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A Discontinuous Galerkin Method for Solving Total Variation Minimization Problems

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To my parents
Stephen Moore and Janet Moore.

Abstract

The minimization of functionals which are formed by an L^2-term and a Total Variation (TV) term play an important role in mathematical imaging with many applications in engineering, medicine and art. The TV term is well known to preserve sharp edges in images.

More precisely, we are interested in the minimization of a functional formed by a discrepancy term and a TV term. The first order derivative of the TV term involves a degenerate term which could happen in flat areas of an image. Many well known methods have been proposed to solve this problem.

In this thesis, we present a relaxed functional associated with the TV minimization problem. The relaxed functionals are well-posed and produce a sequence of solutions minimizing our original TV-functional. The relaxed functional results in an *Iteratively Reweighted Least Squares* method that approximates the TV minimization.

Considering the *Euler-Lagrange equation*, the minimizer of the relaxed functional is equivalent to the solution of a second order elliptic partial differential equation. We discretize this partial differential equation in the framework of Discontinuous Galerkin (DG) Finite Element Method (FEM) with linear functions on each element. Specifically, we consider the *Symmetric Interior Penalty Galerkin* method. The discretization leads to a system of linear equations.

The existence and uniqueness of the solution to the DG variational form of Discontinuous Galerkin and the discrete DG problem is studied, and a-priori error estimates are reported.

The Discontinuous Galerkin Finite Element Method in combination with *iteratively reweighted least squares* method is implemented .

Finally, numerical results are presented that demonstrate the accuracy of the numerical solution using the proposed methods.

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Chapter 1

Introduction

In this thesis we will concentrate on the numerical solution of 2D Total Variation (TV) Minimization problems, as they appear in image processing by a Discontinuous Galerkin (DG) Finite Element Method (FEM) in combination with an Iteratively Reweighted Least Squares (IRLS) method.

TV methods and similar approaches based on regularizations with L^1-norms (and semi-norms) have become a very popular tool in image processing and inverse problems due to peculiar features that cannot be realized with smooth regularizations. TV techniques had particular success due to their ability to realize cartoon-type reconstructions with sharp edges [9]. Within the last decade, there have been an explosion of new developments in this field. The TV methods started with an introduction of a variational denoising model by Rudin, Osher and Fatemi consisting of minimizing total variation among all functions within a variance bound [40]. It was shown to be equivalent to an unconstrained minimization problem of the form:

$$\min_u \left(2\lambda \int_\Omega |\nabla u|\, dx + \|Ku - g\|_{L^2}^2 \right), \tag{1.1}$$

where $u : \Omega \to \mathbb{R}, \Omega \subset \mathbb{R}^n, n = 1, 2,\ K : L^2(\Omega) \to L^2(\Omega), g \in L^2(\Omega)$ and $\lambda > 0$. The Euler-Lagrange-equation of the functional in (1.1) reads as follows :

$$-\lambda \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + K^*(Ku - g) = 0, \tag{1.2}$$

where K^* is the adjoint of K. In the literature, several numerical strategies for efficiently solving (1.2) have been proposed. We only mention the following 3 approaches :

(i) *The fixed point iteration method* [15, 41, 42, 43]: Once the coefficients $1/|\nabla u|$ are fixed at a previous iteration u, various iterative solver techniques have been considered. There exist excellent inner solvers but the outer solver can be slow. Further improvements are still useful.

(ii) *The explicit time marching scheme* [33, 38, 40]: It turns the nonlinear partial differential equation (1.2) into a parabolic equation before using an explicit Euler method to march in time to convergence. The method is quite reliable but often slow as well.

(iii) *The primal-dual (PD) method* [12, 13, 14]: It solves for both the primal and the dual variable together in order to achieve faster convergence with the Newton method (and a constrained optimisation with the dual variable).

In this thesis, we will consider an *Iteratively Reweighted Least Squares* (IRLS) algorithm to solve the TV-minimization problem. Under certain assumptions, the IRLS can be related to the minimization of the L^1-norm of derivatives [24]. By considering the Euler-Lagrange equation, the minimizer of the relaxed functional is equivalent to the solution of a second order Partial Differential Equation(PDE) having the form of a reaction -diffusion equation with a specially chosen diffusion coefficient serving as weights. Introducing suitable boundary conditions, the problem can be formulated in a variational form.

The application of the IRLS to (1.2) results in a double minimization algorithm [24], which will be discussed in more detail later.

The two main ingredient involved in our approach are :

1. An **Iteratively Reweighted Least Squares** algorithm is used to reconstruct a sequence that converge to the solution of the original TV minimization problem. It is known to have linear rate of convergence which can be modified to yield a super linear rate of convergence [24, 32].

2. The **Discontinuous Galerkin Finite Element Method (DGFEM)** is used to discretize the continuous, infinite dimensional problem. For an introduction to DGFEMs, we recommend the book by Rivière [39] or the survey article by Arnoldi, Brezzi, Cockburn and Marrini [4].

We will combine these ingredients to construct a new efficient numerical method for TV minimization problems in the following way :

- Firstly, we analyze the conditions that relate the TV minimization problem to the iteratively reweighted least squares algorithm. We will use results from [17].

- Secondly, we present the discontinuous Galerkin finite element method and a standardized assembling of the elemental matrix [39].

- Finally, we will present numerical experiments using this particular combination of methods and discuss the results.

The rest of the thesis is organized as follows: In Chapter 2, starting from the total variation minimization problem, we will present a link between the TV and the iteratively reweighted least squares method, we then derive the Euler-Lagrange equation of the functional. In Chapter 3, after recapitulating the concepts of Discontinuous Galerkin Finite Element Method (h-version), we will provide some analysis for existence and uniqueness and present also some analysis on error estimates, particularly, a-priori estimates in L^2 and the DG energy norms . In Chapter 4, we present some numerical results to illustrate the efficiency of the combination of DG methods and the IRLS algorithm. Finally, in Chapter 5, we draw some conclusions and discuss some future work.

Chapter 2

Problem Formulation and Analysis

In this chapter, we define the space of functions of bounded variation and give some properties that form the basis of its applications in regularization methods following mostly the expositions in [2, 21, 22, 26]. In solving the minimization functional (1.1), we introduce an associated well-posed relaxed functional following from [25]. Finally, we derive the corresponding Euler-Lagrange equation of the relaxed functional which is a second order elliptic partial differential equation.

The space of functions of bounded variation $BV(\Omega)$ plays an important role in many problems in the calculus of variations. For instance, BV-spaces are used in treating the minimal surface problem [26, 5] and in the theory and numerics of hyperbolic conservation laws. It was introduced to image processing by Rudin, Osher and Fatemi (ROF) [40] and has subsequently found applications in the related field of inverse problems [20]. The interesting feature of the minimization problem from the ROF model is that they are well suited for problems with discontinuous solutions.

2.1 Functions of Bounded Variation

In this section, we denote by Ω a simply connected, bounded, nonempty subset of $\mathbb{R}^n, n = 1, 2, 3$ with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We use the symbol ∇ to denote the gradient of a smooth function $u : \mathbb{R}^n \to \mathbb{R}$, i.e., $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$. Let $C_0^1(\Omega; \mathbb{R}^n)$ denote the space of vector-valued functions $\varphi = (\varphi_1, ..., \varphi_n)$ whose component function φ_i are continuously-differentiable and compactly supported on Ω, i.e., each φ_i vanishes outside some compact subset of Ω. The divergence of φ is given by

$$\operatorname{div} \varphi = \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial x_i}.$$

The Euclidean norm is denoted by $|\cdot|$ and given by $|\varphi(x)| = [\sum_{i=1}^{n} (\varphi_i(x))^2]^{1/2}$ for $\varphi \in C_0^1(\Omega; \mathbb{R}^n)$. The Sobolev space $W^{1,1}(\Omega)$ denotes the closure of $C_0^1(\overline{\Omega})$ with respect to the norm

$$\| u \|_{W^{1,1}(\Omega)} = \int_\Omega \left[|u(x)| + \sum_{i=1}^{n} \left| \frac{\partial u(x)}{\partial x_i} \right| \right] dx,$$

where $|\cdot|$ denotes the modulos.

Definition 2.1 ([26])**.** *The total variation of a function $u \in L^1(\Omega)$ is defined by*

$$\int_\Omega |Du| = \sup \left\{ \int_\Omega u \, div \varphi \, dx : \varphi \in C_0^1(\Omega; \mathbb{R}^n) : \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}. \tag{2.1}$$

Further, we define the space of *functions of bounded variation* as

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \int_\Omega |Du| < \infty \right\}, \tag{2.2}$$

with the *seminorm*

$$|u|_{BV(\Omega)} = \int_\Omega |Du|.$$

The space (2.5) is a Banach space endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{BV(\Omega)}. \tag{2.3}$$

Remark 2.2. *If $u \in W^{1,1}(\Omega)$, then $u \in BV(\Omega)$ and*

$$|u|_{BV(\Omega)} = \int_\Omega |\nabla u| \, dx.$$

Example 2.3. *Let u be defined in $(-1, +1)$ by $u(x) = -1$ if $-1 \leq x < 0$ and $u(x) = +1$ if $0 < x \leq 1$. Then $\int_{-1}^{+1} u \, \varphi' \, dx = -2\varphi(0)$ and $\int_{-1}^{+1} |Du| = 2$. We can remark that Du, the distributional derivative of u, is equal to $2\delta_0$, where δ_0 is the Dirac measure in 0 [5].*

Remark 2.2 and Example 2.3 shows that $W^{1,1}(\Omega)$ is a proper subspace of $BV(\Omega)$.

Variants of the total variation are obtained by changing the norm of the vector φ in (2.1) that is $\|\varphi\|_\infty = \sup_{x \in \Omega} |\varphi(x)|_{\ell^s}$ by different norms on $\mathbb{R}^n$ which yields equivalent but anisotropic versions of the $BV(\Omega)$ seminorm with $1 \leq s \leq \infty$. More precisely, a family of seminorms equivalent to the standard $BV(\Omega)$ seminorm can be defined by

$$\int_\Omega |Du|_{\ell^r} = \sup \left\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C_0^1(\Omega; \mathbb{R}^n) : |\varphi(x)|_{\ell^s} \leq 1, \forall \, x \in \Omega \right\}, \tag{2.4}$$

for $1 \leq r < \infty$ and the Hölder conjugate index s, i.e. $r^{-1} + s^{-1} = 1$.

The distributional derivative Du of a function $u \in BV(\Omega)$ can be identified with a vector valued Radon measure that has *total variation* $\|Du\| = |u|_{BV(\Omega)}$ in the sense of measure theory as the following theorem asserts. For this reason, one sometimes also finds the notation $\|Du\|$ for the BV seminorm.

Theorem 2.4 (Structure theorem for BV functions)**.** *Let $u \in BV(\Omega)$. Then there exist a Radon measure μ on Ω and a μ-measurable function $\sigma\colon \Omega \to \mathbb{R}^n$ such that*

$$|\sigma(x)| = 1 \quad \mu \text{ a.e.},$$
$$\int_\Omega u \, div \varphi \, dx = -\int_\Omega \varphi \cdot \sigma \, d\mu,$$

for any $\varphi \in C_0^1(\Omega, \mathbb{R}^n)$.

Theorem 2.4 essentially follows from the Riesz representation theorem, see [22] for details. Here the crucial difference to Sobolev spaces is that the measure Du need not necessarily be represented as a Lebesgue measurable function.

The following properties of BV functions play a central role in the analysis of total variation minimization problems, see [26] for proofs.

Theorem 2.5 (Lower semicontinuity)**.** *Let $(u_j)_{j\in\mathbb{N}}$ be a sequence of functions in $BV(\Omega)$ which converge in $L^1_{loc}(\Omega)$ to a function u, then*

$$\int_\Omega |Du| \leq \liminf_{j\to\infty} \int_\Omega |Du_j|,$$

where

$$L^1_{loc}(U) = \Big\{ u : U \to \mathbb{R} \mid v \in L^1(V)\,;\, \text{for each } \ V \subset\subset U \Big\}. \tag{2.5}$$

Recall that in a real vector space X, a set $F \subset X$ is called *convex* if, for any $u, v \in F$, $\theta u + (1-\theta)v \in F$ for any $\theta \in [0, 1]$. For functions from such an X to the real numbers, we allow the value $+\infty$, i.e. we consider functions from X to $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

Definition 2.6 (Convexity)**.** *Let X be a real vector space, $F \subset X$ convex, $\varphi : F \to \bar{\mathbb{R}}$. φ is called a convex functional if for any $u, v \in F$,*

$$\varphi(\theta u + (1-\theta)v) \leq \lambda\varphi(u) + (1-\theta)\varphi(v), \tag{2.6}$$

for all $\theta \in [0, 1]$. φ is strictly convex if (2.6) *holds strictly for all $\theta \in (0, 1)$.*

For any $\varphi : X \to \bar{\mathbb{R}}$, the set

$$dom\, \varphi = \{u \in X : \varphi(u) < \infty\}.$$

is called the *effective domain* of φ. A convex function from X to $\bar{\mathbb{R}}$ is called *proper* if it is not identically equal to $+\infty$.

Remark 2.7. *For convex functionals, strong lower semicontinuity implies weak lower semicontinuity (see [5], Theorem 2.1.2), and therefore the $BV-$seminorm is also lower semicontinuous with respect to weak convergence in $L^1(\Omega)$ and consequently, for bounded Ω in $L^p(\Omega), p > 1$ as well.*

Theorem 2.8 (Compactness)**.** *Let in addition $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then*

$$BV(\Omega) \subset\subset L^p(\Omega) \quad for \quad 1 \leq p < p^* = n/(n-1),$$

and

$$BV(\Omega) \subset L^{p^*}(\Omega).$$

These compactness properties are analogous to those of functions in $W^{1,1}(\Omega)$. There is another type of compactness corresponding to a type of convergence that also carries information on the gradient (see [5]).

Definition 2.9 (BV-weak* convergence)**.** *A sequence $(u_j)_{j\in\mathbb{N}} \subset BV(\Omega)$ converges to $u \in BV(\Omega)$ in the BV-weak* topology, denoted by $u_j \overset{*}{\rightharpoonup} u$, if and only if*

$$u_j \to u \quad in \quad L^1(\Omega) \quad and \quad Du_j \overset{M}{\rightharpoonup} Du,$$

where $Du_j \overset{M}{\rightharpoonup} Du$ denotes a weak convergence of measures, which is defined as

$$\int_\Omega \varphi Du_j \to \int_\Omega \varphi Du \quad for\ all \quad \varphi \in C_0(\Omega; \mathbb{R}^n).$$

Theorem 2.10 (BV-weak* compactness.)**.** *Let $(u_j)_{j\in\mathbb{N}} \in BV(\Omega)$ with $\|u_j\|_{BV(\Omega)}$ uniformly bounded, then there exists a subsequence $(u_{j_k})_{k\in\mathbb{N}}$ and $u \in BV(\Omega)$ such that*

$$u_j \overset{*}{\rightharpoonup} u \quad in \quad BV(\Omega).$$

Theorem 2.11 (Approximation)**.** *Let $u \in BV(\Omega)$, then there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that*

$$\lim_{j\to\infty} \int_\Omega |u - u_j|\, dx = 0,$$

and

$$\lim_{j\to\infty} \int_\Omega |Du_j| = \int_\Omega |Du|.$$

The latter result shows a difference to approximation results for Sobolev spaces: we obtain the approximation but not in the BV semi-norm, whereas for Sobolev spaces approximation in the corresponding norm is possible.

The well-known coarea formula relates the total variation of a function to the regularity of its level sets [5, 26].

Theorem 2.12 (Coarea formula)**.** *Let $u \in BV(\Omega)$ and $L_t := \{x \in \Omega : u(x) < t\}$, then L_t has finite perimeter for L^1 a.e. $t \in \mathbb{R}$ and*

$$\int_\Omega |Du| = \int_\mathbb{R} \left(\int_\Omega |D\mathbf{1}_{L_t}| \right) dt.$$

Conversely, $u \in L^1(\Omega)$ and

$$\int_\mathbb{R} \left(\int_\Omega |D\mathbf{1}_{L_t}| \right) dt < \infty,$$

imply $u \in BV(\Omega)$ where $\mathbf{1}_E$ denotes the characteristic function of the set E, defined by $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \notin E$.

Since $BV(\Omega)$ contains discontinuous functions, the standard DG finite element spaces are contained in it [36]. This will be discussed later in Chapter 3 .

2.2 Existence and Uniqueness of Minimizers

As in [21, 40, 22, 26], we consider the minimization in $BV(\Omega)$ of the functional

$$J(u) = \|Ku - g\|^2_{L^2(\Omega)} + 2\lambda \int_\Omega |Du|, \tag{2.7}$$

where $K : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator, $g \in L^2(\Omega)$ is given, and $\lambda > 0$ is a fixed regularization parameter. Several numerical strategies to efficiently perform total variation minimization have been proposed in literature [11, 10, 23, 27, 37]. However, in the following we will only discuss how to adapt an iteratively reweighted least squares algorithm to this particular situation.

In order to guarantee the existence of minimizers for (2.7) we assume that:

> J is *coercive in* $L^2(\Omega)$, *i.e.*, there exists $C > 0$ such that $\{u \in L^2(\Omega) \colon J(u) \le C\}$ is bounded in $L^2(\Omega)$.

For smooth u, one can approximate the TV-term in (2.7) by a smooth, convex functional

$$E_\varepsilon(u) = \int_\Omega \varphi_\varepsilon(|\nabla u|)\, dx, \tag{2.8}$$

where $\varphi_\varepsilon \in C^1(\Omega)$, (i.e, continuously differentiable) and defined as:

$$\varphi_\varepsilon(z) = \begin{cases} \frac{1}{2\varepsilon} z^2 + \frac{\varepsilon}{2} & \text{if} \quad 0 \le |z| \le \varepsilon, \\ |z| & \text{if} \quad \varepsilon \le |z| \le 1/\varepsilon, \\ \frac{\varepsilon}{2} z^2 + \frac{1}{2\varepsilon} & \text{if} \quad |z| \ge 1/\varepsilon. \end{cases}$$

Note that

$$\varphi_\varepsilon(z) \ge |z| \quad \text{and} \quad \lim_{\varepsilon \to 0} \varphi_\varepsilon(z) = |z|, \quad \text{pointwise.}$$

We now consider the following relaxed functional

$$J_\varepsilon(u) = \|Ku - g\|^2_{L^2(\Omega)} + 2\lambda E_\varepsilon(u) = \|Ku - g\|^2_{L^2(\Omega)} + 2\lambda \int_\Omega \varphi_\varepsilon(|\nabla u|)\, dx, \tag{2.9}$$

which approximates J pointwise from above, i.e.,

$$J_\varepsilon(u) \ge J(u), \tag{2.10}$$

and

$$\lim_{\varepsilon \to 0} J_\varepsilon(u) = J(u). \tag{2.11}$$

Since J_ε is convex and smooth, by taking the Euler-Lagrange equations, we have that u_ε is a minimizer for J_ε if and only if

$$-\lambda \operatorname{div}\left(\frac{\varphi_\varepsilon'(|\nabla u|)}{|\nabla u|}\nabla u\right) + K^*(Ku-g) = 0. \tag{2.12}$$

2.3 Euler-Lagrange Equations and a Relaxation Algorithm

In this section we want to provide an algorithm to compute efficiently minimizers of the approximating functionals J_ε. First, we want to derive the Euler-Lagrange equations associated to J_ε. In the following we assume that φ_ε is continuously differentiable and Ω is an open, bounded and connected subset of $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ (see [25]).

Proposition 2.13. *If u is a minimizer in $W^{1,2}(\Omega) = H^1(\Omega)$ of J_ε, then u solves the following of Euler-Lagrange equations*

$$\begin{cases} 0 = -\lambda\, div\left(\frac{\varphi_\varepsilon'(|\nabla u|)}{|\nabla u|}\nabla u\right) + K^*(Ku-g) \quad in \quad \Omega, \\ \frac{\varphi_\varepsilon'(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial \vec{\nu}} = 0 \quad \partial\Omega. \end{cases} \tag{2.13}$$

The equations (2.13) are the necessary condition for the computation of minimizers of J_ε. The nonlinear $\operatorname{div}\left(\frac{\varphi_\varepsilon'(|\nabla u|)}{|\nabla u|}\nabla u\right)$ term constitutes the main complication for the numerical solution of these equations. Sometimes, the second term is not easy to treat.

In order to compute efficiently solutions of (2.13), we introduce a new functional given by:

$$\widetilde{J}(u,w) = \|Ku-g\|^2_{L^2(\Omega)} + 2\lambda\int_\Omega \left(w|\nabla u(x)|^2 + \frac{1}{w}\right)dx, \tag{2.14}$$

where $u \in W^{1,2}(\Omega) := V$ and $w \in L^2(\Omega)$ is such that $\varepsilon \le w \le 1/\varepsilon$ almost everywhere. While the variable u is the function to be reconstructed, the function w is called the *gradient weight.*

For any given $u^{(0)} \in V$ and $w \in L^2(\Omega)$ (for example $w^{(0)} := 1$), we define the following iterative double-minimization algorithm :

$$\begin{cases} u^{(k+1)} = \arg\min_{u\in V} \widetilde{J}(u, w^{(k)}), \\ w^{(k+1)} = \arg\min_{\varepsilon\le w\le \frac{1}{\varepsilon}} \widetilde{J}(u^{(k+1)}, w). \end{cases} \tag{2.15}$$

We have the following convergence result (see [25]). We will include the proof for the sake of completeness of this thesis.

Theorem 2.14. *The sequence $(u^k)_{k\in\mathbb{N}}$ has subsequences that converge to a minimizer $u_\varepsilon := u^{(\infty)}$ of J_ε. If J_ε has a unique minimizer u^*, then $u^{(\infty)} = u^*$ and the full sequence $(u^k)_{k\in\mathbb{N}}$ converges to u^*.*

Proof. The proof we present here follows from [25]. Observe that

$$\widetilde{J}(u^{(k)}, w^{(k)}) - \widetilde{J}(u^{(k+1)}, w^{(k+1)}) = \underbrace{(\widetilde{J}(u^{(k)}, w^{(k)}) - \widetilde{J}(u^{(k+1)}, w^{(k)}))}_{A_k} + \underbrace{(\widetilde{J}(u^{(k+1)}, w^{(k)}) - \widetilde{J}(u^{(k+1)}, w^{(k+1)}))}_{B_k} \geq 0.$$

Therefore $\widetilde{J}(u^{(k)}, w^{(k)})$ is a non-increasing sequence and moreover it is bounded from below, since

$$\inf_{\varepsilon \leq w \leq \frac{1}{\varepsilon}} \int_\Omega \left(w|\nabla u(x)|^2 + \frac{1}{w} \right) dx \geq 0.$$

This implies that $\widetilde{J}(u^{(k)}, w^{(k)})$ converges. Moreover, we can write

$$B_k = \int_\Omega c(w^{(k)}, |\nabla u^{(k+1)}(x)|) - c(w^{(k+1)}, |\nabla u^{(k+1)}(x)|),$$

where $c(t, z) := tz^2 + \frac{1}{t}$. By Taylor's formula, we have

$$c(w^{(k)}, z) = c(w^{(k+1)}, z) + \frac{\partial c}{\partial t}(w^{(k+1)}, z)(w^{(k)} - w^{(k+1)}) + \frac{1}{2}\frac{\partial^2 c}{\partial t^2}(\xi, z)|w^{(k)} - w^{(k+1)}|^2,$$

for $\xi \in \operatorname{conv}(w^{(k)}, w^{(k+1)})$ (the segment between $w^{(k)}$ and $w^{(k+1)}$). By definition of $w^{(k+1)}$, and taking into account that $\varepsilon \leq w^{(k+1)} \leq \frac{1}{\varepsilon}$, we have

$$\frac{\partial c}{\partial t}(w^{(k+1)}, |\nabla u^{(k+1)}(x)|)(w^{(k)} - w^{(k+1)}) \geq 0,$$

and $\frac{\partial^2 c}{\partial t^2}(t, z) = \frac{2}{t^3} \geq 2\varepsilon^3$, for any $t \leq 1/\varepsilon$. This implies that

$$\widetilde{J}(u^{(k)}, w^{(k)}) - \widetilde{J}(u^{(k+1)}, w^{(k+1)}) \geq B_k \geq \varepsilon^3 \int_\Omega |w^{(k)}(x) - w^{(k+1)}(x)|^2 dx,$$

and since $\widetilde{J}(u^{(k)}, w^{(k)})$ is convergent, we have

$$\|w^{(k)} - w^{(k+1)}\|_{L^2(\Omega)} \to 0, \tag{2.16}$$

for $n \to \infty$. Since $u^{(k+1)}$ is a minimizer of $J(u, w^{(k)})$, it solves the following system of variational equation

$$\int_\Omega \left(w^{(k)} \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x) + \tilde{\lambda}(Ku^{(k+1)} - g)(x) K\varphi(x) \right) = 0, \tag{2.17}$$

for all $\varphi \in V$ and $\tilde{\lambda} := \lambda^{-1}$. Therefore we can write

$$\int_\Omega \Big(w^{(k+1)} \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x) + \tilde{\lambda}(K u^{(k+1)} - g)(x) K\varphi(x) \Big)$$
$$= \int_\Omega (w^{(k+1)} - w^{(k)}) \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x),$$

and for $\frac{1}{p} + \frac{1}{q} + \frac{1}{2} = 1$, we have

$$\left| \int_\Omega \Big(w^{(k+1)} \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x) + (K u^{(k+1)} - g)(x) K\varphi(x) \Big) \right|$$
$$\leq \|w^{(k+1)} - w^{(k)}\|_{L^p} \|\nabla u^{(k+1)}\|_{L^q} \|\nabla \varphi\|_{L^2}.$$

By monotonicity of $(J(u^{(k+1)}, w^{(k+1)}))_k$, and since $w^{(k+1)} = \frac{\varphi_\varepsilon'(|\nabla u^{(k+1)}|)}{|\nabla u^{(k+1)}|}$, we have

$$\begin{aligned} \widetilde{J}(u^1, w^0) \geq \widetilde{J}(u^{(k+1)}, w^{(k+1)}) = J_\varepsilon(u^{(k+1)}) \geq J(u^{(k+1)}) &\geq C|\nabla u|(\Omega) \\ &\geq C\|\nabla u^{(k+1)}\|_{L^2(\Omega)}. \end{aligned}$$

Moreover, since $J_\varepsilon(u^{(k+1)}) \geq J(u^{(k+1)})$ and J is coercive, we have that $\|u^{(k+1)}\|_{L^2(\Omega)}$ and $\|\nabla u^{(k+1)}\|_{L^2}$ are uniformly bounded with respect to k. Therefore, using (2.16), we can conclude that

$$\int_\Omega \Big(w^{(k+1)} \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x) + \tilde{\lambda}(K u^{(k+1)} - g)(x) K\varphi(x) \Big) \to 0,$$

for $k \to \infty$, and there exists a subsequence $(u^{(k_p+1)})_{p\in\mathbb{N}}$ that converges in V to a function $u^{(\infty)}$. Since $w^{(k_p+1)} = \frac{\varphi_\varepsilon'(|\nabla u^{(k_p+1)}|)}{|\nabla u^{(k_p+1)}|}$, and by taking the limit for $p \to \infty$, we obtain

$$-\lambda \text{div} \left(\frac{\varphi_\varepsilon'(|\nabla u^{(\infty)}|)}{|\nabla u^{(\infty)}|} \nabla u^{(\infty)} \right) + K^* \left(K u^{(\infty)} - g \right) = 0. \tag{2.18}$$

This is the Euler-Lagranage equation (2.12) associated to the functional J_ε and therefore $u^{(\infty)}$ is a minimizer of J_ε.

Assume now that J_ε has a unique minimizer u^*. Then necessarily $u^{(\infty)} = u^*$. Since every subsequence of $(u^k)_k$ has a subsequence converging to u^*, the full sequence $(u^k)_k$ converges to u^*. □

Since both J_ε and $\widetilde{J}(\cdot, w)$ admit minimizers, their uniqueness is equivalent to the uniqueness of the solutions of the corresponding Euler-Lagrange equations. If uniqueness of the solution is satisfied, then the algorithm (2.15) can be equivalently reformulated as the following two-step iterative procedure: Given $w^{(0)} \in L^\infty(\Omega)$, for $k = 0, 1, ...$ define :

- Find $u^{(k+1)} \in V$:

$$\int_\Omega \Big(w^{(k+1)} \nabla u^{(k+1)}(x) \cdot \nabla \varphi(x) + \tilde{\lambda}(Ku^{(k+1)} - g)(x) K\varphi(x) \Big) = 0, \quad \forall \varphi \in V.$$

- Compute directly $w^{(k+1)}$ by

$$w^{(k+1)} = \varepsilon \vee \frac{1}{|\nabla u^{(k+1)}|} \wedge \frac{1}{\varepsilon} := \min \left(\max \left(\varepsilon, \frac{1}{|\nabla u^{(k+1)}|} \right), \frac{1}{\varepsilon} \right).$$

By a standard fixed point argument, the solution of the equation is unique for $\lambda \sim \varepsilon$. The condition $\lambda \sim \varepsilon$ is acceptable only for those applications where the constraints on the data are weak, e.g., when the data is affected by a strong noise (see [25]).

The following result establishes the convergence of the algorithm.

Theorem 2.15. *Let us assume that $(\varepsilon_j)_{j\in\mathbb{N}}$ is a sequence of positive numbers monotonically converging to zero. The accumulation points of the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ of the minimizers of J_{ε_j} are minimizers of J.*

Remark 2.16. *The proof requires the notion of $\Gamma - Convergence$. The minimizers of a relaxed functional $\bar{J}$ can be approximated by the minimum points of functionals that are defined in $W^{1,2}(\Omega)$ (see [5], Section 2.1.4).*

Chapter 3

DG Finite Element Discretization

In [4], Arnold et al. present a uniform analysis of the Discontinuous Galerkin Finite Element Methods. In this chapter, we start with definitions of some function spaces and derive the standard and the DG variational formulations of our model problem which follows mostly the work of Groosmann, Roos and Stynes [28] and B. Rivière [39]. Furthermore, we present a short introduction to the theory of DG finite element discretization applied to the variational setting of our PDE. Particularly, we investigate the discretization errors which we gain from the discontinuous Galerkin finite element method. Finally, we present the DG-version of the *iteratively re-weighted least squares algorithm* for solving our TV minimization problem.

3.1 Some Basic Function Spaces

We start with the definition of some functions spaces [1].

Definition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. The Lebesgue space $L^p(\Omega)$ is given by*

$$L^p(\Omega) = \left\{ v : \Omega \to \mathbb{R} \,|\, \|v\|_{L^p(\Omega)} < \infty \right\},$$

where the norm is defined by

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v|^p dx \right)^{1/p}, \quad 1 \le p < \infty,$$

and

$$\|v\|_{L^\infty(\Omega)} = ess\,sup \left\{ |v(x)| : x \in \Omega \right\}, \quad for \quad p = \infty.$$

The Lebesgue spaces $L^p(\Omega)$ are Banach spaces. The space $L^2(\Omega)$ is a Hilbert space. The Sobolev spaces $H^s(\Omega)$ are subspaces of $L^2(\Omega)$. They play a fundamental role in the variational treatment of second-order partial differential equations.

Definition 3.2. *For a non-negative integer s, the* ***Sobolev space*** *$H^s(\Omega)$ is defined by*

$$H^s(\Omega) = \left\{ v \in L^2(\Omega) |\; D^\alpha v \in L^2(\Omega) \quad \forall\; |\alpha| \le s \right\},$$

where

$$D^\alpha v = \partial_{x_1}^{\alpha_1}, ..., \partial_{x_n}^{\alpha_n} := \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1, ..., \partial^{\alpha_n} x_n}$$

*denotes the weak derivative of the order α, where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ is a **multi-index** and $|\alpha| = \sum_{i=1}^n \alpha_i$.*

The Sobolev space $H^s(\Omega)$ is equipped with the norm

$$\|v\|_s := \left(\sum_{0 \le |\alpha| \le s} \|D^\alpha v\|^2_{L^2(\Omega)} \right)^{1/2} . \tag{3.1}$$

Correspondingly, a semi-norm on this space is defined as

$$|v|_s := \left(\sum_{|\alpha| = s} \|D^\alpha v\|^2_{L^2(\Omega)} \right)^{1/2} . \tag{3.2}$$

3.2 DG Variational Formulations

In this section, we will derive the standard and the DG variational formulations. Furthermore, we will define some special function spaces needed for the DG formulations. Let us consider the following Neumann problem as model problem: Find u such that

$$-\nabla \cdot (w \nabla u) + \tilde{\lambda} u = \tilde{\lambda} g \quad \text{in} \quad \Omega, \tag{3.3}$$

$$w \nabla u \cdot \vec{\nu} = 0 \quad \text{on} \quad \Gamma = \partial\Omega. \tag{3.4}$$

Here $\vec{\nu}$ is the outer unit normal, $w \in L^\infty(\Omega)$ is assumed to be unifomly, $g \in L^2(\Omega)$, and $\Omega \subset \mathbb{R}^2$ is a bounded polygonal Lipschitz domain. We recall from (2.17) that $\tilde{\lambda} = \lambda^{-1}$, where λ is a positive regularization parameter. The standard variational formulation of the Neumann problem (3.3) - (3.4) reads as follows: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega (w \nabla u \cdot \nabla v + \tilde{\lambda} u v)\, dx = \int_\Omega \tilde{\lambda} g v \, dx \quad \forall v \in H^1(\Omega). \tag{3.5}$$

The existence and uniqueness of a solution of (3.5) immediately follows from Lax-Milgram's lemma.

Let us now derive the DG variational formulation. We start with a decomposition $\mathcal{T}$ of Ω into triangles or rectangles T such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T} \quad \text{and} \quad T_i \cap T_j = \emptyset \quad \text{if} \quad T_i, T_j \in \mathcal{T}, \quad T_i \neq T_j.$$

We assume that $\mathcal{T}$ is a *regular* triangulation, i.e., the intersection of any two elements is either empty or a common vertex or edge. We also assume that the elements are *shape regular* i.e., there exists a constant γ such that

$$\frac{h_T}{\rho_T} \le \gamma, \quad \forall T \in \mathcal{T},$$

where h_T denotes the diameter of the element T and ρ_T is the diameter of the largest ball inscribed in T .

To each element $T \in \mathcal{T}$ we assign a non-negative integer s_T and define the "broken" Sobolev space of the order $\mathbf{s} = \{s_T : T \in \mathcal{T}\}$ by

$$H^{\mathbf{s}}(\Omega;\mathcal{T}) := \{v \in L^2(\Omega) : v|_T \in H^{s_T}(T), \quad \forall T \in \mathcal{T}\}.$$

The associated norm and seminorm are

$$\|v\|_{\mathbf{s},\mathcal{T}} = \left(\sum_{T\in\mathcal{T}} \|v\|^2_{H^{s_T}(T)}\right)^{\frac{1}{2}} \quad \text{and} \quad |v|_{\mathbf{s},\mathcal{T}} = \left(\sum_{T\in\mathcal{T}} |v|^2_{H^{s_T}(T)}\right)^{\frac{1}{2}}, \text{ respectively.}$$

If $s_T = s$ for all $T \in \mathcal{T}$, we write $\|v\|_{s,\mathcal{T}}$ and $|v|_{s,\mathcal{T}}$ instead of $\|v\|_{H^{\mathbf{s}}(\Omega;\mathcal{T})}$ and $|v|_{H^{\mathbf{s}}(\Omega;\mathcal{T})}$. If $v \in H^1(\Omega;\mathcal{T})$ then the composite gradient $\nabla_{\mathcal{T}} v$ of a function v is defined by $(\nabla_{\mathcal{T}} v)|_T = \nabla(v|_T)$, $T \in \mathcal{T}$.

In the following, we assume that each element $T \in \mathcal{T}$ is the affine image of a rectangular reference element $\widehat{T}$ (unit square), i.e., $T = F_T(\widehat{T})$. The finite element space is defined by

$$\mathcal{V}_h(\Omega;\mathcal{T},\mathbf{F}) = \{v \in L^2(\Omega) : v|_T \circ F_T \in \mathbb{Q}_1(\widehat{T})\}, \tag{3.6}$$

where $\mathbf{F} = \{F_T : T \in \mathcal{T}\}$ and $\mathbb{Q}_1(\widehat{T})$ is the space of linear polynomials of degree one in each space direction on $\hat{T}$. Note that the functions in $\mathcal{V}_h \equiv \mathcal{V}_h(\Omega;\mathcal{T},\mathbf{F})$ may be discontinuous across element edges.

Let $\mathcal{E}$ be the set of all edges of the given triangulation $\mathcal{T}$, with $\mathcal{E}_{int} \subset \mathcal{E}$ the set of all interior edges $e \in \mathcal{E}$ in Ω. Set $\Gamma_{int} = \{x \in \Omega : x \in e \quad \text{for some} \quad e \in \mathcal{E}_{int}\}$. Let the elements of $\mathcal{T}$ be numbered sequentially: $T_1, T_2, \ldots$. Then for each $e \in \mathcal{E}_{int}$ there exist indices i and j such that $i > j$ and $e = \overline{T_i} \cap \overline{T_j}$. Set $T := T_i$ and $T' := T_j$. Define the *jump* (which depends on the enumeration of the triangulation) and *average* of each function $v \in H^1(\Omega,\mathcal{T})$ on $e \in \mathcal{E}_{int}$ by

$$[\![v]\!]_e = (v|_{\partial T\cap e} - v|_{\partial T'\cap e}), \qquad \{v\}_e = \frac{1}{2}(v|_{\partial T\cap e} + v|_{\partial T'\cap e}).$$

Furthermore, to each edge $e \in \mathcal{E}_{int}$ we assign a unit normal vector $\vec{\nu}$ directed from T to T'. If instead $e \subset \Gamma$ then we take the outward-pointing unit normal vector $\vec{\nu}$ on Γ. When there is no danger of misinterpretation we omit the indices in $[\![v]\!]_e$ and $\{v\}_e$.

For simplicity, we shall assume that the solution u of (3.5) belongs to $H^2(\Omega) \subset H^2(\Omega;\mathcal{T})$. Let us mention that, for more general problems, it is standard to assume that $u \in H^2(\Omega;\mathcal{T})$ and that both u and $\nabla u \cdot \vec{\nu}$ are continuous across all interior edges, where $\vec{\nu}$ is a normal to the edge. In particular, we have

$$[\![u]\!]_e = 0, \quad \{u\}_e = u, \quad e \in \mathcal{E}_{int}.$$

Multiply the differential equation (3.3) by a (possibly discontinuous) test function $v \in H^1(\Omega, \mathcal{T})$ and integrate over Ω, we obtain

$$\int_\Omega \left(-\nabla \cdot (w\nabla u) + \tilde{\lambda} u\right) v \, dx = \int_\Omega \tilde{\lambda} g v \, dx. \tag{3.7}$$

First, we consider the term $-\nabla \cdot (w\nabla u)$ in (3.7). Let $\vec{\nu}_T$ denote the outward-pointing unit normal to ∂T for each $T \in \mathcal{T}$. Integration by parts and elementary transformations give us

$$\begin{aligned}
\int_\Omega (-\nabla \cdot (w\nabla u)) v \, dx &= \sum_{T\in\mathcal{T}} \int_T (w\nabla u) \cdot \nabla \, v \, dx - \sum_{T\in\mathcal{T}} \int_{\partial T} (w\nabla u \cdot \vec{\nu}_T) \, v \, ds \\
&= \sum_{T\in\mathcal{T}} \int_T (w\nabla u) \cdot \nabla v \, dx - \sum_{e\in\mathcal{E}\cap\Gamma} \int_e (w\nabla u \cdot \vec{\nu}) \, v \, ds \\
&- \sum_{e\in\mathcal{E}_{int}} \int_e \Big(((w\nabla u \cdot \vec{\nu}_T) v)|_{\partial T\cap e} + ((w\nabla u \cdot \vec{\nu}_{T'}) v)|_{\partial T'\cap e} \Big) \, ds.
\end{aligned}$$

Let $\vec{\nu}$ be the unit normal vector points from T to T', then the sum of the integrals over $e \in \mathcal{E}_{int}$ can be written as

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_{int}} \int_e \left(((w\nabla u \cdot \vec{\nu}_T) v)|_{\partial T\cap e} + ((w\nabla u \cdot \vec{\nu}_{T'}) v)|_{\partial T'\cap e} \right) ds \\
&= \sum_{e\in\mathcal{E}_{int}} \int_e \left(((w\nabla u \cdot \vec{\nu}) v)|_{\partial T\cap e} - ((w\nabla u \cdot \vec{\nu}) v)|_{\partial T'\cap e} \right) ds \\
&= \sum_{e\in\mathcal{E}_{int}} \int_e \left(\{w\nabla u \cdot \vec{\nu}\}_e \, [\![v]\!]_e + [\![w\nabla u \cdot \vec{\nu}]\!]_e \{v\}_e \right) ds \qquad (3.8) \\
&= \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla u \cdot \vec{\nu}\}_e \, [\![v]\!]_e \, ds.
\end{aligned}$$

Here the basic relation

$$(w_1 \cdot \vec{\nu}_1) z_1 + (w_2 \cdot \vec{\nu}_2) z_2 = \{w \cdot \vec{\nu}\}[\![z]\!] + [\![w \cdot \vec{\nu}]\!]\{z\}, \tag{3.9}$$

is used in (3.8). Applying the boundary condition and using the abbreviation

$$\sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla u \cdot \vec{\nu}\}_e \, [\![v]\!]_e \, ds = \int_{\Gamma_{int}} \{w\nabla u \cdot \vec{\nu}\} \, [\![v]\!] \, ds, \tag{3.10}$$

we get

$$\int_\Omega (-\nabla \cdot (w\nabla u)) v \, dx = \sum_{T\in\mathcal{T}} \int_T (w\nabla u) \cdot \nabla \, v \, dx - \int_{\Gamma_{int}} \{w\nabla u \cdot \vec{\nu}\} \, [\![v]\!] \, ds. \tag{3.11}$$

Finally, we add the terms

$$\int_{\Gamma_{int}} \sigma [\![u]\!] \, [\![v]\!] \, ds \quad \text{and} \quad \tau \int_{\Gamma_{int}} [\![u]\!] \{w\nabla v \cdot \vec{\nu}\} \, ds, \tag{3.12}$$

where the choices of $\tau = \pm 1, 0$ will define different DG methods (see below). The terms (3.12) vanish for the exact solution $u \in H^1(\Omega, \mathcal{T})$. The *penalty parameter* σ is piecewise constant, i.e.,

$$\sigma|_e = \sigma_e := \kappa h_e^{-1}, \quad e \in \mathcal{E}, \quad \kappa \geq 0.$$

Finally, we obtain

$$\int_\Omega (-\nabla \cdot (w\nabla u))v\, dx = \sum_{T\in\mathcal{T}} \int_T (w\nabla u) \cdot \nabla\, v\, dx - \int_{\Gamma_{int}} \{w\nabla u \cdot \vec{\nu}\}\, [\![v]\!]\, ds.$$
$$+ \tau \int_{\Gamma_{int}} [\![u]\!] \{w\nabla v \cdot \vec{\nu}\}\, ds + \int_{\Gamma_{int}} \sigma [\![u]\!]\, [\![v]\!]\, ds.$$

We can now give the primal formulation of the discontinuous Galerkin method with interior penalties, and its relation to the standard variational formulation (3.5).

Theorem 3.3. *If $u \in H^2(\Omega, \mathcal{T})$ is a solution to (3.5), then u also solves*

$$\mathcal{A}(u, v) = g(v), \quad \forall v \in H^2(\Omega, \mathcal{T}), \tag{3.13}$$

where the linear form is given by the relation

$$g(v) = \sum_{T\in\mathcal{T}} \int_T \tilde{\lambda} g v\, dx, \tag{3.14}$$

and the bilinear form is given by the expression

$$\mathcal{A}(u, v) = \sum_{T\in\mathcal{T}} \left(\int_T w\nabla u \cdot \nabla v\, dx + \int_T \tilde{\lambda} uv\, dx \right) - \int_{\Gamma_{int}} \{w\nabla u \cdot \vec{\nu}\}\, [\![v]\!]\, ds$$
$$+ \tau \int_{\Gamma_{int}} [\![u]\!] \{w\nabla v \cdot \vec{\nu}\}\, ds + \int_{\Gamma_{int}} \kappa h_e^{-1} [\![u]\!] [\![v]\!]\, ds. \tag{3.15}$$

The following choices for τ and κ define some of the well known DG methods :

- $\tau = -1$ and $\kappa \geq \kappa_0$ sufficiently large define the symmetric interior penalty (SIPG) method [4, 3, 44].
- $\tau = +1$ and $\kappa > 0$ define the non-symmetric interior penalty (NIPG) method [4, 39].
- $\tau = +1$ and $\kappa = 0$ define the method of Baumann and Oden [6, 35].

The proof of Theorem 3.3 follows by taking an arbitrary function $v \in H^2(\Omega, \mathcal{T})$, multiplying (3.3) by v and integrate by parts over each element. The desired consistency is achieved by applying the algebraic equality (3.9) and the boundary condition from (3.4) (see [39] for detailed proof).

3.3 Some Basic Properties

Next, we look at some basic properties of these methods (see [39, 28, 19]). More precisely we will consider the SIPG method. Firstly, we state the trace inequality as a requirement for the proof of the **coercivity property** and then proceed with some other properties.

Theorem 3.4. *Let T be a bounded polygonal domain with boundary ∂T and diameter h_T. Let e be an edge and $\vec{\nu}$ a unit outward normal vector to e. Let $p > 0$ be an integer. There exists a constant C independent of h_T such that*

$$\|v\|_{L^2(e)} \leq C\, h_T^{-1/2} \|v\|_{L^2(T)}, \quad \forall v \in \mathbb{Q}_p(T), \quad \forall e \subset \partial T, \tag{3.16}$$

and

$$\|\nabla v \cdot \vec{\nu}\|_{L^2(e)} \leq C\, h_T^{-1/2} \|\nabla v\|_{L^2(T)}, \quad \forall v \in \mathbb{Q}_p(T), \quad \forall e \subset \partial T. \tag{3.17}$$

Theorem 3.5 (*Consistency,* [39])**.** *All three of the above methods are consistent. That is, if the exact solution u to (3.5) is in $H^s(\Omega)$ for some $s > 3/2$ then we have*

$$\mathcal{A}(u, v) = g(v), \quad \forall v \in H^2(\Omega, \mathcal{T}). \tag{3.18}$$

We now look at the continuity and coercivity properties of the bilinear form $\mathcal{A}(\cdot, \cdot)$ with respect to the DG norm:

$$|||v||| = \left(\sum_{T \in \mathcal{T}} \int_T (w \nabla v) \cdot \nabla v \, dx + \int_\Omega \tilde{\lambda} v^2 \, dx + \int_{\Gamma_{int}} \kappa h_e^{-1} [\![v]\!]^2 \, ds \right)^{\frac{1}{2}}, \quad \forall v \in H^1(\Omega, \mathcal{T}). \tag{3.19}$$

We will also show the proof for coercivity for the sake of completeness of the thesis.

Definition 3.6 (*Coercivity*)**.** *The bilinear form $\mathcal{A}(\cdot, \cdot)$ is coercive on $\mathcal{V}_h$ if there exists a constant $C > 0$ such that*

$$\mathcal{A}(v, v) \geq C |||v|||^2 \quad \forall v \in \mathcal{V}_h. \tag{3.20}$$

For the SIPG bilinear form, we have

$$\begin{aligned} \mathcal{A}(v, v) = \sum_{T \in \mathcal{T}} \left(\int_T w (\nabla v)^2 \, dx + \int_T \tilde{\lambda} v^2 \, dx \right) - 2 \int_{\Gamma_{int}} \{w \nabla v \cdot \vec{\nu}\} [\![v]\!] \, ds \\ + \int_{\Gamma_{int}} \kappa h_e^{-1} [\![v]\!]^2 \, ds. \end{aligned} \tag{3.21}$$

By using the Cauchy-Schwarz inequality[1] we obtain an upper bound for the second term

$$\int_e \{w \nabla v \cdot \vec{\nu}\} [\![v]\!] \, ds \leq \|\{w \nabla v \cdot \vec{\nu}\}\|_{L^2(e)} \|[\![v]\!]\|_{L^2(e)}.$$

[1] Cauchy-Schwarz inequality : $\forall x, y \in L^2(\Omega), \quad |(x, y)_\Omega| \leq \|x\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)}$.

Next, we estimate the average of the fluxes for an interior edge e shared by T and T' as follows

$$\|\{w\nabla v\cdot\vec{\nu}\}\|_{L^2(e)} \le \frac{1}{2}\|w\nabla v|_T\|_{L^2(e)} + \frac{1}{2}\|w\nabla v|_{T'}\|_{L^2(e)}.$$

Using the trace inequality (3.17), we find

$$\begin{aligned}\|\{w\nabla v\cdot\vec{\nu}\}\|_{L^2(e)} &\le \frac{1}{2}\|w\nabla v|_T\|_{L^2(e)} + \frac{1}{2}\|w\nabla v|_{T'}\|_{L^2(e)}\\ &\le \frac{C_*}{2}h_T^{-\frac{1}{2}}\|w\nabla v\|_{L^2(T)} + \frac{C_*}{2}h_{T'}^{-\frac{1}{2}}\|w\nabla v\|_{L^2(T')},\end{aligned} \tag{3.22}$$

where C_* is independent of v and h. Also we have

$$|h_e| \le h_T \le h, \quad \forall e\subset\partial T,\, T\in\mathcal{T}.$$

Using (3.22), we obtain

$$\begin{aligned}\int_e\{w\nabla v\cdot\vec{\nu}\}[\![v]\!] &\le \frac{C_*}{2}|h_e|^{\frac{1}{2}}\left(h_T^{-\frac{1}{2}}\|w\nabla v\|_{L^2(T)} + h_{T'}^{-\frac{1}{2}}\|w\nabla v\|_{L^2(T')}\right)\\ &\times\left(|h_e|^{-\frac{1}{2}}\|[\![v]\!]\|_{L^2(e)}\right)\\ &\le \frac{C_*}{2}\left(|h_e|^{\frac{1}{2}}h_T^{-\frac{1}{2}} + |h_e|^{\frac{1}{2}}h_{T'}^{-\frac{1}{2}}\right)\left(\|w\nabla v\|^2_{L^2(T)} + \|w\nabla v\|^2_{L^2(T')}\right)^{\frac{1}{2}}\\ &\times\left(|h_e|^{-\frac{1}{2}}\|[\![v]\!]\|_{L^2(e)}\right)\\ &\le C_*\left(\|w\nabla v\|^2_{L^2(T)} + \|w\nabla v\|^2_{L^2(T')}\right)^{\frac{1}{2}}\left(|h_e|^{-\frac{1}{2}}\|[\![v]\!]\|_{L^2(e)}\right).\end{aligned}$$

A similar bound is obtained if e is a boundary edge. Using the Cauchy-Schwarz inequality and Young's inequality for $\delta > 0$ [2], we obtain

$$\begin{aligned}\sum_{e\in\mathcal{E}_{int}}\int_e\{w\nabla v\cdot\vec{\nu}\}[\![v]\!]\,ds &\le C_*\sum_{e\in\mathcal{E}_{int}}\left(\|w\nabla v|_T\|^2_{L^2(T)} + \|w\nabla v|_{T'}\|^2_{L^2(T')}\right)^{\frac{1}{2}}\left(|h_e|^{-\frac{1}{2}}\|[\![v]\!]\|_{L^2(e)}\right)\\ &\le C_*\left(\sum_{T\in\mathcal{T}}\|w\nabla v\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{int}}|h_e|^{-1}\|[\![v]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\le C_* w^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}}\|w^{\frac{1}{2}}\nabla v\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{int}}|h_e|^{-1}\|[\![v]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\le \frac{\delta}{2}\sum_{T\in\mathcal{T}}\|w^{\frac{1}{2}}\nabla v\|^2_{L^2(T)} + \frac{\widetilde{C}}{2\delta}\sum_{e\in\mathcal{E}_{int}}|h_e|^{-1}\|[\![v]\!]\|^2_{L^2(e)}\end{aligned}$$

We obtain a lower bound for $\mathcal{A}(v,v)$:

$$\mathcal{A}(v,v) \ge (1-\delta)\sum_{T\in\mathcal{T}}\|w^{\frac{1}{2}}\nabla v\|^2_{L^2(T)} + \|\tilde{\lambda}\|_{L^\infty(\Omega)}\|v\|^2_{L^2(\Omega))} + \left(\kappa - \frac{\widetilde{C}}{\delta}\right)\sum_{e\in\mathcal{E}_{int}}|h_e|^{-1}\|[\![v]\!]\|^2_{L^2(e)}$$

[2] Young's inequality : $\forall a,b\in\mathbb{R},\ \forall\delta>0,\quad ab \le \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2$

where the positive constant $\widetilde{C}$ is independent of h_T. We achieve the coercivity result with $C = 1/2$ from (3.20) if we choose $\delta = 1/2$ and $\kappa_* = 1/2 + \widetilde{C}/\delta$.

Definition 3.7 (*Continuity*, [8])**.** *If $\kappa > 0$ for all ∂T, then the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous on $\mathcal{V}_h$ equipped with the energy norm $|||\cdot|||$, if there exists $\widetilde{C} > 0$ such that*

$$\mathcal{A}(u,v) \le \widetilde{C}\, |||u|||\ |||v||| \quad \forall\ u,v \in \mathcal{V}_h.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\mathcal{A}(u,v)| \le & \left|\sum_{T\in\mathcal{T}} \left(\int_T w\nabla u\cdot\nabla v\,dx + \int_T \tilde{\lambda} uv\,dx\right)\right| + \left|\int_{\Gamma_{int}} \{w\nabla u\cdot\vec{\nu}\}[\![v]\!]\,ds\right| \\
& + \left|\int_{\Gamma_{int}} \{w\nabla v\cdot\vec{\nu}\}[\![u]\!]\,ds\right| + \left|\int_{\Gamma_{int}} \kappa h_e^{-1}[\![u]\!][\![v]\!]\,ds\right|. \\
& \le \sum_{T\in\mathcal{T}} \|w^{\frac12}\nabla u\|_{L^2(T)}\|w^{\frac12}\nabla v\|_{L^2(T)} + \|\tilde{\lambda}^{\frac12}u\|_{L^2(\Omega)}\|\tilde{\lambda}^{\frac12}v\|_{L^2(\Omega)} \\
& + \sum_{e\in\mathcal{E}_{int}} \|\{w\nabla u\cdot\vec{\nu}\}\|_{L^2(e)}\,\|[\![v]\!]\|_{L^2(e)} + \sum_{e\in\mathcal{E}_{int}} \|\{w\nabla v\cdot\vec{\nu}\}\|_{L^2(e)}\,\|[\![u]\!]\|_{L^2(e)} \\
& + \sum_{e\in\mathcal{E}_{int}} \kappa h_e^{-1}\,\|[\![u]\!]\|_{L^2(e)}\,\|[\![v]\!]\|_{L^2(e)} \\
& \le C w^{\frac12} \sum_{T\in\mathcal{T}} \left(\|w^{\frac12}\nabla u\|^2_{L^2(T)}\right)^{\frac12}\left(\|w^{\frac12}\nabla v\|^2_{L^2(T)}\right)^{\frac12} + \left(\|\tilde{\lambda}^{\frac12}u\|^2_{L^2(\Omega)}\right)^{\frac12}\left(\|\tilde{\lambda}^{\frac12}v\|^2_{L^2(\Omega)}\right)^{\frac12} \\
& + C w^{\frac12} \sum_{e\in\mathcal{E}_{int}} \left(\|w^{\frac12}\nabla u\|^2_{L^2(T)}\right)^{\frac12}\left(h_e^{-1}\|[\![v]\!]\|^2_{L^2(e)}\right)^{\frac12} + C w^{\frac12} \sum_{e\in\mathcal{E}_{int}} \left(\|w^{\frac12}\nabla v\|^2_{L^2(T)}\right)^{\frac12} \\
& \quad \times \left(h_e^{-1}\|[\![u]\!]\|^2_{L^2(e)}\right)^{\frac12} + \sum_{e\in\mathcal{E}_{int}} \left(\kappa h_e^{-1}\,\|[\![u]\!]\|^2_{L^2(e)}\right)^{\frac12}\left(\kappa h_e^{-1}\,\|[\![v]\!]\|^2_{L^2(e)}\right)^{\frac12} \qquad (3.23) \\
& \le \widetilde{C}\left(\sum_{T\in\mathcal{T}} \|w^{\frac12}\nabla u\|^2_{L^2(T)} + \|\tilde{\lambda}^{\frac12}u\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_{int}} \kappa h_e^{-1}\|[\![u]\!]\|^2_{L^2(e)}\right)^{\frac12} \\
& \quad \times \left(\sum_{T\in\mathcal{T}} \|w^{\frac12}\nabla v\|^2_{L^2(T)} + \|\tilde{\lambda}^{\frac12}v\|^2_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_{int}} \kappa h_e^{-1}\|[\![v]\!]\|^2_{L^2(e)}\right)^{\frac12} \\
& \le \widetilde{C}|||u|||\ |||v||| \qquad (3.24)
\end{aligned}$$

where $\widetilde{C} = \widetilde{C}(w)$. In (3.23), we have used the trace inequality (3.22) to obtain (3.24).

Remark 3.8. *In general, the bilinear form is not continuous on the "broken" space $H^2(\Omega,\mathcal{T})$ with respect to the energy norm [39] and C is a generic constant independent of h and may have different values in different places.*

Combining the coercivity result (3.20) and continuity result (3.23), we have the following theorem that establishes the existence and uniqueness of the discontinuous Galerkin solution.

Theorem 3.9. *Let $\mathcal{A} : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R}$ be a bilinear form of the SIPG* (3.21)*. If κ is uniformly bounded below by a sufficiently large positive constant κ_* for all edges e, then the DG solution exists and is unique, where the DG space $\mathcal{V}_h$ is defined by* (3.6)*.*

Proof. We will refer the reader to Section 2.7.4, [39] for the proof in more detail. □

3.4 The DG Finite Element Equations

A comprehensive introduction to the finite element method can be found in [7, 8, 39]. We present a short outline for the most important features. The discrete problem is given by the Galerkin scheme:

$$\text{Find} \quad u_h \in \mathcal{V}_h : \quad \mathcal{A}(u_h, v_h) = g(v_h), \quad \forall v_h \in \mathcal{V}_h \tag{3.25}$$

The index h stands for the discretization parameter and indicates that we want to achieve convergence of the discrete solution for $h \to 0$.

The space $\mathcal{V}_h$ in (3.6) is of finite dimension, thus it must have a finite basis. For a quadrilateral mesh and bilinear elements, $n = \dim \mathcal{V}_h = 4N_h$ where N_h is the number of elements. Let ϕ_i denote the basis functions, i.e., $\mathcal{V}_h = span\{\phi_i : i = 1, ..., n\}$. Consequently, the solution u_h is of the form

$$u_h(\mathbf{x}) = \sum_{i=1}^{n} u_h^i \phi_i(\mathbf{x}), \quad \text{for} \quad \mathbf{x} \in \overline{\Omega}, \tag{3.26}$$

with coefficients $u_h^i \in \mathbb{R}$. Using the notation

$$\underline{u}_h := (u_h^i)_{i=1}^n \quad \in \mathbb{R}^n, \tag{3.27}$$

$$\underline{g}_h := (g(\phi_i))_{i=1}^n \quad \in \mathbb{R}^n, \tag{3.28}$$

$$\mathcal{A}_h[i,j] := \mathcal{A}(\phi_j, \phi_i), \quad \mathcal{A}_h \in \mathbb{R}^{n \times n}, \tag{3.29}$$

we arrive at the linear algebraic system :

$$\text{Find} \quad \underline{u}_h \in \mathbb{R}^n : \quad \mathcal{A}_h \underline{u}_h = \underline{g}_h, \tag{3.30}$$

which is equivalent to the original discrete problem (3.25).
Since the bilinear form $\mathcal{A}(\cdot,\cdot)$ is positive definite, the matrix $\mathcal{A}_h$ is also positive definite. This is because, for $\underline{u}_h \in \mathbb{R}^n$, we can write

$$\underline{u}_h^T \mathcal{A}_h \underline{u}_h = \mathcal{A}\left(\sum_{j=1}^{n} u_j \phi_j, \sum_{i=1}^{n} u_i \phi_i \right).$$

So the positive definiteness of $\mathcal{A}(\cdot,\cdot)$ implies $\underline{u}_h^T \mathcal{A}_h \underline{u}_h > 0$ and $(\underline{u}_h^T \mathcal{A}_h \underline{u}_h = 0 \iff \underline{u}_h = 0)$. Now, since $\mathcal{A}_h$ is positive definite, it is invertible. Therefore, the linear algebraic system has exactly one solution $\underline{u}_h \in \mathbb{R}^n$. Thus, (3.25) has exactly one solution $u_h \in \mathcal{V}_h$.

3.5 A Priori Error Estimates

In this section, we state approximation results in the space of polynomials of degree less than p in each space direction and also recall the trace inequalities which enables us to prove discretization error estimates, see e.g [39].

Theorem 3.10. *Let T be a rectangle in 2D. Let $v \in H^s(T)$ for $s \geq 1$. Let $p \geq 0$ be an integer. There exists a constant C independent of v and h and a function $\tilde{v} \in \mathbb{Q}_p(T)$ such that*

$$\|\tilde{v} - v\|_{H^q(T)} \leq C\, h^{\min(p+1,s)-q} |v|_{H^s(T)} \quad \forall 0 \leq q \leq s, \tag{3.31}$$

where $h = diam(T)$.

The next result yields an approximation that conserves the average of the normal flux on each edge.

Theorem 3.11. *Let T be a rectangle in 2D. Denote by $\vec{\nu}$ the outward normal to T. Let $v \in H^s(T)$ for $s \geq 2$ and $p > 0$. There exists an approximation $\tilde{v} \in \mathbb{Q}_p(T)$ of v satisfying $\int_e \nabla(\tilde{v} - v) \cdot \vec{\nu} = 0 \quad \forall e \subset \partial T$ and the optimal error bounds*

$$\|\nabla^i(\tilde{v} - v)\|_{L^2(T)} \leq C h^{\min(p+1,s)-i} |v|_{H^s(T)} \quad \forall i = 0, 1, 2, \tag{3.32}$$

We now state and show an *a priori energy error estimate* in the DG-norm.

Theorem 3.12 (Energy error estimate, [39])**.** *Let $\mathcal{A}(\cdot,\cdot)$ be the SIPG bilinear form from Theorem* (3.3) *with $\kappa > 0$. If the exact solution u to the model problem* (3.3) *is in $H^2(\mathcal{T})$ and $u_h \in \mathcal{V}_h$ satisfies* (3.25) *then there exist a constant $\widetilde{C}$ independent of h such that*

$$|||u - u_h||| \leq \widetilde{C}\, h\, \|u\|_{H^2(\mathcal{T})}. \tag{3.33}$$

Proof. We will present the main steps of the proof given in ([39], Sect. 2.8). From Proposition 3.5, we know that

$$\mathcal{A}(u, v) = g(v) \quad \forall v \in H^2(\mathcal{T}).$$

Since $\mathcal{A}(\cdot,\cdot)$ is bilinear and

$$\mathcal{A}(u_h, v) = g(v) \quad \forall v \in \mathcal{V}_h,$$

we get the Galerkin orthogonality

$$\mathcal{A}(u_h - u, v) = 0 \quad \forall v \in \mathcal{V}_h.$$

Let $\chi \in H^2(\mathcal{T})$ denote the error: $\chi := u_h - u$. We have to find an upper bound for $\mathcal{A}(\chi, \chi)$ which is independent of u_h. To achieve this, let $\tilde{u} \in \mathcal{V}_h$ be a function that approximates the exact solution u. Using Galerkin orthogonality, we can write

$$\begin{aligned}\mathcal{A}(\chi,\chi) &= \mathcal{A}(\chi,u_h) - \mathcal{A}(\chi,u) = -\mathcal{A}(\chi,u)\\ &= \mathcal{A}(\chi,\tilde{u}) - \mathcal{A}(\chi,u)\\ &= \mathcal{A}(\chi,\tilde{u}-u).\end{aligned}$$

As $u, \tilde{u} \in C^0(\bar{\Omega})$, we see $[\![\tilde{u}-u]\!] = 0$ on any interior edge and $\tilde{u}$ is a continuous interpolant. Therefore, most terms vanish when $\mathcal{A}(\chi,\tilde{u}-u)$ is expanded. We obtain :

$$\begin{aligned}\mathcal{A}(\chi,\chi) = \mathcal{A}(\chi,\tilde{u}-u) &= \sum_{T\in\mathcal{T}}\int_T \Big(w\nabla\chi\cdot\nabla(\tilde{u}-u) + \tilde{\lambda}(\tilde{u}-u)\chi\Big)\\ &- \sum_{e\in\mathcal{E}_{int}}\int_e [\![\chi]\!]\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\} - \sum_{e\in\mathcal{E}_{int}}\int_e \{w\nabla\chi\cdot\vec{\nu}\}\ [\![\tilde{u}-u]\!]\\ &+ \sum_{e\in\mathcal{E}_{int}}\int_e \kappa h_e^{-1}\ [\![\chi]\!]\ [\![\tilde{u}-u]\!].\end{aligned}$$

Using the Cauchy Schwarz inequality, Young's inequality and the approximation theorems Theorem 3.10 and Theorem 3.11, we obtain a bound for the first term as follows:

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}}\int_T w\nabla\chi\cdot\nabla(\tilde{u}-u)\right| &\le w^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}}\|\nabla(\tilde{u}-u)\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}}\|w^{\frac{1}{2}}\nabla\chi\|^2_{L^2(T)}\right)^{\frac{1}{2}}\\ &\le \frac{C}{2\delta}\|\nabla(\tilde{u}-u)\|^2_{L^2(\mathcal{T})} + \frac{\delta}{2}\|w^{\frac{1}{2}}\nabla\chi\|^2_{L^2(\mathcal{T})}.\\ \left|\sum_{T\in\mathcal{T}}\int_T \tilde{\lambda}(\tilde{u}-u)\chi\right| &\le \|\tilde{\lambda}\|_{L^\infty(\mathcal{T})}\left(\sum_{T\in\mathcal{T}}\|\tilde{u}-u\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}}\|\chi\|^2_{L^2(T)}\right)^{\frac{1}{2}}\\ &\le \frac{C}{2\delta}\|\tilde{\lambda}\|_{L^\infty(\Omega)}\|\tilde{u}-u\|^2_{L^2(\mathcal{T})} + \frac{\delta}{2}\|\chi\|^2_{L^2(\mathcal{T})}.\end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}}\int_T \Big(w\nabla\chi\cdot\nabla(\tilde{u}-u) + \tilde{\lambda}(\tilde{u}-u)\chi\Big)\right| &\le \frac{C}{2\delta}(1+\|\tilde{\lambda}\|_{L^\infty(\Omega)})\|\tilde{u}-u\|^2_{H^1(\mathcal{T})} + \frac{\delta}{2}|||\chi|||^2\\ &\le \widetilde{C}h^2\|u\|^2_{H^2(\mathcal{T})} + \frac{\delta}{2}|||\chi|||^2.\end{aligned}$$

The second term can be bounded by applying the Cauchy Schwarz inequality and

Young's inequality to obtain:

$$\begin{aligned}
\left|\sum_{e\in\mathcal{E}_{int}}\int_e [\![\chi]\!]\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\}\right| &\le \sum_{e\in\mathcal{E}_{int}} \|[\![\chi]\!]\|_{L^2(e)}\|\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\}\|_{L^2(e)}\\
&= \sum_{e\in\mathcal{E}_{int}} h^{-\frac{1}{2}}\|[\![\chi]\!]\|_{L^2(e)} h^{\frac{1}{2}}\|\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\}\|_{L^2(e)}\\
&\le \left(\sum_{e\in\mathcal{E}_{int}} h^{-1}\|[\![\chi]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{int}} h\|\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\}\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\
&\le C\left(\sum_{e\in\mathcal{E}_{int}} h^{-1}\|[\![\chi]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\left(\sum_{T\in\mathcal{T}} hh_T^{-1}\|\nabla(\tilde{u}-u)\|^2_{L^2(T)}\right)^{\frac{1}{2}}\\
&\le \frac{\delta}{2}|||\chi|||^2 + \frac{C}{2\delta}\|\nabla(\tilde{u}-u)\|^2_{L^2(\mathcal{T})}\\
&\le \frac{\delta}{2}|||\chi|||^2 + \widetilde{C}h^2\|u\|^2_{H^2(\mathcal{T})}.
\end{aligned}$$

Using the approximation properties Theorem 3.10 and Theorem 3.11 and the trace inequalities from Theorem 3.4, we obtain the bound

$$\left|\sum_{e\in\mathcal{E}_{int}}\int_e [\![\chi]\!]\{w\nabla(\tilde{u}-u)\cdot\vec{\nu}\}\right| \le \widetilde{C}\,h^2\|u\|^2_{H^2(\mathcal{T})} + \frac{\delta}{2}|||\chi|||^2.$$

In general, when $\tilde{u}$ is not continuous, we proceed similarly for the third and fourth terms with $(\tilde{u}-u)$ unknown on the interfaces. Using *jump norm*, the trace inequalities togetherer with definitions of the trace operators (*average* and *jump*), and using the coercivity result (3.20), we obtain

$$|||\chi|||^2 \le \mathcal{A}(\chi,\chi) = \mathcal{A}(\chi,\tilde{u}-u) \le \widetilde{C}h^2\|u\|^2_{H^2(\mathcal{T})},$$

where $\widetilde{C} = \widetilde{C}(\tilde{\lambda}, w)$. □

Next, we prove an error estimate in the L^2 norm.

Theorem 3.13 (L^2 error estimate)**.** *Assume that Theorem 3.12 holds and that our model problem is H^2-coercive then there exists a constant $\widetilde{C}$ independent of h such that*

$$\|u-u_h\|_{L^2(\Omega)} \le \widetilde{C}h^2\|u\|_{H^2(\mathcal{T})}.$$

Proof. The proof we present follows [39]. Consider the dual problem

$$\begin{aligned}
-\nabla\cdot(w\nabla v) + \tilde{\lambda}v = u - u_h \quad &\text{in} \quad \Omega,\\
w\nabla v\cdot\vec{\nu} = 0 \quad &\text{on} \quad \Gamma = \partial\Omega.
\end{aligned}$$

We assume that $v \in H^2(\Omega)$ and that there is a constant C that depends on Ω such that

$$\|v\|_{H^2(\Omega)} \le C\|u-u_h\|_{L^2(\Omega)}. \tag{3.34}$$

Denote $\chi = u - u_h$,

$$\|\chi\|^2_{L^2(\Omega)} = \sum_{T\in\mathcal{T}} \int_T (-\nabla\cdot(w\nabla v) + \tilde{\lambda} v)\chi.$$

Integrating by part on each element and applying the basic inequality (3.9) yields

$$\begin{aligned}
\|\chi\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}} \Big(\int_T w\nabla v \cdot \nabla\chi \, dx + \int_T \tilde{\lambda} v \chi \, dx \Big) - \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla v\cdot\vec{\nu}\} [\![\chi]\!] \\
&- \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla\chi\cdot\vec{\nu}\} [\![v]\!] \\
&= \sum_{T\in\mathcal{T}} \Big(\int_T w\nabla v \cdot \nabla\chi \, dx + \int_T \tilde{\lambda} v \chi \, dx \Big) - \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla v\cdot\vec{\nu}\} [\![\chi]\!],
\end{aligned}$$

since v is continuous. By subtracting the Galerkin orthogonality equation, we obtain for $\tilde{v} \in \mathcal{V}_h$

$$\begin{aligned}
\|\chi\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}} \int_T \Big(w\nabla\chi\cdot\nabla(\tilde{v}-v) + \tilde{\lambda}(\tilde{v}-v)\chi \Big) \\
&- \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla v\cdot\vec{\nu}\} [\![\chi]\!] + \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla \tilde{v}\cdot\vec{\nu}\} [\![\chi]\!] \\
&- \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla \chi\cdot\vec{\nu}\} [\![\tilde{v}]\!] - \sum_{e\in\mathcal{E}_{int}} \int_e \kappa h_e^{-1} [\![\chi]\!][\![\tilde{v}]\!].
\end{aligned}$$

By noting that $v \in H^2(\Omega)$ and choosing $\tilde{v} \in C^0(\Omega)$, we are left with

$$\begin{aligned}
\|\chi\|^2_{L^2(\Omega)} &= \sum_{T\in\mathcal{T}} \int_T \Big(w\nabla\chi\cdot\nabla(\tilde{v}-v) + \tilde{\lambda}(\tilde{v}-v)\chi \Big) \\
&- \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla v\cdot\vec{\nu}\} [\![\chi]\!] + \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla \tilde{v}\cdot\vec{\nu}\} [\![\chi]\!] \\
&= \sum_{T\in\mathcal{T}} \int_T \Big(w\nabla\chi\cdot\nabla(\tilde{v}-v) + \tilde{\lambda}(\tilde{v}-v)\chi \Big) + \sum_{e\in\mathcal{E}_{int}} \int_e \{w\nabla(\tilde{v}-v)\cdot\vec{\nu}\} [\![\chi]\!].
\end{aligned} \tag{3.35}$$

The first term is bounded in the following way :

$$\begin{aligned}
\Big| \sum_{T\in\mathcal{T}} \int_T w\nabla(\tilde{v}-v)\cdot\nabla\chi \Big| &\leq C \sum_{T\in\mathcal{T}} \|w\nabla(\tilde{v}-v)\|_{L^2(T)} \|\nabla\chi\|_{L^2(T)} \\
&\leq \widetilde{C} \sum_{T\in\mathcal{T}} \|\nabla(\tilde{v}-v)\|_{L^2(T)} \|\nabla\chi\|_{L^2(T)} \\
&\leq \widetilde{C} \, \|\nabla(\tilde{v}-v)\|_{L^2(\Omega)} \|\nabla\chi\|_{L^2(\Omega)}. \\
\Big| \sum_{T\in\mathcal{T}} \int_T \tilde{\lambda}(\tilde{v}-v)\chi \Big| &\leq C \|\tilde{\lambda}\|_{L^\infty(\Omega)} \|\tilde{v}-v\|_{L^2(\Omega)} \|\chi\|_{L^2(\Omega)} \\
&\leq \tilde{C} \, \|\tilde{v}-v\|_{L^2(\Omega)} \|\chi\|_{L^2(\Omega)}.
\end{aligned}$$

By the triangle inequality, we obtain

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}}\int_T\left(w\nabla(\tilde{v}-v)\cdot\nabla\chi+\tilde{\lambda}(\tilde{v}-v)\chi\right)\right| &\leq \widetilde{C}\|\tilde{v}-v\|_{H^1(\Omega)}\|\chi\|_{H^1(\Omega)}\\ &\leq \widetilde{C}\, h\|v\|_{H^2(\Omega)}|||\chi|||. \qquad (3.36)\end{aligned}$$

The last term is bounded by using the Cauchy-Schwarz inequality and by taking advantage of the definition of the penalty parameter :

$$\begin{aligned}\left|\sum_{e\in\mathcal{E}_{int}}\int_e\{w\nabla(\tilde{v}-v)\cdot\vec{\nu}\}[\![\chi]\!]\right| &\leq \sum_{e\in\mathcal{E}_{int}} h_e^{\frac{1}{2}}\|\{w\nabla(\tilde{v}-v)\cdot\vec{\nu}\}\|_{L^2(e)}h_e^{-\frac{1}{2}}\|[\![\chi]\!]\|_{L^2(e)}\\ &\leq \left(\sum_{e\in\mathcal{E}_{int}} h_e\|\{w\nabla(\tilde{v}-v)\cdot\vec{\nu}\}\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\times\left(\sum_{e\in\mathcal{E}_{int}} h_e^{-1}\|[\![\chi]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\leq C\left(\sum_{T\in\mathcal{T}} h_e h_T^{-1}\|\nabla(\tilde{v}-v)\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{int}} h_e^{-1}\|[\![\chi]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\leq \widetilde{C}\left(\sum_{T\in\mathcal{T}} h_T^2\|v\|^2_{L^2(T)}\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}_{int}} h_e^{-1}\|[\![\chi]\!]\|^2_{L^2(e)}\right)^{\frac{1}{2}}\\ &\leq \widetilde{C}\, h\|v\|_{H^2(\Omega)}|||\chi|||. \qquad (3.37)\end{aligned}$$

We substitute (3.36) and (3.37) into (3.35). Therefore, using the bound (3.34) we obtain

$$\begin{aligned}\|\chi\|^2_{L^2(\Omega)} &\leq \widetilde{C}h\|v\|_{H^2(\Omega)}|||\chi|||\\ &\leq \widetilde{C}h\|\chi\|_{L^2(\Omega)}|||\chi|||.\end{aligned}$$

With (3.12), this implies

$$\|\chi\|_{L^2(\Omega)} \leq \widetilde{C}h|||\chi||| \leq \widetilde{C}h^2\|u\|_{H^2(\mathcal{T})}.$$

□

3.6 DG-Version of IRLS Algorithm

In this section, we present the iteratively reweighted least squares algorithm in combination with the DG discretization. The 1D case has already been presented in [34]. Now we are in the position to proceed with the 2D case. Indeed, we have all the ingredients to assemble our numerical scheme into the following algorithm.

Algorithm 3.14 (Double Minimization, [24])**.**

Input: *Data vector* $g, \varepsilon > 0$*, initial gradient weight* $w^{(0)}$ *with* $\varepsilon \leq w^{(0)} \leq 1/\varepsilon$*, number* n_{max} *of outer iterations.*

Parameters: $\tilde{\lambda}$*,* $\kappa > 0$*.*

Output: *Approximation* u^* *of the minimizer of* J_ε*.*

$u_h^{(0)} = 0$

for $n = 0$ ***to*** n_{max} ***do***

Assemble the matrix $\mathcal{A}_h^{(n+1)}$ *from Theorem 3.3*

Compute $u_h^{(n+1)}$ *such that the matrix* $\mathcal{A}_h^{(n+1)} u_h^{(n+1)} = g_h$;

Compute the gradient $\nabla u_h^{(n+1)} = \sum_{k \in \mathcal{N}} u_{h,k}^{(n+1)} \nabla \phi_k$;

$$w^{(n+1)} = \min\left(\max\left(\varepsilon, \frac{1}{|\nabla u_h^{(n+1)}|}\right), \frac{1}{\varepsilon}\right).$$

endfor

$u_h^* := u_h^{(n+1)}$.

Chapter 4

Numerical Results

In this chapter, we present some simulation results obtained with the DG-FEM. In particular, we will show results of our model problem with Dirichlet boundary condition and with Neumann boundary conditions. We present numerical convergence rates for the L^2-norm of the error for a smooth exact solution. We continue with results of the IRLS, particulary a denoising problem, by presenting results for the convergence of the algorithm with varying regularization parameter and mesh size.

4.1 DG for Diffusion Problems

4.1.1 Dirichlet Problem

Let us consider the Dirichlet boundary value problem

$$\begin{cases} -\nabla \cdot (w\nabla u) + \widetilde{\lambda} u = \widetilde{\lambda} g & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{4.1}$$

with $\Omega = (-1,1)^2$, $g(x,y) = -1$ if $x, y < 0, 1$ if $x, y > 0,$ and 0 *otherwise*. Discretizing problem (4.1) by means of the **SIPG** method with penalty parameter $\kappa = 10$ as described in Section 3.1, we obtain the results displayed in Figure 4.1. More precisely, we set $w = 1$. The plots show that increasing the regularization parameter $\widetilde{\lambda}$ yields solutions that approaches g.

4.1.2 Neumann Problem

Let us consider the Neumann boundary value problem

$$\begin{cases} -\nabla \cdot (w\nabla u) + \widetilde{\lambda} u = \widetilde{\lambda} g & \Omega, \\ \nabla u \cdot \vec{\nu} = 0 & \partial\Omega, \end{cases} \tag{4.2}$$

with $\Omega = (-1,1)^2$, $g(x,y) = -1$ if $x, y < 0, 1$ if $x, y > 0,$ and 0 *otherwise*. Discretizing problem (4.2) by means of the **SIPG** method with penalty parameter $\kappa = 10$ as described in Section 3.1, we obtain the results displayed in Figure 4.2. More precisely,

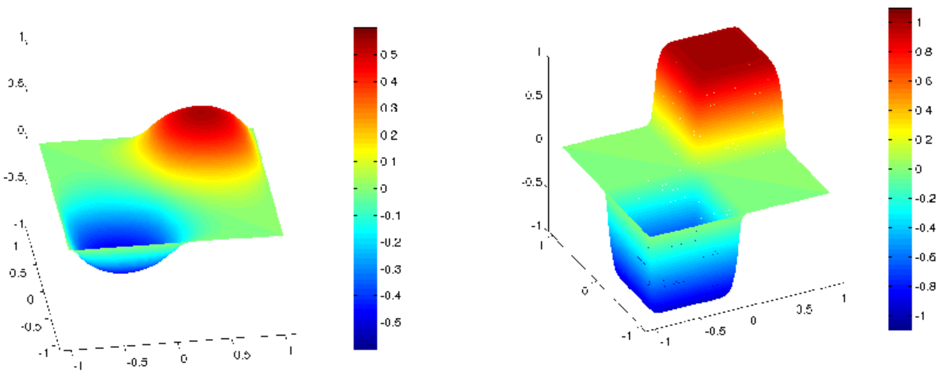

Figure 4.1: DG solution of Dirichlet problem with penalty parameter $\kappa = 10$; $\widetilde{\lambda} = 10$ (left) and $\widetilde{\lambda} = 1000$ (right).

we set $w = 1$. The plots show that increasing the regularization parameter $\widetilde{\lambda}$ yields solutions that approaches g.

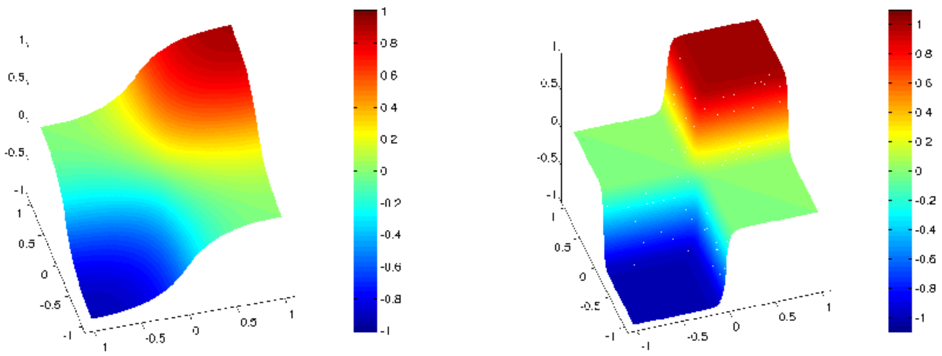

Figure 4.2: DG solution of Neumann problem with penalty parameter $\kappa = 10$; $\widetilde{\lambda} = 10$ (left) and $\widetilde{\lambda} = 1000$ (right).

4.1.3 Poisson Problem with Known Solution

Let us consider the Poisson problem

$$\begin{cases} -\triangle u = g & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.3}$$

on the unit square $(-1,1)^2$. We choose g such that the analytical solution of the problem is given by $u(x,y) = \sin(\pi x)\sin(\pi y)$.

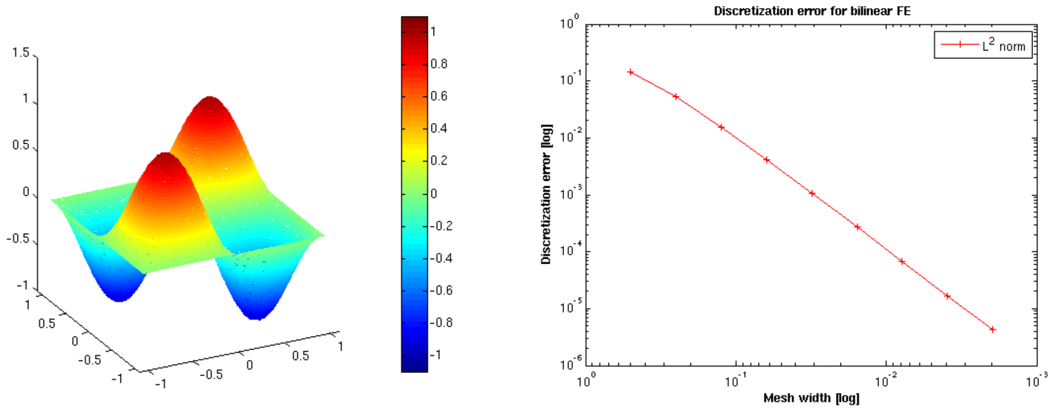

Figure 4.3: DG solution of Poisson problem with penalty parameter $\kappa = 10$ (left) and error for the L^2-norm (right).

n	h	$e_n = \|u - u_h\|$	β
1	1/2	1.4358×10^{-1}	
2	1/4	5.3246×10^{-2}	2.696
3	1/8	1.5578×10^{-2}	3.418
4	1/16	4.1438×10^{-3}	3.759
5	1/32	1.0628×10^{-3}	3.899
6	1/64	2.6871×10^{-4}	3.955
7	1/128	6.7532×10^{-5}	3.979
8	1/256	1.6928×10^{-5}	3.989
9	1/512	4.2367×10^{-6}	3.996

Table 4.1: Numerical error estimates for a smooth function.

Table 4.1 shows the numerical error in the L^2-norm for equation (4.3). The convergence rate given by

$$\beta = \frac{e_n}{e_{n+1}}$$

as predicted by the theory is $\mathcal{O}(h^2)$. This implies halving the mesh size leads to an increse in the error estimate. Table 4.1 demonstrates that as n increases, the convergance rate β increases to a factor 4. The error in the L^2 norm plotted against the mesh size is shown in Figure 4.3 using a penalty parameter of $\kappa = 10$.

4.2 Iteratively Reweighted Least Squares Algorithm

4.2.1 Denoising Problem

For studying the numerical results of the iteratively reweighted least squares (IRLS) Algorithm 3.14, we use a regularization functional

$$\widetilde{J}(u,w) = \frac{\widetilde{\lambda}}{2}\|u-G\|^2_{L^2(\Omega)} + \int_\Omega \left(w|\nabla u|^2 + \frac{1}{w} \right) dx. \tag{4.4}$$

The functional (4.4) is called a *denoising functional* where $G = g + \eta$, $\widetilde{\lambda} > 0$ and η is a uniformly distributed noise. The corresponding Euler-Lagrange equation is given by

$$\begin{cases} -\nabla \cdot (w\nabla u) + \widetilde{\lambda} u = \widetilde{\lambda} G & \Omega, \\ w\nabla u \cdot \vec{\nu} = 0 & \partial\Omega. \end{cases} \tag{4.5}$$

Concerning the numerical work involved in the IRLS algorithm, each iteration requires solving the Euler-Lagrange equation and using the solution to compute new gradient weights w. Figure 4.4 shows the result for 20 iterations of the IRLS algorithm. We set the regularization parameter $\widetilde{\lambda} = 10^2$, the penalty parameter $\kappa = 10$ and $\varepsilon = 10^{-2}$ as in Algorithm 3.14. As the results indicate, there is a sharp improvement in the solution within the first 10 iteration steps. The solution becomes steady till the total number of iteration is reached. This means an approximate solution is reached after few number of iterations.

	$J(u)$				
n	$h = 1/8$	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
1	45.3461	44.2652	51.2620	63.7826	71.7443
2	43.3240	38.5002	49.8223	58.2257	67.1117
3	38.6976	41.6775	47.8067	55.4609	63.6222
5	41.0890	40.0938	45.4033	52.1332	58.6105
10	42.6990	40.4148	42.9642	47.8849	53.0425
20	39.7064	38.1429	40.7648	45.4784	48.5690
30	39.9341	37.2609	41.3725	43.1490	46.5024
40	38.1215	38.7159	40.3228	43.0523	44.8862
50	42.8382	39.6733	40.8136	43.2327	44.2599

Table 4.2: Results for the functional $J(u)$ from (4.6) for 50 iterations and decreasing mesh size .

The Table 4.2 shows the values of the convergence of the iteratively reweighted least squares algorithm. For a fixed $\tilde{\lambda} = 10^4$ and penalty parameter $\kappa = 10$, the denoising problem (4.5) is solved for 50 iterations with the iteratively reweighted least squares algorithm. The solution is used in computing the functional

$$J(u^{(n)}) = \frac{\widetilde{\lambda}}{2}\|u^{(n)} - G\|^2_{L^2(\Omega)} + \int_\Omega |\nabla u^{(n)}| \tag{4.6}$$

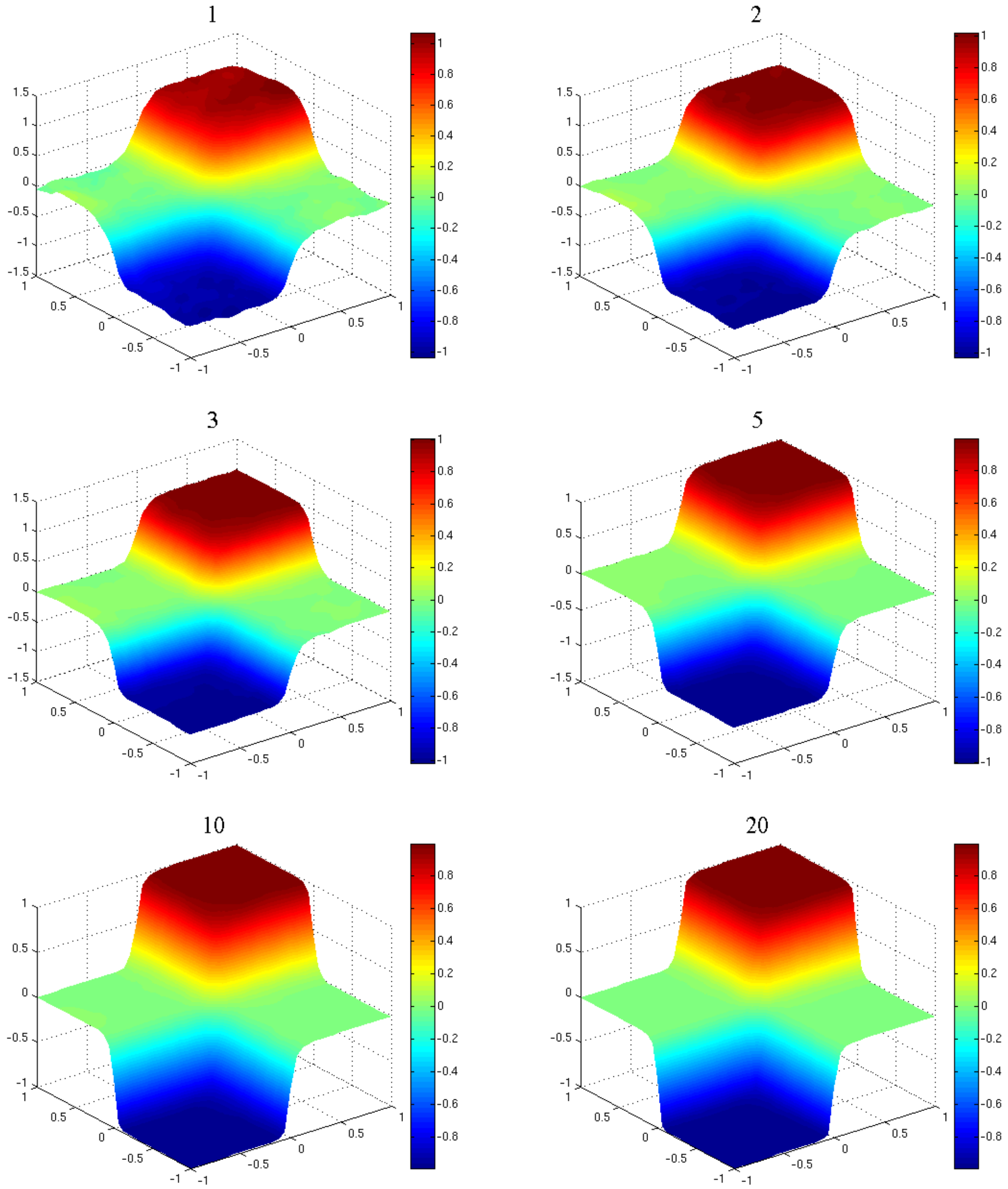

Figure 4.4: Results of the iteratively reweighted least squares algorithm at iterations 1 and 2 (above) and iterations 3 and 5 (middle) and 10 and 20 (below).

for varying mesh sizes h.

In the Table 4.3, the results for different regularization parameter $\widetilde{\lambda}$ are computed. We set $\varepsilon = 10^{-1}$.

	$J(u)$		
n	$\tilde{\lambda} = 10^2$	$\tilde{\lambda} = 10^3$	$\tilde{\lambda} = 10^4$
1	35.7395	46.7507	63.1128
2	35.5685	44.2403	56.8716
3	34.9391	42.6369	54.2316
5	34.7120	41.4692	51.4730
10	35.1925	39.2793	47.5556
20	34.6311	38.8595	44.6642
30	34.5625	38.1292	43.6768
40	34.5749	37.9654	42.9581
50	34.8817	38.2415	42.1900

Table 4.3: Convergence results for a fixed mesh $h = 2^{-6}$ computed for 50 iterations with different $\widetilde{\lambda}$.

Figure 4.5 shows the value of $J(u^{(n)})$ for the first 50 iterations for $\tilde{\lambda} = 10^2$ (above), $\tilde{\lambda} = 10^3$ (middle) and $\tilde{\lambda} = 10^4$ (below). It can be observed that the functional $J(u^{(n)})$ converges to a minimum with increasing iteration. The minimum is achieved and remains almost the same with regularization parameters 10^3 and 10^4. As the regularization parameter increases, the solution converges whiles a decrease in ε (figures from left to right) does not make any significant difference. When $\widetilde{\lambda} = 10^2$, the functional $J(u^{(n)})$ has an irregular behaviour for both $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-1}$. This means the regularization parameter $\widetilde{\lambda}$ must be larger than 10^2 to achieve good results.

4.2.2 Diffusion Problem, TV-Minimization Problem and IRLS

We consider an example of our model problem (4.5). In this example, we set the noise $\eta = 0$, $\varepsilon = 10^{-2}$, $\kappa = 10$ and the regularization parameter $\widetilde{\lambda} = 100$. In Figure 4.6, the results for some iterations are plotted. The results show the efficiency of the IRLS algorithm to yield sharp edges for problems without noise.

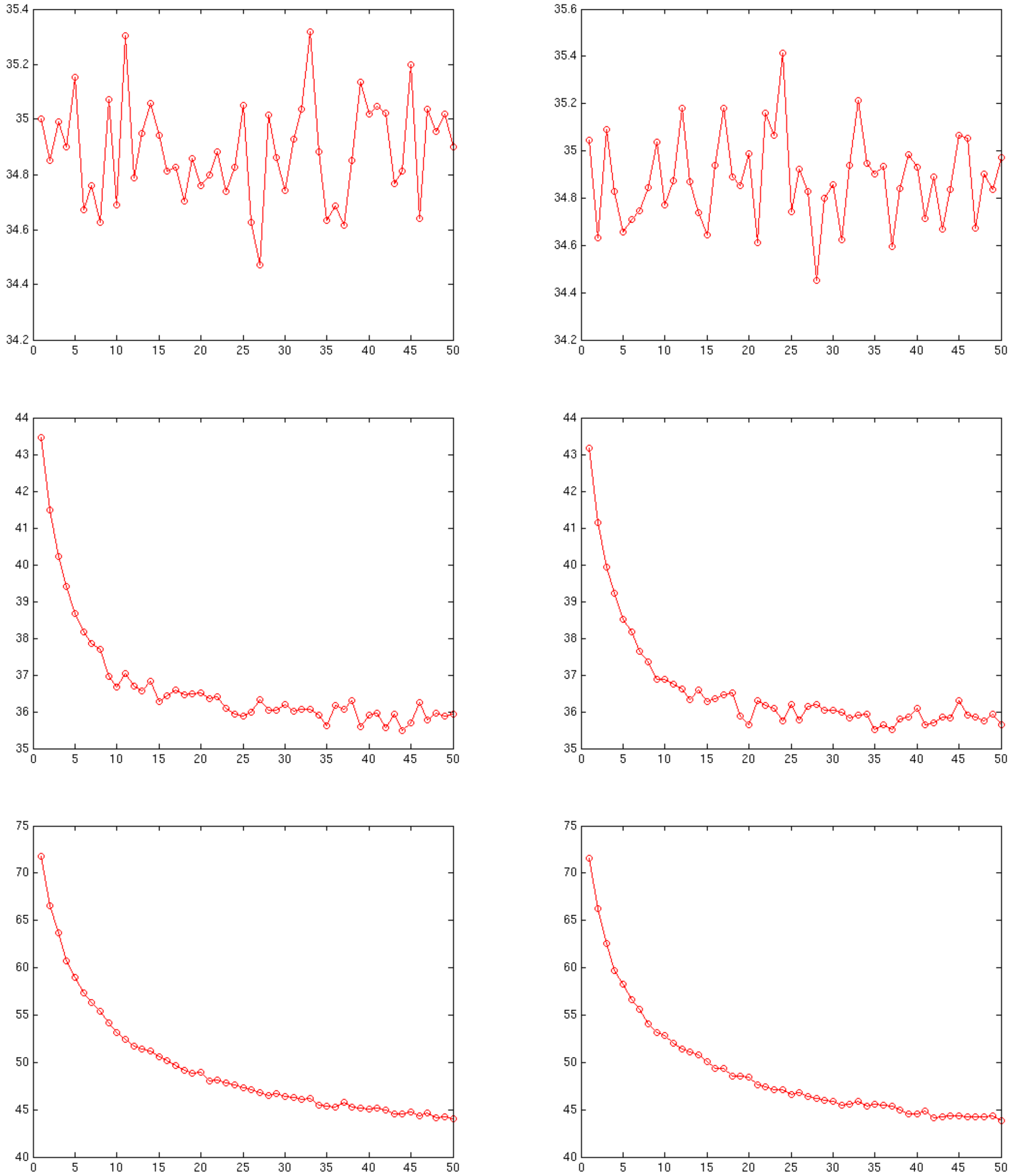

Figure 4.5: Convergence for the regularization functional plotted against the number of iterations. The results on the left correspond to $\varepsilon = 10^{-1}$ and on the right $\varepsilon = 10^{-3}$. The plot shows the values of $J(u^{(n)})$ for the first 50 iterations for $\widetilde{\lambda} = 10^2$ (above), $\widetilde{\lambda} = 10^3$ (middle) and $\widetilde{\lambda} = 10^4$ (below).

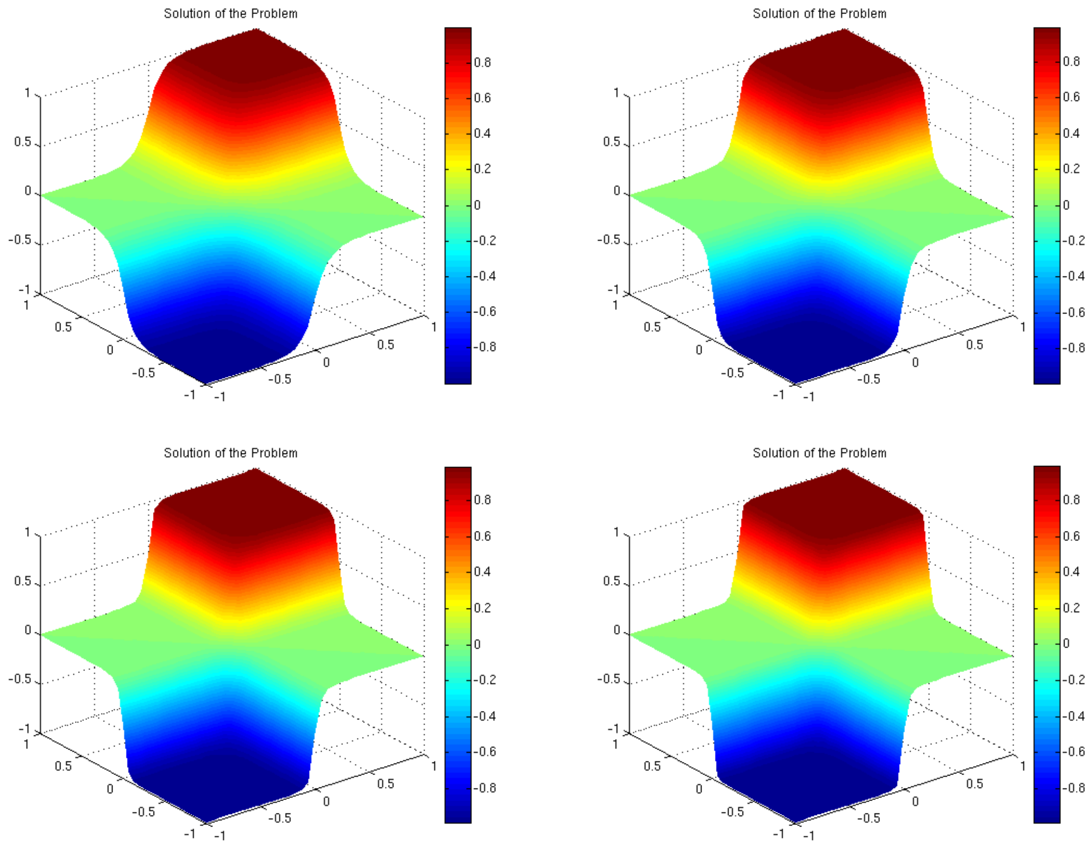

Figure 4.6: Results of the IRLS for the varying diffusion coefficients at iterations 1 and 3 (above) and iterations 10 and 20 (below).

Chapter 5

Conclusion and Outlook

5.1 Conclusion

In this thesis, we solved a total variation minimization problem by using a discontinuous Galerkin finite element method. In Chapter 1, we considered the Rudin, Osher and Fattemi (ROF) model. In general, the first order optimality condittion of the model yields an ill-posed non-linear second order partial differential equation. Considering the space of Bounded Variation of functions and some of its relevant properties, we showed that the existence and uniqueness of minimizers of the ill-posed functional can be achieved by a relaxation algorithm called *iteratively reweighted least squares algorithm* which is also referred to as *double minimization algorithm* in some books and articles.

The well-posed second order partial differential equation equation obtained from the *iteratively reweighted least squares algorithm* is discretized by a standard discontinuous Galerkin (DG) finite element method. This is particularly useful since the space of bounded variation functions contain discontinuous functions. We presented some theory on the primal formulation of a discontinuous Galerkin finite element method. We showed the existence and uniqueness of the DG solution and also presented a-priori error estimates in the energy norm and L^2-norm.

Finally, we have discussed some numerical results obtained by DG discretization method. In Chapter 4, two examples are discussed and the L^2 error are computed for a known problem. Results from the *iteratively reweighted least squares algorithm* are also presented with an application to a denoising problem. The rate of convergence of the iteratively reweighted least squares algorithm is also presented.

5.2 Outlook

This thesis provides a starting point for preparing theoritical results for a discontinuous Galerkin finite element method for solving total variation minimization problems. The

DG discretization has high degrees of freedom and requires efficient solvers. The work can be continued in the following way

1. *Robust multilevel solver.*
   At each step of the IRLS, we have to solve a diffusion problem with varying diffusion coefficient w between ε and $1/\varepsilon$. Here we need robust iterative solvers and an appropriate stopping criteria in order to achieve optimal results. One candidate for such a robust solver is the algebraic multilevel solver proposed by Kraus and Tomar [31] and Kraus [29].

2. *Harmonic mean for DG formulation.*
   In the formulation of the DG, the harmonic mean can be used for averaging of the flux. This approach is particularly useful since the diffusion coefficients can be discontinuous [18].

3. *Kraus–Tomar Assembling.*
   The assembling process can also be improved by reducing the total number of degrees of freedom. The Kraus-Tomar assembling of the discontinuous Galerkin finite element method reduces the total degree of freedom compared to the standard assembling. It is useful since the coefficients (gradient weights) can be discontinuous [31].

4. *Analysis and Simluation of 3D Problems.*
   The work can be carried on to higher dimension.

List of Notations and Function Spaces

Notation

$\mathbb{N}, \mathbb{R}, \mathbb{R}^n$	natural numbers, real numbers, n-dimensional Euclidean space
$\mathcal{L}(X, Y)$	space of bounded linear operator between Banach spaces X and Y
X^*	dual space of the Banach space X
$V \subset U$	V is a subset of U; for Banach spaces U and V. It also denotes continuous embedding, i.e. the identity mapping $I : V \to U$ is continuous
$V \subset\subset U$	V is compactly contained in U, i.e. $\overline{V} \subset U$ and $\overline{V}$ is compact; for Banach spaces U and V. It also denotes compact embedding, i.e. $I : V \to U$ is compact.
$\mathbf{1}_V$	characteristic function of the set V, i.e. $\mathbf{1}_V(x) = 1$ if $x \in V$, 0 otherwise
Du	distributional derivative of u
$\operatorname{div} g$	divergence of the vector-valued function g
∇u	classical or weak gradient of the scalar functional u
$\mathcal{X}_V$	characteristic function of the set V in the sense of convex analysis
∂V	boundary of the set V
K^*	adjoint in $\mathcal{L}(Y^*, X^*)$ of $K \in \mathcal{L}(X, Y)$
$\lvert x \rvert_{\ell^p}$	ℓ^p norm of $x \in \mathbb{R}^n, 1 \le p \le \infty$
$\lvert\alpha\rvert, \lvert x\rvert$	modulos of $\alpha \in \mathbb{R}$ or Euclidean norm of $x \in \mathbb{R}^n$
Id	identity matrix, $Id \in \mathbb{R}^{m\times n}$
$\lVert v \rVert_V$	norm of v in the Banach space V
$\mathcal{O}$	Landau symbol
$\triangle$	Laplacian operator, $\triangle u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$, for $u : \mathbb{R}^n \to \mathbb{R}$

Function Spaces

Let $\Omega \subset \mathbb{R}^n$ be open. See [21] as general reference.

$C(\Omega,\mathbb{R}^n)$	continuous functions on Ω with values in $\mathbb{R}^n$, denoted by $C(\Omega)$ for $n=1$
$C_0(\Omega,\mathbb{R}^n)$	functions in $C(\Omega,\mathbb{R}^n)$ with copmact support in Ω
$C^k(\Omega,\mathbb{R}^n)$	k times continuously differentiable functions on Ω with values in $\mathbb{R}^n$ denoted by $C^k(\Omega)$ for $n=1$
$C_0^k(\Omega,\mathbb{R}^n)$	functions in $C^k(\Omega,\mathbb{R}^n)$ with compact support in Ω
$C^\infty(\Omega,\mathbb{R}^n)$	infinitely differentiable functions in Ω with values in $\mathbb{R}^n$, denoted by $C^\infty(\Omega)$ for $n=1$
$C_0^\infty(\Omega,\mathbb{R}^n)$	functions in $C^\infty(\Omega)$ for $n=1$
$L^p(\Omega,\mathbb{R}^n)$	Lebesgue space, $1\le p\le\infty$, on Ω with values in $\mathbb{R}^n$, denoted by $L^p(\Omega)$ for $n=1$
$W^{k,p}(\Omega)$	Sobolev space of functions with $k-$th order weak derivatives in $L^p(\Omega)$
$W_0^{k,p}(\Omega)$	closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$H^k(\Omega)$, $H_0^k(\Omega)$	abbreviations for the Hilbert spces $W^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$
$BV(\Omega)$	space of functions of bounded variation on Ω

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Linz, August 2011

Stephen Edward Moore