# A convergent adaptive Algorithm for Poisson's Equation

B. Endtmayer [1]

[1] Johann Radon Institute for Computational and Applied Mathematics (RICAM)

Seminar on Numerical Analysis:
Adaptive Finite Element Methods,
Oct. 22, 2019

# Preliminaries

# Conforming Triangulation

- $\Omega \subset \mathbb{R}^2$: bounded, polygonal boundary

## Conforming Triangulation

We say $\mathcal{T}$ is a conforming triangulation of $\Omega$ iff the following properties hold:

- $\mathcal{T}$ is a decomposition of $\Omega$ into a set of $N_{\mathcal{T}}$ open triangles $T_j$,
- $\overline{\Omega} = \cup_{j=1}^{N_{\mathcal{T}}} \overline{T_j}$,
- $\overline{T_i} \cap \overline{T_j}$ is $\{$ empty, a vertex, a common edge $\}$.

# Refinement of $\mathcal{T}$

## Refinement of $\mathcal{T}$

We say $\mathcal{T}_{k+1}$ is a refinement of a triangulation $\mathcal{T}_k$ iff

- we decompose a a subset of triangles in $\mathcal{T}_k$ into subtriangles,
- the resulting decomposition is a again a triangulation of $\Omega$.

# Basis functions and FE Spaces

## Basis functions and FE Spaces

Let $\mathcal{N}_k$ be the set of nodes in the mesh $\mathcal{T}_k$.

- $V_k := span\{\{\psi_q^k\}_{q\in\mathcal{N}_k}\}$,
- $V_{0,k} := span\{\{\psi_q^k\}_{q\in\mathcal{N}_k\cap\partial\Omega}\}$,
- $\psi_q^k(q') = c_q\delta_{qq'}$ for all $q, q' \in \mathcal{N}_k$.

# Sequence of meshes property

Let $\{\mathcal{T}_k\}_{k\geq 0}$ be a sequence of meshes.
We assume that all triangles are of regular shape, i.e

$$\frac{d_T}{d_{T'}} \leq \sigma_0,$$

independent of $k \geq 0$.

# Neighbours

The set of neighbours of a triangle $T$ is given by

$$\omega_T := \{T' \in \mathcal{T}_k : \overline{T} \cap \overline{T'} \neq \emptyset\}.$$

This implies that

$$\frac{d_{\omega_T}}{d_T} \leq \sigma_1.$$

# $L^2$ Projection $P_k^1$

## Estimates for the $L^2$ Projection from $H_0^1$ to $V_{0,k}$

There are constants $C_0(\omega_T)$ and $C_1(\omega_T)$ such that

$$\|v - P_k^1 v\|_{L^2(T)} \leq C_0(\hat{\omega}_T) d_{\omega_T} \|\nabla v\|_{L^2(\omega_T)},$$

$$\|\nabla(v - P_k^1 v)\|_{L^2(T)} \leq C_{1,m}(\hat{\omega}_T) d_{\omega_T}^m \|\nabla^{m+1} v\|_{L^2(\omega_T)} \qquad m \in \{0,1\}.$$

# Poincare inequality

## Poincare inequality

For $u \in H_0^1(\Omega)$ it holds

$$\|u\|_{L^2(\Omega)} \leq C_P(\Omega)\|\nabla u\|_{L^2(\Omega)}.$$

# The model problem

# The model problem

Find $u \in H_0^1$ such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } H^{-1}(\Omega) \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

# The model problem

Find $u \in H_0^1$ such that

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega f \phi \qquad \forall \phi \in H_0^1(\Omega).$$

# The discrete model problem

Find $u_k \in V_{0,k}$ such that

$$\int_\Omega \nabla u_k \nabla \phi_k = \int_\Omega f_k \phi_k \qquad \forall \phi \in V_{0.k},$$

where $f_k$ is some suitable approximation of $f$. We assume $f_k(q) = f(q)$ for all $q \in \mathcal{N}_k$.

# Error reduction for linear finite elements

# We want to

## Solve $\rightarrow$ Estimate $\rightarrow$ Mark $\rightarrow$ Refine

$$\mathcal{T}_0 \quad \leq \quad \mathcal{T}_1 \quad \leq \quad \mathcal{T}_2 \quad \leq \quad \mathcal{T}_3 \leq \ldots$$

$$\|\nabla(u-u_0)\|^2_{L^2(\Omega)} \quad \geq \|\nabla(u-u_1)\|^2_{L^2(\Omega)} \quad \geq \|\nabla(u-u_2)\|^2_{L^2(\Omega)} \quad \geq \|\nabla(u-u_3)\|^2_{L^2(\Omega)} \geq \ldots$$

# The goal?

$$\exists \kappa \in (0,1) : \kappa \|\nabla(u - u_k)\|^2_{L^2(\Omega)} \geq \|\nabla(u - u_{k+1})\|^2_{L^2(\Omega)}$$

- $\mathcal{T}_h$, $\mathcal{T}_H$ conforming triangulations
- $T_h$ refinement of $\mathcal{T}_H$
- $V_H \subseteq V_h$ the corresponding spaces
- $u_h, u_H$ the corresponding solutions on $V_h, V_H$
- $e_h := u - u_h$, $e_H := u - u_H$

# Fineness of of the triangulation

## Fineness of of the triangulation $\mathcal{T}_h$

We say a triangulation $\mathcal{T}_h$ has fineness $\mu$ with respect to $\varepsilon$ if there are positive weights $(w_1, w_2)$ for the approximation $f_h$ such that

$$\max\{w_1\|f - f_{h'}\|_{L^2(\Omega)}, w_2\|h' f_{h'}\|_{L^2(\Omega)}\} \leq \mu\varepsilon,$$

for any refinement $\mathcal{T}_{h'}$ of $\mathcal{T}_h$.

## Lemma 1

If

- $\mathcal{T}_H$ has fineness $\mu$ w.r.t. $\varepsilon$, $w_1 = C_P$ (Poincare constant), $w_2 > 0$,
- $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$,
- there is a constant $C_e > 0$ such that

$$\|\nabla e_H\|_{L^2(\Omega)} \geq \frac{\varepsilon}{C_e},$$

then

$$\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} - 4C_e\mu(1+6C_e\mu)\|\nabla e_H\|^2_{L^2(\Omega)}.$$

Proof: Blackboard

Some blackboard computations ...

# Poincare inequality

## Poincare inequality

For $u \in H_0^1(\Omega)$ it holds

$$\|u\|_{L^2(\Omega)} \leq C_P(\Omega)\|\nabla u\|_{L^2(\Omega)}.$$

# Fineness of of the triangulation

## Fineness of of the triangulation $\mathcal{T}_h$

We say a triangulation $\mathcal{T}_h$ has fineness $\mu$ with respect to $\varepsilon$ if there are positive weights $(w_1, w_2)$ for the approximation $f_h$ such that

$$\max\{w_1\|f - f_{h'}\|_{L^2(\Omega)}, w_2\|h' f_{h'}\|_{L^2(\Omega)}\} \leq \mu\varepsilon,$$

for any refinement $\mathcal{T}_{h'}$ of $\mathcal{T}_h$.

## Lemma 1

If

- $\mathcal{T}_H$ has fineness $\mu$ w.r.t. $\varepsilon$, $w_1 = C_P$ (Poincare constant), $w_2>0$,
- $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$,
- there is a constant $C_e > 0$ such that

$$\|\nabla e_H\|_{L^2(\Omega)} \geq \frac{\varepsilon}{C_e},$$

then

$$\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} - 4C_e\mu(1+6C_e\mu)\|\nabla e_H\|^2_{L^2(\Omega)}.$$

- $\|\nabla \psi_q^k\|^2_{L^2(\Omega)} \leq \sigma_\psi$
- $\mathcal{E}_{0,H}$ the set of interior edges in $\mathcal{T}_H$
- $\mathcal{R}_{0,H/2}$ be the set of midpoints on the interior edges.
- $\mathcal{R}_h \subset \mathcal{R}_{0,H/2}$ be the set of additional nodes $\mathcal{T}_h$.
- $[\partial_n v_H]_E$ be the jump of the normal derivative across the edge $E \in \mathcal{E}_{0,H}$.

## Lemma 2

If

- $\mathcal{R}_h \subset \mathcal{R}_{0,H/2}$,
- for $q \in \mathcal{R}_h$ let $E_q \in \mathcal{E}_{0,H}$ with $q \in E_q$,

then

$$\sum_{q \in \mathcal{R}_h} d_{E_q} \| [\partial_n u_H]_E \|^2_{L^2(E_q)} \leq 24\sigma_\psi^2 \| \nabla (u_h - u_H) \|^2_{L^2(\Omega)} + \hat{C}_2^2 \| h f_h \|^2_{L^2(\Omega)},$$

$$\sum_{q \in \mathcal{R}_{0,H/2}} d_{E_q} \| [\partial_n u_H]_E \|^2_{L^2(E_q)} \leq 24\sigma_\psi^2 \| \nabla e_H \|^2_{L^2(\Omega)} + \hat{C}_2^2 \| H f \|^2_{L^2(\Omega)},$$

where $\hat{C}_2^2 = 12(1 + \sigma_0^2)$.

### Lemma 3

For $\mathcal{T}_H$ it holds

$$\|\nabla e_H\|_{L^2(\Omega)} \leq \hat{C}_3 \Big( \sum_{E \in \mathcal{E}_{0,H}} d_E \|[\partial_n u_H]_E\|^2_{L^2(E)} \Big)^{\frac{1}{2}} + \hat{C}_0 \|H f_H\|_{L^2(\Omega)} + C_P \|f - f_H\|_{L^2(\Omega)},$$

where $\hat{C}_3$ only depends on the shape of the triangles.

# Error Estimates

# An error estimator

Lemma 2 and Lemma 3 suggest the following error estimate:

$$\eta_E^2 := d_E \|[\partial_n u_H]_E\|_{L^2(E)}^2$$

for $E \in \mathcal{E}_{0,H}$ or

$$\eta_T^2 := \frac{1}{2} \sum_{E \subseteq \partial T \cap \partial\Omega} d_E \|[\partial_n u_H]_E\|_{L^2(E)}^2.$$

# Assumption on Refinement Strategy

- if a triangle is marked, then all 3 edges are divided!

- $\mathcal{A} \subset \mathcal{T}$
- $\eta^2_{\mathcal{A}} := \sum_{T\in\mathcal{A}} \eta^2_T$
- $\mathcal{T}_h$ is the resulting mesh by marking all triangles in $\mathcal{A}$
- $\eta^2_{\mathcal{A}} \leq \sum_{q\in\mathcal{R}_h} \eta^2_{E_q}$
- $\eta^2_{\mathcal{T}} = \sum_{q\in\mathcal{R}_{0,H/2}} \eta^2_{E_q}$

# The goal?

$$\kappa\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)}$$

## Lemma 1

If

- $\mathcal{T}_H$ has fineness $\mu$ w.r.t. $\varepsilon$, $w_1 = C_P$ (Poincare constant), $w_2 > 0$,
- $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$,
- there is a constant $C_e > 0$ such that

$$\|\nabla e_H\|_{L^2(\Omega)} \geq \frac{\varepsilon}{C_e},$$

then

$$\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} - 4C_e\mu(1+6C_e\mu)\|\nabla e_H\|^2_{L^2(\Omega)}.$$

## Lemma 1

If

- $\mathcal{T}_H$ has fineness $\mu$ w.r.t. $\varepsilon$, $w_1 = C_P$ (Poincare constant), $w_2 > 0$,
- $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$,
- there is a constant $C_e > 0$ such that

$$\|\nabla e_H\|_{L^2(\Omega)} \geq \frac{\varepsilon}{C_e},$$

then

$$\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} - 4C_e\mu(1+6C_e\mu)\|\nabla e_H\|^2_{L^2(\Omega)}.$$

## Lemma 2

If

- $\mathcal{R}_h \subset \mathcal{R}_{0,H/2}$,
- for $q \in \mathcal{R}_h$ let $E_q \in \mathcal{E}_{0,H}$ with $q \in E_q$,

then

$$\sum_{q \in \mathcal{R}_h} d_{E_q} \|[\partial_n u_H]_E\|^2_{L^2(E_q)} \le 24\sigma_\psi^2 \|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} + \hat{C}_2^2 \|hf_h\|^2_{L^2(\Omega)},$$

$$\sum_{q \in \mathcal{R}_{0,H/2}} d_{E_q} \|[\partial_n u_H]_E\|^2_{L^2(E_q)} \le 24\sigma_\psi^2 \|\nabla e_H\|^2_{L^2(\Omega)} + \hat{C}_2^2 \|Hf\|^2_{L^2(\Omega)}$$

where $\hat{C}_2^2 = 12(1+\sigma_0^2)$.

- $\mathcal{A} \subset \mathcal{T}$
- $\eta_{\mathcal{A}}^2 := \sum_{T \in \mathcal{A}} \eta_T^2$
- $\mathcal{T}_h$ is the resulting mesh by marking all triangles in $\mathcal{A}$
- $\eta_{\mathcal{A}}^2 \leq \sum_{q \in \mathcal{R}_h} \eta_{E_q}^2$
- $\eta_{\mathcal{T}}^2 = \sum_{q \in \mathcal{R}_{0,H/2}} \eta_{E_q}^2$

## Lemma 2

If

- $\mathcal{R}_h \subset \mathcal{R}_{0,H/2}$,
- for $q \in \mathcal{R}_h$ let $E_q \in \mathcal{E}_{0,H}$ with $q \in E_q$,

then

$$\eta_{\mathcal{A}}^2 \leq \sum_{q \in \mathcal{R}_h} d_{E_q} \|[\partial_n u_H]_E\|_{L^2(E_q)}^2 \leq 24\sigma_\psi^2 \|\nabla(u_h - u_H)\|_{L^2(\Omega)}^2 + \hat{C}_2^2 \|hf_h\|_{L^2(\Omega)}^2,$$

$$\eta_{\mathcal{T}}^2 = \sum_{q \in \mathcal{R}_{0,H/2}} d_{E_q} \|[\partial_n u_H]_E\|_{L^2(E_q)}^2 \leq 24\sigma_\psi^2 \|\nabla e_H\|_{L^2(\Omega)}^2 + \hat{C}_2^2 \|Hf\|_{L^2(\Omega)}^2,$$

where $\hat{C}_2^2 = 12(1 + \sigma_0^2)$.

# Marking strategy

$$\eta_{\mathcal{A}} \geq (1-\theta^*)\eta_{\mathcal{T}}$$

## Theorem 1

Let us choose the previous marking strategy. If

- $\mu^* > 0$ and depends only on $\theta^*$, $\sigma_0$ and $H_{max}/C_P$,
- $\mathcal{T}_H$ has fineness $\mu \leq \mu^*$ with respect to $\varepsilon$, $w_1 = C_P$, $w_2 = max\{\hat{C}_0, \hat{C}_2\}$,

then there exists a $\kappa \in (0,1)$ such that for the resulting triangulation $\mathcal{T}_h$ holds

$$\|\nabla e_h\|^2_{L^2(\Omega)} \leq \kappa \|\nabla e_H\|^2_{L^2(\Omega)},$$

or $\eta_{\mathcal{T}} \leq \varepsilon$.

## Lemma 2

If

- $\mathcal{R}_h \subset \mathcal{R}_{0,H/2}$,
- for $q \in \mathcal{R}_h$ let $E_q \in \mathcal{E}_{0,H}$ with $q \in E_q$,

then

$$\eta_{\mathcal{A}}^2 \leq \sum_{q \in \mathcal{R}_h} d_{E_q} \|[\partial_n u_H]_E\|_{L^2(E_q)}^2 \leq 24\sigma_\psi^2 \|\nabla(u_h - u_H)\|_{L^2(\Omega)}^2 + \hat{C}_2^2 \|hf_h\|_{L^2(\Omega)}^2,$$

$$\eta_{\mathcal{T}}^2 = \sum_{q \in \mathcal{R}_{0,H/2}} d_{E_q} \|[\partial_n u_H]_E\|_{L^2(E_q)}^2 \leq 24\sigma_\psi^2 \|\nabla e_H\|_{L^2(\Omega)}^2 + \hat{C}_2^2 \|Hf\|_{L^2(\Omega)}^2,$$

where $\hat{C}_2^2 = 12(1 + \sigma_0^2)$.

## Lemma 3

For $\mathcal{T}_H$ it holds

$$\|\nabla e_H\|_{L^2(\Omega)} \leq \hat{C}_3 \Big( \sum_{E \in \mathcal{E}_{0,H}} d_E \|[\partial_n u_H]_E\|^2_{L^2(E)} \Big)^{\frac{1}{2}} + \hat{C}_0 \|H f_H\|_{L^2(\Omega)} + C_P \|f - f_H\|_{L^2(\Omega)},$$

where $\hat{C}_3$ only depends on the shape of the triangles.

## Lemma 1

If

- $\mathcal{T}_H$ has fineness $\mu$ w.r.t. $\varepsilon$, $w_1 = C_P$ (Poincare constant), $w_2 > 0$,
- $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$,
- there is a constant $C_e > 0$ such that

$$\|\nabla e_H\|_{L^2(\Omega)} \geq \frac{\varepsilon}{C_e},$$

then

$$\|\nabla e_H\|^2_{L^2(\Omega)} \geq \|\nabla e_h\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla(u_h - u_H)\|^2_{L^2(\Omega)} - 4C_e\mu(1 + 6C_e\mu)\|\nabla e_H\|^2_{L^2(\Omega)}.$$

## Theorem 1

Let us choose the previous marking strategy. If

- $\mu^* > 0$ and depends only on $\theta^*$, $\sigma_0$ and $H_{max}/C_P$,
- $\mathcal{T}_H$ has fineness $\mu \leq \mu^*$ with respect to $\varepsilon$, $w_1 = C_P$, $w_2 = max\{\hat{C}_0, \hat{C}_2\}$,

then there exists a $\kappa \in (0,1)$ such that for the resulting triangulation $\mathcal{T}_h$ holds

$$\|\nabla e_h\|^2_{L^2(\Omega)} \leq \kappa \|\nabla e_H\|^2_{L^2(\Omega)},$$

or $\eta_{\mathcal{T}} \leq \varepsilon$.

Holds also for the residual error estimator
$\eta_T^R := (\eta_T^2 + s_0^2 \|Hf_H\|^2_{L^2(T)})^{\frac{1}{2}}$.
($\eta_T^R := (\eta_T^2 + s_0^2 \|H(\Delta u_H + f_H)\|^2_{L^2(T)})^{\frac{1}{2}}$ for higher order elements)

# Locally equivalent error estimators

## Locally equivalent

We say an error estimator is locally equivalent to the residual error estimator if

- sets $S'_T, S''_T \subset \mathcal{T}$,
- constants $C'_{\tilde{\eta}}, C''_{\tilde{\eta}}$,

such that for all $T \in \mathcal{T}$ holds

- $\tilde{\eta}_T \leq C'_{\tilde{\eta}} \eta^R_{S'_T}$,
- $\eta^R_T \leq C''_{\tilde{\eta}} \tilde{\eta}_{S''_T}$.

# Modified refinement strategy

- if a triangle $T$ is marked, then all 3 edges of all triangles contained in $S'_T$ are divided.

### Theorem 2

Let us choose the previous marking strategy. If

- $\tilde{\eta}$ is locally equivalent to $\eta^R$

then Theorem 1 also holds for $\tilde{\eta}$, with different constants.

# Initial Mesh

# Construction of an initial mesh with fineness $\mu$

$$\eta_T^0 := \max\{w_1\|f - f_h\|_{L^2(T)}, w_2\|hf_h\|_{L^2(T)}\},$$

- use same marking and refinement strategy as for $\eta$
- stop wenn $\eta_{\mathcal{T}_0}^0 \leq \mu\varepsilon$.

# References

W. Dörfler.
A convergent adaptive algorithm for Poisson's equation.
*SIAM J. Numer. Anal.*, 33(3):1106–1124, 1996.

# Thank you for your Attention!