}(\ln(\ln(h^{-1})))$ smoothing sweeps per level are sufficient to achieve an h-independent behavior of the preconditioned system provided that there exists a spectrally equivalent preconditioner for the modified Schur complement with spectral equivalence constants independent of h.

Keywords : Boundary value problems, Finite element method, Domain decomposition, Preconditioning, Parallel iterative solvers.

1 Introduction

This paper concerns with the use of hierarchical bases (see Yserentant [27], Xu [26], Oswald [22]) in domain decomposition methods (see for reference Bramble/Pasciak/Schatz [3], Smith [24], Smith/Widlund [23]) together with iterative subdomain solvers (Börgers [2], Haase/Langer/Meyer [10]) for solving second order partial differential equations. In contradiction to the work of Bramble/Pasciak/Xu [4] we use the idea of the hierarchical/multilevel bases just locally.

In Haase et al [9, 10, 11, 12] the parallelization and preconditioning of the Conjugate Gradient (cg) method on the basis of a non-overlapping Domain Decomposition (DD) approach was proposed. In Sections 2 and 3 we review some of the results of these papers. The DD preconditioner proposed contains three components which can be chosen in order to adapt the preconditioner to the problem under consideration as well as possible. One component is a (modified) Schur-complement preconditioner that has been studied by the DD community very intensively [3, 6, 17]. Another component is a preconditioner for the local homogeneous Dirichlet problems arising in each subdomain. The most sensitive part is the basis transformation matrix transforming the nodal f.e. basis on the interfaces into the approximate discrete harmonic basis [10]. In order to

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construct the last component, we use local multigrid methods together with a nonzero initial guess. The initial guess obtained from a hierarchical extension technique [12] which will be described briefly in Section 4. It follows from the estimates given therein that $\mathcal{O}(\ln(\ln h^{-1}))$ multigrid iterations are necessary to achieve an h-independent behavior of the condition number $\kappa(C^{-1}K)$ of the preconditioned system.

In Section 5 of this paper, we combine the ideas of the hierarchical extension with the multigrid idea of an exact solver on the coarsest grid and additional smoothing sweeps at the remaining levels. In this case no additional multigrid iterations are needed and an h-independent condition number $\kappa(C^{-1}K)$ will be obtained when $q = \ln(\ln h^{-1})$ smoothing sweeps are performed on each level. The estimate for the condition number includes the preconditioner proposed in [12] and the use of the exact harmonic extension for defining the basis transformation [16]. Because of the smoothing components therein the new extension technique works also in the case of changing coefficients in the interior of the domain and now represents the pde-harmonic extension, i.e. the harmonic extension with respect to the given differential operator.

In Section 6 the special structure of the hierarchical extension operator plus smoothing leads directly to an algorithmical improvement of the well known ASM(Additive Schwarz Method)-DD-preconditioner [10] in which the inner preconditioner (a multigrid V-cycle) will be defined by means of the new extension operator, i.e., more precisely, that the necessary transposed of the new extension procedure may be used as the backslash part of the corresponding multigrid V-cycle. So half of the algorithmical operations necessary for performing a multigrid V-cycle as an inner preconditioner can be saved. The numerical results in Section 7 show that the new DD-precondioner is faster then the ones proposed earlier and works for a wider range of operators.

2 The Non overlapping DD FEM

We consider the symmetric, $\mathbb{V}_0$–elliptic and $\mathbb{V}_0$–bounded variational problem

$$\text{find } \; u \in \mathbb{V}_0 \;=\; \overset{\circ}{\mathbb{H}}{}^1(\Omega) \;:\; \int_\Omega \lambda(x)\, \nabla^T u(x)\, \nabla v(x)\, dx \;=\; \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in \mathbb{V}_0\,, \tag{2.1}$$

arising from the weak formulation of a scalar second–order, symmetric and uniformly bounded elliptic boundary value problem (b.v.p.) given in a plane bounded domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma = \partial\Omega$. The technique described in Section 4 and [12] requires $\lambda(x) = \lambda_i = \text{const} > 0 \quad \forall x \in \overline{\Omega}_i$. The more general coefficients $\lambda(x) \geq \lambda_0 > 0 \quad \forall x \in \overline{\Omega}$ are feasible for the technique described in Section 5. However all the results carry over to systems of symmetric elliptic 2.nd order pdes resulting in symmetric $\mathbb{V}_0$-elliptic and $\mathbb{V}_0$-bounded bilinear forms.

As in the finite element substructuring technique, we decompose Ω into p non-overlapping subdomains Ω_i $(i = 1, 2, \dots, p)$ such that $\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i$, and each subdomain Ω_i into Courant's linear triangular finite elements δ_r such that this discretization process results in a conform triangulation of Ω . In the following, the indices "C" and "I" correspond to the nodes belonging to the coupling boundaries (interfaces) $\Gamma_C = \bigcup_{i=1}^{p} \partial\Omega_i \setminus \Gamma_D$ and to the interior $\Omega_I = \bigcup_{i=1}^{p} \Omega_i$ of the subdomains, respectively, where Γ_D is that part of $\partial\Omega$ where Dirichlet–type boundary conditions are given. Boundaries with Neumann boundary conditions will be handled as coupling boundaries.

Define the usual f.e. nodal basis

$$\Phi = [\Phi_C, \Phi_I] = \left[\psi_1, \cdots, \psi_{N_C}, \psi_{N_C+1}, \cdots, \psi_{N_C+N_{I,1}}, \cdots, \psi_{N=N_C+N_I}\right], \tag{2.2}$$

where the first N_C basis functions belong to Γ_C, the next $N_{I,1}$ to Ω_1, the next $N_{I,2}$ to Ω_2 and so on such that $N_I = N_{I,1} + N_{I,2} + \ldots + N_{I,p}$. The f.e. subspace

$$\mathbb{V} \;=\; \mathbb{V}_h \;=\; \mathrm{span}(\Phi) \;=\; \mathrm{span}(\Phi V) \;=\; \mathbb{V}_C + \mathbb{V}_I \subset \mathbb{V}_0 \tag{2.3}$$

can be represented as direct sum of the subspaces $\mathbb{V}_C = \mathrm{span}(\Phi V_C)$ and $\mathbb{V}_I = \mathrm{span}(\Phi V_I)$ with

$$V \;=\; (V_C\; V_I) \;=\; I \;=\; \begin{pmatrix} I_C & O \\ O & I_I \end{pmatrix}_{N\times N} ,V_C \;=\; \begin{pmatrix} I_C \\ O \end{pmatrix}_{N\times N_C} \text{ and } V_I = \begin{pmatrix} O \\ I_I \end{pmatrix}_{N\times N_I} .$$

The f.e. isomorphism between the f.e. function $u \in \mathbb{V}$ and the corresponding vector $\underline{u} = (\underline{u}_C^T, \underline{u}_I^T)^T \;\in \mathbb{R}^N$ of the nodal parameters is given by

$$\mathbb{V} \;\ni\; u \;=\; \Phi V \underline{u} \;=\; \Phi \underline{u} \quad \stackrel{\Phi}{\longleftrightarrow} \quad \underline{u} \in \mathbf{R}^N \quad . \tag{2.4}$$

Once the basis Φ is chosen, the f.e. approximation leads to a large–scale sparse system

$$K\,\underline{u} \;=\; \underline{f} \tag{2.5}$$

of finite element equations with the symmetric and positive definite stiffness matrix K. Because of the arrangement of the basis functions made above, the system (2.5) can be rewritten in the block form

$$\begin{pmatrix} K_C & K_{CI} \\ K_{IC} & K_I \end{pmatrix} \begin{pmatrix} \underline{u}_C \\ \underline{u}_I \end{pmatrix} = \begin{pmatrix} \underline{f}_C \\ \underline{f}_I \end{pmatrix} \quad , \tag{2.6}$$

where $K_I = \mathrm{blockdiag}\,(K_{I,i})_{i=1,2,\ldots,p}$ is block diagonal.

3 The ASM-DD Preconditioner

Now we use the Parallel Preconditioned Conjugate Gradient method for solving (2.5)–(2.6) on parallel computers. The data distribution and the parallelization of the cg–method is described in [10]. The crucial point is the preconditioning equation

$$C\,\underline{w} \;=\; \underline{r} \tag{3.1}$$

which must fit into the DD parallelization concept proposed earlier in [9, 10].

In [9, 10], the ASM–DD preconditioner

$$C = \begin{pmatrix} I_C & K_{CI} B_I^{-T} \\ O & I_I \end{pmatrix} \begin{pmatrix} C_C & O \\ O & C_I \end{pmatrix} \begin{pmatrix} I_C & O \\ B_I^{-1} K_{IC} & I_I \end{pmatrix} \tag{3.2}$$

was derived on a purely algebraic basis. This preconditioner contains the three components C_C, $C_I = \mathrm{diag}\,(C_{I,i})_{i=1,2,\ldots,p}$ and $B_I = \mathrm{diag}\,(B_{I,i})_{i=1,2,\ldots,p}$, which can be freely chosen in order to adapt the preconditioner to the specialties of the problem under consideration, see also [10]. As Schur complement preconditioner C_C the BPS [3] and the S(chur)-BPX [25] are used. In [9, 10] the inner preconditioners $C_{I,i} = K_{I,i}\,(I_{I,i} - M_{I,i})^{-1}$ and the basis transformations $B_{I,i} = K_{I,i}\,\left(I_{I,i} - \overline{M}_{I,i}\right)^{-1}$ were defined via some cheap multigrid iteration operators $M_{I,i}$ and $\overline{M}_{I,i}$.

The upper and lower bounds for the condition number $\kappa(C^{-1}K)$ of the preconditioned system given in Haase et al [10] were improved by Cheng [5]. The following theorem reproduces Cheng‘s result in the notation used by the author in previous papers.

THEOREM 3.1. *Denote the Schur complement by $S_C = K_C - K_{CI}K_I^{-1}K_{IC}$, its perturbation by $T_C = K_{CI}(K_I^{-1} - B_I^{-T})K_I(K_I^{-1} - B_I^{-1})K_{IC}$ and let the symmetric and positive definite block preconditioners C_C and C_I satisfy the spectral equivalence inequalities*

$$\underline{\gamma}_C C_C \;\leq\; S_C \;\leq\; \overline{\gamma}_C C_C \quad \text{and} \quad \underline{\gamma}_I C_I \;\leq\; K_I \;\leq\; \overline{\gamma}_I C_I \tag{3.3}$$

with some positive constants $\underline{\gamma}_C$, $\overline{\gamma}_C$, $\underline{\gamma}_I$ and $\overline{\gamma}_I$. Further denote the spectral radius of $S_C^{-1}T_C$ by $\mu = \rho(S_C^{-1}T_C)$. Then the condition number $\kappa(C^{-1}K)$ of the preconditioned system using the ASM-DD preconditioner (3.2) is bounded from above

$$\begin{aligned}\kappa(C^{-1}K) \;\leq\; & \frac{1}{4\underline{\gamma}_C\underline{\gamma}_I} \cdot \left[(1+\mu)\underline{\gamma}_C + \underline{\gamma}_I + \sqrt{((1+\mu)\underline{\gamma}_C + \underline{\gamma}_I)^2 - 4\underline{\gamma}_C\underline{\gamma}_I}\,\right] \\ & \cdot \left[(1+\mu)\overline{\gamma}_C + \overline{\gamma}_I + \sqrt{((1+\mu)\overline{\gamma}_C + \overline{\gamma}_I)^2 - 4\overline{\gamma}_C\overline{\gamma}_I}\,\right]\end{aligned} \tag{3.4}$$

and below

$$\begin{aligned}\kappa(C^{-1}K) \;\geq\; & \frac{1}{4(1+\mu)\overline{\gamma}_C\overline{\gamma}_I} \cdot \left[(1+\mu)(\underline{\gamma}_C + \underline{\gamma}_I) + \sqrt{(1+\mu)^2(\underline{\gamma}_C + \underline{\gamma}_I)^2 - 4(1+\mu)\underline{\gamma}_C\underline{\gamma}_I}\,\right] \\ & \cdot \left[(1+\mu)\overline{\gamma}_C + \overline{\gamma}_I + \sqrt{((1+\mu)\overline{\gamma}_C + \overline{\gamma}_I)^2 - 4\overline{\gamma}_C\overline{\gamma}_I}\,\right] .\end{aligned} \tag{3.5}$$

The non singular matrix B_I , which is not supposed to be neither symmetric nor positive definite, can be interpreted as a part of the basis transformation matrix

$$\widetilde{V} \;=\; \left(\widetilde{V}_C\, \widetilde{V}_I\right) \;=\; \begin{pmatrix} I_C & O \\ -B_I^{-1}K_{IC} & I_I \end{pmatrix} \tag{3.6}$$

transforming the nodal basis Φ in the approximate discrete harmonic basis $\widetilde{\Phi} = \Phi\widetilde{V}$ and μ can be defined by the angle between $\widetilde{\mathbb{V}}_C = \operatorname{span}(\Phi\widetilde{\mathbb{V}}_C)$ and $\widetilde{\mathbb{V}}_I = \mathbb{V}_I = \operatorname{span}(\Phi\widetilde{V}_I)$, [9, 10]. More precisely,

$$\sqrt{\frac{\mu}{1+\mu}} \;=\; \cos \sphericalangle \left(\widetilde{\mathbb{V}}_C, \widetilde{\mathbb{V}}_I\right) . \tag{3.7}$$

The remaining paper follows the alternative interpretation that the function $\Phi\left(\begin{smallmatrix} \underline{u}_C \\ -B_I^{-1}K_{IC}\underline{u}_C \end{smallmatrix}\right)$ is an extension of the function $u_C = \Phi V_C \underline{u}_C$ on Γ_C into the interior.

Nepomnyaschikh proved in [20] that one can construct a norm–preserving extension $\Phi\left(\begin{smallmatrix} \underline{u}_C \\ E_{IC}\underline{u}_C \end{smallmatrix}\right)$ of u_C such that the inequality

$$\left\| \Phi \begin{pmatrix} \underline{u}_C \\ E_{IC}\underline{u}_C \end{pmatrix} \right\|_{H^1(\Omega)} \;\leq\; c_E \;\|\, u_C \,\|_{H^{\frac{1}{2}}(\Gamma_C)} \tag{3.8}$$

holds for all $u_C = \Phi V_C \underline{u}_C$ and $\underline{u}_C \in \mathbb{R}^{N_C}$, where $E_{IC} :\; \mathbb{R}^{N_C} \to \mathbb{R}^{N_I}$ denotes the corresponding extension operator and c_E is an h–independent positive constant. Replacing $-B_I^{-1}K_{IC}\underline{u}_C$ by $E_{IC}\underline{u}_C$, we obtain an h–independent bound $1+\mu = \widetilde{c}_E^{\,2}$, where $\widetilde{c}_E$ is defined by c_E and by the norm equivalence constants between the $\mathbb{H}^1(\Omega)$– and the K–energy norm on the one hand and the $\mathbb{H}^{\frac{1}{2}}(\Gamma_C)$ and the S_C–energy norm on the other hand. In [16], the splitting $\mathbb{V} = \widetilde{\mathbb{V}}_C + \widetilde{\mathbb{V}}_I$, with $\widetilde{\mathbb{V}}_C = \operatorname{span}(\Phi\widetilde{\mathbb{V}}_C)$, $\widetilde{\mathbb{V}}_I = \mathbb{V}_I = \operatorname{span}(\Phi\widetilde{V}_I)$, and

$$\widetilde{V}_C \;=\; \begin{pmatrix} I_C \\ E_{IC} \end{pmatrix}, \tag{3.9}$$

was used in order to derive asymptotically optimal ASM–Preconditioners.

Let us return to some algorithmical aspects for solving the preconditioning equation (3.1) and to some modifications of the basic preconditioning algorithm (Algorithm 1) given by the ASM–DD preconditioner (3.2). The basic preconditioning Algorithm 1 can be rewritten in the form $\underline{w} = C^{-1}\underline{r}$ as

Algorithm 1 : The ASM-DD Preconditioner [9, 10]
$\underline{\mathbf{w}}_C = C_C^{-1} \sum_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} - K_{CI,i} B_{I,i}^{-T} \underline{r}_{I,i}\right)$ $\underline{\mathbf{w}}_{I,i} = C_{I,i}^{-1}\underline{r}_{I,i} - B_{I,i}^{-1} K_{IC,i}\underline{\mathbf{w}}_{C,i} \qquad ; \; i = 1, 2, \ldots, p$
Determined by the user : $C_C =\Gamma$, $C_I =\Gamma$, $B_I =\Gamma$

where $A_i = \left(\begin{smallmatrix} A_{C,i} & A_{CI,i} \\ A_{IC,i} & A_{I,i} \end{smallmatrix}\right)$ denotes the subdomain connectivity matrix which is used for a convenient notation only. The subdomain f.e. assembling process which is connected with nearest neighbour communication stands behind this notation [18, 9, 10].

Other DD-preconditioners and modifications of Algorithm 1 can be found in [11, 8]. The symmetric MSM(Multiplicative Schwarz Method)-DD preconditioner can be rewritten as an ASM-DD preconditioner (3.2) with a very special structure of the components B_I, C_I (see [8]). Therefore this symmetric MSM-DD preconditioner can be parallelized in the same way as the ASM one. The use of that symmetric MSM-DD preconditioner together with an extension procedure proposed in [12] is out of scope of this work.

4 The Hierarchical Extension Operator

This section follows closely the joint work of HLMN [12] describing some basic statements for the following section.

Let us now construct a very simple and easily implementable, almost norm preserving extension procedure $E_{IC} : \mathbb{R}^{N_C} \to \mathbb{R}^{N_I}$ on the basis of the hierarchical transformation technique proposed by Yserentant in [27] for preconditioning finite element equations. It is obviously sufficient to construct the extension for each subdomain $\overline{\Omega}_i$ separately. So, we omit the subindex i and describe the extension $E_{\overline{\Omega}\Gamma}$ of a piecewise linear function given on the boundary Γ $(= \Gamma_{C,i})$ of Ω to some piecewise linear function on $\overline{\Omega}$ $(= \overline{\Omega}_i)$. Emphasize that throughout this section Ω and Γ play the rule of the subdomain Ω_i and of the subdomain boundary $\Gamma_i = \partial\Omega_i$, respectively. For simplicity, we suppose that Ω is a polygonal plane domain and, of course, bounded.

Now, starting with a coarse grid triangulation $\overline{\Omega}_0^h$ we are forced to introduce explicitly a sequence of finer and finer triangulation

$$\overline{\Omega}_k^h = \cup_{i=1}^{M_k} \overline{\tau}_i^{(k)}, \quad k = 0, 1, \ldots, \ell,$$

where the triangles $\tau_i^{(k+1)}$ are generated by subdividing triangles $\tau_i^{(k)}$ into four congruent subtriangles connecting the midpoints of the edges (red subdivision) [1]. In the following, the indices "ℓ" and "0" correspond to fine and coarse level quantities, respectively. Denote by $x_i^{(k)}$, ${\scriptstyle i=1,2,\ldots,L_k}$ the nodes of the triangulation Ω_k^h.

Introduce now the spaces $\mathbb{W}_k$ and $\mathbb{V}_k$ of finite element functions. The space $\mathbb{W}_k$ consists of real-valued functions which are continuous on Ω and linear on the triangles in Ω_k^h. The space $\mathbb{V}_k$ is the space of traces on Γ of functions from $\mathbb{W}_k$:

$$\mathbb{V}_k = \{\varphi^h : \quad \varphi^h = u^h|_\Gamma, \text{ with } u^h \in \mathbb{W}_k\}.$$

We will consider the usual norms of the Sobolev spaces $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^{\frac{1}{2}}(\Gamma)$, respectively, in the finite element subspaces $\mathbb{W}_\ell$ and $\mathbb{V}_\ell$, too.

Our goal is the construction of some norm-preserving explicit extension operator $E_{\overline{\Omega}\Gamma}$ from $\mathbb{V}_\ell$ into $\mathbb{W}_\ell$:

$$E_{\overline{\Omega}\Gamma} : \mathbb{V}_\ell \to \mathbb{W}_\ell.$$

As was mentioned above, the basis of our construction is the hierarchical decomposition of the space V_ℓ which was suggested by Yserentant [27, 28]:

$$\varphi^h = I_0\varphi^h + \sum_{k=1}^{l}(I_k - I_{k-1})\varphi^h \quad \forall \varphi^h \in \mathbb{V}_\ell,$$

where $I_k\varphi^h \in \mathbb{V}_k$ denotes the finite element interpolant. Introduce the notation

$$\begin{aligned} \varphi_0^h &= I_0\varphi^h, \\ \varphi_k^h &= (I_k - I_{k-1})\varphi^h, \quad k = 1, \dots, \ell, \end{aligned}$$

and define the extension $u_k^h \in \mathbb{W}_k \setminus \mathbb{W}_{k-1}$ of the function φ_k^h in the following way:

$$u_0^h(x_i^{(0)}) = \begin{cases} \varphi_0^h(x_i^{(0)}) & , x_i^{(0)} \in \Gamma, \\ \overline{\varphi} & , x_i^{(0)} \notin \Gamma, \end{cases} \tag{4.1a}$$

$$u_k^h(x_i^{(k)}) = \begin{cases} \varphi_k^h(x_i^{(k)}) & , x_i^{(k)} \in \Gamma, \\ 0 & , x_i^{(k)} \notin \Gamma. \end{cases} \quad {\scriptstyle k=\overline{1,\ell}} \tag{4.1b}$$

Here $\overline{\varphi}$ is, for instance, the mean value of the function φ_0^h on Γ:

$$\overline{\varphi} = \tfrac{1}{N_0} \sum_{i=1}^{N_0} \varphi_0^h(x_i^{(0)}), \tag{4.2}$$

where N_0 denotes the number of nodes $x_i^{(0)}$ on Γ. We assume that the nodes $x_i^{(k)}$ are enumerated at first on Γ (in the natural order) and then inside Ω. Set

$$E_{\overline{\Omega}\Gamma}\varphi^h = u^h \equiv u_0^h + u_1^h + \cdots + u_\ell^h. \tag{4.3}$$

and supposing that $\mathbb{W}$ is a piecewise-linear finite element space on Ω_h the estimate

$$\|E_{\overline{\Omega}\Gamma}\varphi^h\|_{H^1(\Omega)} \le c \cdot \ell \cdot \|\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)} , \tag{4.4}$$

holds with some positive constant c, independent of h, which was proved for the extension operator (4.3) in [12].

Changing into the hierarchical basis $\widehat{\Phi} = \Phi\,\widehat{V}$ the basis transformation $\widehat{V} = \left(\begin{smallmatrix} Q_C & 0 \\ Q_{IC} & Q_I \end{smallmatrix}\right)$ will be used, where Q_C, Q_I and Q_{IC} are the matrix representations of the hierarchical interpolation from the coarsest to the finest grid. So we can write the hierarchical extension operator $E_{IC,\ell}$ on the finest grid as

$$E_{IC,\ell} := [\, Q_I\, B_{IC,0} + Q_{IC} \,]\; Q_C^{-1} , \tag{4.5}$$

where the harmonic extension on the coarsest grid $B_{IC,0}$ has to be chosen with respect to (4.2).

5 The Hierarchical Extension Operator plus smoothing

Using the same hierarchical basis as in Section 4 we will construct a norm preserving extension procedure $\widehat{E}_{IC} : \mathbb{R}^{N_C} \to \mathbb{R}^{N_I}$ again via a norm-preserving explicit extension operator $\widehat{E}_{\overline{\Omega}\Gamma}$ from $\mathbb{V}_\ell$ into $\mathbb{W}_\ell$:

$$\widehat{E}_{\overline{\Omega}\Gamma} : \mathbb{V}_\ell \to \mathbb{W}_\ell,$$

including additional linear smoothing operators $S_k : \mathbb{W}_k \to \mathbb{W}_k$, realized via the discrete smoothing operator (= iteration operator of the affine linear smoothing iteration [14]) $S_{I,k} : \mathbb{R}^{N_{I,k}} \to \mathbb{R}^{N_{I,k}}$, $k{=}\overline{1,\ell}$ fulfilling

$$\begin{array}{lll} \| S_{I,k}\, \underline{v}^h \|_{K_I} \le \overline{\varrho}_k \| \underline{v}^h \|_{K_I} & \forall \Phi_I \underline{v}^h \in \mathbb{W}_k \setminus \mathbb{W}_{k-1} \overset{\text{high frequencies}}{\Longleftrightarrow} & \| S_{I,k_{high}} \|_{K_I} \le \overline{\varrho}_k < 1 \\ \| S_{I,k}\, \underline{v}^h \|_{K_I} \le \| \underline{v}^h \|_{K_I} & \forall \Phi_I \underline{v}^h \in \mathbb{W}_{k-1}\,,\, \mathbb{W}_k \overset{\text{low frequencies}}{\Longleftrightarrow} & \| S_{I,k_{low}} \|_{K_I} \le 1 \quad , \end{array} \tag{5.1}$$

where the number of smoothing sweeps is denoted by ν_k. Additionally we require smoothing factors $\overline{\varrho}_k$ independent of h and ℓ. Please note the operators S_k are defined via the discrete representation $S_{I,k}$ in the interior of the domain which is extended to 0 on the boundary.

From the mesh theorem for mesh functions [19] there exists the exact harmonic extension $w_k^h \in \mathbb{W}_k$ so that for any $I_k\varphi^h \in \mathbb{V}_k$

$$\begin{aligned} w_k^h|_\Gamma &= I_k\varphi^h, \\ \|w_k^h\|_{H^1(\Omega)} &\le c_1 \|I_k\varphi^h\|_{H^{\frac{1}{2}}(\Gamma)} \qquad k{=}\overline{0,\ell} \end{aligned} \tag{5.2}$$

is valid with some h-independent positive constant c_1.

The exact harmonic extension $w_k^h \in \mathbb{W}_k$ $(k{=}\overline{1,\ell})$ can be split into the direct sum of the exact harmonic extension on the next coarser grid $w_{k-1}^h \in \mathbb{W}_{k-1}$ and the high frequency part $\widetilde{w}_k^h \in \mathbb{W}_k \setminus \mathbb{W}_{k-1}$

$$w_k^h \;=\; w_{k-1}^h \oplus \widetilde{w}_k^h \quad . \tag{5.3}$$

Lemma 5.1. *Let u_k^h the simple extension of φ_k^h defined in (4.1a) and $\widetilde{w}_k^h$ the high frequency part of the exact harmonic extension of φ_k^h, see (5.3). Then there exists an h-independent constant c_2 so that the following estimate is valid.*

$$\| u_k^h - \widetilde{w}_k^h \|^2_{H^1(\Omega)} \;\le\; c_2 \;\| \varphi \|^2_{H^{1/2}(\Gamma)} \tag{5.4}$$

Proof. First, we have to show the relation

$$\| u_k^h \|^2_{H^1(\Omega)} \;\le\; 2^k\, C_3 \;\| \varphi_k^h \|^2_{L_2(\Gamma)} \quad . \tag{5.5}$$

In the following we omit the superscript h and denote the bilinear basis function ψ from (2.2) belonging to the grid node $x_i^{(k)}$ by ψ_i^k. The traces of ψ_i^k $(i{=}i1,i2)$ on an edge $\gamma_j = \{x | x := x_{i1}^{(k)} + \xi(x_{i2}^{(k)} - x_{i1}^{(k)})\,;\, \xi \in [0, h_k = 2^{-k}]\}$ of the boundary are $\psi_{i1}^k|_{\gamma_j} = \frac{\xi}{h_k}$ and $\psi_{i2}^k|_{\gamma_j} = \frac{h_k - \xi}{h_k}$. Let us denote the finite elements near the boundary Γ by δ_Γ, the nodes indices belonging to a linear triangular element δ_Γ by ω_δ and the maximal number of elements a node belongs to by n_{max}. Due to the fact that u_k is only nonzero in the nodes $x_i^{(k)}$ belonging to the boundary we estimate the left side of the inequality above :

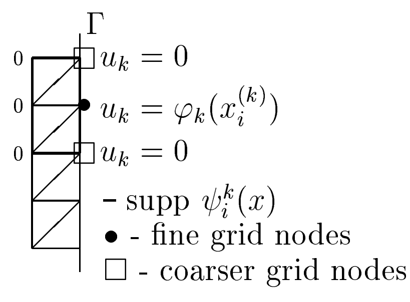


Figure 1: Region near $x_i^{(k)}$

$$
\begin{aligned}
\| u_k \|^2_{H^1(\Omega)} &= \int\limits_\Omega \left(|u_k|^2 + |\nabla u_k|^2\right) ds \;=\; \sum_{\delta_\Gamma} \int\limits_{\delta_\Gamma} \left(|u_k|^2 + |\nabla u_k|^2\right) ds \\
&= \sum_{\delta_\Gamma} \int\limits_{\delta_\Gamma} \left(\Big| \sum_{i\in\omega_\delta} u_k(x_i^{(k)}) \cdot \psi_i^k \Big|^2 + \Big| \sum_{i\in\omega_\delta} u_k(x_i^{(k)}) \cdot \nabla\psi_i^k \Big|^2 \right) ds \\
&\leq \sum_{\delta_\Gamma} \sum_{i\in\omega_\delta} |u_k(x_i^{(k)})|^2 \cdot \underbrace{\sum_{i\in\omega_\delta} \| \psi_i^k \|_{H^1(\delta_\Gamma)}}_{\leq c} \;\leq\; c \sum_{\delta_\Gamma} \sum_{i\in\omega_\delta\,,\, x_i^{(k)}\in\Gamma} \left(\varphi_k(x_i^{(k)})\right)^2 \\
&\leq n_{max}\, c \sum_{x_i^{(k)}\in\Gamma} \left(\varphi_k(x_i^{(k)})\right)^2 \quad .
\end{aligned}
$$

Compared with the lower bound of the right hand side in (5.5)

$$
\begin{aligned}
\| \varphi_k \|^2_{L_2(\Gamma)} = \int\limits_\Gamma (\varphi_k)^2 ds &= \sum_{\gamma_j} \int\limits_{\gamma_j} \left(\varphi_k(x_{i1}^{(k)}) \cdot \psi_{i1}^k|_{\gamma_j} + \varphi_k(x_{i2}^{(k)}) \cdot \psi_{i2}^k|_{\gamma_j} \right)^2 d\xi \\
&= \sum_{\gamma_j} \int\limits_0^{h_k} \left(\varphi_k(x_{i1}^{(k)}) \tfrac{\xi}{h_k} + \varphi_k(x_{i2}^{(k)}) \tfrac{h_k-\xi}{h_k} \right)^2 d\xi \\
&= \tfrac{h_k}{3} \sum_{\gamma_j} \left(\varphi_k^2(x_{i1}^{(k)}) + \varphi_k(x_{i1}^{(k)}) \varphi_k(x_{i2}^{(k)}) + \varphi_k^2(x_{i2}^{(k)}) \right) \\
&\geq \tfrac{h_k}{3} \sum_{\gamma_j} \tfrac{1}{2} \left(\varphi_k^2(x_{i1}^{(k)}) + \varphi_k^2(x_{i2}^{(k)}) \right) \;=\; \tfrac{1}{3} h_k \sum_{x_i^{(k)}\in\Gamma} \left(\varphi_k(x_i^{(k)})\right)^2 \quad .
\end{aligned}
$$

the h-independent constant C_3 will be achieved. In case of an equidistant triangular mesh we get the bounded constant $C_3 = 3 \cdot 3 \cdot (2 + \frac{7}{12} h_k^2) \leq \frac{93}{4}$.
According to Nepomnyaschikh [19] there exists a positive constant C_2 so that

$$2^k \| \varphi_k^h \|^2_{L_2(\Gamma)} \;\leq\; C_2 \, \| \varphi^h \|^2_{H^{1/2}(\Gamma)}$$

holds. Together with (5.5) the above inequality leads directly to the statement

$$\| u_k^h \|^2_{H^1(\Omega)} \;\leq\; C_2 C_3 \, \| \varphi^h \|^2_{H^{1/2}(\Gamma)} \quad . \tag{5.6}$$

Using inequality (5.6) and the obvious relation $\| \widetilde{w}_k^h \|_{H^1(\Omega)} \leq \| u_k^h \|_{H^1(\Omega)}$ for the exact harmonic extension $\widetilde{w}_k^h$ the remaining proof is trivial

$$\| u_k^h - \widetilde{w}_k^h \|^2_{H^1(\Omega)} \leq \left(\| u_k^h \|_{H^1(\Omega)} + \| \widetilde{w}_k^h \|_{H^1(\Omega)} \right)^2 \;\leq\; 4 \, \| u_k^h \|^2_{H^1(\Omega)} \leq\; 4\, C_2\, C_3 \, \| \varphi \|^2_{H^{1/2}(\Gamma)} \quad . \tag{5.7}$$

■

Now we are in the position to define the extension $\widehat{u}_k^h \in \mathbb{W}_k$ of the function $I_k \varphi^h$ in the following recursive way:

$$\widehat{u}_0^h \;=\; w_0^h \tag{5.8a}$$

$$\widehat{u}_k^h \;=\; S_k^{\nu_k} \left(u_k^h + \widehat{u}_{k-1}^h \right) \,+\, (I - S_k^{\nu_k})\, w_k^h \qquad {\scriptstyle k=\overline{1,\ell}} \quad , \tag{5.8b}$$

where u_k^h $({\scriptstyle k\geq 1})$ represents the extension defined in (4.1b). Equation (5.8a) requires a coarse grid solver for obtaining the discrete harmonic extension on the lowest level. So we achieve at the hierarchical extension operator plus smoothing

$$\widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \;=\; \widehat{u}_\ell^h \tag{5.9}$$

Please note that in contradiction to (4.3) the new extension is defined recursively and no longer as a sum.

THEOREM 5.2. *Applying on each level k $({\scriptstyle k=\overline{1,\ell}})$ ν_k times the smoothing operator S_k defined in (5.1) then there exists a positive constant c, independent of h, such that*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq c \left(1 + \sqrt{\ell}\cdot \sqrt{\sum_{k=1}^{\ell} \overline{\varrho}_k^{2\nu_k}} \right) \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)}$$

holds, where the hierarchical extension operator plus smoothing $\widehat{E}_{\overline{\Omega}\Gamma}$ was defined in (5.9).

Proof. We have to estimate

$$\begin{aligned}
\| \widehat{u}_\ell^h \|_{H^1(\Omega)} &= \| \widehat{u}_\ell^h - w_\ell^h + w_\ell^h \|_{H^1(\Omega)} \leq \| \widehat{u}_\ell^h - w_\ell^h \|_{H^1(\Omega)} + \| w_\ell^h \|_{H^1(\Omega)} \\
&\overset{(5.8b)}{\leq} \| S_\ell^{\nu_\ell} \left(u_\ell^h + \widehat{u}_{\ell-1}^h - w_\ell^h \right) \|_{H^1(\Omega)} + \| w_\ell^h \|_{H^1(\Omega)} \\
&\overset{(5.3)}{=} \| S_{\ell_{high}}^{\nu_\ell} \left(u_\ell^h - \widetilde{w}_\ell^h \right) + S_{\ell_{low}}^{\nu_\ell} \left(\widehat{u}_{\ell-1}^h - w_{\ell-1}^h \right) \|_{H^1(\Omega)} + \| w_\ell^h \|_{H^1(\Omega)} \\
&\vdots \\
&\overset{(5.3)}{=} \left\| \sum_{k=1}^{\ell} \prod_{j=k}^{\ell} S_j^{\nu_j} \cdot \left(u_k^h - \widetilde{w}_k^h \right) + \prod_{k=1}^{\ell} S_k^{\nu_k} \cdot \left(\widehat{u}_0^h - w_0^h \right) \right\|_{H^1(\Omega)} + \| w_\ell^h \|_{H^1(\Omega)} \\
&\leq \left\| \sum_{k=1}^{\ell} \prod_{j=k}^{\ell} S_j^{\nu_j} \cdot \left(u_k^h - \widetilde{w}_k^h \right) \right\|_{H^1(\Omega)} + \left\| \prod_{k=1}^{\ell} S_k^{\nu_k} \cdot \left(\widehat{u}_0^h - w_0^h \right) \right\|_{H^1(\Omega)} + \| w_\ell^h \|_{H^1(\Omega)}
\end{aligned}$$

<u>First term :</u>
Using the Cauchy-inequality, and the obvious relation $2ab \leq a^2 + b^2$ we are able to estimate for

an arbitrary function v^h (defined in the same way as u^h)

$$
\begin{aligned}
\| v^h - v_0^h \|^2_{H^1(\Omega)} &= \| v_1^h + \ldots + v_\ell^h \|^2_{H^1(\Omega)} = \left(\sum_{k=1}^{\ell} v_k^h \,,\, \sum_{k=1}^{\ell} v_k^h \right)_{L_2(\Omega)} + \left(\sum_{k=1}^{\ell} \nabla v_k^h \,,\, \sum_{k=1}^{\ell} \nabla v_k^h \right)_{L_2(\Omega)} \\
&= \sum_{k=1}^{\ell} \left[\left(\nabla v_k^h \,,\, \nabla v_k^h \right)_{L_2(\Omega)} + \left(v_k^h \,,\, v_k^h \right)_{L_2(\Omega)} \right] \\
&\qquad + 2 \sum_{k=1}^{\ell-1} \sum_{j=k+1}^{\ell} \left[\left(\nabla v_k^h \,,\, \nabla v_j^h \right)_{L_2(\Omega)} + \left(v_k^h \,,\, v_j^h \right)_{L_2(\Omega)} \right] \\
&\le \sum_{k=1}^{\ell} \| v_k^h \|^2_{H^1(\Omega)} + 2 \sum_{k=1}^{\ell-1} \sum_{j=k+1}^{\ell} \underbrace{\| v_k^h \|_{H^1(\Omega)} \| v_j^h \|_{H^1(\Omega)}}_{\le 1/2 \left(\|v_k^h\|^2_{H^1(\Omega)} + \|v_j^h\|^2_{H^1(\Omega)} \right)} \\
&\le \sum_{k=1}^{\ell} \| v_k^h \|^2_{H^1(\Omega)} + \sum_{k=1}^{\ell-1} \sum_{j=k+1}^{\ell} \| v_k^h \|^2_{H^1(\Omega)} + \sum_{k=1}^{\ell-1} \sum_{j=k+1}^{\ell} \| v_j^h \|^2_{H^1(\Omega)} \\
&= \sum_{k=1}^{\ell} \| v_k^h \|^2_{H^1(\Omega)} + \sum_{k=1}^{\ell-1} (\ell - k) \cdot \| v_k^h \|^2_{H^1(\Omega)} + \sum_{j=2}^{\ell} (j-1) \cdot \| v_j^h \|^2_{H^1(\Omega)} \\
&= \ell \sum_{k=1}^{\ell} \| v_k^h \|^2_{H^1(\Omega)}
\end{aligned}
$$

Now, substituting $v_k^h = \prod_{j=k}^{\ell} S_j^{\nu_j} \left(u_k^h - \widetilde{w}_k^h \right)$, taking into account the norm equivalence

$$
\underline{c}^2 \, \| \Phi \underline{v}^h \|^2_{H^1(\Omega)} \le \| \underline{v} \|^2_K = a\left(\Phi \underline{v}, \Phi \underline{v} \right) \le \overline{c}^2 \, \| \Phi \underline{v}^h \|^2_{H^1(\Omega)} \qquad \forall \underline{v} \in \mathbb{R}^N \tag{5.10}
$$

and using Lemma 5.1 we are in the position to estimate

$$
\begin{aligned}
\left\| \sum_{k=1}^{\ell} \prod_{j=k}^{\ell} S_j^{\nu_j} (u_k^h - \widetilde{w}_k^h) \right\|^2_{H^1(\Omega)} &\le \ell \sum_{k=1}^{\ell} \left\| \prod_{j=k}^{\ell} S_j^{\nu_j} (u_k^h - \widetilde{w}_k^h) \right\|^2_{H^1(\Omega)} \le \ell / \underline{c}^2 \sum_{k=1}^{\ell} \left\| \prod_{j=k}^{\ell} S_{I,j}^{\nu_j} (\underline{u}_k^h - \underline{\widetilde{w}}_k^h) \right\|^2_{K_I} \\
&\overset{(5.1)}{\le} \ell / \underline{c}^2 \sum_{k=1}^{\ell} \overline{\varrho}_k^{2\nu_k} \cdot \| \underline{u}_k^h - \underline{\widetilde{w}}_k^h) \|^2_{K_I} \le \ell \frac{\overline{c}^2}{\underline{c}^2} \sum_{k=1}^{\ell} \overline{\varrho}_k^{2\nu_k} \cdot \| u_k^h - \widetilde{w}_k^h \|^2_{H^1(\Omega)} \\
&\overset{(5.4)}{\le} \ell c_2 \frac{\overline{c}^2}{\underline{c}^2} \, \| \varphi^h \|^2_{H^{1/2}(\Gamma)} \sum_{k=1}^{\ell} \overline{\varrho}_k^{2\nu_k} \, .
\end{aligned}
$$

<u>Second term :</u> This term vanishes because of (5.8a).
<u>Third term :</u> The relation (5.2) with $k = \ell$ and $I_\ell \varphi^h \equiv \varphi^h$ is used.
Combining the estimates for the three terms results in

$$
\| \widehat{u}_\ell^h \|_{H^1(\Omega)} \le \left(c_1 + \sqrt{c_2} \, \frac{\overline{c}}{\underline{c}} \, \sqrt{\ell} \cdot \sqrt{\sum_{k=1}^{\ell} \overline{\varrho}_k^{2\nu_k}} \right) \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)}
$$

■

REMARK 5.3. *The estimate for the hierarchical extension plus smoothing in Theorem 5.2 includes the following special cases :*

1. *No smoothing, i.e.* $\varrho_k \equiv 1 \quad \forall k \geq 1$ *: (see estimate (4.4))*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c\,(\ell+1)\cdot \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)}$$

2. *On each level the exact harmonic extension, i.e.* $\varrho_k \equiv 0 \quad \forall k \geq 1$ *: (see estimate (3.8))*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c_1 \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)}$$

REMARK 5.4 (slash-cycle). *Assuming* $\overline{\varrho}_k \leq \overline{\varrho} < 1$ *and* $\nu_k = \nu$ $(k=\overline{1,\ell})$ *the estimate of Theorem 5.2 changes into*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c\,[\,1 \,+\, \ell\cdot\overline{\varrho}^{\nu}\,]\; \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)} \quad , \tag{5.11}$$

i.e., it is sufficient to perform $\mathcal{O}(\ln \ell) = \mathcal{O}(\ln(\ln h))$ *smoothing sweeps of* $S_{I,k}$ *on each level to achieve an* h- *and* ℓ-*independent bound for* $\widehat{E}_{\overline{\Omega}\Gamma}$.

REMARK 5.5 (generalized slash-cycle). *Assuming* $\overline{\varrho}_k \leq \overline{\varrho} < 1$ *and* $\nu_{k-1} := 2\cdot\nu_k$ $(k=\overline{1,\ell})$, $\nu_\ell := \nu$ *the estimate of Theorem 5.2 changes into*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c\left[1 \,+\, \sqrt{\ell}\cdot\overline{\varrho}^{\nu}\cdot\sqrt{\frac{1-\overline{\varrho}^{2\nu\ell}}{1-\overline{\varrho}^{2\nu}}}\right]\; \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)} \quad . \tag{5.12}$$

The rather rough estimate $\sqrt{(1-\overline{\varrho}^{2\nu\ell})/(1-\overline{\varrho}^{2\nu})} < \sqrt{1/(1-\overline{\varrho}^{2\nu})}$ *leads to the conclusion that it is sufficient to start with* $\mathcal{O}(\ln\sqrt{\ell})$ *smoothing sweeps of* $S_{I,\ell}$ *on the finest level to achieve an* h- *and* ℓ-*independent bound for* $\widehat{E}_{\overline{\Omega}\Gamma}$, *i.e., at most half of the smoothing sweeps of the slash-cycle on the finest grid are needed.*

REMARK 5.6 (sharper estimate). *Sharpen the requirements for the low frequencies of the smoother* $S_{I,k}$ *in definition (5.1) into* $\| S_{I,k_{low}} \|_{K_I} \leq \sigma_k \leq 1$ *the estimate in Theorem 5.2 changes into*

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c\left(1 \,+\, \sqrt{\ell}\cdot\sqrt{\sum_{k=1}^{\ell}\overline{\varrho}_k^{2\nu_k}\prod_{j=k+1}^{\ell}\sigma_j^{2\nu_j}}\right)\; \| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)}$$

Performing just on the finest grid ℓ *the iteration operator* $S_{I,k}$, *i.e.* $\nu_k = 0 \quad \forall k = \overline{1,\ell-1}$ *and* $\nu_\ell = \nu$, *the term under the square root changes to* $(\ell - 1 + \overline{\varrho}^{2\nu})\sigma^{2\nu}$. *Let* $\max\{\overline{\varrho},\sigma\} \leq \eta \leq 1$ *we achieve the same estimate as in [12] for using the hierarchical extension (4.3) as an initial guess for the following iteration procedure* $M_I \equiv S_{I,k}$ *with* $\| M_I \|_{K_I} \leq \eta$

$$\| \widehat{E}_{\overline{\Omega}\Gamma}\varphi^h \|_{H^1(\Omega)} \leq \; c\,(\,1 \,+\, \ell\cdot\eta^{\nu})\;\| \varphi^h \|_{H^{\frac{1}{2}}(\Gamma)} \quad .$$

Now we change into the hierarchical basis $\widehat{\Phi} = \Phi\widehat{V}$ and use the basis transformation $\widehat{V} = \left(\begin{smallmatrix} Q_C & 0 \\ Q_{IC} & Q_I \end{smallmatrix}\right)$, with $Q_C = Q_{C,\ell}\,Q_{C,\ell-1}\cdots Q_{C,1}$, $Q_{IC} = Q_{IC,\ell}\,Q_{IC,\ell-1}\cdots Q_{IC,1}$ and

$Q_I = Q_{I,\ell}\, Q_{I,\ell-1} \cdots Q_{I,1}$ where $Q_{C,k}, Q_{IC,k}, Q_{I,k}$ are the matrix representations of the hierarchical interpolation from the level $k-1$ to the level k ($k=\overline{1,\ell}$). According to (5.8a) and (5.8b) we define the hierarchical extension operator $\widehat{E}_{IC,\ell}$ recursively as

$$\begin{aligned} \widehat{E}_{IC,1} &= S_{I,1}^{\nu_1} \left[Q_{I,1}\, K_{I,0}^{-1} (-K_{IC,0}) + Q_{IC,1} \right] Q_{C,1}^{-1} \,, \\ \widehat{E}_{IC,k} &= S_{I,k}^{\nu_k} \left[Q_{I,k}\, \widehat{E}_{IC,k-1} + Q_{IC,k} \right] Q_{C,k}^{-1} \qquad \forall k = \overline{2,\ell} \quad , \end{aligned} \tag{5.13}$$

where on the coarsest grid the exact discrete harmonic extension ($-K_{I,0}^{-1} K_{IC,0}$) was chosen.

COROLLARY 5.7 (Fixed number of smoothing sweeps). *Assume that we use preconditioners C_C and C_I so that the spectral equivalence inequalities (3.3) are fulfilled with h-independent constants $\underline{\gamma}_C$, $\overline{\gamma}_C$, $\underline{\gamma}_I$ and $\overline{\gamma}_I$ (e.g. perform a symmetric local multigrid cycle to define $C_I = K_I(I_I - M_I)^{-1}$ with $\| M_I \|_{K_I} \le \eta < 1$, $\eta \ne \eta(h)$). The basis transformation (3.6) is defined via the hierarchical extension procedure $\widehat{E}_{IC} : \mathbb{R}^{N_C} \longrightarrow \mathbb{R}^{N_I}$ described in (5.13). If there exists an h-independent constant $\overline{\varrho} \ne \overline{\varrho}(h)$ bounding the constants $\overline{\varrho}_k$ in (5.1) then for a fixed number of smoothing sweeps with the operators $S_{I,k}$ (5.1) the condition number of the preconditioned system $\kappa(C^{-1}K)$ behaves like*

$$\mathcal{O}(\ell^2) \;\le\; \kappa(C^{-1}K) \;\le\; \mathcal{O}(\ell^4) \;=\; \mathcal{O}(\ln^4(h^{-1})) \quad . \tag{5.14}$$

Proof. The norm equivalence inequalities (5.10) and

$$\underline{c}_C \,\| \Phi_C \underline{v}_C^h \|_{H^{1/2}(\Gamma_C)} \le \| \underline{v}_C \|_{S_C} \le \overline{c}_C \,\| \Phi_C \underline{v}_C^h \|_{H^{1/2}(\Gamma_C)} \qquad \forall \underline{v}_C \in \mathbb{R}^{N_C}$$

together with the estimate (5.11) lead directly to

$$\left\| \begin{pmatrix} \underline{v}_C \\ E_{IC}\underline{v}_C \end{pmatrix} \right\|_K \;\le\; \underbrace{\overline{c}\,\underline{c}_C^{-1}\, c\, (1 + \ell \overline{\varrho}^{\nu})}_{\widehat{c}_E := \widehat{c}_E(\ell,\overline{\varrho},\nu)} \,\| \underline{v}_C \|_{S_C)} \quad .$$

Another simple calculation results in the following equation of the spectral radius $\mu = \varrho(S_C^{-1}T_C)$

$$\begin{aligned} \mu &= \max_{\underline{v}_C \in \mathbb{R}^{N_C}\setminus\{\emptyset\}} \frac{\| \widehat{E}_{IC}\underline{v}_C + K_I^{-1} K_{IC}\underline{v}_C \|_{K_I}^2}{\| \underline{v}_C \|_{S_C}^2} = \max_{\underline{v}_C \in \mathbb{R}^{N_C}\setminus\{\emptyset\}} \frac{\left\| \begin{pmatrix} \underline{v}_C \\ \widehat{E}_{IC}\underline{v}_C \end{pmatrix} - \begin{pmatrix} \underline{v}_C \\ -K_I^{-1} K_{IC}\underline{v}_C \end{pmatrix} \right\|_{K_I}^2}{\| \underline{v}_C \|_{S_C}^2} \\ &= \max_{\underline{v}_C \in \mathbb{R}^{N_C}\setminus\{\emptyset\}} \frac{\left\| \begin{pmatrix} \underline{v}_C \\ \widehat{E}_{IC}\underline{v}_C \end{pmatrix} \right\|_{K_I}^2 - 2 \,\| \underline{v}_C \|_{S_C}^2 + \| \underline{v}_C \|_{S_C}^2}{\| \underline{v}_C \|_{S_C}^2} = \widehat{c}_E^{\,2} - 1 \quad . \end{aligned}$$

Both deductions applied to the upper (3.4) and lower (3.5) bound of the condition number $\kappa(C^{-1}K)$ results in the statement of the corollary. ■

THEOREM 5.8 (Final Result). *Under the assumptions of Corollary 5.7 it is sufficient to perform $\nu = \mathcal{O}(\ln(\ln h^{-1})) = \mathcal{O}(\ln \ell)$ smoothing sweeps with the iteration operators $S_{I,k}$ on each level $k=\overline{1,\ell}$ to achieve an h-independent $\mu = \varrho(S_C^{-1}T_C)$ and hence the condition number of the preconditioned system behaves like $\kappa(C^{-1}K) = \mathcal{O}(1)$.*

Proof. Follows directly from the proof of Corollary 5.7 . ■

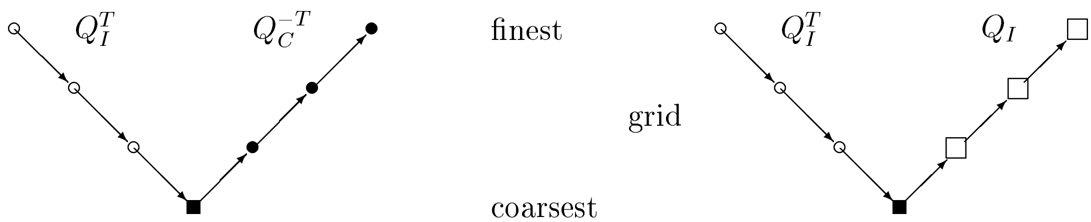
6 An algorithmical improvement of the ASM-DD-preconditioner

Substituting in Algorithm 1 the basis transformation operator $-B_{I,i}^{-1}K_{IC,i}$ by the new hierarchical extension operator $\widehat{E}_{IC,\ell,i}$ (5.13) the algorithm can be rewritten into

Algorithm 1b : The algorithmical improved ASM-DD Preconditioner
$\underline{\mathbf{w}}_C \;=\; C_C^{-1} \sum\limits_{i=1}^{p} A_{C,i}^T \left(\underline{r}_{C,i} + \widehat{E}_{CI,\ell,i}^T \, \underline{r}_{I,i} \right)$
$\underline{\mathbf{w}}_{I,i} \;=\; C_{I,i}^{-1} \, \underline{r}_{I,i} + \widehat{E}_{CI,\ell,i} \, \underline{\mathbf{w}}_{C,i} \qquad ; \; i = 1,2,\ldots,p$

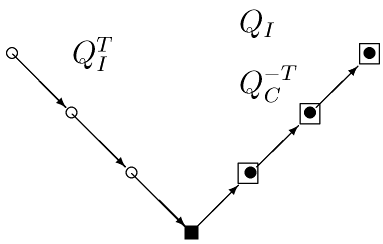
Neglecting the new definition of the basis transformation operator B_I this algorithm seams similar to the old one. Normally we use for the definition of C_I a multigrid method, see [10, 12]. But $\widehat{E}_{IC,\ell}$ is defined in the same recursive way as the V-cycle multigrid operator [13] and possesses the same components (coarse grid solver, smoother, interpolation, restriction). Observing that in the algorithm above the operators $\widehat{E}_{CI,\ell}^T$ and C_I are applied to the same vector $\underline{r}_I$ the two operations

$\underline{v}_C \;=\; \widehat{E}_{IC}^T \, \underline{r}_I \qquad\qquad \underline{v}_I \;=\; C_I^{-1} \, \underline{r}_I$



$\circ \;=\; (S_{I,k}^T)^{\nu_k} \qquad\qquad \square \;=\; (S_{I,k})^{\nu_k}$

may use the same restriction, presmoothing and coarse grid solver for the inner nodes. Thus, both operators can be combined in the implementation.



The resulting V-cycle multigrid operator C_I will be defined by the choice of the components (mostly the smoothing) of the new extension operator $E_{IC,\ell}$ and is positive definite and symmetric. So this premise of Theorem 3.1 is fulfilled automatically.

In comparison to a non sophisticated implementation presmoothing, restriction and coarse grid solver for the multigrid cycle are saved. This leads to the very good behavior of Algorithm 1b with respect to the CPU-time for solving equation (2.5).

7 Numerical results

In this section we arrange the following abbreviations :

E_{IC}	hierarchical extension operator (4.5).
$\widehat{E}_{IC}(s)$	hierarchical extension operator (5.13) with s Gauss-Seidel-smoothing sweeps on each level.
$\widehat{E}_{IC}(s)gen.$	hierarchical extension operator (5.13) with s smoothing sweeps on the highest level and doubling of the Gauss-Seidel-smoothing sweeps on each lower level.
Vsk / Wsk	multigrid V/W-cycle with s (lexicographically forward) pre- and k (lexicographically backward) post-Gauss-Seidel-smoothing sweeps.
$\widehat{V}ss$	multigrid-V-cycle defined by means of the extension operator $\widehat{E}_{IC}(s)$.

All calculations were done on a 16 processor Parsytec POWER-XPLORER with 32 MByte memory per node. The sometimes appearing Algorithm 4 denotes the MSM-DD-preconditioner, see [12] for reference, and is just included for the sake of comparing the new algorithm with the fastest of the older ones. All examples were solved with the preconditioned parallelized cg using algorithm 1 or algorithm 1b as preconditiong step until an accuracy of 10^{-6} was achieved.

Test example : To check the theoretical results given in Section 5 we consider the problem

$$\begin{aligned} -\mathrm{div}(\lambda(x)\,\nabla\,u(x)) &= f(x) \qquad \text{in}\,\Omega\,=\,(0,1)\times(0,0.5) \\ u &= 0 \qquad\quad\;\; \text{on}\,\Gamma=\partial\Omega\;\;, \end{aligned}$$

with the given solution $u(x_1,x_2)=(\sin(i\pi x_1)+\sin(j\pi x_1))\cdot(\sin(i\pi x_2)+\sin(j\pi x_2))$, the coefficient function $\lambda(x_1,x_2)=4.1\,+\,(\sin(i\pi x_1)+\sin(j\pi x_1))\cdot(\sin(i\pi x_2)+\sin(j\pi x_2))$ and the proper right hand side $f(x_1,x_2)$. In the numerical experiments $i=2$ and $j=56$ were chosen.
The domain Ω was subdivided into two squares each mapped onto one processor. Algorithm 1 was used as DD-preconditioner with the Dryja preconditioner [6] as Schur complement preconditioner C_C and an exact solver for C_I. The accuracy was measured in the K-energy norm of the error $\|\,\underline{u}-\underline{u}^{iter}\,\|_K$ so that the condition number of the preconditioned system $\kappa(C^{-1}K)$ could be estimated. The initial grid ($\ell=0$) with the discretization parameter $h_0=\frac{1}{4}$ produced by an automatic mesh generator differs from the often used triangular mesh based on an equidistant rectangular grid.

level	0		1		2		3		4		5		6	
B_{IC}	κ	iter	κ	iter	κ	iter	κ	iter	κ	iter	κ	iter	κ	iter
$\widehat{E}_{IC}(0)$	1.00	1	1.90	7	3.29	13	5.64	16	8.70	20	13.17	25	22.17	33
$\widehat{E}_{IC}(1)$	1.00	1	1.42	5	1.89	7	2.47	9	3.10	11	4.10	13	5.49	15
$\widehat{E}_{IC}(2)$	1.00	1	1.29	5	1.64	7	1.98	8	2.30	9	2.71	10	3.69	12
$\widehat{E}_{IC}(3)$	1.00	1	1.25	5	1.38	5	1.74	7	1.91	7	2.30	9	2.95	10
$\widehat{E}_{IC}(4)$	1.00	1	1.20	4	1.35	5	1.60	6	1.75	7	2.03	8	2.58	9
$\widehat{E}_{IC}(4)$	1.00	1	1.20	4	1.35	5	1.60	6	1.75	7	2.03	8	2.58	9
$3W33\cdot K_{IC}$	1.00	1	1.00	2	1.00	2	1.00	2	1.00	2	1.00	2	1.00	2

Table 1: $\kappa(C^{-1}K)$ and # cg-iterations for the test example using 2 processors, Algorithm 1

The last row in Table 1 indicates that all spectral equivalence constants $\underline{\gamma}_C,\overline{\gamma}_C,\underline{\gamma}_I,\overline{\gamma}_I$ in (3.3) are equal 1. So, the estimates (3.4) and (3.5) simplify and we obtain bounds for $\mu=\varrho(S_C^{-1}T_C)$ depending on the condition number κ :

$$\sqrt{\kappa}+\tfrac{1}{\sqrt{\kappa}}\;\;\leq\;\;\mu\;\;\leq\;\;\kappa+\tfrac{1}{\kappa} \qquad .$$

Following the proof of Corollary 5.7 and taking into account the observation that $\kappa(C^{-1}K) = 1$ for $\ell = 0$ the estimates

$$\sqrt{\sqrt{\kappa} + \tfrac{1}{\sqrt{\kappa}} - 1} \;\leq\; \widehat{c}_E = 1 + \widehat{c}\,\ell\,\overline{\varrho}^{\nu} \;\leq\; \sqrt{\kappa + \tfrac{1}{\kappa} - 1} \tag{7.1}$$

are valid with a constant $\widehat{c} > 0$, $\widehat{c} \neq \widehat{c}(\ell, \overline{\varrho})$.

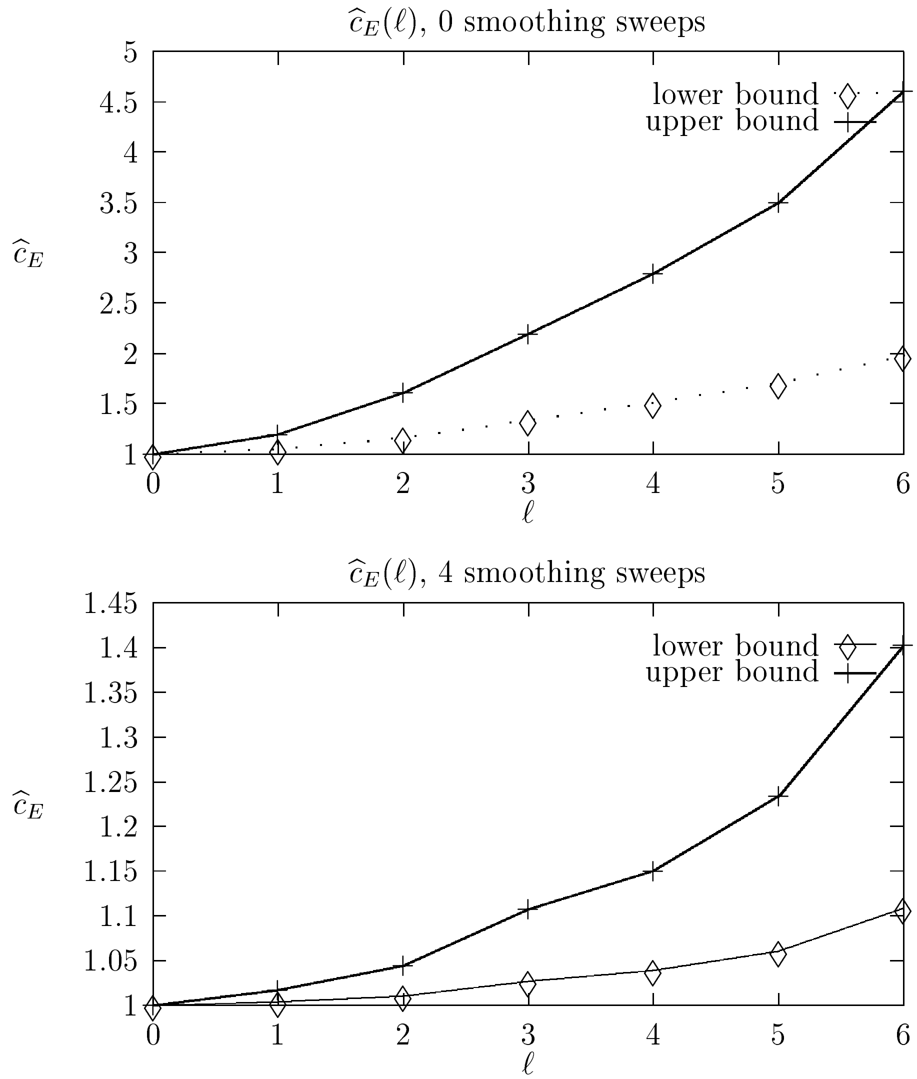


Figure 2: Lower and upper bounds of $\widehat{c}_E$ (7.1) deduced from Table 1

We see from the pictures in Figure 1 that there exists a $\widehat{c}$ so that for a constant ν the linear expression for $\widehat{c}_E$ realizes the estimates (7.1) for all levels observed. The behavior of $\widehat{c}_E$ for the remaining rows in Table 1 is similar. Please note that $\overline{\varrho}$ in (7.1) is an upper bound for the smoothing factors ϱ_k of the smoothing procedure on the levels $k = \overline{1, \ell}$ so that the measured behavior of the condition number κ might be better then the predicted one for certain levels.
Additionally, the asymptotic behavior of the bounds for $\widehat{c}_E(\kappa)$ in (7.1) just can be seen for $\kappa > 20$. Using our good hierarchical extension plus smoothing we never reach that range.
All the theoretical results are valid under the assumption that the smoothing factor $\overline{\varrho}$ is constant for all smoothing sweeps. But in practice the first smoothing sweeps result in a better factor $\overline{\varrho}$ then the remaining ones. So, the condition numbers for the extension with one or two smoothing sweeps achieved from the experiment are better then the one predicted from the theory.

Looking at the iteration numbers in Table 1 it seems to be sufficient to perform at most $\mathcal{O}(\ell)$ smoothing sweeps per level to achieve constant iteration numbers. Together with the discussion above this validates the theoretical results in the Theorems 5.2 and 5.8.

Electrical machine : As a more challenging example we calculated the magnetic potential in an electrical machine with a rather complex geometry (see also [15]).

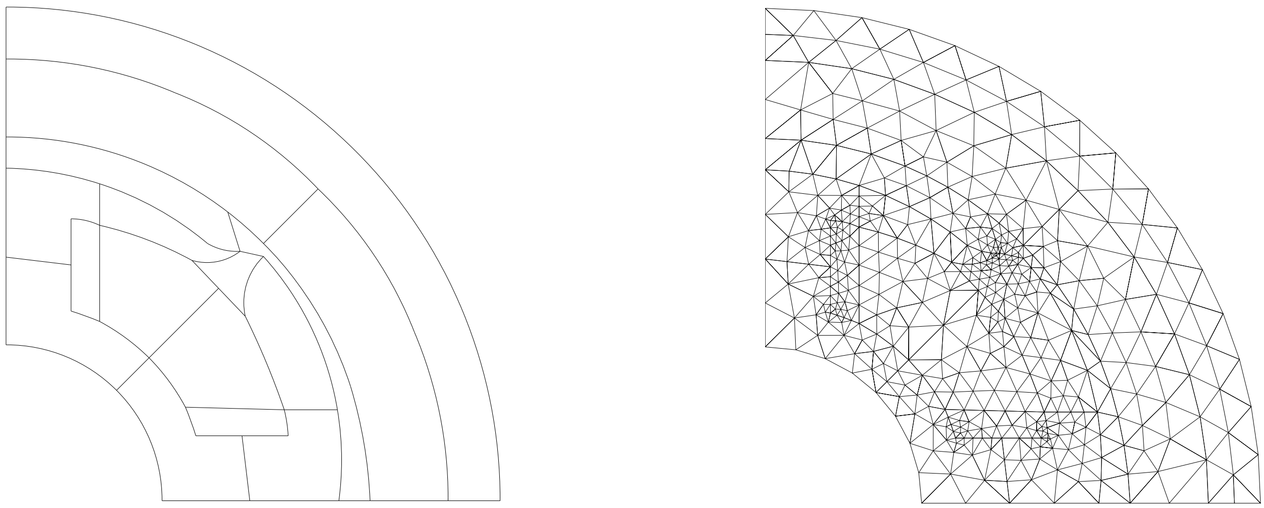


Figure 3: Material adapted decomposition and mesh on level 0 of the electrical machine

The pde is similar to (2.1) with coefficients $a(x)$ constant in the material domains but with large jumps between the materials. The material adapted decomposition of the domain into 16 subdomains was done by the code ADDPre (see [7]). The mesh of level 4 includes 93 377 unknowns and the mesh of level 5 solves the problem on 374 129 unknowns. In the DD-preconditioner S-BPX [25] was used as Schur complement preconditioner C_C. The inner preconditioner C_I was chosen freely (Algorithms 1 and 4) or was defined implicitly (Algorithm 1b). The accuracy was measured in the $KC^{-1}K$-energy norm of the error.

Alg.	B_{IC}	C_I	grid 4		grid 5	
			# iter.	sec.	# iter.	sec.
$\boxed{1}$	E_{IC}	V11	> 200		> 200	
$\boxed{1}$	$\widehat{E}_{IC}(1)$	V11	58	18.0	65	71
$\boxed{1}$	$\widehat{E}_{IC}(2)$	V22	39	16.7	41	66
$\boxed{1b}$	$\widehat{E}_{IC}(1)$	$\widehat{V}11$	58	14.4	65	52.6
$\boxed{1b}$	$\widehat{E}_{IC}(2)$	$\widehat{V}22$	39	13.7	41	49.1
$\boxed{1}$	$V11\,\widehat{E}_{IC}(1)$	V11	51	25.0	53	99
$\boxed{1}$	$\widehat{E}_{IC}(1)$gen.	V11	53	17.4	54	63.2
$\boxed{1b}$	$\widehat{E}_{IC}(1)$gen.	$\widehat{V}11$gen.	43	13.4	42	46.0
$\boxed{4}$	$\widehat{E}_{IC}(0)$	V01	55	15.9	57	57.1

Table 2: Electrical machine, 16 processors

We see from Table 2 that the new extension technique together with the new Algorithm 1b is even

faster then the best older preconditioner in Algorithm 4 (symmetric MSM [12]). The rows 2-5 indicate that in comparison with Algorithm 1 the Algorithm 1b saves up to 25 % of the arithmetical work. Additionally there is no need to perform an additional multigrid step after the extension $\widehat{E}_{IC}$ (rows 4 and 6). The good behavior of the extension with generalized smoothing sweeps in row 8 validates Remark 5.4.
In general the iteration numbers for the electrical machine are higher then for the test example. One reason is the use of rather cheap multigrid cycles defining the inner preconditioner C_I. On the other hand our Schur complement preconditioner C_C has no longer spectral equivalence constants $\underline{\gamma}_C, \overline{\gamma}_C$ near 1.

8 Conclusions

The presented extension technique is a cheap method for approximating the pde-harmonic extension, i.e. a harmonic extension appropriate to the pde, in a 2-dimensional domain accurately. Due to the smoothing sweeps and the exact discrete harmonic extension on the coarsest grid the technique described works also on rather general symmetric elliptic operators, e.g.

$$\begin{aligned} -\mathrm{div}(\lambda(x)\nabla u(x)) + b(x)u(x) &= 0 \qquad \forall x \in \Omega_i \\ u(x) &= g(x) \quad \forall x \in \partial\Omega_i \end{aligned}$$

with $\lambda(x) \geq \lambda_0 > 0$ and $b(x) \geq 0\ \forall x \in \Omega_i$ and on the linear elasticity equation.

Using the same hierarchical splitting of the f.e. space $\mathbb{V}$ in the 3D-case results in $\mu = \varrho(S_C^{-1}T_C) = \mathcal{O}(h^{-1})$. In [21], Nepomnyaschikh constructed a norm-preserving extension operator using a BPX-like splitting of the f.e. space $\mathbb{V}$ so that even in the 3D-case $\kappa(C^{-1}K) = \mathcal{O}(1)$ can be achieved. The combination of his approach together with smoothing sweeps and the exact discrete harmonic extension on the coarsest grid will improve the constant in the estimate given in [21]. Implementing a similar improvement as in Section 6 should again result in a very fast ASM-DD-preconditioner for 3D case.

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