Analysis and Numerical Solution of the k-ϵ Turbulence Model with Non-Standard Boundary Conditions

Diplomarbeit

zur Erlangung des akademischen Grades eines

Diplom-Ingenieurs

in der Studienrichtung Technische Mathematik

an der

Technisch-Naturwissenschaftlichen Fakultät

der

Johannes Kepler Universität Linz

eingereicht von

Christoph Reisinger

und

Markus Wabro

bei

A. Univ.-Prof. Dipl.-Ing. Dr. Walter Zulehner

Linz, im Juni 1999

Eidesstattliche Erklärung

Hiermit erklären wir an Eides statt, die vorliegende Diplomarbeit selbständig verfaßt, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und uns auch sonst keiner unerlaubten Hilfe bedient zu haben.

Diese Diplomarbeit wurde bisher weder im In- noch im Ausland als Prüfungsarbeit vorgelegt.

Christoph Reisinger Markus Wabro

Linz, Juni 1999

Arbeitsschwerpunkte

Christoph Reisingers Schwerpunkt lag bei der analytischen Behandlung des Problems, zu finden in Kapitel 2, Markus Wabro beschäftigte sich vornehmlich mit der Diskretisierung, was in etwa Kapitel 3 entspricht. Die Programmierarbeit teilt sich zu gleichen Teilen auf die beiden Autoren auf, und auch die numerischen Tests aus Kapitel 4 wurden gemeinsam durchgeführt. Ebenso wurde das für die folgenden Betrachtungen grundlegende Kapitel 1 in Zusammenarbeit erstellt.

Eine strikte Trennung der Beiträge ist jedoch schwer möglich, da für eine erfolgreiche Bearbeitung der umfangreichen und vielschichtigen Aufgabenstellung eine enge Kooperation nötig war.

"I am an old man now, and when I die and go to heaven there are two matters on which I hope enlightenment. One is quantum electrodynamics, and the other one is the turbulent motion of fluids. And about the former one I am rather optimistic.

Horace Lamb, 1932

Preface

This thesis forms the theoretical background for the initial stage of an industrial project on flow simulation in combustion engines. An overview over the entire solution procedure from the development of the model to the numerical solution of the discretised system is given.

The underlying model that we use consists of the Navier-Stokes equations coupled with a common two-equation turbulence model, the k-ϵ model. The aim of the first chapter is to give a derivation thereof, in great part following the presentation of Mohammadi and Pironneau [MP94]. We add suitable boundary conditions, and particularly the behaviour at solid walls is studied in great detail. As the k-ϵ model fails in close vicinity to solid walls, we exclude these boundary layers from the computational domain and set up conditions at the new boundary using a well-established boundary-layer model.

The second chapter is devoted to existence and uniqueness questions of the system, particularly the Navier-Stokes system with turbulent diffusion supplemented with a wall-law (which leads to a non-linear boundary condition of third order) and various inflow and outflow conditions. The general statement is that the same results as for the Navier-Stokes system with constant viscosity and homogeneous Dirichlet conditions are still valid in the stationary case. In an earlier paper Parés [Par92] investigated Smagorinsky's model and succeeded to show strong regularity results due to the special structure of the eddy viscosity. To our knowledge, time-dependent problems are still an open problem, as there are difficulties in estimating the energy entering through the inflow duct — and so is the analysis of the k-ϵ system, although several attempts have been made (see [MP94]).

The numerical solution procedure presented in the third chapter is based on a finite element discretisation of both the Navier-Stokes part and the k-ϵ-system. A stabilising technique for the conforming P_1-P_1 element is used for the Navier-Stokes equations, the convection term is stabilised by the streamline diffusion method. For the k- and ϵ-equations we employ a stable semi-implicit multi-step scheme in combination with upwinding, in order to guarantee positivity of the solutions.

The numerical tests give satisfactory results for simple geometries and not too high Reynolds numbers, but for strongly dominating convection the nonlinear coupling of both parts leads to bad convergence.

We want to thank our thesis advisor Prof. Zulehner for many fruitful suggestions and countless interesting discussions, but also continuous encouragement throughout the project. Our special thank also goes to Ferdinand Kickinger for his helpfulness and sup-

port during the work with his mesh generator and AMG solver NAOMI. Finally we thank Carlos Parés for supplying us with his work on related subjects, Pavel Solin and Gisbert Stoyan for helpful comments on preliminary versions of the manuscript.

Christoph Reisinger Markus Wabro

Linz, June 1999

Notation

We generally use standard characters for scalar values and scalar functions (p, q, ...) and boldface characters for vectors and vector valued functions ($\mathbf{u}$, $\mathbf{v}$, ...). The components of vectors are denoted by $(u_1, \dots, u_N)^T = \mathbf{u}$, where N is the space dimension (generally $N = 2, 3$). For matrices we use boldface capital letters ($\mathbf{A}$, ...).

$\mathcal{G}$ is an open, connected subset of $\mathbb{R}^N$, $\partial\mathcal{G}$ its boundary.

Operators

$\mathbf{u} \cdot \mathbf{v}$	$= \sum_{i=1}^N u_i v_i$ (scalar product).
$\mathbf{u} \otimes \mathbf{v}$	$= (u_i v_j)_{i,j=1,\dots,N}$ (tensor product).
$\mathbf{A} : \mathbf{B}$	$= \sum_{i,j=1}^N A_{ij} B_{ij}$.
$\partial_j p$	$= \frac{\partial p}{\partial x_j}$ (partial derivative of p with respect to x_j).
$\partial_j \mathbf{u}$	$= (\partial_j u_i)_{i=1,\dots,N}$.
$\partial_t p$	$= \frac{\partial p}{\partial t}$ (partial derivative of p with respect to t) .
$\partial_t \mathbf{u}$	$= (\partial_t u_i)_{i=1,\dots,N}$.
∇p	$= (\partial_i p)_{i=1,\dots,N}$ (gradient of p).
$\nabla \mathbf{u}$	$= (\partial_i u_j)_{i,j=1,\dots,N}$.
$\nabla \cdot \mathbf{u}$	$= \sum_{i=1}^N \partial_i u_i$ (divergence of $\mathbf{u}$).
$\nabla \times \mathbf{u}$	$= (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$, (curl of $\mathbf{u}$ in three dimensions).
$\mathbf{u} \cdot \nabla \varphi$	$= \sum_{j=1}^N u_j \partial_j \varphi$.
$\mathbf{u} \cdot \nabla \mathbf{v}$	$= (\sum_{j=1}^N u_j \partial_j v_i)_{i=1,\dots,N}$.
$D_t \varphi$	$= \partial_t \phi + \mathbf{u} \cdot \nabla \varphi$ (total derivative of scalar ϕ in a flow field $\mathbf{u}$).
$D_t \mathbf{v}$	$= \partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v}$.

Function spaces

$C(\mathcal{G})$	space of continuous functions on $\mathcal{G}$.
$C^k(\mathcal{G})$	space of functions with continuous k-th derivative on $\mathcal{G}$.
$C_0^\infty(\mathcal{G})$	space of infinitely smooth functions with compact support in $\mathcal{G}$.
$C^\infty(\bar{\mathcal{G}})$	space of infinitely smooth functions on $\bar{\mathcal{G}}$.
$L^p(\mathcal{G})$	Lebesgue space of measureable functions q with finite norm $\|q\|_{0,p} = \left(\int_{\mathcal{G}} \|q\|^p\right)^{\frac{1}{p}}$.

$W_p^k(\mathcal{G})$	Sobolev space of functions with k-th derivatives in $L^p(\mathcal{G})$.
$H^k(\mathcal{G})$	$= W_2^k(\mathcal{G})$.
$H^{\frac{1}{2}}(\partial\mathcal{G})$	space of traces of functions in $H^1(\mathcal{G})$.
$H_0^1(\mathcal{G})$	the closure of $C_0^\infty(\mathcal{G})$ in $H^1(\mathcal{G})$.
$H^{-1}(\mathcal{G})$	the dual space of $H_0^1(\mathcal{G})$.

Contents

Chapter 1

Physical modelling

In a typical case of the problems we want to solve we are facing the following situation: A fluid (e. g. a mixture of air and petrol) enters an experimental setup (e. g. the piston of an engine) through an aperture (e. g. driven by a specified pressure drop), moves through the setup in turbulent motion and exits again at one or more outflow ducts.

The aim of this chapter is to develop a closed set of equations together with appropriate boundary conditions, which govern the turbulent flow of this fluid. The system of underlying equations — namely the Navier-Stokes equations — is well-known and well-studied, but the high degree of irregularity for strongly turbulent flows still doesn't allow reasonable 'direct' numerical simulations for high Reynolds numbers. That's when turbulence modelling gets involved.

We first give a short summary of typical properties of turbulent fluid flow, which we shall then use to derive the k-ϵ model, first introduced by Launder and Spalding [LS72], which is a reasonably good approximation in many practical cases and more suitable for numerical computation than the original problem, because small scale fluctuations have been 'filtered out'. In the derivation we basically follow the approach of Mohammadi and Pironneau, [MP94]. At solid walls the model has to be coupled with a so called wall law — analytical expressions describing the flow — in order to 'bridge' boundary layers, for which the model hypotheses are not valid. Finally we supplement our equations with inflow and outflow conditions corresponding to various physical scenarios, which we believe to be of interest in technical applications.

1.1 Discussion on turbulent flow

1.1.1 General properties

At a quick glance turbulent flow is characterized by chaotic movement of particles as opposed to the ordered structure of laminar flow. This occurs, when inertial forces are large compared to viscous forces so internal friction is no longer able to damp out chaotic fluctuations origining in - ever present and inevitable - disturbances (e. g. at the inflow,

boundary, etc.).

The transition from laminar flow to turbulence was first investigated on a systematic basis by Osborne Reynolds (1842-1912). To this end he introduced the dimensionless *Reynolds number*

$$Re = \frac{\rho L V}{\mu},$$

where ρ is the density of the fluid, L and V a characteristic length and velocity, respectively, and μ the viscosity of the fluid. In his famous experiments he observed that

1. flow through a geometrically similar setup is physically similar, if and only if Re is identical (cf. 1.1.2) and

2. turbulence occurs, when Re exceeds a specific threshold (depending on the geometry).

Qualitatively, turbulent flow has several characteristic features, such as

- a large range of frequencies and wave lengths,
- disparity in length scales,
- irregularity,
- energy cascades,
- dissipation,
- mixing.

It is observed that large eddies brake into smaller eddies and at the end of the spectrum energy is dissipated by internal friction. On the macroscopic level, diffusion is increased — compared to the laminar case — as a result of 'virtual stresses' caused by small scale fluctuations. Reynolds conjectured that turbulent flow can be described similar to laminar flow with the (constant) viscosity replaced by a variable term — this is indeed the case, as we shall see later.

Turbulence is called *homogeneous*, if all moments (and particularly means) of the velocity and its gradients are spatially constant. It is called *isotropic*, if all moments are invariant by rotation.

As our starting point for a mathematical description we take the Navier-Stokes equations for incompressible flow. A mathematically rigorous derivation from fundamental physical principles and conservation laws can be found in [Fei93].

1.1.2 Navier-Stokes equations

We denote by $\mathbf{u}$ the velocity of the fluid, p is the static pressure. Then the instationary flow of incompressible, Newtonian fluids is governed by

$$\rho \frac{\partial}{\partial t}\mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{1.2}$$

(1.1) expresses Newton's law of motion, (1.2) the conservation of mass. The underlying physical assumption for these equations to hold are incompressibility and Stokes' hypothesis for the stress tensor,

$$\mathbf{T}(\mathbf{u}, p) = -p\mathbf{I} + \mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right),$$

for incompressible Newtonian fluids. A 'good' turbulence model should preserve fundamental principles of Newtonian mechanics and similarity properties of Navier-Stokes equations.

Physical similarity

With the choice of scales $\mathbf{u} = V\mathbf{u}^*$, $\mathbf{x} = L\mathbf{x}^*$, $t = L/Vt^*$ and $p = V^2 \rho p^*$ we get the dimensionless formulation

$$\partial_{t^*}\mathbf{u}^* + \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \nu \Delta^* \mathbf{u}^*, \tag{1.3}$$

$$\nabla^* \cdot \mathbf{u}^* = 0, \tag{1.4}$$

where

$$\boxed{\nu := \frac{1}{Re} = \frac{\mu}{\rho L V}.}$$

For simplicity in notation we will omit the stars again in the following.

Frame invariance

Navier-Stokes equations arise from Newtonian mechanics, which is frame invariant. Consequently they are invariant under

- translations $\mathbf{x} = \mathbf{y} + \mathbf{z}$, $\mathbf{z} \in \mathbb{R}^N$ const.,
- Galilean transformations $\mathbf{x} = \mathbf{y} + \mathbf{w}t$, $\mathbf{w} \in \mathbb{R}^N$ const., and
- time independent rotations $\mathbf{x} = \mathbf{M}\mathbf{y}$, $\mathbf{M} \in \mathbb{R}^{N \times N}$, where $\mathbf{M}^T = \mathbf{M}^{-1}$.

Standard boundary conditions for Navier-Stokes are no-slip conditions $\mathbf{u} = \mathbf{0}$ at solid walls, which 'close' the system (1.3), (1.4) and the existence of (for two space dimensions and under stronger assumptions also in three dimensions unique) solutions can be shown.

1.1.3 Why turbulence modelling?

Navier-Stokes equations describe correctly the flow for **all** values of ν. However, if ν becomes very small, regularity of the solution is lost. In numerical simulations todays computers cannot handle as many grid points as are necessary to resolve small eddies.

To see why, we study the distribution of turbulent kinetic energy to the different modes, i. e. we look at the *kinetic energy density* E as a function of the wave number $\kappa = |\mathbf{k}|$, the modulus of the wave vector. For homogeneous isotropic stationary turbulence and κ in the *initial range* $V\nu^{-1} \ll \kappa \ll \epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}}$ dimensional analysis gives *Kolmogorov's law*

$$E(\kappa) \approx 1.5\epsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$$

with the *dissipation rate of turbulent kinetic energy* ϵ. Viscosity affects only the end of the spectrum, initial conditions the beginning. According to this, the smallest eddies are of order of magnitude $\nu^{\frac{3}{4}}$, and the number of grid points that is necessary to resolve these structures is

$$n \sim Re^{\frac{3N}{4}},$$

when N is the space dimension. In many important applications $Re = O(10^6)$ yielding $n = O(10^{13.5})$.

Anyway, often it is not the motion of each and every small eddy, which is interesting, but only the effect of turbulence on quantities of engineering significance is relevant. The idea is now to calculate some sort of averaged flow field that is smoother than the actual flow, which drastically reduces the number of grid points required — the effects of small scale fluctuations are estimated by different *ad hoc* models, the derivation of which we will be concerned with in sections 1.2 and 1.3.

1.1.4 Boundary layers

For high Reynolds numbers the behaviour of the flow is dominated by the convective part. Near the wall, however, the velocity is small due to interaction with the wall (no-slip conditions) yielding a small 'local Reynolds number', and the properties of the flow in this region are very different from the main stream.

In a thin *boundary layer* with width $\delta \sim \nu$ the flow passes from one regime to the other. This gives two major computational difficulties, namely

- different simplifications are required for near wall and other regions and
- strong gradients appear in close vicinity of the wall.

We will circumvent these difficulties by excluding the boundary layer from the computational domain and set up an auxiliary boundary condition at the interface using an appropriate boundary layer model (see 1.4).

1.2 Eddy viscosity models

As we pointed out before, the aim of turbulence modelling is to gain regularity for the solution of our problem. Now the idea is to decompose the velocity (pressure) field via

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}', \quad p = \bar{p} + p',$$

where $\mathbf{u}'$ is the non relevant (non computable) part, $\bar{\mathbf{u}}$ the mean part and to look for $\bar{\mathbf{u}}$ instead of $\mathbf{u}$. The interactions between $\mathbf{u}'$ and $\mathbf{u}'$, $\mathbf{u}'$ and $\bar{\mathbf{u}}$ etc. have to be modeled.

So far, so good. But how do we get the decomposition? And which equations does $\bar{\mathbf{u}}$ fulfil? The answers to these questions are found in 1.2.1 resp. 1.2.2. It will be seen that the equations for $\bar{\mathbf{u}}$ are similar to the original equations for $\mathbf{u}$ with additional stresses, which are usually modeled proportional to the original stresses with a turbulent 'eddy viscosity', which gives name to this class of models.

1.2.1 Filters

We first need a suitable rule $\langle . \rangle$ defined on an appropriate function space, say $\mathcal{W}$, to obtain a 'filtered' quantity $\bar{w}$ from its actual values w via

$$\langle w \rangle =: \bar{w}$$

for any $w \in \mathcal{W}$. Several approaches are at hand, which give rise to the following definitions:

1. *Space averaging:* In a first attempt one might think of eliminating small spatial fluctuations by replacing the actual value of w at a specific point by the average over a ball with radius r around this point,

$$\langle w \rangle_B (\mathbf{x}, t) = \frac{1}{|B|} \int_{B(\mathbf{x},r)} w(\mathbf{y}, t)\, d\mathbf{y},$$

   where r is large compared to turbulent length scales, but small with respect to the macroscopic dimensions of the setup.

2. *Time averaging:* Similarly we can integrate in time over an interval of length T_1,

$$\langle w \rangle_T (\mathbf{x}, t) = \frac{1}{T_1} \int_{t-T_1}^{t} w(\mathbf{x}, \tau)\, d\tau,$$

   with T_1 considerably larger than turbulent time scales, but small in comparison to the overall duration of the process.

3. *Space-time averaging:* Combining these ideas leads to

$$\langle\langle w \rangle\rangle = \langle \langle w \rangle_B \rangle_T = \langle \langle w \rangle_T \rangle_B .$$

4. *Fourier (low pass) filter:* Another possibility to cut off small wavelengths in the spectrum is to leave out these components in a Fourier decomposition,

$$\langle w \rangle_F = F^{-1} \pi_N F(w).$$

F denotes the discrete Fourier transform, π_N the truncation operator.

5. *Statistical average:* Finally one might take into account that turbulent fluctuations origin in small disturbances in the initial conditions and study the average over the evolution of random initial data, i. e. $\bar{\mathbf{u}}$ is viewed as the expected value of the distribution of $\mathbf{u}$ due to a (given) distribution of initial data. So let ω be the induced probability measure, then

$$\langle w \rangle_E (\mathbf{x}, t) = \int w(\mathbf{x}, t, \omega) \, d\omega.$$

We will expect several properties of filters:

(a) Linearity:

$$\langle v + \lambda w \rangle = \langle v \rangle + \lambda \langle w \rangle \quad \forall v, w \in \mathcal{W}, \lambda \in \mathbb{R}.$$

(b) $\langle . \rangle$ maintains differentiability and

$$\begin{aligned} \langle \partial_t w \rangle &= \partial_t \langle w \rangle \quad \forall w \in \mathcal{W}, \\ \langle \nabla w \rangle &= \nabla \langle w \rangle \quad \forall w \in \mathcal{W}. \end{aligned}$$

(c) Double averaging gives average, i. e.

$$\langle \langle w \rangle \rangle = \langle w \rangle \quad \forall w \in \mathcal{W}.$$

Remark. This is necessary for $\langle w' \rangle = 0$.

More generally we may require

(d)

$$\langle v \langle w \rangle \rangle = \langle v \rangle \langle w \rangle \quad \forall v, w \in \mathcal{W}.$$

Unfortunately the simple space and time filters do not fulfil properties (c) and (d). We summarize the results about the properties of the listed filters as stated in [MP94] (see also Appendix A) ('X' stands for 'fulfils', 'O' for 'does not fulfil') :

	(a)	(b)	(c)	(d)
1.	X	X	O	O
2.	X	X	O	O
3.	X	X	O	O
4.	X	X	X	O
5.	X	X	X	X

1.2.2 Averaged equations — the Reynolds stresses

We will now apply a filter, for which all four properties hold, to the Navier-Stokes equations to obtain equations for the averaged quantities. Due to (a) - (c) we can just replace p by $\bar{p}$ and $\mathbf{u}$ by $\bar{\mathbf{u}}$ in the linear terms (mixing is only an effect of the non-linearity). Furthermore

$$\begin{aligned}\langle \mathbf{u} \cdot \nabla \mathbf{u} \rangle &= \langle (\bar{\mathbf{u}} + \mathbf{u}') \cdot \nabla (\bar{\mathbf{u}} + \mathbf{u}') \rangle \\ &= \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \langle \mathbf{u}' \rangle + \langle \mathbf{u}' \rangle \cdot \nabla \bar{\mathbf{u}} + \langle \mathbf{u}' \cdot \nabla \mathbf{u}' \rangle \\ &= \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle ,\end{aligned}$$

so the averaged equations read

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -\nabla \bar{p} + \nu \Delta \bar{\mathbf{u}} - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle \tag{1.5}$$

$$\nabla \cdot \bar{\mathbf{u}} = 0. \tag{1.6}$$

Turbulence modelling is the problem of approximating the term $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ in *Reynolds' equation* (1.5) in terms of known or at least computable quantities, particularly without the use of $\mathbf{u}'$, which we intended to get rid of in the first place. Common to all models is the assumption — based on experiments and symmetry — that the additional virtual *Reynolds stresses* only depend on the original stress tensor,

$$-\langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \mathbf{R}(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T). \tag{1.7}$$

Reasonable models have to retain the invariance properties of the Navier-Stokes system. Translation and Galilean invariance are automatically fulfilled. For rotational invariance we notice that with $\mathbf{x} = \mathbf{M}\mathbf{y}$

$$\nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} \mathbf{u}^T = \mathbf{M} \left(\nabla_{\mathbf{y}} \mathbf{v} + \nabla_{\mathbf{y}} \mathbf{v}^T \right) \mathbf{M}^T,$$

so it is obviously necessary (and also sufficient) for invariance that

$$\mathbf{R} \left(\mathbf{M} \left(\nabla_{\mathbf{y}} \mathbf{v} + \nabla_{\mathbf{y}} \mathbf{v}^T \right) \mathbf{M}^T \right) = \mathbf{M}\mathbf{R} \left(\nabla_{\mathbf{y}} \mathbf{v} + \nabla_{\mathbf{y}} \mathbf{v}^T \right) \mathbf{M}^T.$$

Definition 1.1. A function $\mathbf{A} : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is called invariant, iff it transforms as

$$\tilde{\mathbf{A}}(\mathbf{M}\mathbf{y}) = \mathbf{M}\mathbf{A}(\mathbf{y})\mathbf{M}^T$$

under rotations. $\mathbf{M}$ is an arbitrary rotation matrix, $\tilde{\mathbf{A}}$ the transformed matrix (more precisely: the representation of $\mathbf{A}$ with respect to the transformed basis).

Theorem 1.2. *If $\mathbf{A} : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ is symmetric and invariant under rotations, then $\mathbf{R} \circ \mathbf{A}$ is invariant for $\mathbf{R} : \mathbb{R}^{N \times N} \to \mathbb{R}^{N \times N}$, iff*

$$\mathbf{R}(\mathbf{A}) = \sum_{j=0}^{N-1} a_j \mathbf{A}^j, \tag{1.8}$$

where all a_j are invariant under rotations.

Proof. It is obvious that all functions of the form (1.8) are invariant. For the 'only if' part we assume that $\mathbf{R} \circ \mathbf{A}$ is invariant and proceed in two steps:

1. The eigenspaces of $\mathbf{A}$ and $\mathbf{R}(\mathbf{A})$ are identical: We define two sets of rotation matrices $\mathbf{Q}^{(i)}$ and $\mathbf{Q}^{(ij)}$ by

$$Q^{(i)}_{lm} = \begin{cases} \delta_{lm} & l \neq i \\ -\delta_{lm} & l = i \end{cases}$$

and

$$Q^{(ij)}_{lm} = \begin{cases} \delta_{lm} & m \notin \{i, j\} \\ \delta_{lj} & m = i \\ \delta_{li} & m = j \end{cases} .$$

Now if $\mathbf{M}$ is a matrix of normalized orthogonal eigenvectors of $\mathbf{A}$ with corresponding eigenvalues λ_k then for $\mathbf{Q} = \mathbf{Q}^{(i)}$ also $\mathbf{MQ}$ is such a matrix. If $\lambda_i = \lambda_j$ then this is also valid for $\mathbf{Q} = \mathbf{Q}^{(ij)}$. This means $(\mathbf{MQ})^T \mathbf{A} (\mathbf{MQ}) = \mathbf{M}^T \mathbf{A} \mathbf{M} = \Lambda$ and

$$\mathbf{Q}^T \mathbf{M}^T \mathbf{R}(\mathbf{A}) \mathbf{M} \mathbf{Q} = \mathbf{R}(\mathbf{Q}^T \mathbf{M}^T \mathbf{A} \mathbf{M} \mathbf{Q}) = R(\mathbf{M}^T \mathbf{A} \mathbf{M}) = \mathbf{M}^T \mathbf{R}(\mathbf{A}) \mathbf{M}. \tag{1.9}$$

We want to show that $\mathbf{D} := \mathbf{M}^T \mathbf{R}(\mathbf{A}) \mathbf{M}$ is diagonal and the elements on positions i and j are equal for $\lambda_i = \lambda_j$. This is pretty straightforward: From (1.9), $\mathbf{Q}^{(i)} \mathbf{D} \mathbf{Q}^{(i)} = \mathbf{D}$, and

$$\left(\mathbf{Q}^{(i)} \mathbf{D} \mathbf{Q}^{(i)}\right)_{lm} = \begin{cases} D_{lm} & l = m = i \vee (l \neq i \wedge m \neq i) \\ -D_{lm} & \text{else} \end{cases}$$

we get that $\mathbf{D}$ has non-zero entries in the i-th row and column only in the diagonal. Since this is valid for all i, $\mathbf{D}$ is diagonal, e. g. $\mathbf{D} = \operatorname{diag}(\mu_i)$, and now from

$$\left(\mathbf{Q}^{(ij)} \mathbf{D} \mathbf{Q}^{(ij)}\right)_{lm} = \begin{cases} \mu_i \delta_{jm} & l = j \\ \mu_j \delta_{im} & l = i \\ \mu_l \delta_{lm} & \text{else} \end{cases}$$

we get that $\mu_i = \mu_j$, if $\lambda_i = \lambda_j$.

2. Let $\mathbf{p}_i$ be the eigenvectors of $\mathbf{A}$ resp. $\mathbf{R}(\mathbf{A})$ corresponding to eigenvalues λ_i resp. μ_i, then

$$\mathbf{A}^j = \sum_{i=1}^{N} \lambda_i^j \mathbf{p}_i \mathbf{p}_i^T \Rightarrow \sum_{j=0}^{d-1} a_j \mathbf{A}^j = \sum_{i=1}^{N} \left(\sum_{j=0}^{N-1} a_j \lambda_i^j \right) \mathbf{p}_i \mathbf{p}_i^T$$

and

$$\mathbf{R}(\mathbf{A}) = \sum_{i=1}^{N} \mu_i \mathbf{p}_i \mathbf{p}_i^T .$$

Consequently a_j can be found by solving the linear system $\sum_{j=0}^{N-1} a_j \lambda_i^j = \mu_i$, which has a solution because of 1. ($\lambda_i = \lambda_j \Rightarrow \mu_i = \mu_j$).

□

Corollary 1.3. *It is a necessary consequence of invariance that*

- *in 2D*

$$\mathbf{R}(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T) = a(|\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T|)\mathbf{I} + \mu(|\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T|)(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)$$

  and

- *in 3D*

$$\mathbf{R}(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T) = a\mathbf{I} + \mu(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T) + \lambda(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)^2,$$

  where a, μ, λ are functions of $|\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T|$ and $|(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T)^2|$ only.

Proof. We apply Theorem 1.2 to $\mathbf{A} := \nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T$. The coefficients a_j in the proof depend on the eigenvalues λ_i, hence on N independent invariants of $\mathbf{A}$, which can also be tr$\mathbf{A}$, $|\mathbf{A}|$ and (for N=3) $|\mathbf{A}^2|$. The desired results follow immediately with tr$\mathbf{A} = 2\nabla\cdot\bar{\mathbf{u}} = 0$. □

1.2.3 Eddy viscosity models and the turbulent kinetic energy

Fundamental for this class of models is Boussinesq's assumption (neglecting the quadratic term in 3D)

$$\mathbf{R} = -\frac{2}{3}k\mathbf{I} + \nu_T\left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right) \tag{1.10}$$

with the *turbulent viscosity* ν_T and the *turbulent kinetic energy*

$$k := \frac{1}{2}\left\langle|\mathbf{u}'|^2\right\rangle. \tag{1.11}$$

Remark. The first term is necessary for the model to be consistent in the three dimensional case in the sense that

$$\text{tr}\mathbf{R} = -\text{tr}\langle\mathbf{u}'\otimes\mathbf{u}'\rangle = -\left\langle|\mathbf{u}'|^2\right\rangle = -2k.$$

In fact we should replace $\frac{2}{3}k$ by $\frac{2}{N}k$ in general (and particularly by k in two dimensions), but this is not done usually.

This leads to the following equation for $\bar{\mathbf{u}}$:

$$\boxed{\partial_t\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} = -\nabla\left(\bar{p} + \frac{2}{3}k\right) + \nabla\cdot\left[(\nu+\nu_T)\left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right)\right]} \tag{1.12}$$

Remark. 1. So far the only simplification inherent to the model is Boussinesq's assumption (1.10), which — for two space dimensions — can be deduced from assumption (1.7).

2. k can formally be included in the pressure term, but it has to be noticed that in that case boundary conditions in $\bar{p}$ (e. g. if we want to specify a pressure drop) have to be formulated in $\bar{p}+\frac{2}{3}k$, which is the static pressure augmented with turbulent dynamic pressure.

The remaining problem is to find an expression for ν_T, and there is a cascade of models with a wide range of complexity. A popular representative of the class of eddy viscosity models is the k-ϵ model, which forms a compromise between generality and complexity.

1.3 The k-ϵ model

The k-ϵ model is based on the ansatz

$$\boxed{\nu_T = c_\mu \frac{k^2}{\epsilon},} \tag{1.13}$$

where

$$\epsilon := \nu \left\langle \nabla \mathbf{u}' : \nabla \mathbf{u}' \right\rangle = \nu \sum_{i,j} \left\langle \left(\frac{\partial u_i'}{\partial x_j} \right)^2 \right\rangle \tag{1.14}$$

is the *dissipation rate of turbulent kinetic energy*. For homogeneous turbulence it is assumed (see [MP94]) that

$$\left\langle \nabla \mathbf{u}' : \nabla \mathbf{u}'^T \right\rangle = \sum_{j,k} \left\langle \frac{\partial u_k'}{\partial x_j} \frac{\partial u_j'}{\partial x_k} \right\rangle = 0$$

and hence

$$\begin{aligned} \epsilon &= \nu \left\langle \left(\nabla \mathbf{u}' + \nabla \mathbf{u}'^T \right) : \nabla \mathbf{u}' \right\rangle \\ &= \frac{\nu}{2} \left\langle \left| \nabla \mathbf{u}' + \nabla \mathbf{u}'^T \right| \right\rangle \\ &= \nu \left\langle |\nabla \times \mathbf{u}'|^2 \right\rangle . \end{aligned}$$

The last equation is obtained by ($\epsilon_{ijk} = e_i(e_j \times e_k)$)

$$\begin{aligned} |\nabla \times \mathbf{u}'|^2 &= \sum_i \left(\sum_{j,k} \epsilon_{ijk} \frac{\partial u_k'}{\partial x_j} \right)^2 \\ &= \sum_{j,k,l,m} \sum_i \epsilon_{ijk} \epsilon_{ilm} \frac{\partial u_k'}{\partial x_j} \frac{\partial u_m'}{\partial x_l} \\ &= \sum_{j,k} \left(\frac{\partial u_k'}{\partial x_j} \right)^2 - \sum_{j,k} \frac{\partial u_k'}{\partial x_j} \frac{\partial u_j'}{\partial x_k}, \end{aligned}$$

where we have used the identity $\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

Contrary to *algebraic models* (*0-equation models*), where algebraic expressions for k and possibly ϵ are used and no additional partial differential equations need to be solved, and *1-equation models*, where one equation is added to the original system, the k-ϵ model employs PDEs for both k and ϵ. *2-equation models* are nowadays *state-of-the-art*.

1.3.1 Derivation of the equation for k

Subtracting the averaged momentum equation from the original one gives an equation for the fluctuations,

$$\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \bar{\mathbf{u}} + (\mathbf{u}' + \bar{\mathbf{u}}) \cdot \nabla \mathbf{u}' + \nabla p' - \nu \Delta \mathbf{u}' - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \mathbf{0}. \tag{1.15}$$

Multiplying with $\mathbf{u}'$, inserting the identities

$$\begin{aligned} \mathbf{u}'(\mathbf{u}' \cdot \nabla \bar{\mathbf{u}}) &= (\mathbf{u}' \otimes \mathbf{u}') : \nabla \bar{\mathbf{u}}, \\ \mathbf{u}' \left[(\bar{\mathbf{u}} + \mathbf{u}') \cdot \nabla \mathbf{u}' \right] &= (\bar{\mathbf{u}} + \mathbf{u}') \cdot \nabla \frac{1}{2} \mathbf{u}'^2 \end{aligned}$$

and applying $\langle . \rangle$ leads to

$$\partial_t \left\langle \frac{\mathbf{u}'^2}{2} \right\rangle + \langle \mathbf{u}' \otimes \mathbf{u}' \rangle : \nabla \bar{\mathbf{u}} + \left\langle (\bar{\mathbf{u}} + \mathbf{u}') \cdot \nabla \frac{\mathbf{u}'^2}{2} \right\rangle + \nabla \cdot \langle p' \mathbf{u}' \rangle - \nu \langle \mathbf{u}' \Delta \mathbf{u}' \rangle = 0.$$

With the definition of k and Reynolds' hypothesis this yields

$$\partial_t k + \bar{\mathbf{u}} \cdot \nabla k = \mathbf{R} : \nabla \bar{\mathbf{u}} - \left\langle \mathbf{u}' \cdot \nabla \frac{\mathbf{u}'^2}{2} \right\rangle - \nabla \cdot \langle p' \mathbf{u}' \rangle + \nu \langle \mathbf{u}' \Delta \mathbf{u}' \rangle . \tag{1.16}$$

The first term on the right side of (1.16), which is referred to as *production rate P of turbulent kinetic energy*, is given by

$$\begin{aligned} \mathbf{R} : \nabla \bar{\mathbf{u}} &= -\frac{2}{3} k \mathbf{I} : \nabla \bar{\mathbf{u}} + \nu_T \left(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right) : \nabla \bar{\mathbf{u}} \\ &= \frac{\nu_T}{2} \left| \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right|^2 . \end{aligned}$$

The next terms are modelled as a diffusion process, i. e.

$$- \left\langle \mathbf{u}' \cdot \nabla \frac{\mathbf{u}'^2}{2} \right\rangle - \nabla \cdot \langle p' \mathbf{u}' \rangle = \nabla \cdot (\mu_k \nabla k) ,$$

where μ_k is the dimensionally correct combination $c_k \frac{k^2}{\epsilon}$. Experiments show that $c_k = c_\mu$ approximately and therefore we replace it right away. Finally the last term can be rewritten by

$$\nu \langle \mathbf{u}' \Delta \mathbf{u}' \rangle = \frac{\nu}{2} \left\langle \Delta \mathbf{u}'^2 \right\rangle - \nu \langle \nabla \mathbf{u}' : \nabla \mathbf{u}' \rangle = \nu \Delta k - \epsilon.$$

Rearranging we get

$$\partial_t k + \bar{\mathbf{u}} \cdot \nabla k = \nabla \cdot \left[(c_\mu \frac{k^2}{\epsilon} + \nu) \nabla k \right] + \frac{c_\mu}{2} \frac{k^2}{\epsilon} \left| \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right|^2 - \epsilon,$$

where ϵ is the dissipation rate of turbulent kinetic energy, an equation for which has yet to be derived.

1.3.2 Derivation of the equation for ϵ

Here we will need to utilize a very strong instrument for modelling certain terms: *ergodicity*. Heuristically this means that under certain conditions (e. g. a strongly mixing process) it is allowed to replace statistical averaging with spatial or time averaging, thus we can use partial integration in space. Unfortunately the applicability of ergodicity to the Navier-Stokes equations is still an open problem, we are not able to prove it formally, but take it as a modelling hypothesis.

We start again with (1.15)

$$\partial_t \mathbf{u}' + \mathbf{u}' \nabla \bar{\mathbf{u}} + (\mathbf{u}' + \bar{\mathbf{u}}) \nabla \mathbf{u}' + \nabla p' - \nu \Delta \mathbf{u}' - \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \mathbf{0}.$$

Taking the curl leads to

$$\partial_t \nabla \times \mathbf{u}' + \nabla \times (\mathbf{u}' \nabla \bar{\mathbf{u}}) + \nabla \times [(\bar{\mathbf{u}} + \mathbf{u}') \nabla \mathbf{u}'] + \underbrace{\nabla \times \nabla p'}_{0} - \nu \underbrace{\nabla \times \Delta \mathbf{u}'}_{\Delta \nabla \times \mathbf{u}'} - \nabla \times \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle = 0. \tag{1.17}$$

With easy but lengthy calculation we can show that

$$\begin{aligned} \nabla \times (\mathbf{u}' \nabla \bar{\mathbf{u}}) + \nabla \times [(\bar{\mathbf{u}} + \mathbf{u}') \nabla \mathbf{u}'] &= (\mathbf{u}' \nabla) \nabla \times \bar{\mathbf{u}} - (\nabla \times \mathbf{u}' \nabla) \bar{\mathbf{u}} \\ &\quad + (\bar{\mathbf{u}} + \mathbf{u}') \nabla \nabla \times \mathbf{u}' - \nabla \times (\bar{\mathbf{u}} + \mathbf{u}') \nabla \mathbf{u}'. \end{aligned}$$

We now denote

$$\begin{aligned} \omega' &:= \nabla \times \mathbf{u}', \\ \omega &:= \nabla \times \bar{\mathbf{u}}, \end{aligned}$$

and get from (1.17)

$$\partial_t \omega' + \mathbf{u}' \nabla \omega - \omega' \nabla \bar{\mathbf{u}} + (\bar{\mathbf{u}} + \mathbf{u}') \nabla \omega' - (\omega + \omega') \nabla \mathbf{u}' - \nu \Delta \omega' + \nabla \times \nabla \cdot \mathbf{R} = 0.$$

Multiplying with $2\nu\omega'$ and applying $\langle . \rangle$ leads to

$$2\nu \left\langle \omega' (\partial_t \omega' + (\bar{\mathbf{u}} + \mathbf{u}') \nabla \omega' + \mathbf{u}' \nabla \omega - (\omega + \omega') \nabla \mathbf{u}' - \omega' \nabla \bar{\mathbf{u}} - \nu \Delta \omega') \right\rangle = 0.$$

We now simplify some of the terms above.

- $\langle 2\nu\omega'\partial_t\omega'\rangle = \partial_t \langle \nu\omega'^2\rangle = \partial_t\epsilon$
- $2\nu\omega'(\bar{\mathbf{u}}+\mathbf{u}')\nabla\omega' = \nu(\bar{\mathbf{u}}+\mathbf{u}')\nabla\omega'^2$
- $$\begin{aligned} & \nabla\times(\mathbf{u}'\times\omega) = \omega\nabla\mathbf{u}' - \mathbf{u}'\nabla\omega \\ \Longrightarrow\quad & \omega'\nabla\times(\mathbf{u}'\times\omega) = \omega'\omega\nabla\mathbf{u}' - \omega'\mathbf{u}'\nabla\omega \\ \Longrightarrow\quad & 2\nu\omega'\mathbf{u}'\nabla\omega - 2\nu\omega'\omega\nabla\mathbf{u}' = \omega'\nabla\times(\mathbf{u}'\times\omega) \end{aligned}$$
- A simple calculation shows $\mathbf{a}(\mathbf{a}\cdot\nabla\mathbf{b}) = (\mathbf{a}\otimes\mathbf{a}):\nabla\mathbf{b}$

$$\Longrightarrow\quad \begin{cases} -2\nu\omega'\omega'\nabla\mathbf{u}' = -2\nu(\omega'\otimes\omega'):\nabla\mathbf{u}' \\ -2\nu\omega'\omega'\nabla\bar{\mathbf{u}} = -2\nu(\omega'\otimes\omega'):\nabla\bar{\mathbf{u}} \end{cases}.$$

Combining these results we get

$$\begin{aligned} \partial_t\epsilon + \left\langle (\bar{\mathbf{u}}+\mathbf{u}')\nabla(\nu\omega'^2)\right\rangle - 2\nu\langle\omega'\nabla\times(\mathbf{u}'\times\omega)\rangle - 2\nu\langle\omega'\otimes\omega'\rangle:\nabla\bar{\mathbf{u}} \\ -2\nu\langle(\omega'\otimes\omega'):\nabla\mathbf{u}'\rangle - 2\nu^2\langle\omega'\Delta\omega'\rangle = 0. \end{aligned} \quad (1.18)$$

Ergodicity yields

$$\langle\omega'\Delta\omega'\rangle = -\left\langle|\nabla\omega'|^2\right\rangle.$$

Analogously to the k-equation, $\left\langle\mathbf{u}'\nabla\nu\omega'^2\right\rangle$ is modelled as a diffusion term $-\nabla\cdot(\mu_\epsilon\nabla\epsilon)$ with $\mu_\epsilon = c_\epsilon\frac{k^2}{\epsilon}$.

Furthermore we have (intensively using ergodicity, referred to as 'erg')

$$\begin{aligned} \langle\omega'\nabla\times(\mathbf{u}'\times\omega)\rangle &\overset{\text{erg}}{=} \langle(\mathbf{u}'\times\omega)\nabla\times\omega'\rangle \overset{\text{erg}}{=} -\langle\mathbf{u}'\times\omega\Delta\mathbf{u}'\rangle \\ &= -\left\langle \begin{pmatrix} u_2'\omega_3 - u_3'\omega_2 \\ u_3'\omega_1 - u_1'\omega_3 \\ u_1'\omega_2 - u_2'\omega_1 \end{pmatrix} \begin{pmatrix} \Delta u_1' \\ \Delta u_2' \\ \Delta u_3' \end{pmatrix} \right\rangle \\ &= \omega_1\langle u_2'\Delta u_3' - u_3'\Delta u_2'\rangle + \omega_2\langle u_3'\Delta u_1' - u_1'\Delta u_3'\rangle + \omega_3\langle u_1'\Delta u_2' - u_2'\Delta u_1'\rangle \end{aligned}$$

Using

$$\langle u_1'\Delta u_2' - u_2'\Delta u_1'\rangle \overset{\text{erg, Green}}{=} \frac{1}{|B(\mathbf{x},r)|}\int_{\partial B(\mathbf{x},r)}\left(u_1'\frac{\partial u_2'}{\partial\mathbf{n}} - u_2'\frac{\partial u_1'}{\partial\mathbf{n}}\right) = 0 \quad (1.19)$$

we get

$$\langle\omega'\nabla\times(\mathbf{u}'\times\omega)\rangle = 0.$$

The last equality in (1.19) holds, because we assume isotropy of $\mathbf{u}'$ to neglect all odd boundary integrals after integration by parts. [1] Thus we have

$$\partial_t \epsilon + \bar{\mathbf{u}} \nabla \epsilon - 2\nu \langle \omega' \otimes \omega' \rangle : \nabla \bar{\mathbf{u}} - \nabla \cdot (\mu_\epsilon \nabla \epsilon) - 2\nu \langle (\omega' \otimes \omega') : \nabla \mathbf{u}' \rangle + 2\nu^2 \left\langle |\nabla \omega'|^2 \right\rangle = 0.$$

Analogously to $\mathbf{R}$ the term $-2\nu \langle \omega' \otimes \omega' \rangle$ is modeled — using a Reynolds-type hypothesis — by $a\mathbf{I} - c_1 k(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T)$. This leads to

$$-2\nu \langle \omega' \otimes \omega' \rangle : \nabla \bar{\mathbf{u}} = (a \underbrace{\nabla \cdot \bar{\mathbf{u}}}_{0} - c_1 k \underbrace{(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T) : \nabla \bar{\mathbf{u}}}_{\frac{1}{2}|\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T|^2}).$$

The previous to last term is neglected as a third order moment,

$$\langle (\omega' \otimes \omega') : \nabla \mathbf{u}' \rangle = 0.$$

Finally $2\nu^2 \left\langle |\nabla \omega'|^2 \right\rangle$ is assumed to be a function of k and ϵ only and is chosen proportional to the (dimensionally correct) combination ϵ^2/k. Inserting everything in (1.18) gives the desired equation for ϵ as

$$\partial_t \epsilon + \bar{\mathbf{u}} \cdot \nabla \epsilon = \nabla \cdot \left(c_\epsilon \frac{k^2}{\epsilon} \nabla \epsilon \right) + \frac{c_1}{2} k |\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T|^2 - c_2 \frac{\epsilon^2}{k}.$$

1.3.3 Choice of constants

The model contains four undetermined constants c_μ, c_1, c_2 and c_ϵ, which are chosen to fit the model to experiments of well understood cases.

Measurements of the decay of homogeneous turbulence (e. g. behind a grid) — i. e. in the absence of turbulent production — and shear layers in local equilibrium (e. g. behind a backward facing step) — i. e. where production and dissipation balance – yield

$$c_\mu = 0.09, \quad c_1 = 0.126, \quad c_2 = 1.92.$$

For more details we refer the interested reader to [MP94].

Finally c_ϵ is chosen for the model to reproduce the logarithmic behaviour of the velocity near solid walls (c. f. 1.4.2) with $E = \frac{1}{2} \left| \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right|^2 = \frac{u_\tau^2}{\kappa^2 y^2}$ and

$$\epsilon = \frac{u_\tau^3}{\kappa y}, \qquad k = \frac{u_\tau^2}{\sqrt{c_\mu}}, \tag{1.20}$$

which is experimentally well established. y is the normal distance to the wall, the parameters are explained in 1.4.2. For flow parallel to the wall (say in x-direction) and k independent of x the k-ϵ equations reduce to

$$\begin{aligned} -\frac{\partial}{\partial y} \left(c_\mu \frac{k^2}{\epsilon} \frac{\partial k}{\partial y} \right) - c_\mu \frac{k^2}{\epsilon} E + \epsilon &= 0, \\ -\frac{\partial}{\partial y} \left(c_\epsilon \frac{k^2}{\epsilon} \frac{\partial \epsilon}{\partial y} \right) - c_1 k E + c_2 \frac{\epsilon^2}{k} &= 0. \end{aligned}$$

[1] In fluids with large Reynolds numbers, turbulence has a *locally isotropic structure* (except near boundaries). This means, that the turbulent oscillations ($\mathbf{u}'$) in small regions have no preferred direction.

The first equation is fulfilled identically for solutions of the form (1.20) for all values of c_ϵ, the second equation if and only if ($\kappa = 0.41$)

$$c_\epsilon = \frac{\sqrt{c_\mu}}{\kappa^2}(c_2 c_\mu - c_1) \approx 0.084.$$

Today most often — and also here — $c_\epsilon = 0.07$ is used.

1.3.4 Summary and discussion

Using physical reasoning we have derived the system of equations

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = -\nabla \bar{p} + \nabla \cdot \left((\nu + \nu_T) \left(\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right) \right) - \frac{2}{3} \nabla k, \tag{1.21}$$

$$\nabla \cdot \bar{\mathbf{u}} = 0, \tag{1.22}$$

where $\nu > 0$,

$$\nu_T = c_\mu \frac{k^2}{\epsilon} \tag{1.23}$$

with $c_\mu = 0.09$. k, ϵ satisfy

$$\partial_t k + \bar{\mathbf{u}} \cdot \nabla k = \frac{c_\mu}{2} \frac{k^2}{\epsilon} \left| \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right|^2 + \nabla \cdot \left(c_\mu \frac{k^2}{\epsilon} \nabla k \right) - \epsilon, \tag{1.24}$$

$$\partial_t \epsilon + \bar{\mathbf{u}} \cdot \nabla \epsilon = \frac{c_1}{2} k \left| \nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T \right|^2 + \nabla \cdot \left(c_\epsilon \frac{k^2}{\epsilon} \nabla \epsilon \right) - c_2 \frac{\epsilon^2}{k} \tag{1.25}$$

with $c_1 = 0.126$, $c_2 = 1.92$ and $c_\epsilon = 0.07$.

Justification

We have used the following hypotheses.

1. Reynolds hypothesis for $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ (function of $\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T$, k, ϵ only).
2. Convection by random fields produces diffusion for the mean.
3. Ergodicity to replace means by spatial averages when necessary.
4. Isotropy of $\mathbf{u}'$ to neglect all odd boundary integrals after integration by parts.
5. Reynolds hypothesis for $\langle \omega' \otimes \omega' \rangle$ (function of $\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T$, k, ϵ only).
6. Quasi Gaussian turbulence so as to neglect $\langle (\omega' \otimes \omega') : \nabla \mathbf{u}' \rangle$.
7. A closure hypothesis to model $\nu^2 \langle |\nabla \omega'|^2 \rangle$ by $c_2 \epsilon^2 / k$.
8. The "coefficient of proportionality" between $\langle \mathbf{u}' \otimes \mathbf{u}' \rangle$ and $\nabla \bar{\mathbf{u}} + \nabla \bar{\mathbf{u}}^T$ is $\nu_T = c_\mu k^2 / \epsilon$.

1.4 Boundary conditions — solid boundary

It is easily observed that viscous fluids have velocity **0** at solid walls. The problem is that some assumptions used to derive the k-ϵ-equations are not fulfilled in these near wall regions and indeed the model fails. Now obviously we can either modify the equations appropriately in these parts of the domain to be able to use the same boundary conditions at the same boundary anyway, or we introduce a new boundary inside the original area, where the model is assumed valid and derive new boundary conditions there. Both approaches are found in the literature.

Some authors use so called *Low-Reynolds-Number-Models*. They replace the constants c_μ, c_1, c_2 and c_ϵ with functions f_μ, f_1, f_2 and f_ϵ. These functions should be chosen in a way to reflect the right behaviour of the flow near the wall. A comparison of different approaches can be found in [FLB93]. The problems with these models consist in the fact that there are strong velocity gradients in near-wall regions, which requires adaptive refinement, but then again it might happen that all the refinement is done is this region alone. The result would be a dissatisfying level of accuracy in the interesting inner region.

Therefore we take the second way. We will develop an algebraic relation between the velocity and the normal distance to the wall, a so called *wall function* or *wall law*. This will be used to establish a nonlinear boundary condition of the third kind at the interface between the inner domain and the boundary layer.

Our considerations will be based on a special ansatz described in the following section.

1.4.1 Prandtl's mixing length ansatz

Prandtl's mixing length ansatz is a simple model for the Reynolds stresses in the turbulent boundary layer. We explain the case of flow parallel to the x-axes. To model $\tau_t := -\rho\overline{u_1' u_2'}$

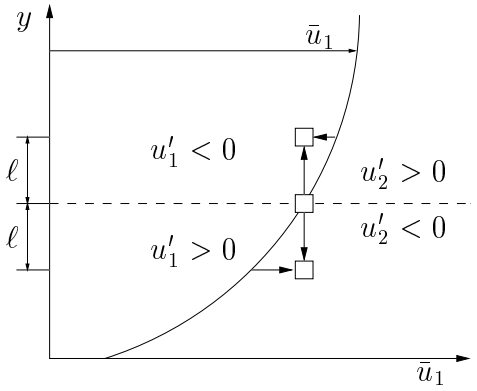

Figure 1.1: Fluctuating fluid-element □ and profile of velocity.

we look at a fluid-element □ in Figure 1.1, which is currently at position y and moves with mean velocity $\bar{u}_1(y)$. (It should be noticed, that because of the structure of turbulent flow such an element should not be viewed as a particle, but more as a 'lump'.) Because of the

fluctuation u_2' the element covers a distance of $\pm\ell$ in y direction until it is mixed with the layers at $y \pm \ell$.[2]

If we assume that it keeps its impulse, we get a measure for the fluctuation velocity in x-direction as

$$\Delta u_1 = \bar{u}_1(y + \ell) - \bar{u}_1(y).$$

With Taylor-expansion and truncation after the linear term we get

$$\Delta u_1 \approx \ell \frac{\partial \bar{u}_1}{\partial y}.$$

We now look at two different fluid elements, which fluctuate from the positions $y + \ell$ resp. $y - \ell$ to position y. If the faster element lies behind the slower, they collide and drift to the sides. In the other case the elements immedeatly move away from each other. In both cases fluid mass has to fill the arising gap, and because of continuity we can assume that u_1' and u_2' have the same order of magnitude and get $|\overline{u_1' u_2'}| = (\Delta u_1)^2$.

Thus we set

$$\tau_t = \rho \ell^2 \left| \frac{\partial \bar{u}_1}{\partial y} \right| \frac{\partial \bar{u}_1}{\partial y}.$$

Note that the sign has been chosen in a way that the Reynolds stress points in the direction of the positive velocity gradient.

ℓ has to be modelled depending on the specific situation, e. g. $\ell = \kappa y$ near solid walls, where y is the normal distance to the wall and $\kappa = 0.41$ the *von Karman constant.*

More details about this ansatz can be found in [Tru80].

1.4.2 Derivation of a wall law

We start with Reynolds' equations

$$\partial_t \bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + \nabla \bar{p} - \nu \Delta \bar{\mathbf{u}} + \nabla \cdot \langle \mathbf{u}' \otimes \mathbf{u}' \rangle = \mathbf{0}, \tag{1.26}$$

which, as has to be noticed, do not contain any simplifications yet. If we look at the first coordinate we get (omitting bars for the means)

$$\begin{aligned}(\partial_t u_1) + (u_1 \partial_1 u_1 + u_2 \partial_2 u_1 + u_3 \partial_3 u_1) + (\partial_1 \bar{p}) \\ - \nu(\partial_{11} u_1 + \partial_{22} u_1 + \partial_{33} u_1) + (\partial_1 \overline{u_1'^2} + \partial_2 \overline{u_1' u_2'} + \partial_3 \overline{u_1' u_3'}) = 0\end{aligned}$$

We now consider stationary flow in the positive x-direction parallel to the wall according to Figure 1.2. Thus the time derivatives vanish and so do the derivatives in x and z direction and the velocities in y and z direction. What's left is

$$-\nu \partial_{22} u_1 + \partial_2 \overline{u_1' u_2'} = 0,$$

[2] ℓ corresponds to the mean free path in kinetic gas-theory.

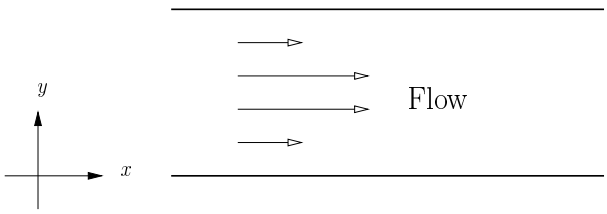

Figure 1.2: Parallel stationary flow

which expresses the balance of viscous and turbulent stresses. Integration and multiplication with ρ leads to

$$\underbrace{\mu\partial_2 u_1}_{=:\bar{\tau}_v} \underbrace{-\rho\overline{u_1' u_2'}}_{\tau_t} =: \bar{\tau}_w = \text{const.}$$

with viscous stress $\bar{\tau}_v$, turbulent stress τ_t and wall shear stress $\bar{\tau}_w$. We now look at three regions.

1. *Viscous sublayer* ($0 \le y \le \delta_0$): Here the first term dominates and we have

$$\mu\partial_2 u_1 = \rho u_\tau^2 \quad \Rightarrow \quad \underbrace{\frac{u_1}{u_\tau}}_{u^+} = \underbrace{\frac{\rho u_\tau y}{\mu}}_{y^+}.$$

2. *Transition zone* ($\delta_0 \le y \le \delta_t$): Complicated models are required to give accurate approximations. For small/large distances these expressions asymptotically turn into the two other cases (Figure 1.3). We refer the reader to [SG97] and do not go into details here (see the remark below).

3. *Turbulent wall layer* ($\delta_t \le y \le \delta$): Here the second term dominates. Prandtl's mixing length ansatz (see section 1.4.1) yields

$$\tau_t = \rho\ell^2 |\partial_2 u_1| \partial_2 u_1,$$

where $\ell = \kappa y$. $\kappa = 0.41$ is the *von Karman* constant. Using this we get

$$(\kappa y)^2 (\partial_2 u_1)^2 = u_\tau^2 \tag{1.27}$$

$$\Longrightarrow \quad u_1 = \frac{u_\tau}{\kappa} \ln y + \tilde{\beta}$$

$$\Longrightarrow \quad \underbrace{\frac{u_1}{u_\tau}}_{u^+} = \frac{1}{\kappa} \ln \underbrace{\frac{\rho u_\tau y}{\mu}}_{y^+} + \beta. \tag{1.28}$$

The integration constant β depends on the roughness of the wall. Typical values are in between 5 for smooth and about 7 for rough walls.

Combining these results we get u^+ as shown in Figure 1.3.

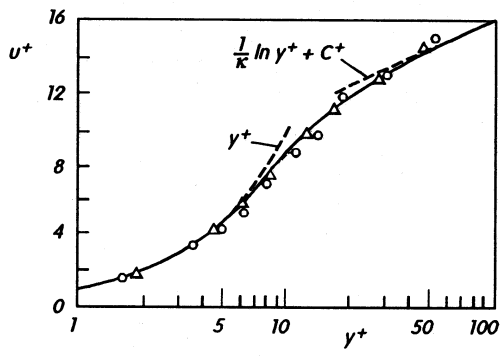

Figure 1.3: u^+ plotted against y^+. The triangles and circles mark results of experiments. (Courtesy of [SG97])

Remark. We assume a two-layer structure of the boundary layer and switch from the linear law to the logarithmic wall-law at some value y_0^+. For continuity of the velocity we have to require

$$y_0^+ = \frac{1}{\kappa} \ln y_0^+ + \beta \quad \Rightarrow \quad y_0^+ = 11.45$$

for $\beta = 5.5$. In fact the linear law is a good approximation only for $y^+ \leq 5$, the logarithmic law only for $20 \leq y^+ \leq 100$.

Some authors work with more complicated wall-laws, which are obtained from more sophisticated models for τ_t (see for example [SG97]). However it can not be expected that much improvement (in accuracy) is achieved as a boundary condition for the k-ϵ model, because the model is not suitable to simulate flows in regions closer to the wall than the logarithmic layer — indeed it has been constructed to turn into the logarithmic wall law in this limit.

1.4.3 2D boundary condition

We will now use the results from the previous section to set up boundary conditions for $\bar{\mathbf{u}}$, k and ϵ at the interface.

Boundary condition for $\bar{\mathbf{u}}$

The wall-function

$$u^+(y^+) = \begin{cases} y^+ & 0 \leq y^+ \leq y_0^+ \\ \frac{1}{\kappa} \ln y^+ + \beta & y_0^+ \leq y^+ \leq 100 \end{cases} \tag{1.29}$$

with $u^+ = u_1/u_\tau$, $y^+ = y u_\tau/\nu$ and $y_0^+ = 11.45$ contains the unknown parameter u_τ, which will be different at every point of the boundary. Thus we use (1.27) to express u_τ in terms

of $\frac{\partial u_1}{\partial y}$ and insert this in (1.28) for the log-layer, which gives

$$u_1 = \kappa y \frac{\partial u_1}{\partial y} \left(\frac{1}{\kappa} \ln \left(\frac{\kappa y^2}{\nu} \left| \frac{\partial u_1}{\partial y} \right| \right) + \beta \right). \tag{1.30}$$

Expressing the interval of validity of (1.30) in $\frac{\partial u_1}{\partial y}$ gives

$$y_0^+ \leq \frac{\kappa y^2 \frac{\partial u_1}{\partial y}}{\nu} \leq 100.$$

In the sub-layer obviously

$$u_1 = y \frac{\partial u_1}{\partial y}, \tag{1.31}$$

which is valid for

$$0 \leq \frac{\kappa y^2 \frac{\partial u_1}{\partial y}}{\nu} \leq \kappa {y_0^+}^2.$$

Note that these intervals have non-empty intersection. Introducing the abbreviations $x := \frac{\kappa y e^{\kappa\beta}}{\nu} u_1$, $z := \frac{\kappa y^2 e^{\kappa\beta}}{\nu} \frac{\partial u_1}{\partial y}$ gives the relations

$$x = \begin{cases} z & 0 \leq z \leq \kappa {y_0^+}^2 e^{\kappa\beta} \\ z \ln z & y_0^+ e^{\kappa\beta} \leq z \leq 100 e^{\kappa\beta} \end{cases},$$

which are shown in Figure 1.4 (thick lines). In order to set up appropriate boundary conditions we need to express z as a function of x. Considering Figure 1.4 a natural choice would be to take the linear law for $x \leq x_2 = \kappa {y_0^+}^2 e^{\kappa\beta}$, otherwise the logarithmic law. However this function is not continuous, which is physically not sensible and gives mathematical difficulties. Sophisticated boundary layer models interpolate smoothly between the viscous sub-layer and the log-layer, but are more complicated to deal with (practically). Therefore we extend the logarithmic wall function also to smaller values of x and switch to the linear law at the point of intersection (z_1, x_1) with $x_1 = z_1 = e \approx 2.72$. For large x it is impossible to find a general expression for z, as there cannot be an universal law at large distances from the wall, but still we have to prescribe values in order to pose boundary conditions. For analytical convenience we linearise the wall-law at $z_4 = 100 e^{\kappa\beta}$. This seems justified, if we check *a posteriori* that the artificial boundary has been chosen in the logarithmic region.

So far we have only considered flow in the positive x-direction, but for the general case we just have to extend z to an odd function on $\mathbb{R}$. Hence we define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(z) := \begin{cases} z & |z| \leq z_1 \\ z \ln |z| & z_1 \leq |z| \leq z_4 \\ z_4 \ln z_4 + (\ln z_4 + 1)(z - z_4) & z \geq z_4 \\ -z_4 \ln z_4 + (\ln z_4 + 1)(z + z_4) & z \leq -z_4 \end{cases}, \tag{1.32}$$

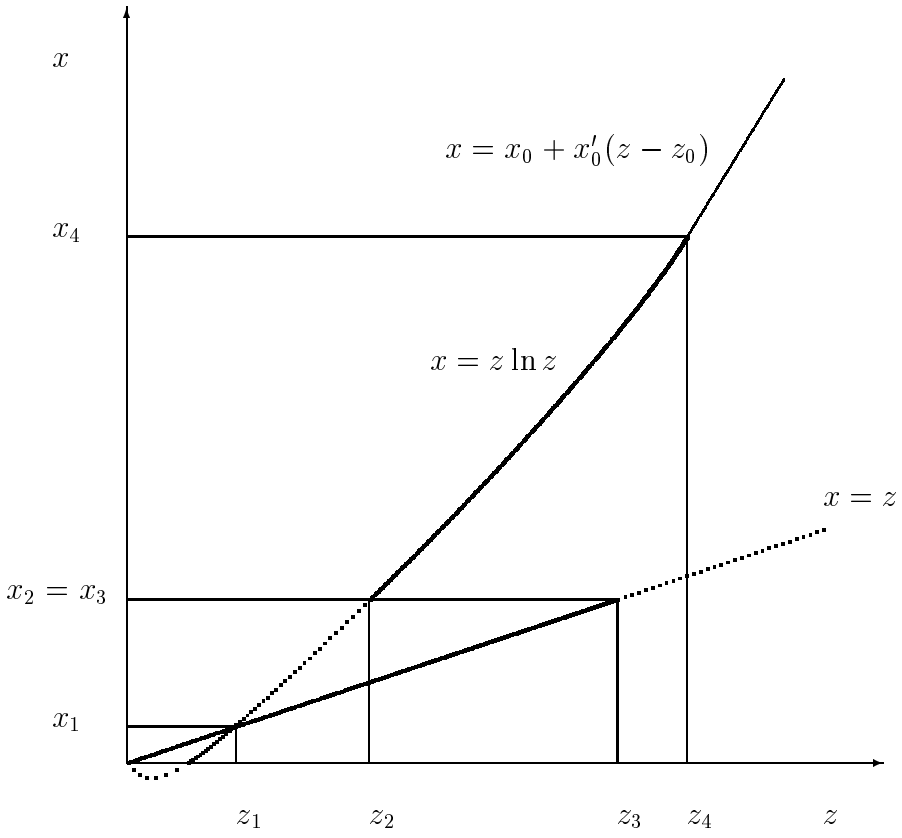

Figure 1.4: Relation between z and x.

which is also seen in Figure 1.4.

Since $\psi : \mathbb{R} \to \mathbb{R}$ is one-to-one, we can define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(z) := \frac{\nu}{\kappa\delta^2} e^{-\kappa\beta} \psi^{-1}(\frac{\kappa\delta}{\nu} e^{\kappa\beta} z).$$

For a more general (curved) boundary we have to use local orthogonal coordinates. Due to the smoothness of the boundary Γ there exist $\mathbf{n}$, $\mathbf{s}$ orthogonal respectively parallel to the boundary. We assume that if we replace $u_1 \to \mathbf{u} \cdot \mathbf{s}$, $\frac{\partial u_1}{\partial y} \to -\frac{\partial \mathbf{u} \cdot \mathbf{s}}{\partial \mathbf{n}}$ the above considerations are still valid. Particularly we assume

$$\bar{\mathbf{u}} \cdot \mathbf{n} = 0 \tag{1.33}$$

in the entire boundary layer, for the viscous sub-layer

$$\bar{\mathbf{u}} \cdot \mathbf{s} = -\delta \frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}}$$

and for the log-layer

$$\bar{\mathbf{u}} \cdot \mathbf{s} = -\kappa\delta \frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}} \left(\frac{1}{\kappa} \ln \left(\frac{\kappa\delta^2}{\nu} \left| \frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}} \right| \right) + \beta \right). \tag{1.34}$$

δ is the normal distance to the wall.

The complete boundary condition takes the form

$$\frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}} + \phi(\bar{\mathbf{u}} \cdot \mathbf{s}) = 0,$$

which is a non-linear boundary condition of the third kind.

Remark. In [MP94] we find a different approach, namely (1.34) is replaced by

$$\frac{\bar{\mathbf{u}} \cdot \mathbf{s}}{\sqrt{\nu \left| \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{n}} \right|}} - \frac{1}{\kappa} \ln \left(\delta \sqrt{\frac{1}{\nu} \left| \frac{\partial \bar{\mathbf{u}}}{\partial \mathbf{n}} \right|} \right) + \beta = 0. \tag{1.35}$$

We believe that this boundary condition is not consistent with the model in the way that (if viewed as an ODE for $\bar{\mathbf{u}} \cdot \mathbf{s}$) it does not lead to a logarithmic law.

Boundary conditions for k and ϵ

For k and ϵ we have seen in 1.3.3 that

$$\epsilon = \frac{u_\tau^3}{\kappa\delta}, \qquad k = \frac{u_\tau^2}{\sqrt{c_\mu}}$$

at the interface. Thus the according 2D boundary conditions are

$$\begin{aligned} k &= c_\mu^{-\frac{1}{2}} \kappa^2 \delta^2 \left| \frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}} \right|^2 = c_\mu^{-\frac{1}{2}} \kappa^2 \delta^2 \phi(|\bar{\mathbf{u}} \cdot \mathbf{s}|)^2, \\ \epsilon &= \kappa^2 \delta^2 \left| \frac{\partial \bar{\mathbf{u}} \cdot \mathbf{s}}{\partial \mathbf{n}} \right|^3 = \kappa^2 \delta^2 \phi(|\bar{\mathbf{u}} \cdot \mathbf{s}|)^3. \end{aligned}$$

1.4.4 General formulation of boundary condition

Boundary condition for $\bar{\mathbf{u}}$

For high Reynolds numbers experiments of flow over a flat plate show that the streamlines above a specific point are nearly parallel in near wall regions and therefore it is possible to extend the above considerations to the three dimensional case also. In analogy to (1.32) we define $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\Psi(\mathbf{z}) = \psi(|\mathbf{z}|)\frac{\mathbf{z}}{|\mathbf{z}|}.$$

Lemma 1.4. *Ψ is continuous and one-to-one in $\mathbb{R}^N$.*

Proof. Continuity is obvious. $\tilde{\psi} : |\mathbf{z}| \to |\Psi(\mathbf{z})|$ is well defined and strictly increasing on $[0, \infty[$, $\lim_{|\mathbf{z}|\to\infty} |\Psi(\mathbf{z})| = \infty$. Since $\Psi(\mathbf{z})$ has the same direction as $\mathbf{z}$, the result follows. □

Hence we can define $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\Phi(\mathbf{z}) := \frac{\nu}{\kappa\delta^2} e^{-\kappa\beta} \Psi^{-1}\left(\frac{\kappa\delta}{\nu} e^{\kappa\beta}\mathbf{z}\right)$$

and demand

$$\mathbf{n} \cdot \mathbf{T}(\bar{\mathbf{u}}, p)\mathbf{s}_i + (\nu + \nu_T)\Phi(\bar{\mathbf{u}}) \cdot \mathbf{s}_i = 0, \tag{1.36}$$

where $\{\mathbf{s}_i : i = 1, \ldots, N-1\}$ is a set of orthonormal tangential vectors of Γ and

$$\mathbf{T}(\mathbf{u}, p) = -p\mathbf{I} + (\nu + \nu_T)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$$

the stress tensor. It is easily seen that (1.36) is equivalent to the wall law, if $\mathbf{T}$ is represented in local coordinates $\mathbf{n}$, $\mathbf{s}_i$ and $\mathbf{u}$ is split up via $\mathbf{u} = \mathbf{u_n} + \mathbf{u_s}$, where $\mathbf{u_s} \cdot \mathbf{n} = 0$. Note that $\frac{\partial \mathbf{u_n}}{\partial \mathbf{s}_i} = 0$.

Boundary conditions for k and ϵ

Again we express the boundary conditions for k and ϵ as

$$k = c_\mu^{-\frac{1}{2}} \kappa^2 \delta^2 \phi(|\bar{\mathbf{u}}|)^2, \tag{1.37}$$

$$\epsilon = \kappa^2 \delta^2 \phi(|\bar{\mathbf{u}}|)^3. \tag{1.38}$$

1.5 Boundary conditions — artificial boundaries

Since the fluid enters and leaves the domain through some openings in the wall, we have to cut off the outer regions by artificial boundaries in order to concentrate on the interesting part. Hence we have to supply information about the behaviour at these sections. We will discuss three major types of boundary conditions, but all sorts of combinations are possible. To this end we divide the boundary $\partial\Omega$ into distinct sections Γ, which denotes the solid boundary, and S_1 to S_3 for Dirichlet, pressure and natural boundary conditions, respectively.

From now on we will omit bars, since we are only dealing with averaged quantities.

1.5.1 Dirichlet conditions

At an inflow boundary, here S_1, a natural choice would be to specify the velocity of the incoming fluid,

$$\mathbf{u}|_{S_1} = \mathbf{u}_0.$$

The same seems sensible for k and ϵ,

$$\begin{aligned} k|_{S_1} &= k_0, \\ \epsilon|_{S_1} &= \epsilon_0. \end{aligned}$$

1.5.2 Natural boundary conditions

Following [HRT94] one might consider natural boundary conditions (also referred to as *'do nothing'* boundary conditions) for the velocity of the form

$$\mathbf{n} \cdot \mathbf{T}(\mathbf{u}, p) = \mathbf{0}$$

in the static pressure formulation or similarly in the total pressure formulation, in components

1.
$$\begin{aligned} \left[p - 2(\nu + \nu_T) \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{n}} \right]_{S_3} &= 0, \\ \left[\frac{\partial \mathbf{u} \cdot \mathbf{s}_i}{\partial \mathbf{n}} + \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{s}_i} \right]_{S_3} &= 0 \end{aligned}$$

   or (with total pressure)

2.
$$\begin{aligned} \left[p + \frac{1}{2}|\mathbf{u}|^2 + \frac{2}{3}k - 2(\nu + \nu_T) \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{n}} \right]_{S_3} &= 0, \\ \left[\frac{\partial \mathbf{u} \cdot \mathbf{s}_i}{\partial \mathbf{n}} + \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{s}_i} \right]_{S_3} &= 0. \end{aligned}$$

This has the physical interpretation that the normal stresses vanish at the (e. g. outflow) boundary.

Natural boundary conditions for k and ϵ (i. e. for the Laplace operator) are homogeneous Neumann conditions

$$\begin{aligned} \left. \frac{\partial k}{\partial \mathbf{n}} \right|_{S_3} &= 0, \\ \left. \frac{\partial \epsilon}{\partial \mathbf{n}} \right|_{S_3} &= 0. \end{aligned}$$

1.5.3 Pressure drop problems

In some situations it might also be interesting to specify a certain pressure drop between several parts of the boundary, say $S_2 = \bigcup_j S_{2j}, j = 0, \ldots, r$.

1. Static pressure drop: We require

$$\begin{aligned} \left[p - 2(\nu + \nu_T) \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{n}} \right]_{S_{2j}} &= P_j, \\ \mathbf{u} \cdot \mathbf{s}_i|_{S_2} &= \mathbf{u}_i \cdot \mathbf{s}_i, \quad i = 1, \ldots, N-1 \end{aligned}$$

   with given pressures $P_j, j = 0, \ldots, r$, given tangential velocities $\mathbf{u}_i$ and an orthonormal set of tangential vectors $\mathbf{s}_i$.

2. Total pressure drop: The same with p replaced by $p + \frac{1}{2}|\mathbf{u}|^2 + \frac{2}{3}k$.

We note that this is a combination of Dirichlet conditions for the tangential component and natural conditions for the normal component. For orthogonal intersections of S_{2j} with the solid boundary and for $\nu_T = 0$ P_j is the average pressure over S_{2j}. This is seen from

$$\int_{S_{2j}} \frac{\partial \mathbf{u} \cdot \mathbf{n}}{\partial \mathbf{n}} = - \int_{S_{2j}} \nabla_{\mathbf{s}} \cdot \mathbf{u}_{\mathbf{s}} = - \int_{\partial S_{2j}} \mathbf{u}_{\mathbf{s}} \cdot \mu = 0$$

with the normal vector μ on ∂S_{2j}.

Chapter 2

Analysis of the model

In this chapter we present a partial analysis of the model. More precisely, we will analyse stationary Navier-Stokes equations with variable viscosity and boundary conditions as discussed in the previous sections.

In a first step we look at the associated Stokes problem, because the properties of the Stokes operator derived there can be used later, but also stronger results can be shown here — we will be able to prove existence and uniqueness.

For Navier-Stokes ellipticity of the problem can only be shown (in the general case) for sufficiently small data, which results in restrictions on the boundary conditions and the right hand side.

2.1 Definition of the steady state problem

We assume that Ω is an open, connected subset of $\mathbb{R}^N$, $N = 2, 3$ with its boundary $\partial\Omega \in C^{0,1}$ made of four smooth (e. g. C^1) subsets Γ, S_1, S_2, S_3, such that

$$\begin{aligned} \partial\Omega &= \bar{\Gamma} \cup \bar{S}_1 \cup \bar{S}_2 \cup \bar{S}_3, \\ \operatorname{meas}_{\mathbb{R}^{N-1}}(\Gamma \cup S_1) &> 0, \\ \Gamma \cap S_i &= \emptyset \quad \forall i = 1, 2, 3, \\ S_i \cap S_j &= \emptyset \quad \forall i, j = 1, 2, 3,\ i \neq j. \end{aligned}$$

The connected components of S_2 are denoted by S_{2j} for $j = 0, \ldots, r$.

2.1.1 Classical formulation

We will be concerned with the following problems: For given $k \in C^1(\Omega) \cap C^0(\bar{\Omega})$, $\nu_T \in C^1(\Omega)$, find $\mathbf{u} \in C^2(\Omega)^N \cap C^1(\bar{\Omega})^N$, $p \in C^1(\Omega) \cap C^0(\bar{\Omega})$, such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot \left((\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \right) - \nabla \frac{2}{3} k \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

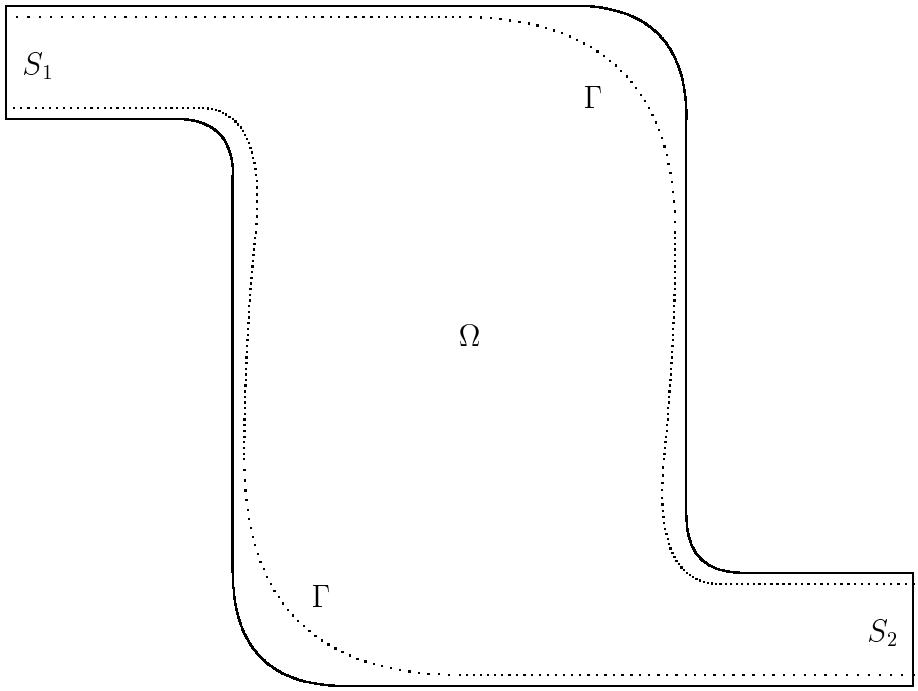

Figure 2.1: Flow region and computational domain.

with the wall-law

$$\left.\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= 0 \\ \mathbf{n}\cdot\mathbf{T}(p,\mathbf{u})\cdot\mathbf{s}_i + (\nu+\nu_T)\Phi(\mathbf{u})\cdot\mathbf{s}_i &= 0 \end{aligned}\right\} \quad \text{on } \Gamma \tag{2.3}$$

($\mathbf{n}$ is the outward unit normal to the boundary and $\{\mathbf{s}_i : i = 1,\ldots,N-1\}$ an orthonormal set of tangential vectors) and

$$\mathbf{u} = \mathbf{u}_0 \qquad \text{on } S_1, \tag{2.4}$$

$$\left.\begin{aligned} p - 2(\nu+\nu_T)\tfrac{\partial \mathbf{u}\cdot\mathbf{n}}{\partial \mathbf{n}} &= P_j \\ \mathbf{u}\cdot\mathbf{s}_i &= \mathbf{u}_i\cdot\mathbf{s}_i \end{aligned}\right\} \qquad \text{on } S_{2j},\ i = 1,2,\ 0 \le j \le r, \tag{2.5}$$

$$\mathbf{T}(p,\mathbf{u})\cdot\mathbf{n} = 0 \qquad \text{on } S_3. \tag{2.6}$$

An equivalent formulation of (2.1) is

$$\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot(\nabla\mathbf{u})^T = -\nabla\underbrace{\left(p + \frac{1}{2}|\mathbf{u}|^2 + \frac{2}{3}k\right)}_{=:\bar{p}} + \nabla\cdot\left((\nu+\nu_T)\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)\right) \tag{2.7}$$

with the total pressure $\bar{p}$. The respective boundary conditions are obtained by replacing p by $\bar{p}$ in the above formulae.

2.1.2 Weak formulation

Assumption: There exists $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^N$, such that

$$\begin{aligned} \mathbf{g} &= \mathbf{u}_0 && \text{on } S_1, && (2.8)\\ \mathbf{g}\cdot\mathbf{s}_i &= \mathbf{u}_i\cdot\mathbf{s}_i && \text{on } S_{2j},\ i=1,2,\ 0\le j\le r, && (2.9)\\ \mathbf{g}\cdot\mathbf{n} &= 0 && \text{on } \Gamma && (2.10)\end{aligned}$$

and the compatibility condition

$$\int_{\partial\Omega} \mathbf{g}\cdot\mathbf{n} = 0 \tag{2.11}$$

holds. This implies (cf. 2.2.2) the existence of $\mathbf{U}_0 \in H^1(\Omega)^N$ with

$$\begin{aligned} \nabla\cdot\mathbf{U}_0 &= 0 && \text{in } \Omega, && (2.12)\\ \mathbf{U}_0 &= \mathbf{u}_0 && \text{on } S_1, && (2.13)\\ \mathbf{U}_0\cdot\mathbf{s}_i &= \mathbf{u}_i\cdot\mathbf{s}_i && \text{on } S_{2j},\ i=1,2,\ 0\le j\le r, && (2.14)\\ \mathbf{U}_0\cdot\mathbf{n} &= 0 && \text{on } \Gamma. && (2.15)\end{aligned}$$

Conversely the trace operator maps $H^1(\Omega)^N$ into $H^{\frac{1}{2}}(\partial\Omega)^N$ and $\int_{\partial\Omega}\mathbf{u}\cdot\mathbf{n}=0$, if $\nabla\cdot\mathbf{v}=0$, so it is natural to assume that the boundary conditions can be fulfilled. Now let

$$\begin{aligned} X &:= \{\mathbf{v}\in H^1(\Omega)^N : \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma, \mathbf{v}=0 \text{ on } S_1, \mathbf{v}\cdot\mathbf{s}_i=0 \text{ on } S_2\} && (2.16)\\ V &:= \{\mathbf{v}\in X : \nabla\cdot\mathbf{v}=0 \text{ in } \Omega\} && (2.17)\end{aligned}$$

and

$$\begin{aligned} X(\mathbf{U}_0) &:= \{\mathbf{v}\in H^1(\Omega)^N : \mathbf{v}-\mathbf{U}_0\in X\} && (2.18)\\ V(\mathbf{U}_0) &:= \{\mathbf{v}\in X(\mathbf{U}_0) : \nabla\cdot\mathbf{v}=0\}. && (2.19)\end{aligned}$$

Furthermore we introduce

$$W := \{\mathbf{g}\in H^{\frac{1}{2}}(\partial\Omega)^N : \mathbf{g}\cdot\mathbf{n}=0 \text{ on } \Gamma, \mathbf{g}=0 \text{ on } S_1, \mathbf{g}\cdot\mathbf{s}_i=0 \text{ on } S_2, \int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}=0\},$$

the range of the trace operator applied on V.

1. Weak formulation with distributions

Find

$$\begin{aligned} \mathbf{u} &\in X(\mathbf{U}_0),\\ p &\in M,\end{aligned}$$

(where $\mathbf{U}_0$ is constructed as above,) such that

$$\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\mathbf{T}(\mathbf{u},p) = -\nabla\frac{2}{3}k \qquad \text{in } H^{-1}(\Omega)^N, \tag{2.20}$$

$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } L^2(\Omega)^N \tag{2.21}$$

with

$$\langle \mathbf{n}\cdot\mathbf{T}(\mathbf{u},p),\mathbf{v}\rangle = -\int_\Gamma(\nu+\nu_T)\Phi(\mathbf{u})\cdot\mathbf{v} - \sum_{j=0}^{r} P_j\int_{S_{2j}}\mathbf{v}\cdot\mathbf{n} \quad \forall\mathbf{v}\in X. \tag{2.22}$$

$\mathbf{n}\cdot\mathbf{T}\in W^*$ is defined by

$$\langle \mathbf{n}\cdot\mathbf{T}(\mathbf{u},p),\mu\rangle \;=\; \int_\Omega(\nabla\cdot\mathbf{T}(\mathbf{u},p))\mathbf{v} + \int_\Omega \mathbf{T}(\mathbf{u},p):\nabla\mathbf{v} \tag{2.23}$$

for $\mu\in W$ and $\mathbf{v}\in X(\mathbf{U}_0)$, such that $\mathbf{v}|_{\partial\Omega}=\mu$.

Similarly we look for $\bar{p}\in M$, such that

$$\begin{array}{rcll} \mathbf{u}\cdot\nabla\mathbf{u} - \nabla\cdot\mathbf{T}(\mathbf{u},\bar{p}) &=& 0 & \text{in } H^{-1}(\Omega)^N \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } L^2(\Omega)^N \end{array}$$

with appropriate boundary conditions.

Remark. $\mathbf{n}\cdot\mathbf{T}$ is well defined by (2.23), because from (2.20) it follows that $\nabla\cdot\mathbf{T}\in L^{\frac{4}{3}}(\Omega)^N$, since $H^1(\Omega)^N$ is continuously embedded in $L^4(\Omega)^N$ for $N\le 4$, and for the same reason the first integral in (2.23) exists and defines a continuous linear functional. Obviously both factors in the second integral are in $L^2(\Omega)^N$ and continuity follows easily.

For smooth solutions (2.22) is equivalent to the classical boundary conditions (2.3) to (2.6) presented earlier.

2. Weak formulation without solenoidal functions

Find

$$\begin{array}{rcl} \mathbf{u} &\in& X(\mathbf{U}_0), \\ p,\bar{p} &\in& M, \end{array}$$

such that

$$\begin{array}{lcll} a_0(\mathbf{u},\mathbf{v}) + \bar{a}_1(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},\bar{p}) &=& \langle\bar{\mathbf{F}},\mathbf{v}\rangle & \forall\mathbf{v}\in X, \\ b(\mathbf{u},q) &=& 0 & \forall q\in M \end{array} \tag{2.24}$$

for the total pressure formulation and

$$\begin{array}{lcll} a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{u};\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &=& \langle\mathbf{F},\mathbf{v}\rangle & \forall\mathbf{v}\in X, \\ b(\mathbf{u},q) &=& 0 & \forall q\in M \end{array} \tag{2.25}$$

for the static pressure formulation. Here

$$\begin{aligned} a_0(\mathbf{u},\mathbf{v}) &= \frac{1}{2}\int_\Omega (\nu+\nu_T)\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right):\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right) \\ &\quad + \int_\Gamma (\nu+\nu_T)\Phi(\mathbf{u})\cdot\mathbf{v} \end{aligned} \tag{2.26}$$

$$a_1(\mathbf{w};\mathbf{u},\mathbf{v}) = (\mathbf{w}\cdot\nabla\mathbf{u},\mathbf{v}) \tag{2.27}$$

$$\bar{a}_1(\mathbf{w};\mathbf{u},\mathbf{v}) = (\mathbf{w}\cdot\nabla\mathbf{u}-\mathbf{w}\cdot(\nabla\mathbf{u})^T,\mathbf{v}) \tag{2.28}$$

$$b(\mathbf{u},q) = -(\nabla\cdot\mathbf{u},q) \tag{2.29}$$

$$\langle\mathbf{F},\mathbf{v}\rangle = -\sum_{j=0}^{r} P_j\int_{S_{2j}}\mathbf{v}\cdot\mathbf{n}+\frac{2}{3}\int_\Omega \nabla k\cdot\mathbf{v} \tag{2.30}$$

$$\langle\bar{\mathbf{F}},\mathbf{v}\rangle = -\sum_{j=0}^{r} P_j\int_{S_{2j}}\mathbf{v}\cdot\mathbf{n}. \tag{2.31}$$

3. Weak formulation with solenoidal functions

Find

$$\mathbf{u}\in V(\mathbf{U}_0),$$

such that

$$a_0(\mathbf{u},\mathbf{v})+\bar{a}_1(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle\bar{\mathbf{F}},\mathbf{v}\rangle \quad \forall\mathbf{v}\in V \tag{2.32}$$

for the total pressure formulation and

$$a_0(\mathbf{u},\mathbf{v})+a_1(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle\mathbf{F},\mathbf{v}\rangle \quad \forall\mathbf{v}\in V \tag{2.33}$$

for the static pressure formulation (with a_0 etc. as in 2.).

Remark. It is clear that $\mathbf{F},\bar{\mathbf{F}}\in V^*$, because

$$\left|\int_{S_{2j}}\mathbf{v}\cdot\mathbf{n}\right| \le |S_{2j}|\|\mathbf{v}\|_{0,S_{2j}} \le \gamma|S_{2j}|\|\mathbf{v}\|_1 \quad \forall 0\le j\le r,$$

so e. g.

$$\|\mathbf{F}\|_{V^*} \le \frac{2}{3}|k|_1+\gamma|S_2|\max_{j=0,\dots,r}|P_j|.$$

Theorem 2.1. *Formulations 1. to 3. are equivalent.*

Proof. We take $\mathbf{f}:=-\frac{2}{3}\nabla k$ resp. $\mathbf{f}:=\mathbf{0}$ and consider the convective term in the static pressure formulation. The other case follows analogously. We first show that every solution of 1. is a solution of 2. Applying (2.20) to a test function $\mathbf{v}\in X$ gives

$$\langle\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}\rangle-\langle\nabla\cdot\mathbf{T}(\mathbf{u},p),\mathbf{v}\rangle = \langle\mathbf{f},\mathbf{v}\rangle.$$

Inserting $\nabla \cdot \mathbf{T}$ from (2.23), using the definition of $\mathbf{T}$ and boundary condition (2.22) gives

$$(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}) + \frac{1}{2} \int_\Omega (\nu + \nu_T) \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \int_\Gamma (\nu + \nu_T) \Phi(\mathbf{u}) \cdot \mathbf{v} - \int_\Omega p \nabla \cdot \mathbf{v} = -\sum_{j=0}^{r} P_j \int_{S_{2j}} \mathbf{v} \cdot \mathbf{n} + (\mathbf{f}, \mathbf{v}),$$

that is $\mathbf{v}$ solves 2.

Obviously every solution of 2. solves 3. ($\mathbf{u} \in X(\mathbf{U}_0)$ and $b(\mathbf{u}, q) = 0$ for all q implies that $\mathbf{u} \in V(\mathbf{U}_0)$ and for $\mathbf{v} \in V$ equation (2.25) reduces to (2.33)).

Finally let $\mathbf{u}$ be a solution of 3. Then for all $\mathbf{v} \in \mathcal{V}$,

$$\mathcal{V} := \{\mathbf{v} \in H_0^1(\Omega)^N : \nabla \cdot \mathbf{v} = 0\},$$

the definition of the distributional divergence yields

$$\langle \mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \left((\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \right) - \mathbf{f}, \mathbf{v} \rangle = 0,$$

where $\langle . \rangle$ is the duality pairing between H_0^1 and H^{-1}. Therefore

$$\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \left((\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \right) - \mathbf{f} \in \mathcal{V}^0$$

with

$$\mathcal{V}^0 = \{\mathbf{f} \in H^{-1}(\Omega)^N : \langle \mathbf{f}, \mathbf{v} \rangle = 0 \ \forall \mathbf{v} \in \mathcal{V}\}.$$

Consequently there exists $p \in M$, such that

$$\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \cdot \left((\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \right) - \mathbf{f} = -\nabla p$$

in $H^{-1}(\Omega)^N$, which is equivalent to (2.20). (2.21) and $\mathbf{u} \in X(\mathbf{U}_0)$ are clear, because $\mathbf{u} \in V(\mathbf{U}_0)$.

Now

$$\begin{aligned} \langle \mathbf{n} \cdot \mathbf{T}(\mathbf{u}, p), \mu \rangle &= \int_\Omega (\nabla \cdot \mathbf{T}(\mathbf{u}, p)) \mathbf{v} + \int_\Omega \mathbf{T}(\mathbf{u}, p) : \nabla \mathbf{v} \\ &= \int_\Omega (\nabla \cdot \mathbf{T}(\mathbf{u}, p)) \mathbf{v} + \frac{1}{2} \int_\Omega (\nu + \nu_T) \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \end{aligned}$$

for $\mu \in W$ and $\mathbf{v} \in V(\mathbf{U}_0)$, such that $\mathbf{v}|_{\partial\Omega} = \mu$. Inserting (2.20) (which we have shown already) and (2.33) we get

$$\langle \mathbf{n} \cdot \mathbf{T}(\mathbf{u}, p), \mu \rangle = -\int_\Gamma \Phi(\mathbf{u}) \cdot \mu - \sum_j P_j \int_{S_{2j}} \mu \cdot \mathbf{n}$$

for all $\mu \in W$, which concludes the proof.

□

2.2 Analysis of the associated Stokes problem

We will start by discussing the case of homogeneous boundary data and later reduce the general case to this one.

2.2.1 Homogeneous case

Let V as in (2.17), then the homogeneous Stokes problem reads: For given $\mathbf{F} \in V^*$ find $\mathbf{u} \in V$, such that

$$(\mathrm{VP}_0) \qquad \boxed{a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V,}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \int_\Omega (\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) : \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) + \int_\Gamma (\nu + \nu_T) \Phi(\mathbf{u}) \cdot \mathbf{v}.$$

Obviously a is well defined on $V \times V$ (due to the Lipschitz-continuity of Φ), linear and continuous with respect to the second argument. Hence we can define a non-linear operator $A : V \to V^*$ by

$$\langle A(\mathbf{u}), \mathbf{v} \rangle = a(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Then (VP_0) takes the form

$$A(\mathbf{u}) = \mathbf{F}. \tag{2.34}$$

Now the plan is to apply Lax-Milgram theory to (2.34) and to this end we will show that A has the following properties:

Definition 2.2. 1. An operator $A : V \to V^*$ is called **Lipschitz-continuous**, if there exists $\mu_2 > 0$, such that

$$\|A(\mathbf{u}) - A(\mathbf{v})\|_{V^*} \leq \mu_2 \|\mathbf{u} - \mathbf{v}\| \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

2. A is called **strongly monotone**, if there exists $\mu_1 > 0$, such that

$$\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq \mu_1 \|\mathbf{u} - \mathbf{v}\|^2 \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Lemma 2.3. *Φ is Lipschitz-continuous and strongly monotone in $\mathbb{R}^N$.*

Proof. We first note that Ψ is differentiable everywhere except for $\|\mathbf{x}\| = e$ (the intersection of the two layers, which formally should be dealt with separately in the following discussion, but it is easily seen that the same results are still valid). From

$$\frac{\partial}{\partial x_j} \left(\frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|} x_i \right) = \frac{x_j}{\|\mathbf{x}\|} \frac{\psi'(\|\mathbf{x}\|)\|\mathbf{x}\| - \psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|^2} x_i + \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|} \delta_{ij}$$

we see that the Jacobian of Ψ is given by

$$\mathbf{J}_\Psi(\mathbf{x}) = \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\left(\mathbf{I} + \left(\frac{\psi'(\|\mathbf{x}\|)\|\mathbf{x}\|}{\psi(\|\mathbf{x}\|)} - 1\right)\frac{\mathbf{x}\otimes\mathbf{x}}{\|\mathbf{x}\|^2}\right).$$

It is easily seen that

$$0 \le \mu(\xi) := \frac{\psi'(\xi)\xi}{\psi(\xi)} - 1 \le 1 \quad \forall \xi > 0.$$

From $0 < \underline{c} \le \frac{\psi(\xi)}{\xi} \le \overline{c} < \infty$ (for properly chosen $\underline{c}$, $\overline{c}$) it follows that the eigenvalues of $\mathbf{J}_\Psi$ are bounded below by $\underline{c}$ and above by $2\overline{c}$. Therefore $\mathbf{J}_\Psi$ is invertible, which implies that Φ is differentiable and its Jacobian is bounded uniformly. Consequently Φ is Lipschitz-continuous.

Furthermore there is a uniform lower bound for the smallest eigenvalue, $\exists \underline{\lambda} > 0$, such that

$$\inf_{\mathbf{x}\in\mathbb{R}^N} \lambda_{\min}(\mathbf{J}_\Phi(\mathbf{x})) \ge \underline{\lambda}.$$

From this we easily deduce that

$$\langle \Phi(\mathbf{u}) - \Phi(\mathbf{v}), \mathbf{u} - \mathbf{v}\rangle = \int_0^1 \langle (\mathbf{u}-\mathbf{v})\cdot \mathbf{J}_\Phi\left(\mathbf{v} + t(\mathbf{u}-\mathbf{v})\right), \mathbf{u}-\mathbf{v}\rangle \, dt \ge \underline{\lambda}\|\mathbf{u}-\mathbf{v}\|^2.$$

□

We will need some form of Korn's inequality, which is a simple consequence of the following powerful result proved in [DL76].

Theorem 2.4. *Let Ω be a bounded domain with Lipschitz-continuous boundary. Given $m \in \mathbb{Z}$, $p > 1$ and $f \in W^{m-1,p}(\Omega)$, such that $\partial_i f \in W^{m-1,p}(\Omega)$ for $i = 1, \dots, N$, we have that $f \in W^{m,p}(\Omega)$ and there exists C independent of f, such that*

$$\|f\|_{m,p}^p \le C\left(\|f\|_{m-1,p}^p + \sum_{i=1}^N \|\partial_i f\|_{m-1,p}^p\right).$$

Let us now introduce

$$\|\mathbf{v}\|_E^2 := \|\mathbf{v}\|_{0,2,\Gamma}^2 + \|\nabla\mathbf{v} + \nabla\mathbf{v}^T\|_{0,2,\Omega}^2.$$

Corollary 2.5. *$\|.\|_E$ is a norm on H^1, which is equivalent to $\|.\|_1$.*

Proof. Let

$$E := \{\mathbf{v} \in L^2(\Omega)^N : \varepsilon_{ij}(\mathbf{v}) \in L^2(\Omega), 1 \le i,j \le N\}.$$

Now

$$\frac{\partial^2 v_i}{\partial x_j \partial x_k} = \frac{\partial \varepsilon_{ik}(\mathbf{v})}{\partial x_j} + \frac{\partial \varepsilon_{ij}(\mathbf{v})}{\partial x_k} - \frac{\partial \varepsilon_{jk}(\mathbf{v})}{\partial x_i} \in H^{-1}(\Omega) \quad \forall \mathbf{v} \in E,$$

which yields (with Theorem 2.4) that $\partial_i v_j \in L^2(\Omega)$, i. e. $\mathbf{v} \in H^1(\Omega)^N$. Hence $\|.\|_E$ is well defined on E and it is easily seen that it is a norm (note that $\|\nabla \mathbf{v} + \nabla \mathbf{v}^T\|_{0,2,\Omega} = 0$ implies that $\mathbf{v}$ is a rigid body rotation and together with $\mathbf{v} = \mathbf{0}$ on Γ it follows that $\mathbf{v} = \mathbf{0}$, if $\|\mathbf{v}\|_E = 0$), which turns E into a Banach space. Since $H^1(\Omega)^N$ is embedded continuously in E, the equivalence of the two norms follows from the Open Mapping Theorem. □

Theorem 2.6. *Let $\nu > 0$,*

- *$\nu_T \in L^\infty(\Omega)$ with $\operatorname{ess\,inf} \nu_T \geq 0$, such that*
- *$\nu_T|_\Gamma \in L^\infty(\Gamma)$ and $\operatorname{ess\,inf} \nu_T|_\Gamma \geq 0$.*

If

1. *Φ is Lipschitz-continuous and*
2. *Φ is strongly monotone,*

then

1. *A is Lipschitz-continuous and*
2. *A is strongly monotone.*

Proof. 1. The linear term is continuous, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

$$\left| \int_\Omega (\nu + \nu_T) \nabla(\mathbf{u} - \mathbf{v}) : \left(\nabla \mathbf{w} + \nabla \mathbf{w}^T \right) \right| \leq 2(\nu + \|\nu_T\|_{\infty,\Omega}) \|\mathbf{u} - \mathbf{v}\| \|\mathbf{w}\|,$$

and due to the Lipschitz-continuity of Φ

$$\begin{aligned} \left| \int_\Gamma (\nu + \nu_T) \left(\Phi(\mathbf{u}) - \Phi(\mathbf{v}) \right) \cdot \mathbf{w} \right| &\leq (\nu + \|\nu_T\|_{\infty,\Gamma}) \underbrace{\|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\|_{2,\Gamma}}_{\leq \overline{c} \|\mathbf{u} - \mathbf{v}\|_{2,\Gamma}} \|\mathbf{w}\|_{2,\Gamma} \\ &\leq (\nu + \|\nu_T\|_{\infty,\Gamma}) \overline{c} \gamma^2 \|\mathbf{u} - \mathbf{v}\| \|\mathbf{w}\|, \end{aligned}$$

where $\overline{c}$ is the Lipschitz constant of Φ and γ comes from the trace operator. Hence

$$|\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{w} \rangle| \leq \left(2(\nu + \|\nu_T\|_{\infty,\Omega}) + \overline{c} \gamma^2 (\nu + \|\nu_T\|_{\infty,\Gamma}) \right) \|\mathbf{u} - \mathbf{v}\| \|\mathbf{w}\|.$$

2. With the conditions imposed on ν_T we get for $\mathbf{u}, \mathbf{v} \in V$, $\mathbf{w} := \mathbf{u} - \mathbf{v} \in V$

$$\int_\Omega (\nu + \nu_T) \left(\nabla \mathbf{w} + \nabla \mathbf{w}^T\right) : \left(\nabla \mathbf{w} + \nabla \mathbf{w}^T\right) \geq \nu(\varepsilon(\mathbf{w}), \varepsilon(\mathbf{w}))$$

and due to the strong monotonicity of Φ

$$\int_\Gamma (\nu + \nu_T) \left(\Phi(\mathbf{u}) - \Phi(\mathbf{v})\right) \cdot (\mathbf{u} - \mathbf{v}) \geq \nu \underline{c} \int_\Gamma (\mathbf{u} - \mathbf{v})^2.$$

This implies

$$\langle A(\mathbf{u}) - A(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq \nu \min(1, \underline{c}) \left\{ (\varepsilon(\mathbf{u} - \mathbf{v}), \varepsilon(\mathbf{u} - \mathbf{v})) + \int_\Gamma (\mathbf{u} - \mathbf{v})^2 \right\},$$

where the right side is elliptic because of Corollary 2.5.

□

Remark. 1. Theorem 2.6 is still valid, if we replace $\Phi(\mathbf{u})$ by $\Phi(\mathbf{u}+\mathbf{U}_0)$ with a L^2-function $\mathbf{U}_0$ on Γ, which is seen from

$$|\Phi(\mathbf{u} + \mathbf{U}_0) - \Phi(\mathbf{v} + \mathbf{U}_0)| \leq \overline{c} |(\mathbf{u} + \mathbf{U}_0) - (\mathbf{v} + \mathbf{U}_0)| = \overline{c} |\mathbf{u} - \mathbf{v}|$$

and

$$\langle \Phi(\mathbf{u} + \mathbf{U}_0) - \Phi(\mathbf{v} + \mathbf{U}_0), \mathbf{u} - \mathbf{v} \rangle \geq \underline{c} |(\mathbf{u} + \mathbf{U}_0) - (\mathbf{v} + \mathbf{U}_0)|^2 = \underline{c} |\mathbf{u} - \mathbf{v}|^2,$$

where $|.|$, $\langle ., . \rangle$ denote the Euclidean norm and inner product in $\mathbb{R}^N$.

2. With $\Phi(\mathbf{0}) = \mathbf{0}$ it follows that $A(\mathbf{0}) = \mathbf{0}$ and thus $\exists \alpha > 0$, such that

$$\langle A(\mathbf{v}), \mathbf{v} \rangle \geq \alpha \nu \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V. \tag{2.35}$$

Now the following theorem establishes the well-posedness of the homogeneous Stokes-problem.

Theorem 2.7 (Lax-Milgram). *Let*

1. $\mathbf{F} \in V^*$,
2. $A : V \to V^*$
   - *strongly monotone and*
   - *Lipschitz-continuous.*

Then $A\mathbf{u} = \mathbf{F}$ *has a unique solution* $\mathbf{u} \in V$. *If* $\mathbf{u}_1$, $\mathbf{u}_2$ *are solutions with right sides* $\mathbf{F}_1$ *resp.* $\mathbf{F}_2$, *then*

$$\|\mathbf{u}_1 - \mathbf{u}_2\| \leq \frac{1}{\mu_1} \|\mathbf{F}_1 - \mathbf{F}_2\|.$$

Proof. See [Zei77]. □

2.2.2 Inhomogeneous case

We now show that the stationary problem with non-homogeneous Dirichlet conditions can be reduced to the homogeneous case. In [GR86], p. 24, we find

Lemma 2.8. *For given* $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^N$ *with* $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0$ *there exists* $\mathbf{U}_0 \in H^1(\Omega)^N$*, such that*

$$\begin{aligned} \nabla \cdot \mathbf{U}_0 &= 0, \\ \mathbf{U}_0|_{\partial\Omega} &= \mathbf{g} \end{aligned}$$

and $\|\mathbf{U}_0\|_1 \leq C\|\mathbf{g}\|_{\frac{1}{2},\partial\Omega}$ *with* C *independent of* $\mathbf{g}$*.*

Due to our compatibility assumptions (see 2.1.2) we can thus construct $\mathbf{U}_0 \in H^1(\Omega)^N$, such that (2.12) to (2.15) are fulfilled. The Stokes problem then reads: Find $\mathbf{u} \in V(\mathbf{U}_0)$, such that

$$a(\mathbf{u}, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V. \tag{2.36}$$

With $\mathbf{u} = \mathbf{U}_0 + \hat{\mathbf{u}}$ this gives

$$a(\mathbf{u}, \mathbf{v}) = \underbrace{\frac{1}{2}\int_\Omega (\nu + \nu_T)\left(\nabla\hat{\mathbf{u}} + \nabla\hat{\mathbf{u}}^T\right) : \left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) + \int_\Gamma (\nu + \nu_T)\Phi(\hat{\mathbf{u}} + \mathbf{U}_0) \cdot \mathbf{v}}_{=:\hat{a}(\hat{\mathbf{u}},\mathbf{v})}$$
$$+ \frac{1}{2}\int_\Omega (\nu + \nu_T)\left(\nabla\mathbf{U}_0 + \nabla\mathbf{U}_0^T\right) : \left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right).$$

So if we define

$$\langle \hat{\mathbf{F}}, \mathbf{v} \rangle := \langle \mathbf{F}, \mathbf{v} \rangle - \frac{1}{2}\int_\Omega (\nu + \nu_T)\left(\nabla\mathbf{U}_0 + \nabla\mathbf{U}_0^T\right) : \left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$$

we have the same formal setting as in the homogeneous case with

$$\hat{a}(\hat{\mathbf{u}}, \mathbf{v}) = \langle \hat{\mathbf{F}}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V.$$

Obviously $\hat{\mathbf{F}}$ is continuous and together with the remark after Theorem 2.6 it is easily seen that the above theory is still applicable. We have proved the following:

Theorem 2.9. *For all right hand sides the Stokes problem (2.36) with boundary conditions (2.13) to (2.15) has a unique solution.*

Remark. Dirichlet conditions at some part of the boundary, i. e. $\text{meas}_{\mathbb{R}^{N-1}}(S_1) > 0$, facilitate the analysis and grant existence and uniqueness under weaker assumptions.

Monotonicity (instead of strong monotonicity) of the wall-law is sufficient to establish strong monotonicity of A, since an inequality of Korn's type can be applied directly in the proof of Theorem 2.6.

This means that (from the mathematical point of view) we can work with the 'original' wall law also at large distances from the wall.

2.3 The stationary Navier-Stokes problem

In this section we will discuss existence and uniqueness of solutions of the stationary Navier-Stokes problems again first for the homogeneous problem and then for the general case.

2.3.1 Homogeneous case

Let V as in (2.17). For given $\mathbf{F} \in V^*$ find $\mathbf{u} \in V$, such that

$$(\overline{\text{VP}}) \qquad \boxed{a_0(\mathbf{u}, \mathbf{v}) + \bar{a}_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \bar{\mathbf{F}}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V}$$

for the problems involving total pressure or

$$(\text{VP}) \qquad \boxed{a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V}$$

for the problems with static pressure. a_0, a_1, $\bar{a}_1$, $\mathbf{F}$ and $\bar{\mathbf{F}}$ are defined in(2.26) to (2.31).

We first check that the additional terms a_1 resp. $\bar{a}_1$ make sense.

Lemma 2.10. *The trilinear forms $a_1(.;.,.)$ and $\bar{a}_1(.;.,.)$ are well defined and continuous on $H^1(\Omega)^N \times H^1(\Omega)^N \times H^1(\Omega)^N$.*

Proof. For $N \leq 4$ $H^1(\Omega)^N$ is continuously embedded into $L^4(\Omega)^N$, so $(\mathbf{w} \cdot \nabla \mathbf{u})\mathbf{v} \in L^1$, $(\mathbf{w} \cdot \nabla \mathbf{u}^T)\mathbf{v} \in L^1$ and the integrals are well defined. Applying Cauchy-Schwarz's inequality twice gives $|(\mathbf{w} \cdot \nabla \mathbf{u}, \mathbf{v})| \leq |\nabla \mathbf{u}|_{0,2} \|\mathbf{v}\|_{0,4} \|\mathbf{w}\|_{0,4} \leq c|\mathbf{u}|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1$. The same holds for $|(\mathbf{w} \cdot (\nabla \mathbf{u})^T, \mathbf{v})|$ and continuity follows. □

As for the Stokes problem we can again define operators $\mathcal{A} : V \to V^*$, $\bar{\mathcal{A}} : V \to V^*$ by

$$\langle \bar{\mathcal{A}}(\mathbf{u}), \mathbf{v} \rangle = a_0(\mathbf{u}, \mathbf{v}) + \bar{a}_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in V$$

and

$$\langle \mathcal{A}(\mathbf{u}), \mathbf{v} \rangle = a_0(\mathbf{u}, \mathbf{v}) + a_1(\mathbf{u}; \mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in V.$$

Since we have added convection, Lipschitz continuity of the involved operators is no longer given and Lax-Milgram theory (← Banach's fixed-point theorem) cannot be applied. To prove existence we will first show the existence of solutions of the restriction of the problem to finite-dimensional subspaces, where due to compactness Brouwer's fixed-point theorem can be applied. Then we show that a subsequence of these solutions converges

weakly to a solution of the original problem. This requires continuity in the sense that for all $\mathbf{v} \in V$ and $\mathbf{u} \in V$, $\mathbf{u}_n \in V$, $n \in \mathbb{N}$

$$\text{weak} \lim_{n\to\infty} \mathbf{u}_n = \mathbf{u} \quad \Rightarrow \quad \begin{cases} \lim_{n\to\infty} \langle \mathcal{A}(\mathbf{u}_n), \mathbf{v}\rangle = \langle \mathcal{A}(\mathbf{u}), \mathbf{v}\rangle \\ \lim_{n\to\infty} \langle \bar{\mathcal{A}}(\mathbf{u}_n), \mathbf{v}\rangle = \langle \bar{\mathcal{A}}(\mathbf{u}), \mathbf{v}\rangle \end{cases}.$$

An operator $\mathcal{A}$ with this property is called *sequentially weakly continuous.*

Lemma 2.11. *For all* $\mathbf{v} \in V$ $\langle \mathcal{A}(.), \mathbf{v}\rangle$ *and* $\langle \bar{\mathcal{A}}(.), \mathbf{v}\rangle$ *are sequentially weakly continuous.*

Proof. Let $\mathbf{u}_m \to \mathbf{u}$ weakly in $H^1(\Omega)^N$, $\mathbf{v} \in C^\infty(\bar{\Omega})^N$. Then partial integration yields

$$\begin{aligned} (\mathbf{u}_m \cdot \nabla \mathbf{u}_m, \mathbf{v}) &= \int_{\partial\Omega} (\mathbf{u}_m \cdot \mathbf{n})(\mathbf{u}_m \cdot \mathbf{v}) - (\mathbf{u}_m \cdot \nabla \mathbf{v}, \mathbf{u}_m) \\ &\to \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})(\mathbf{u} \cdot \mathbf{v}) - (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{u}) \\ &= (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v}), \end{aligned}$$

where the limit can be taken, because due to the compactness of the embedding of $H^1(\Omega)^N$ into $L^2(\Omega)^N$ and $L^2(\partial\Omega)^N$ $\mathbf{u}_m \to \mathbf{u}$ strongly in $L^2(\Omega)^N$ and $L^2(\partial\Omega)^N$. Because of the continuity of the trilinear form and the density of $C^\infty(\bar{\Omega})^N$ in $H^1(\Omega)^N$ $(\mathbf{u}_m \cdot \nabla \mathbf{u}_m, \mathbf{v}) \to (\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{v})$ for all $\mathbf{v} \in H^1(\Omega)^N \supset V$.

The same steps can be carried out to prove $(\mathbf{u}_m \cdot (\nabla \mathbf{u}_m)^T, \mathbf{v}) \to (\mathbf{u} \cdot (\nabla \mathbf{u})^T, \mathbf{v})$ for all $\mathbf{v} \in V$.

By a similar argument, again using compact embeddings, it can be shown that $\mathbf{u} \to \int_\Gamma \Phi(\mathbf{u}) \cdot \mathbf{v}$ is sequentially weakly continuous for all $\mathbf{v} \in V$, because

$$\begin{aligned} \left| \int_\Gamma \Phi(\mathbf{u}) \cdot \mathbf{v} - \int_\Gamma \Phi(\mathbf{u}_m) \cdot \mathbf{v} \right| &\leq \int_\Gamma |\Phi(\mathbf{u}) - \Phi(\mathbf{u}_m)||\mathbf{v}| \\ &\leq \bar{c} \int_\Gamma |\mathbf{u} - \mathbf{u}_m||\mathbf{v}| \\ &\leq \gamma \bar{c} \|\mathbf{u} - \mathbf{u}_m\|_{0,2,\Gamma} \|\mathbf{v}\|_{0,2,\Gamma}. \end{aligned}$$

Since the linear terms are sequentially weakly continuous *per definitionem*, the desired result follows. □

Now the analysis can be carried out as in [GR86], pp. 278, which is based on a corollary to Brouwer's fixed-point theorem, which we state without proof.

Theorem 2.12. *Let* C *denote a non-void, convex and compact subset of a finite-dimensional space and let* Ψ *be a continuous mapping from* C *into* C*. Then* Ψ *has at least one fixed point.*

Corollary 2.13. *Let* V *be a finite-dimensional Hilbert space,* $\Psi : V \to V$ *a continuous mapping with the property*

$$\exists \mu \geq 0 : (\Psi(\mathbf{v}), \mathbf{v}) \geq 0 \quad \forall \mathbf{v} \in V : \|\mathbf{v}\| = \mu.$$

Then $\exists \mathbf{v} \in V$ *with* $\|\mathbf{v}\| \leq \mu$, *such that*

$$\Psi(\mathbf{v}) = \mathbf{0}.$$

From this we deduce the following result, which is slightly more general than the one found in [GR86], p. 280:

Theorem 2.14. *Let* V *be a separable Hilbert space,* $\mathbf{F} \in V^*$, $\mathcal{A} : V \to V^*$, *such that*

1. $\exists \rho, \alpha > 0$:

   (a) $\langle \mathcal{A}(\mathbf{v}), \mathbf{v} \rangle \geq \alpha \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V : \|\mathbf{v}\| \leq \rho,$

   (b) $\|\mathbf{F}\|_{V^*} \leq \alpha \rho$

   and

2. $\forall \mathbf{v} \in V : \langle \mathcal{A}(.), \mathbf{v} \rangle$ *is sequentially weakly continuous.*

Then there exists $\mathbf{u} \in V$, *such that* $\mathcal{A}(\mathbf{u}) = \mathbf{F}$.

Proof. We take a basis $\{\mathbf{w}_i\}$ of V, $V_m := \text{span}\{\mathbf{w}_i : 1 \leq i \leq m\}$ to define finite dimensional subproblems

$$(\mathrm{P_m}) \qquad \boxed{\begin{array}{l} \text{Find } \mathbf{u}_m \in V_m, \text{ such that} \\ \langle \mathcal{A}(\mathbf{u}_m), \mathbf{v} \rangle = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_m. \end{array}}$$

In order to apply the above lemma we define $\Psi_m : V_m \to V_m$ by

$$(\Psi_m(\mathbf{v}), \mathbf{w}_i) = \langle \mathcal{A}(\mathbf{v}), \mathbf{w}_i \rangle - \langle \mathbf{F}, \mathbf{w}_i \rangle \quad \forall\ 1 \leq i \leq m,$$

i. e. zeros of Ψ_m are solutions of $(\mathrm{P_m})$ (and vice versa). From

$$\begin{aligned} (\Psi_m(\mathbf{v}), \mathbf{v}) &= \langle \mathcal{A}(\mathbf{v}), \mathbf{v} \rangle - \langle \mathbf{F}, \mathbf{v} \rangle \\ &\geq (\alpha \|\mathbf{v}\| - \|\mathbf{F}\|_{V^*}) \|\mathbf{v}\| \quad \forall \mathbf{v} \in V_m : \|\mathbf{v}\| \leq \rho \end{aligned}$$

it follows that the preliminaries of Corollary 2.13 are fulfilled with $\mu = \|\mathbf{F}\|_{V^*}/\alpha \leq \rho$ and so $(\mathrm{P_m})$ has a solution $\mathbf{u}_m \in V_m$ with $\|\mathbf{u}_m\| \leq \|\mathbf{F}\|_{V^*}/\alpha$.

Since the sequence $\{\mathbf{u}_m\}$ is bounded, there exists a weakly convergent subsequence $\{\mathbf{u}_{m_p}\}$ with (weak) limit $\mathbf{u} \in V$. Due to 2. $\langle \mathcal{A}(\mathbf{u}_{m_p}), \mathbf{v} \rangle \to \langle \mathcal{A}(\mathbf{u}), \mathbf{v} \rangle$ for all $\mathbf{v} \in V$ as $p \to \infty$ and

$$\langle \mathcal{A}(\mathbf{u}), \mathbf{w}_i \rangle = \langle \mathbf{F}, \mathbf{w}_i \rangle \quad \forall i \geq 1.$$

Since $\{\mathbf{w}_i\}$ is a Hilbert basis

$$\langle \mathcal{A}(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{F}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V,$$

i. e. $\mathcal{A}(\mathbf{u}) = \mathbf{F}$. □

Lemma 2.15. *If, in addition to the hypothesis of Theorem 2.14, there exists $\tilde{\alpha}$, such that*

$$\langle \mathcal{A}(\mathbf{u}_1), \mathbf{u}_1 - \mathbf{u}_2 \rangle - \langle \mathcal{A}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle \geq \tilde{\alpha} \|\mathbf{u}_1 - \mathbf{u}_2\|^2 \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in V \tag{2.37}$$

then the solution is unique.

Proof. Let $\mathbf{u}_1$, $\mathbf{u}_2$ be two solutions, then $\langle \mathcal{A}(\mathbf{u}_1), \mathbf{u}_1 - \mathbf{u}_2 \rangle - \langle \mathcal{A}(\mathbf{u}_2), \mathbf{u}_1 - \mathbf{u}_2 \rangle = 0$ and (2.37) yields $\mathbf{u}_1 = \mathbf{u}_2$. □

Theorem 2.16. *1. The homogeneous problem for the stationary Navier-Stokes equations in total pressure formulation has a unique solution.*

2. The homogeneous problem for the stationary Navier-Stokes equations in static pressure formulation has a solution for $\|\mathbf{F}\| \leq c\nu^2$ with $c = c(\Omega)$.

Proof. We begin with the first statement. For existence we check the prerequisites of Theorem 2.14.

1. $\bar{a}_1(\mathbf{v}; \mathbf{v}, \mathbf{v}) = 0$, so $\langle \bar{\mathcal{A}}(\mathbf{v}), \mathbf{v} \rangle = a_0(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V$. Hence there are no restrictions on $\mathbf{F}$.

2. Lemma 2.11.

Uniqueness follows from Lemma 2.15.

For the static pressure formulation we get from Theorem 2.6 (or rather the proof thereof) and Lemma 2.10

$$\begin{aligned} \langle \mathcal{A}(\mathbf{v}), \mathbf{v} \rangle &\geq a_0(\mathbf{v}, \mathbf{v}) - |a_1(\mathbf{v}; \mathbf{v}, \mathbf{v})| \\ &\geq \nu c_1 \|\mathbf{v}\|^2 - c_2 \|\mathbf{v}\|^3 \end{aligned}$$

with $c_1 = \min(1, \underline{c}) c_F$ and $c_2 > 0$. A simple calculation shows that

$$\nu c_1 \|\mathbf{v}\|^2 - c_2 \|\mathbf{v}\|^3 \geq \alpha \|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V : \|\mathbf{v}\| \leq \rho$$

for $\rho = \rho_0 = \nu \frac{c_1}{2c_2}$, $\alpha = \alpha_0 = \nu \frac{c_1}{2}$, which also yield an optimal upper bound for $\|\mathbf{F}\|$ in the context of Theorem 2.14. Now the rest follows as above from Theorem 2.14. □

2.3.2 Inhomogeneous case

Here the homogenization requires more consideration than for the Stokes problem.

Let again $\mathbf{u} = \mathbf{U}_0 + \mathbf{w}$, where $\mathbf{U}_0 \in H^1(\Omega^N)$ is a solenoidal function satisfying the inhomogeneous boundary conditions, then

$$\langle \mathcal{A}(\mathbf{u}), \mathbf{v} \rangle = a_0(\mathbf{w} + \mathbf{U}_0, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{w}, \mathbf{v}) + a_1(\mathbf{U}_0; \mathbf{w}, \mathbf{v}) + a_1(\mathbf{w}; \mathbf{U}_0, \mathbf{v}) + a_1(\mathbf{U}_0; \mathbf{U}_0, \mathbf{v}).$$

This leads to the definitions

$$\begin{aligned}
\langle \hat{\mathcal{A}}(\mathbf{w}), \mathbf{v}\rangle &:= \hat{a}_0(\mathbf{w},\mathbf{v}) + \hat{a}_1(\mathbf{w};\mathbf{w},\mathbf{v}),\\
\hat{a}_0(\mathbf{w},\mathbf{v}) &:= \frac{1}{2}\int_\Omega (\nu+\nu_T)\left(\nabla\mathbf{w}+\nabla\mathbf{w}^T\right) : \left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right) + \int_\Gamma (\nu+\nu_T)\Phi(\mathbf{w}+\mathbf{U}_0)\cdot\mathbf{v}\\
&\qquad + a_1(\mathbf{U}_0;\mathbf{w},\mathbf{v}) + a_1(\mathbf{w};\mathbf{U}_0,\mathbf{v}),\\
\hat{a}_1(\mathbf{w};\mathbf{w},\mathbf{v}) &:= a_1(\mathbf{w};\mathbf{w},\mathbf{v}),\\
\langle \hat{\mathbf{F}}, \mathbf{v}\rangle &:= \langle \mathbf{F},\mathbf{v}\rangle - \frac{1}{2}\int_\Omega (\nu+\nu_T)\left(\nabla\mathbf{U}_0+\nabla\mathbf{U}_0^T\right) : \left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right) - a_1(\mathbf{U}_0;\mathbf{U}_0,\mathbf{v}).
\end{aligned}$$

Obviously $\hat{\mathbf{F}}$ is continuous, but we have to ensure that $\hat{a}_0$ is still strongly monotone.

From $|a_1(\mathbf{v};\mathbf{U}_0,\mathbf{v}) + a_1(\mathbf{U}_0;\mathbf{v},\mathbf{v})| \le c\|\mathbf{U}_0\|_1\|\mathbf{v}\|_1^2$ and Lemma 2.8 we see that for sufficiently small boundary data $\mathbf{g}$ there will still be a (unique) solution.

Theorem 2.17. *If there exists $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^N$, such that conditions (2.8) to (2.11) hold, then the following existence and uniqueness results are valid:*

1. *The inhomogeneous problem for the stationary Navier-Stokes equations in total pressure formulation has a unique solution for $\|\mathbf{g}\|_{\frac{1}{2},\partial\Omega} \le c_1\nu$ with $c_1 = c_1(\Omega)$.*
2. *The inhomogeneous problem for the stationary Navier-Stokes equations in static pressure formulation has a solution for $\|\mathbf{g}\|_{\frac{1}{2},\partial\Omega} \le c_1\nu$ and $\|\mathbf{F}\| \le c_2\nu^2$ with $c_i = c_i(\Omega)$.*

Remark. The condition on $\mathbf{F}$ means in our context of pressure drop problems that $c_3|k|_1 + c_4 \max_{j=0,\dots,r}|P_j| \le \nu^2$.

2.4 The non-stationary Navier-Stokes problem

2.4.1 Classical formulation

We will be concerned with the following problems: For given $k \in C^{0,1}((0,T)\times\Omega) \cap C^{0,0}((0,T)\times\bar{\Omega})$, $\nu_T \in C^{0,1}((0,T)\times\Omega)$, find $\mathbf{u} \in C^{1,2}((0,T)\times\Omega)^N \cap C^{0,1}([0,T)\times\bar{\Omega})^N$, $p \in C^{0,1}((0,T)\times\Omega) \cap C^{0,0}((0,T)\times\bar{\Omega})$ with initial condition

$$\mathbf{u}(0,.) = \mathbf{u}^0,$$

such that

$$\begin{aligned}
\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + \nu\Delta\mathbf{u} + \nabla\cdot\left(\nu_T\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\right) - \nabla\frac{2}{3}k\\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}$$

with the wall-law

$$\left.\begin{aligned}
\mathbf{u}\cdot\mathbf{n} &= 0\\
\mathbf{n}\cdot\mathbf{T}(p,\mathbf{u})\cdot\mathbf{s}_i + (\nu+\nu_T)\Phi(\mathbf{u})\cdot\mathbf{s}_i &= 0
\end{aligned}\right\} \quad \text{on } \Gamma$$

and

$$\left.\begin{aligned} \mathbf{u} &= \mathbf{u}_0 \\ p-(\nu+\nu_T)\tfrac{\partial \mathbf{u}\cdot\mathbf{n}}{\partial n} &= P_j(t) \\ \mathbf{u}\cdot\mathbf{s}_i &= \mathbf{u}_i\cdot\mathbf{s}_i \\ \mathbf{T}(p,\mathbf{u})\cdot\mathbf{n} &= 0 \end{aligned}\right\} \quad \begin{aligned} &\text{on } S_1, \\ &\text{on } S_{2j},\ i=1,2,\ 0\le j\le r, \\ &\text{on } S_3 \end{aligned}$$

or equivalently

$$\begin{aligned} \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot(\nabla\mathbf{u})^T &= -\nabla\bar{p}+\nu\Delta\mathbf{u}+\nabla\cdot\left(\nu_T\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\right) \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

with boundary conditions as above with p replaced by $\bar{p}$.

2.4.2 Weak formulation

For given $\mathbf{F},\bar{\mathbf{F}}\in L^2(0,T;V^*)$, $\nu_T\in L^\infty(\Omega)$, $\Phi:\mathbb{R}^N\to\mathbb{R}^N$ find

$$\mathbf{u}\in L^2(0,T;V)\cap L^\infty(0,T;H)$$

with

$$\mathbf{u}(0)=\mathbf{u}^0\in H:=\overline{\{\mathbf{v}\in C^\infty(\bar{\Omega})^N:\nabla\cdot\mathbf{v}=0\}}^{L^2}$$

and

$$V:=\{\mathbf{v}\in H^1(\Omega)^N:\nabla\cdot\mathbf{v}=0,\mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma,\mathbf{v}=\mathbf{0} \text{ on } S_1,\mathbf{v}\cdot\mathbf{s}=0 \text{ on } S_{2j}\},$$

such that

$$(\overline{\mathrm{VP}}(\mathrm{t})) \qquad \boxed{\tfrac{d}{dt}(\mathbf{u},\mathbf{v}) + a_0(\mathbf{u},\mathbf{v}) + \bar{a}_1(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle\bar{\mathbf{F}},\mathbf{v}\rangle \quad \forall\mathbf{v}\in V}$$

for the problems involving total pressure or

$$(\mathrm{VP}(\mathrm{t})) \qquad \boxed{\tfrac{d}{dt}(\mathbf{u},\mathbf{v}) + a_0(\mathbf{u},\mathbf{v}) + a_1(\mathbf{u};\mathbf{u},\mathbf{v}) = \langle\mathbf{F},\mathbf{v}\rangle \quad \forall\mathbf{v}\in V}$$

for the problems with static pressure. a_0 etc. are defined in (2.26) to (2.31).

2.5 The k-ϵ system

As for the analysis for the Navier-Stokes system (with given ν_T) there are attempts for the k-ϵ equations (with given velocity $\mathbf{u}$),

$$\begin{aligned}
\partial_t k + \mathbf{u} \cdot \nabla k &= \frac{c_\mu}{2} \frac{k^2}{\epsilon} \left| \nabla \mathbf{u} + \nabla \mathbf{u}^T \right|^2 + \nabla \cdot \left(c_\mu \frac{k^2}{\epsilon} \nabla k \right) - \epsilon, \\
\partial_t \epsilon + \mathbf{u} \cdot \nabla \epsilon &= \frac{c_1}{2} k \left| \nabla \mathbf{u} + \nabla \mathbf{u}^T \right|^2 + \nabla \cdot \left(c_\epsilon \frac{k^2}{\epsilon} \nabla \epsilon \right) - c_2 \frac{\epsilon^2}{k}
\end{aligned}$$

with parameters as defined in the previous chapter.

Unfortunately the results here are not as satisfying as above.

2.5.1 Existence and uniqueness

We know no existence or uniqueness results for the k-ϵ system itself.

Some attempts concentrate on a simpler model, which is formulated in the transformed functions $\varphi = \epsilon^2/k^3$, $\theta = k/\epsilon$ and contains some simplifications (see also section 3.4). For this system we find an existence result in [MP94], uniqueness is still an open problem.

2.5.2 Positivity

Positivity of k and ϵ is an essential physical as well as a mathematical requirement.

In [MP94] we find a proof based on a transformation to the k-θ system (with θ defined as above). Assuming the existence of smooth solutions, positivity of k and θ can be shown by maximum-principle-like ideas. The fact that ϵ is greater than zero now follows directly from $\epsilon = k/\theta$.

We think that there is a gap in an essential part of the proof, which we have not been able to fill yet. Thus positivity of k and ϵ seems to be plausible, but not guaranteed.

Chapter 3

Discretisation and numerical analysis

The discretisation is done in two steps. First we proceed with the finite element approximation of the Navier-Stokes part

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p + \nabla \cdot \left((\nu + \nu_T) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T \right) \right), \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

where ν_T is assumed to be a known non-negative function, then — again with the use of finite elements — we discretise the k- and ϵ-equations

$$\begin{aligned} \partial_t k + \mathbf{u} \cdot \nabla k &= \frac{c_\mu}{2} \frac{k^2}{\epsilon} \left| \nabla \mathbf{u} + \nabla \mathbf{u}^T \right|^2 + \nabla \cdot \left(c_\mu \frac{k^2}{\epsilon} \nabla k \right) - \epsilon, \\ \partial_t \epsilon + \mathbf{u} \cdot \nabla \epsilon &= \frac{c_1}{2} k \left| \nabla \mathbf{u} + \nabla \mathbf{u}^T \right|^2 + \nabla \cdot \left(c_\epsilon \frac{k^2}{\epsilon} \nabla \epsilon \right) - c_2 \frac{\epsilon^2}{k} \end{aligned}$$

with given $\mathbf{u}$.

3.1 Preliminary considerations

The finite element approximation of saddle point problems (as the Stokes system) is a well-studied topic.

We use a triangularisation $\Delta_h := \{\delta_r : r \in \mathbf{R}_h\}$ of Ω with an index set $\mathbf{R}_h$ and choose finite element spaces X_h and M_h for the primary variables $\mathbf{u}$ and p (for definition of X, M see the previous chapter). For M_h we use the space

$$M_h = \{p \in M : p|_{\delta_r} \in P_l(\delta_r), r \in \mathbf{R}_h\}$$

of piecewise polynomials of degree l.

For the choice of X_h there are generally two possibilities

- *Conforming elements:*

$$X_h = \{\mathbf{v} \in X : v_i|_{\delta_r} \in P_k(\delta_r), r \in \mathbf{R}_h\}.$$

- *Non-conforming elements:* Here we require continuity only on Gauss points on all triangle edges (e. g. centers of edges):

$$X_h = \{\mathbf{v} : v_i|_{\delta_r} \in P_k(\delta_r), r \in \mathbf{R}_h, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma, \mathbf{v} = \mathbf{0} \text{ on } S_1, \mathbf{v} \cdot \mathbf{s}_i = 0 \text{ on } S_2,$$
$$\mathbf{v} \text{ continuous at the Gauss points on all triangle sides}\}$$

Analogously to the continuous problem we define

$$V_h := \{\mathbf{v} \in X_h : b(\mathbf{v}, q) = 0 \ \forall q \in M_h\}.$$

From the analysis of the continuous problem it is evident that the following properties of the pairing of spaces (X_h, M_h) suffice to grant a unique solution for the finite-dimensional problem and the usual approach is to verify these conditions:

1.
$$a(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha \|\mathbf{v}_h\|^2 \quad \forall \mathbf{v}_h \in V_h$$

and

2.
$$\inf_{p_h \in M_h} \sup_{\mathbf{v}_h \in X_h} \frac{b(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\| \|p_h\|} \geq \beta.$$

If β can be chosen independent of h, then usually (e. g. for homogeneous Dirichlet boundary conditions) convergence can be shown.

For non-conforming elements (e. g. P_1-P_0) the inf-sup condition can easily be proven, they have also some advantages from the computational point of view. Unfortunately they cannot be used here, because in general $a(.,.)$ is not elliptic on V_h, as the following counterexample shows.

Example 3.1. Assume that on the square $[-2, 2] \times [-2, 2]$ we have a special mesh as in Figure 3.1.

We define a piecewise linear vector function $\mathbf{u}$, which is $\mathbf{0}$ outside regions I to IV (and therefore at the boundary) and
in I

$$\begin{aligned} u_1 &= x_2 - 1, \\ u_2 &= -x_1 + 1, \end{aligned}$$

in II

$$\begin{aligned} u_1 &= -x_2 + 1, \\ u_2 &= x_1 + 1, \end{aligned}$$

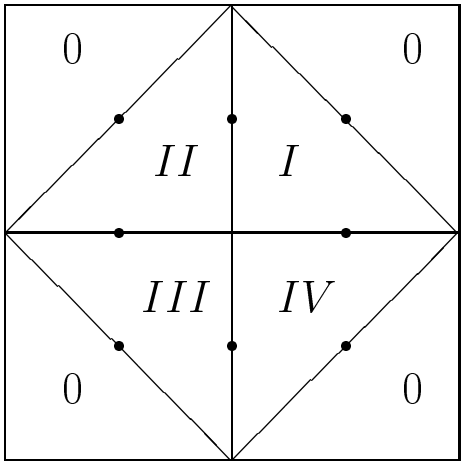

Figure 3.1: Non-conforming finite element on square.

in III

$$\begin{aligned} u_1 &= x_2 + 1, \\ u_2 &= -x_1 - 1, \end{aligned}$$

in IV

$$\begin{aligned} u_1 &= -x_2 - 1, \\ u_2 &= x_1 - 1. \end{aligned}$$

It is easy to check that $\mathbf{u}$ is continuous at the Gauß points (= midpoints of the edges), $\nabla \cdot \mathbf{u} = 0$ piecewise and $\varepsilon(\mathbf{u}) = \mathbf{0}$ piecewise also. Hence $\mathbf{0} \neq \mathbf{u}$, but $a(\mathbf{u}, \mathbf{u}) = 0$, the consequence of this being that no non-conforming finite elements of any degree can be used for the stress tensor formulation of the Stokes problem.

From the computational point of view we want to keep the space dimensions — and therefore the degree of the polynomials — low. It is known that the P_1-P_0 element is not stable (i. e. does not fulfil condition 2. with β independent of h), but it can be stabilised by a least squares method. For the implementation and for algebraic multigrid methods (c. f. section 3.6.3) it is also convenient to use the same elements for velocity and pressure, and for that reason we choose a stabilisation technique for the conforming P_1-P_1 element, first introduced by [HFB86].

Remark. We linearise the wall-law at each point of the boundary at the value of the previous time step. Due to the monotonicity of the wall function ellipticity of the (now bilinear) form is preserved.

In the following section we derive the method and prove convergence for the Stokes problem.

3.2 Spatial discretisation of the Stokes problem

In [FS91] the bilinear form associated with the saddle-point problem of the standard mixed formulation is modified to become coercive and convergence is shown for the following transformed systems:
Find $\mathbf{u}_h \in X_h$ and $p_h \in M_h$ such that

$$A(\mathbf{u}_h, p_h; \mathbf{v}, q) = \tilde{F}(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in X_h \times M_h, \tag{3.1}$$

with

$$\begin{aligned} A(\mathbf{u}, p; \mathbf{v}, q) &= a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q) - \alpha c(p, q), \\ \tilde{F}(\mathbf{v}, q) &= (f, \mathbf{v}) - \alpha \sum_{r \in \mathbf{R}_h} h_{\delta_r}^2 (f, \nabla q)_{\delta_r}, \\ c(p, q) &= \sum_{r \in \mathbf{R}_h} h_{\delta_r}^2 (\nabla p, \nabla q)_{\delta_r} \end{aligned}$$

This formulation is obtained by adding up the weak formulations of the mass equation and the momentum equation (leading to an equation equivalent to the original problem) and subtracting the strong formulation of the momentum equation on each element, tested with ∇q and weighted with the mass of each element. The second derivatives vanish for our FEM space on each element, so if we assume element-wise constant ν_T the diffusion terms vanish.

Remark. (MINI element). Another possibility for the stabilisation of the P_1-P_1 element leads to the so called MINI element. The discretised velocity space is extended by bubble functions, i. e. we replace X_h by

$$\tilde{X}_h := \{\mathbf{v} \in X : \ \mathbf{v}|_{\delta_r} = \mathbf{u}|_{\delta_r} + b_{\delta_r} \bar{\alpha}_{\delta_r}, \ \mathbf{u} \in X_h, \bar{\alpha}_{\delta_r} \in \mathbb{R}^N\},$$

with $b_{\delta_r}(\mathbf{x}) = \lambda_1 \lambda_2 \lambda_3$, where the λ_i are the barycentric coordinates of $\mathbf{x}$ with respect to δ_r.

In [BF91] we now find the interesting fact that if we eliminate the bubble functions at element level, the resulting discretisation is equivalent to (3.1) with α fixed element-wise.

This connection gives hints for a 'good' choice of α and for example it is seen that α should be of order $\mathcal{O}(1/\nu)$.

Theorem 3.2. *Suppose the solution of (VP_0) satisfies $\mathbf{u} \in H^2(\Omega)^N$ and $p \in H^2(\Omega)$. Then for $\alpha > 0$ (3.1) has a unique solution satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C(h|\mathbf{u}|_2 + h^2|p|_2). \tag{3.2}$$

Remark. We have formulated the theorem in a version adapted to our problem. A more general formulation for higher order elements can be found in [FS91]. In general α has to be smaller than some constant C_I, which comes from the inverse inequality

$$C_I \sum_{r \in \mathbf{R}_h} h_{\delta_r}^2 \|\nabla \cdot \varepsilon(\mathbf{v})\|_{0,\delta_r}^2 \leq \|\varepsilon(\mathbf{v})\|_0^2, \quad \mathbf{v} \in X_h.$$

In our case this inequality is fulfilled for arbitrary C_I, because the left hand side is zero.

To prove the theorem it will be necessary to present some intermediate results. First, we define

$$|q|_h = (\alpha c(q,q))^{1/2}$$

and get the following estimates via scaling arguments

$$|q|_h \leq C\|q\|_0, \quad q \in M_h, \tag{3.3}$$
$$\inf_{q \in M_h} |p-q|_h \leq Ch^2|p|_2, \quad p \in H^2(\Omega). \tag{3.4}$$

We first show the boundedness of A with respect to the norm

$$\|(\mathbf{u},p)\| = \left(\|\mathbf{u}\|_1^2 + \|p\|_0^2\right)^{1/2}.$$

Lemma 3.3. *There is a positive constant C, such that for all $(\mathbf{u},p) \in X_h \times M_h$ and $(\mathbf{v},q) \in X_h \times M_h$ we have*

$$A(\mathbf{u},p;\mathbf{v},q) \leq C\|(\mathbf{u},p)\| \cdot \|(\mathbf{v},q)\|.$$

Proof. With the Cauchy-Schwarz inequality we get

$$\begin{aligned} A(\mathbf{u},p;\mathbf{v},q) \leq & \bar{C}\left(\|\mathbf{u}\|_1^2 + \|p\|_0^2 + |p|_h^2\right)^{1/2} \cdot \left(\|\mathbf{v}\|_1^2 + \|q\|_0^2 + |q|_h^2\right)^{1/2} + \\ & + \int_\Gamma (\nu + \nu_T)\Phi(\mathbf{u}\cdot\mathbf{s})\mathbf{v}\cdot\mathbf{s}, \end{aligned}$$

and with

$$\begin{aligned} \int_\Gamma (\nu+\nu_T)\Phi(\mathbf{u}\cdot\mathbf{s})\mathbf{v}\cdot\mathbf{s} &\leq \tilde{C}_1\|\nu+\nu_T\|_{\infty,\Gamma}\int_\Gamma (\mathbf{u}\cdot\mathbf{s})(\mathbf{v}\cdot\mathbf{s}) \\ &\leq \tilde{C}_2\|\mathbf{u}\|_1\|\mathbf{v}\|_1 \end{aligned}$$

and the Cauchy-Schwarz inequality for vectors we get

$$A(\mathbf{u},p;\mathbf{v},q) \leq \tilde{C}_3\left(\|\mathbf{u}\|_1^2 + \|p\|_0^2 + |p|_h^2\right)^{1/2} \cdot \left(\|\mathbf{v}\|_1^2 + \|q\|_0^2 + |q|_h^2\right)^{1/2}.$$

The estimate (3.3) completes the proof. □

To show a stability inequality we first need the following lemma. The proof (even for the restricted space, where it is assumed that homogeneous Dirichlet conditions are imposed at some part of the boundary) can be found in [FS91].

Lemma 3.4. *There exist constants C_1 and C_2, such that for all $p \in M_h$*

$$\sup_{0 \neq \mathbf{v} \in X_h} \frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1} \geq C_1\|p\|_0 - C_2|p|_h.$$

Now we can prove a stability lemma.

Lemma 3.5. *There exists a positive constant C_3 such that for $(\mathbf{u}, p) \in X_h \times M_h$ we have*

$$\sup_{\substack{(\mathbf{v},q)\in X_h\times M_h\\(\mathbf{v},q)\neq(\mathbf{0},0)}} \frac{A(\mathbf{u},p;\mathbf{v},q)}{\|(\mathbf{v},q)\|} \geq C_3\|(\mathbf{u},p)\|.$$

Proof. Let $\mathbf{u}$ and p be fixed. With the results of section 2.2 we immediately get

$$A(\mathbf{u},p;\mathbf{u},-p) \geq \tilde{C}_4\|(\mathbf{u},p)\|^2 + |p|_h^2.$$

Now, let $\mathbf{w} \in X_h$ be a function, for which the supremum of Lemma 3.4 is attained and assume $\|\mathbf{w}\|_1 = \|p\|_0$. We get

$$\begin{aligned}
A(\mathbf{u},p;-\mathbf{w},0) &= -\frac{1}{2}((\nu+\nu_T)\varepsilon(\mathbf{u}),\varepsilon(\mathbf{w})) - \int_\Gamma (\nu+\nu_T)\Phi(\mathbf{u}\cdot\mathbf{s})\mathbf{w}\cdot\mathbf{s} + (\nabla\cdot\mathbf{w},p)\\
&\geq -\tilde{C}_5\|\mathbf{u}\|_1\|\mathbf{w}\|_1 - \int_\Gamma (\nu+\nu_T)\Phi(\mathbf{u}\cdot\mathbf{s})\mathbf{w}\cdot\mathbf{s} + (\nabla\cdot\mathbf{w},p)\\
&\geq -\tilde{C}_5\|\mathbf{u}\|_1\|\mathbf{w}\|_1 - \tilde{C}_6\|\mathbf{u}\|_1\|\mathbf{w}\|_1 + C_1\|p\|_0^2 - C_2\|p\|_0|p|_h\\
&\geq -\tilde{C}_7\|\mathbf{u}\|_1\|p\|_0 + C_1\|p\|_0^2 - C_2\|p\|_0|p|_h.
\end{aligned}$$

The formula

$$\frac{\beta}{2}x^2 + \frac{1}{2\beta}y^2 \geq xy, \quad \forall\beta > 0,$$

yields

$$\begin{aligned}
A(\mathbf{u},p;-\mathbf{w},0) &\geq -\tilde{C}_8\|\mathbf{u}\|_1^2 - \tilde{C}_9|p|_h^2 + \left(-\frac{\beta}{2}\tilde{C}_7 + C_1 - \frac{\beta}{2}C_2\right)\|p\|_0^2\\
&\geq -\tilde{C}_8\|\mathbf{u}\|_1^2 - \tilde{C}_9|p|_h^2 + \tilde{C}_{10}\|p\|_0^2
\end{aligned}$$

with $\tilde{C}_{10} > 0$, if β is chosen small enough. We choose γ with $0 < \gamma < \min\{\tilde{C}_4/\tilde{C}_8, 1/\tilde{C}_9\}$ and set $(\mathbf{v},q) = (\mathbf{u} - \gamma\mathbf{w}, -p)$. Combining the results from above gives

$$\begin{aligned}
A(\mathbf{u},p;\mathbf{v},q) &= A(\mathbf{u},p;\mathbf{u}-\gamma\mathbf{w},-p) = A(\mathbf{u},p;\mathbf{u},-p) + \gamma A(\mathbf{u},p;-\mathbf{w},0)\\
&\geq (\tilde{C}_4 - \gamma\tilde{C}_8)\|\mathbf{u}\|_1^2 + (\tilde{C}_4 + \gamma\tilde{C}_{10})\|p\|_0^2 + (1-\gamma\tilde{C}_9)|p|_h^2\\
&\geq \tilde{C}_{11}\|(\mathbf{u},p)\|^2.
\end{aligned}$$

For our special $(\mathbf{v},q)$ we have

$$\begin{aligned}
\|(\mathbf{v},q)\|^2 &\leq (\|\mathbf{u}\|_1 + \gamma\|\mathbf{w}\|_1)^2 + \|p\|_0^2 \leq 2\|\mathbf{u}\|_1^2 + 2\gamma^2\|\mathbf{w}\|_1^2 + \|p\|_0^2\\
&= 2\|\mathbf{u}\|_1^2 + (1+2\gamma^2)\|p\|_0^2 \leq \tilde{C}_{12}\|(\mathbf{u},p)\|^2,
\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.2. Existence and uniqueness follow directly from the Lemmata 3.3 and 3.5 and the theorem of Lax and Milgram. Now, let $\tilde{\mathbf{u}} \in X_h$ be the interpolant of $\mathbf{u}$ and $\tilde{p} \in M_h$ the L^2-projection of p. From Lemma 3.5, the formula $\sqrt{x^2+y^2} \geq \sqrt{1/2}(x+y)$ and scaling in $(\mathbf{v}, q)$ we get the existence of $(\mathbf{v}, q) \in X_h \times M_h$ such that

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + \|\tilde{p} - p_h\|_0 \leq A(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q) \tag{3.5}$$

and

$$\|\mathbf{v}\|_1^2 + \|q\|_0^2 \leq \tilde{C}_{13}, \tag{3.6}$$

for fixed $\tilde{C}_{13} > \sqrt{2}/C_3$. With easy but lengthy calculation we can show

$$A(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q) \leq \tilde{C}_{14} A(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q). \tag{3.7}$$

Since

$$A(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q) \leq \tilde{C}_{15} \left(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \|p - \tilde{p}\|_0^2 + |p - \tilde{p}|_h^2\right)^{1/2} \cdot \left(\|\mathbf{v}\|_1^2 + \|q\|_0^2 + |q|_h^2\right)^{1/2}$$

we get, using (3.3), (3.5), (3.6) and (3.7),

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + \|\tilde{p} - p_h\|_0 \leq \tilde{C}_{16} \left(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1 + \|p - \tilde{p}\|_0 + |p - \tilde{p}|_h\right).$$

The estimate of Theorem 3.2 now follows from standard interpolation estimates. □

3.3 Discretisation of the total derivative

In [MP94] a method of characteristics is used for a stable discretisation of the total derivative. However, the implementation is rather complicated in our context and hence we decided for a streamline upwinding scheme (SUPG = *Streamline Upwinding Petrov Galerkin*).

In applying the upwinding idea to finite elements, one would want to use test functions with more weight upstream than downstream. This can be realized (and in that case we can still work on the same elements) by testing the momentum equation on each element with $\mathbf{v}_h + \beta_h \mathbf{u}_h \cdot \nabla \mathbf{v}_h$ instead of $\mathbf{v}_h$,

$$\begin{aligned}(\mathbf{u}_{h,t} + \mathbf{u}_h \cdot \nabla \mathbf{u}_h, \mathbf{v}_h + \beta_h \mathbf{u}_h \cdot \nabla \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h + \beta_h \mathbf{u}_h \cdot \nabla \mathbf{v}_h) + \\ (\nabla p_h, \mathbf{v}_h + \beta_h \mathbf{u}_h \cdot \nabla \mathbf{v}_h) = (f, \mathbf{v}_h + \beta_h \mathbf{u}_h \cdot \nabla \mathbf{v}_h).\end{aligned}$$

β_h is a parameter of magnitude $\mathcal{O}(h)$ with h the discretisation parameter. A semi-implicit time discretisation takes the form

$$\begin{aligned}(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\tau} + \mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}, \mathbf{v}_h + \beta_h \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h) + a(\mathbf{u}_h^{n+1}, \mathbf{v}_h + \beta_h \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h) + \\ (\nabla p_h^{n+1}, \mathbf{v}_h) + (\nabla p_h^{n+1}, \beta_h \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h) = (f^{n+1}, \mathbf{v}_h + \beta_h \mathbf{u}_h^n \cdot \nabla \mathbf{v}_h).\end{aligned}$$

Practically it turns out that among all the additional terms only the one of the form $\beta_h(\mathbf{w}\nabla\mathbf{u}, \mathbf{w}\nabla\mathbf{v})$ with 'old' velocity $\mathbf{w}$ is important, which can be viewed as the weak formulation of a diffusion term

$$\beta_h \nabla \cdot (\mathbf{w} \otimes \mathbf{w} \nabla \mathbf{u})$$

with diffusion only along the streamline.

3.4 The k- and ϵ-equations

We also use piecewise linear conforming finite elements to discretise our second 'subproblem'

$$D_t k = c_\mu \frac{k^2}{\epsilon} E + \nabla \cdot \left(c_\mu \frac{k^2}{\epsilon} \nabla k \right) - \epsilon,$$
$$D_t \epsilon = c_1 k E + \nabla \cdot \left(c_\epsilon \frac{k^2}{\epsilon} \nabla \epsilon \right) - c_2 \frac{\epsilon^2}{k},$$

with $D_t = \partial/\partial t + \mathbf{u} \cdot \nabla$ and given $E > 0$. According to some authors (and also our own experience gained throughout the work on this thesis) it is very difficult to obtain a stable scheme for this set of convection-diffusion equations (plus production and dissipation), which moreover preserves positivity of k and ϵ. We will follow the suggestions in [MP94].

3.4.1 First method: A semi-implicit multi-step scheme utilising an auxiliary system

We introduce the new variables φ and θ defined by

$$\varphi = \epsilon^2/k^3, \quad \theta = k/\epsilon.$$

Substitution of k and ϵ in the above equations leads to the φ-θ system

$$\begin{aligned} D_t\theta - \mathrm{diff}_\theta - E(c_\mu - c_1)\theta^2 + 1 - c_2 = 0, \\ D_t\varphi - \mathrm{diff}_\varphi + (3c_\mu - 2c_1)E\varphi\theta + (2c_2 - 3)\frac{\varphi}{\theta} = 0, \end{aligned} \tag{3.8}$$

where diff_θ and diff_φ are complicated diffusion terms.[1]

We now suppose that at a given time step the Navier-Stokes equations have been solved with the old value of ν_T. Now the k-ϵ system is solved via a two step algorithm. First we take a time step using only the convection, production and dissipation terms of the φ-θ system. Production and dissipation terms are split into implicit and explicit terms to obtain linear systems only. We notice that only positive entries are added to the diagonals of the system matrices. Then we transform the φ and θ solutions to k and ϵ and perform a diffusion step. This leads to the following semi-discretised scheme.

Algorithm 3.6.

1. Compute

$$\theta_h^m = k_h^m/\epsilon_h^m, \quad \varphi_h^m = \epsilon_h^{m2}/k_h^{m3}$$

[1] In [MP94] these terms are replaced by simpler ones. For this system an existence result can be derived.

2. Solve

$$(\partial_t \theta_h^{m+1/2}, w_h) + (\mathbf{u} \cdot \nabla \theta_h^{m+1/2}, w_h) + (\theta_h^m \theta_h^{m+1/2} E_h^m, w_h)[c_1 - c_\mu] = (c_2 - 1, w_h).$$

3. Solve

$$(\partial_t \varphi_h^{m+1/2}, w_h) + (\mathbf{u} \cdot \nabla \varphi_h^{m+1/2}, w_h) + ([3c_\mu - 2c_1] E_h^m \theta_h^m \varphi_h^{m+1/2}, w_h) + \\ + ([2c_2 - 3] \frac{1}{\theta_h^{m+1/2}} \varphi_h^{m+1/2}, w_h) = 0.$$

4. Compute

$$k_h^{m+1/2} = \frac{1}{\varphi_h^{m+1/2} \left(\theta_h^{m+1/2}\right)^2}, \quad \epsilon_h^{m+1/2} = \frac{k_h^{m+1/2}}{\theta_h^{m+1/2}}.$$

5. Solve

$$(\partial_t k_h^{m+1}, w_h) + c_\mu (\frac{k_h^{m2}}{\epsilon_h^m} \nabla k_h^{m+1}, \nabla w_h) = 0.$$

6. Solve

$$(\partial_t \epsilon_h^{m+1}, w_h) + c_\epsilon (\frac{k_h^{m2}}{\epsilon_h^m} \nabla \epsilon_h^{m+1}, \nabla w_h) = 0.$$

Remark. The discretisation of the time derivatives $\partial_t \theta_h^{m+1/2}$ resp. $\partial_t \varphi_h^{m+1/2}$ should use the values of θ_h^m and $\theta_h^{m+1/2}$ resp. φ_h^m and $\varphi_h^{m+1/2}$. The discretisation of the time derivatives $\partial_t k_h^{m+1}$ resp. $\partial_t \epsilon_h^{m+1}$ should use the values of $k_h^{m+1/2}$ and k_h^{m+1} resp. $\epsilon_h^{m+1/2}$ and ϵ_h^{m+1}. Nevertheless the size of δt, the discretisation parameter of time, should not be divided by two as the two step method might indicate.

3.4.2 Second method: A semi-implicit multi-step scheme in k and ϵ only

The previous method turned out to be sensitive to small errors in the solution of the Navier-Stokes system (e. g. due to too early interruption of the convergence process), because they appear in the denominator of the boundary conditions for θ. Therefore we keep the multistep idea but use it on a system in k and ϵ only.

This leads to the following algorithm.

Algorithm 3.7.

1. Solve

$$(\partial_t k_h^{m+1/2}, w_h) + (\mathbf{u} \cdot \nabla k_h^{m+1/2}, w_h) + (k_h^{m+1/2} \frac{\epsilon_h^m}{k_h^m}, w_h) = c_\mu (\frac{k_h^{m2}}{\epsilon_h^m} E_h^m, w_h).$$

2. Solve

$$(\partial_t \epsilon_h^{m+1/2}, w_h) + (\mathbf{u} \cdot \nabla \epsilon_h^{m+1/2}, w_h) + c_2 (\epsilon_h^{m+1/2} \frac{\epsilon_h^m}{k_h^m}, w_h) = c_1 (k_h^m E_h^m, w_h).$$

3. Solve

$$(\partial_t k_h^{m+1}, w_h) + c_\mu (\frac{k_h^{m2}}{\epsilon_h^m} \nabla k_h^{m+1}, \nabla w_h) = 0.$$

4. Solve

$$(\partial_t \epsilon_h^{m+1}, w_h) + c_\epsilon (\frac{k_h^{m2}}{\epsilon_h^m} \nabla \epsilon_h^{m+1}, \nabla w_h) = 0.$$

A critical problem now is the stabilisation of the convective parts. The streamline diffusion method used in the Navier-Stokes equations is not applicable, because it does not guarantee the positivity of k and ϵ. In [Pir89] and [MP94] the method of characteristics is used, but unfortunately its implementation is rather complicated. We use a method based on the same idea: *upwinding*.

3.4.3 Upwinding

The upwinding idea for finite volume methods is adapted to finite elements. We will apply it to conforming triangular elements in 2D, but the procedure can be extended easily to non-conforming elements or different types of polygons or polyhedra.

A secondary net is generated in the following way. In each triangle we connect the center of mass with the midpoints of the three edges by lines and get a box around each vertex as in Figure 3.2.

We now discretise a convective term $\mathbf{u} \nabla \psi$.

$$\begin{aligned}(\mathbf{u} \nabla \psi, w) &= \int_\Omega u_1 \partial_1 \psi w + u_2 \partial_2 \psi w \\ &= \int_\Omega \partial_1 (u_1 \psi) w + \partial_2 (u_2 \psi) w - \underbrace{\int_\Omega \psi w \nabla \cdot \mathbf{u}}_{=0}\end{aligned}$$

The testing function w is now 'lumped' on the boxes and partial integration is used.

$$\begin{aligned}\int_\Omega \partial_1 (u_1 \psi) w + \partial_2 (u_2 \psi) w &\approx \sum_i w(P_i) \int_{B_i} \nabla \cdot (\mathbf{u} \psi) \\ &= \sum_i w(P_i) \int_{\partial B_i} \psi \mathbf{u} \cdot \mathbf{n}\end{aligned}$$

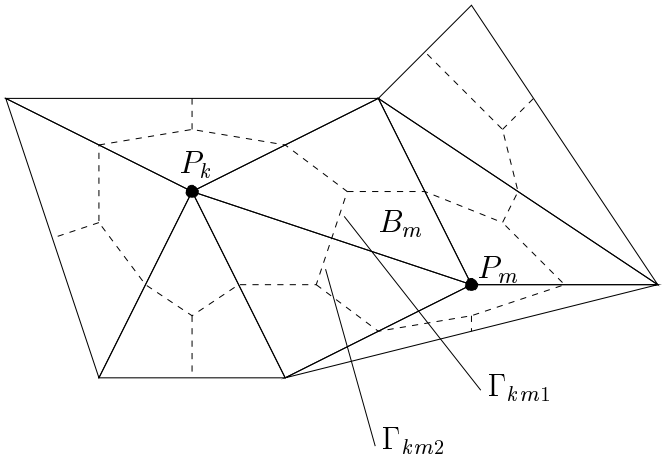

Figure 3.2: Part of a triangle-FEM net and its secondary net. B_m is the box around the vertex P_m.

As shown in Figure 3.2 we divide ∂B_m into segments Γ_{ml_ij}, where the l_i are the numbers of the neighbouring nodes of P_m and $j = 1, 2$. We proceed in a way that ψ is approximated with its value "against the flow":

$$\int_{\Gamma_{km1}} \psi \mathbf{u} \cdot \mathbf{n} \approx \psi_{km1} \int_{\Gamma_{km1}} \mathbf{u} \cdot \mathbf{n}_{mk1},$$

where

$$\psi_{km1} = \begin{cases} \psi(P_m) & \text{if } \int_{\Gamma_{km1}} \mathbf{u} \cdot \mathbf{n}_{mk1} \geq 0 \\ \psi(P_k) & \text{if } \int_{\Gamma_{km1}} \mathbf{u} \cdot \mathbf{n}_{mk1} < 0 \end{cases}$$

and $\mathbf{n}_{mk1}$ is the normal vector of Γ_{km1} directed from P_m to P_k.

The theoretical aspects of this method can be found in [Fei93] (for non-conforming elements) and [BBF90].

Using this technique the generated matrices have an important property.

Definition 3.8. A n-dimensional matrix $\mathbf{C} = (C_{ij})$ is called *M-matrix* iff

1. $C_{ii} > 0$ for all $i = 1, \ldots, n$ and
2. $C_{ij} \leq 0$ for all $i, j = 1, \ldots, n$, $i \neq j$ and
3. $\mathbf{C}^{-1}$ exists and $\mathbf{C}^{-1} \geq 0$ (component-wise).

Remark. The third property in Definition 3.8 is equivalent to the fact that $\mathbf{C}$ is inversely monotonous, i. e.

$$\mathbf{x} \leq \mathbf{y} \quad \Rightarrow \quad \mathbf{C}^{-1}\mathbf{x} \leq \mathbf{C}^{-1}\mathbf{y}$$

(component-wise). For $\mathbf{x} = \mathbf{0}$ we see that if our convection matrices were M-matrices, we could ensure positivity for φ and θ resp. k and ϵ for pure convection. [BBF90] show this property, when the centers of the circumcircles are used for the construction of the secondary net instaed of the centers of mass.

Lumping of the production and dissipation terms preserves the M-matrix property,[2] and because of a discrete maximum principle for diffusion terms (for constant ν_T) we can expect positivity for k and ϵ.

3.5 Approximation of the boundary layer

In some implementations the outermost layer of cells of the grid is taken as approximation of the physical boundary layer, which means, roughly speaking, that δ, denoting the width of the boundary layer, is set to the discretisation parameter,

$$\delta = h.$$

Sometimes the implementors distinct, whether $h(= \delta)$ corresponds to the viscous sub-layer or the turbulent wall-layer, sometimes they do not, which is particularly bad, because in that case an additional boundary condition for the continuous problem is implicitly assumed.

In this section we show which conditions these are. Even if they are sensible for the velocity, this does not change the fact that the model is not valid at small distances to the wall. We therefore choose δ dependent on the point of the boundary, but **independent** of the discretisation parameter.

3.5.1 No distinction of layer

We assume that δ corresponds to the turbulent wall layer.

Use of boundary condition (1.34)

We have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} &= 0, \\ \mathbf{u} \cdot \mathbf{s} &= -\kappa\delta \frac{\partial \mathbf{u} \cdot \mathbf{s}}{\partial \mathbf{n}} \left(\frac{1}{\kappa} \ln \left(\frac{\kappa\delta^2}{\nu} \left| \frac{\partial \mathbf{u} \cdot \mathbf{s}}{\partial \mathbf{n}} \right| \right) + \beta \right). \end{aligned}$$

For $\delta \to 0$ we get the boundary condition

$$\mathbf{u} = \mathbf{0},$$

[2] This is indeed sensible, because the production and dissipation operators are multiplicative and not differential operators.

(unless $\partial\mathbf{u}\cdot\mathbf{s}/\partial\mathbf{n} \to \infty$,) which seems reasonable at first sight. Nevertheless we get the following contradiction: Assume (w. l. o. g.) $\mathbf{u}\cdot\mathbf{s} > 0$ near the wall and $\partial\mathbf{u}\cdot\mathbf{s}/\partial\mathbf{n} \neq 0$. Thus $\partial\mathbf{u}\cdot\mathbf{s}/\partial\mathbf{n} > 0$, as for small δ the logarithm changes its sign and goes to $-\infty$. This contradicts the fact that $\mathbf{u} \to 0$. Consequently we would have to pose the additional boundary condition $\partial\mathbf{u}\cdot\mathbf{s}/\partial\mathbf{n} = 0$, which is one too much in general.

Use of boundary condition (1.35)

We have

$$\mathbf{u}\cdot\mathbf{s} = \frac{1}{\kappa}\sqrt{\nu\left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|}\underbrace{\ln\left(\delta\sqrt{\frac{1}{\nu}\left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|}\right)}_{\ln\sqrt{\frac{1}{\nu}}+\ln\delta+\frac{1}{2}\ln\left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|} - \beta\sqrt{\nu\left|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}\right|}.$$

If we want the boundary conditions not to diverge to infinity for $\delta \to 0$ we need to have the additional boundary condition

$$\frac{\partial\mathbf{u}}{\partial\mathbf{n}} = \mathbf{0}$$

and also get

$$\mathbf{u} = \mathbf{0},$$

which is again one condition too much.

3.5.2 Distinction of layers

For $\delta \to 0$ we arrive sooner or later at the viscous sub-layer. Here we could switch to the linear wall function (c. f. section 1.4.3) which leads to the sensible Dirichlet boundary condition

$$\mathbf{u} = \mathbf{0}.$$

From the mathematical point of view we get no problems here, but it has to be noticed again, that in the viscous sub-layer the k-ϵ model is no longer valid and the numerical results could contradict physical reality.

3.6 Solver for the linear systems

3.6.1 GMRES — a Krylov space method

In this section we describe the method we use to solve the arising linear systems of the form

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{(n,n)}$ is not necessarily symmetric or positive definite, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^n$, the *Generalised Minimal Residual Method*, *GMRES* for short. We will only provide a compact overview, details can be found in [Vos93].

In this section we use $\|.\|$ for the Euclidean norm.

Definition 3.9. The k-th Krylov space of $\mathbf{v}$ and $\mathbf{A}$, denoted by $\mathcal{K}_k(\mathbf{v}, \mathbf{A})$, is defined as

$$\mathcal{K}_k(\mathbf{v}, \mathbf{A}) := \text{span}\{\mathbf{v}, \mathbf{A}\mathbf{v}, \ldots, \mathbf{A}^{k-1}\mathbf{v}\}.$$

The method is based on the Arnoldi process, which produces an orthonormal basis $\{\mathbf{q}^1, \ldots, \mathbf{q}^k\}$ of $\mathcal{K}_k(\mathbf{q}^1, \mathbf{A})$ and a representation $\mathbf{H}_k \in \mathbb{R}^{(k,k)}$ of the orthogonal projection of $\mathbf{A}$ to $\mathcal{K}_k(\mathbf{q}^1, \mathbf{A})$.

The k-th iterate of our solution procedure $\mathbf{x}^k$ is constructed as

$$\mathbf{x}^k := \mathbf{x}^0 + \mathbf{Q}_k \mathbf{y}^k,$$

with $\mathbf{y}^k \in \mathbb{R}^k$ chosen such that

$$\|\mathbf{r}^k\| = \|\mathbf{b} - \mathbf{A}(\mathbf{x}^0 + \mathbf{Q}_k \mathbf{y}^k)\| = \min.$$

It can be shown that this minimum can be calculated with a QR-factorisation using Givens reflections.

GMRES is an exact method, which stops at the solution after finitely many steps, but the serious problem is its enormous memory consumption, if we have large linear systems because all the basis vectors $\mathbf{q}^i$ have to be stored. Thus there are two widely used possibilities of modification. One is to restart the GMRES method periodically after $m+1$ steps, the other to truncate the basis, i. e. to store only the previous m basis vectors $\mathbf{q}^k, \ldots, \mathbf{q}^{k-m+1}$. A new vector $\mathbf{q}^{k+1}$ in the Arnoldi process is then chosen to be orthogonal only to these m vectors. We use the second method, denoted as GMRES(m).

The following algorithm shows the full GMRES(m) method including a preconditioner $\mathbf{M}$. For the moment, let $\mathbf{M}$ equal $\mathbf{I}$, we will describe our choice of $\mathbf{M}$ in sections 3.6.2 and 3.6.3.

Algorithm 3.10. Preconditioned GMRES(m). Iterative Solution of $\mathbf{Ax} = \mathbf{b}$, with preconditioner $\mathbf{M}$.

Choose starting solution $\mathbf{x}^0$.
$\mathbf{q}^1 = \mathbf{M}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}^0)$
$\mathbf{z}^1 = \|\mathbf{q}^1\|$
$\mathbf{q}^1 = \mathbf{q}^1/\mathbf{z}^1$
repeat
  begin
    for $k = 1$ **to** m **do**
      begin
        $\mathbf{q}^{k+1} = \mathbf{M}^{-1}\mathbf{A}\mathbf{q}^k$

for $i = 1$ **to** k **do**
begin
$h_{ik} = (\mathbf{q}^i)^T \mathbf{q}^{k+1}$
$\mathbf{q}^{k+1} = \mathbf{q}^{k+1} - h_{ik}\mathbf{q}^i$
end
$h_{k+1,k} = \|\mathbf{q}^{k+1}\|$
$\mathbf{q}^{k+1} = \mathbf{q}^{k+1}/h_{k+1,k}$
end
for $k = 1$ **to** m **do**
begin
$cc = \sqrt{h_{kk}^2 + h_{k+1,k}^2}$
$c = h_{kk}/cc$
$s = h_{k+1,k}/cc$
$h_{kk} = cc$
for $i = k+1$ **to** m **do**
$$\begin{pmatrix} h_{k,i} \\ h_{k+1,i} \end{pmatrix} = \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \begin{pmatrix} h_{k,i} \\ h_{k+1,i} \end{pmatrix}$$
$$\begin{pmatrix} z_k \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \begin{pmatrix} z_k \\ 0 \end{pmatrix}$$
end
$y_m = z_m/h_{mm}$
for $i = m$ **downto** 1 **do**
$y_i = \left(z_i - \sum_{j=i+1}^{m} h_{ij} y_j\right) / h_{ii}$
$\mathbf{x}^m = \mathbf{x}^0 + \sum_{i=1}^{m} y_i \mathbf{q}^i$
$\mathbf{r}^m = \mathbf{M}^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}^m)$
$\mathbf{x}^0 = \mathbf{x}^m$, $\mathbf{r}^0 = \mathbf{r}^m$
$z_1 = \|\mathbf{r}^0\|$
$\mathbf{q}^1 = \mathbf{r}^0/z_1$
end
until $|z_1| <$ tolerance

3.6.2 Preconditioning

If we want to solve the linear system arising from the Navier-Stokes equations using the GMRES(m) method we face the following well known problem.

The convergence rate of an iterative solver depends strongly on the condition number of the matrix, which gets worse as the discretisation parameter gets small. Thus we use preconditioning, a well known technique preventing this, which can be found in any standard textbook on numerical solution of partial differential equations, e. g. [GR94]. Here we only give an informal motivation.

We solve

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b}$$

instead of solving the original system. If $\mathbf{M}$ is 'similar' to $\mathbf{A}$ then $\mathbf{M}^{-1}\mathbf{A}$ is 'similar' to $\mathbf{I}$. Thus if the inversion of $\mathbf{M}$ is computationally cheap, the solution procedure of the system should get much faster.

One possibility for preconditioning is to process some steps of an iterative method for a preconditioning matrix, e. g. $\mathbf{A}$ itself.

3.6.3 Algebraic multigrid

We use an algebraic multigrid method (AMG) as preconditioner for GMRES(m), but up to now as a black-box like tool. Thus, we wont dwell upon the details but introduce the idea roughly. The interested reader may be refered e. g. to [Hac85] and [RS87].

The multigrid idea in general is to smoothen the high frequencies of the error of a specific iterate on a fine grid and perform a correction step on a corser grid, utilising recursively the same idea again.

In standard multigrid methods (MGM) *a priori* a cascade of fine to coarse grids is known, and the computer program has to manage them all, which leads to an additional cost of memory.

To avoid this, AMG was developed. Here, only the finest grid is known, coarsening is carried out no longer on grid basis but on matrix (interaction) basis. Thus, it is very pleasant that we have used elements of equal order for the velocity and pressure approximation for the Navier-Stokes equations. Otherwise the fitting of the pressure nodes to the right velocity nodes after a coarsening step would have been a non-trivial task (actually there is no more grid information).

Chapter 4

Results and conclusions

4.1 Numerical results

Using the methods and techniques from the previous chapters we have developed a computer program (written in C++) based on Ferdinand Kickinger's mesh generator and AMG solver NAOMI ([LZK99]).

We will now report about the performance of the algorithm for some classical test problems with simple geometries. It is generally noticed that the simple iteration between the k-ϵ equations and the Navier-Stokes system in the numerical algorithm does not lead to satisfactory convergence for high Reynolds numbers.

Although we restrict ourselves to stationary problems for the moment, we solve a time-dependent system with (temporarily) constant boundary data. We start with small Reynolds numbers and gradually increase Re during the first time steps, as there are still problems with the solver for the linear systems for small ν

4.1.1 Flow between two flat plates

The computational domain is the rectangle $[0, 5] \times [0, 1]$. We use a Dirichlet boundary with parabolic velocity data for the inflow, the wall-law at the plates and 'do-nothing' at the outflow. The Reynolds number is set to 10000. After some experiments it turned out that 0.05 is a good choice for δ, the thickness of the wall layer, i. e. we checked *a posteriori* that all the velocities at the boundary correspond to the logarithmic layer.

k and ϵ at the inflow are set to small values, e. g. $k = 0.07$, $\epsilon = 0.02$, which appeared to be the order of magnitude of stationary solutions for low Reynolds numbers far away from the inflow boundary (independent of the inflow conditions).

Physical experiments show that the parabolic inflow velocity profile should flatten in case of turbulent flow (it stays parabolic in the laminar case). This is also what our results show (Figure 4.1).

From the results of k and ϵ in Figure 4.2 we can infer that turbulence is stronger near the walls.

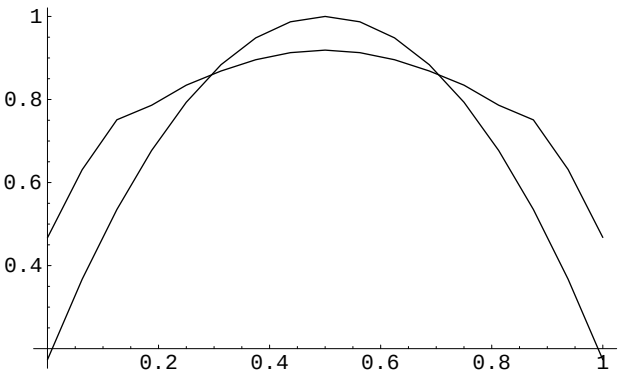

Figure 4.1: Profile of velocity of flow between two flat plates at $Re = 10000$ compared to laminar flow (parabolic). (The blip in the turbulent profile is due to a postprocessing problem)

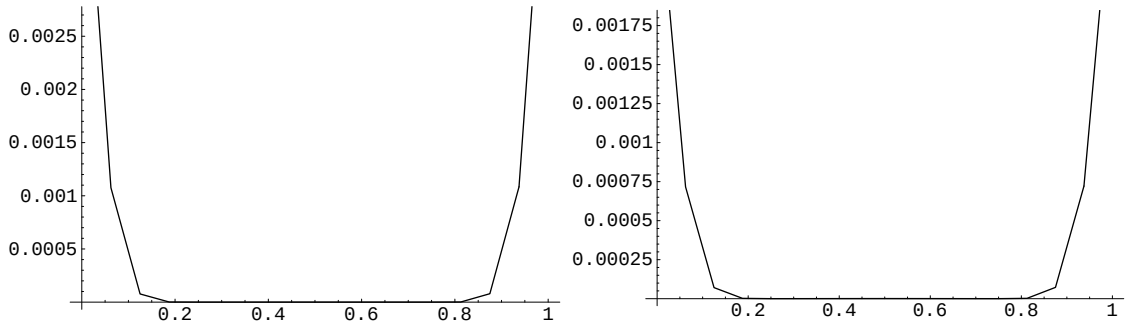

Figure 4.2: Profile of k and ϵ of flow between two flat plates at $Re = 10000$.

When experimenting in this case we also noticed the following effect. The pressure, which should decrease linearly in x-direction, looses this behaviour near the boundary and gets flatter (see Figure 4.3). We think that this is a result of the stabilising part of our system, where we actually add Δp to the conservation of mass. Now the natural boundary conditions for the Laplacian are homogeneous Neumann conditions, violating the linear behaviour of the pressure. This hypothesis is also supported by the fact, that this phenomenon disappears for finer grids (where the influence of the stabilisation term decreases quadratically in the discretisation parameter).

4.1.2 Driven cavity

Consider a fluid in a square, which is driven by flow from left to right at $Re = 550$ along the top boundary. k and ϵ are both set to 10^{-5}. (The problem of prescribing values for k and ϵ could be avoided by using a wall-law at the top boundary also.)

We compare Stokes flow with homogeneous Dirichlet conditions at the remaining walls with according Navier-Stokes flow, Navier-Stokes flow with the wall-law and the solution

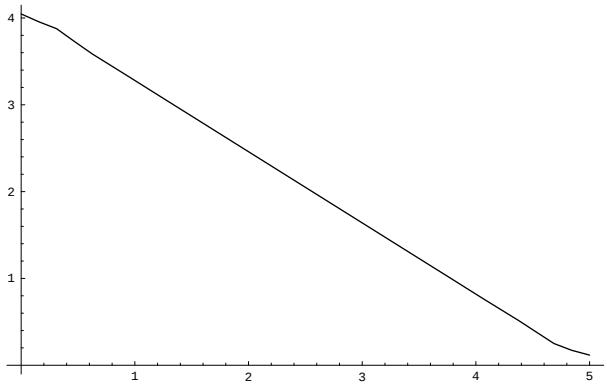

Figure 4.3: The pressure changes its linear behaviour near the boundaries. (Stokes case without turbulence model).

of the complete k-ϵ system (with wall-law), all shown in Figures 4.4 to 4.7.

Again we assume constant width ($\delta = 0.05$) of the boundary layer. This is not quite justified because of a wide range of velocities at this artificial boundary, and consequently some cells near the bottom corners are in the viscous sub-layer. However turbulence is weak in the bottom part of the cavity and errors in k and ϵ (due to the incorrect approximation of the boundary layer) are not noticable in the momentum equations.

The eddy viscosity ν_T shown in Figure 4.10 is considerably larger than $\nu \approx 0.0018$ and the effect is a strong exchange of impulse across the stream lines, which seems to be overestimated. This could be due to a general problem with eddy viscosity models, which are based on the assumption that turbulence occurs in regions with high velocity gradients. This is generally, but not always true. In particular there appear large gradients at the top of the cavity for Navier-Stokes flow, but nevertheless the motion is still laminar (at least in the middle). This leads to high production of turbulent kinetic energy yielding strong turbulent diffusion.

4.2 Conclusions and prospects

Contrary to engineering practice, where often the continuous model and the discretisation thereof get mixed up, which generally is not consistent for $h \to 0$, we have shown that it is possible to use a model, which is in agreement with the physical background, derive mathematical theory and finally produce encouraging numerical results. In particular this approach leads to convergence of the solutions of the discretised problems (which is confirmed by our numerical tests) for $h \to 0$.

Sure, there is still much to be done:

- **Physical modelling:** The k-ϵ model has several know weaknesses (see [MP94]), but still it is widely used, because in many cases it produces reasonable results. Of course there are more involved models, as for example the Reynolds stress models, but the question is wether the considerable increase of complexity is justified.

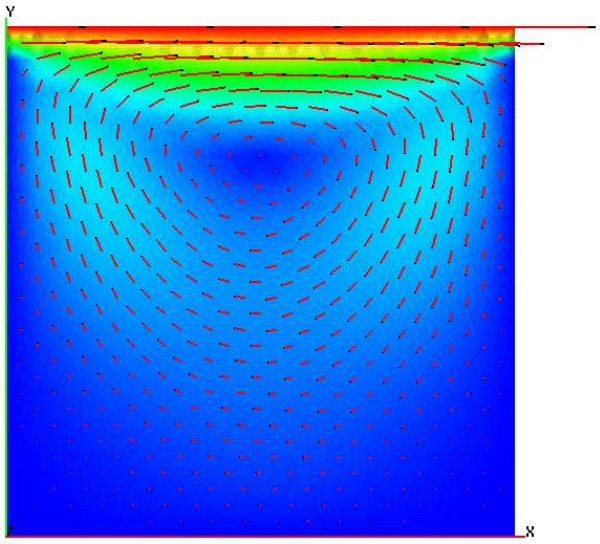

Figure 4.4: Stokes flow in a driven cavity.

Still we are not sure, which handling of the wall-law is the best. Maybe a 'better' wall-law would lead to more accurate results.

- **Analysis:** It seems that no stronger results can be shown for the stationary Navier-Stokes system with the techniques presented in chapter 2 in a context as general as here. Maybe some advance could be made for the non-stationary equations.

  An extensive analysis of the k-ϵ system is still an open problem, but we are not very optimistic about much progress in this field in the near future.

- **Numerics:** In a next step the program should support three dimensional geometries. The mesh generator is already capable of this, therefore we will have to deal with an adaption of the matrix generation to this problem, which is more a matter of technical details than of fundamental numerical principles.

  Besides, we do not know how the techniques work for more complex geometries in connection with high Reynolds numbers, which are important for practical applications. Maybe a 'tuning' of the solver (GMRES with AMG preconditioning) will get essential in this case.

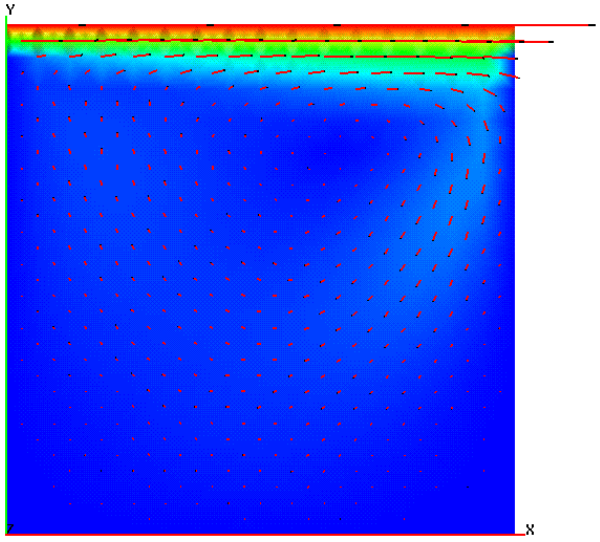

Figure 4.5: Navier-Stokes flow at $Re = 550$ with Dirichlet conditions at the boundary.

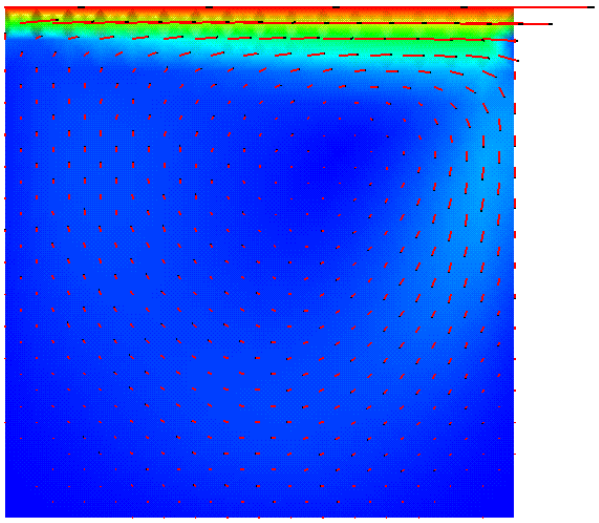

Figure 4.6: Navier-Stokes flow at $Re = 550$ with a wall-law at the solid boundary.

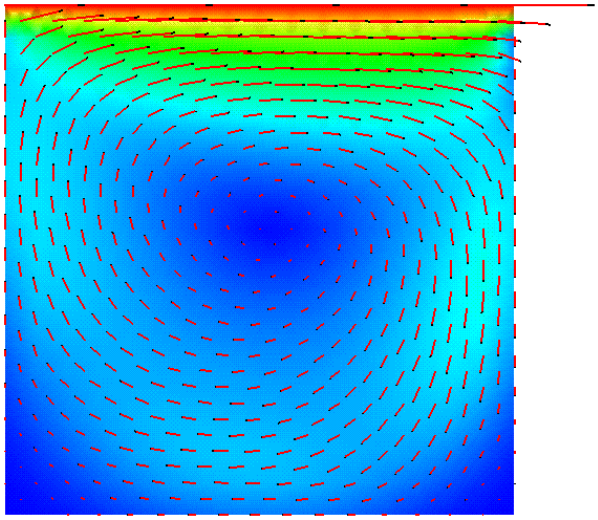

Figure 4.7: k-ϵ-solution at $Re = 550$ with wall-law.

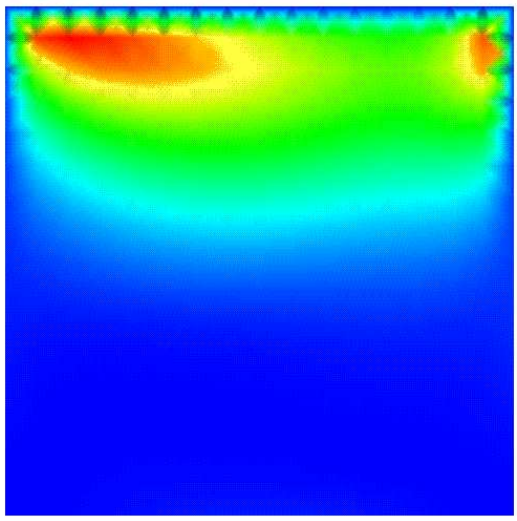

Figure 4.8: Turbulent kinetic energy k.

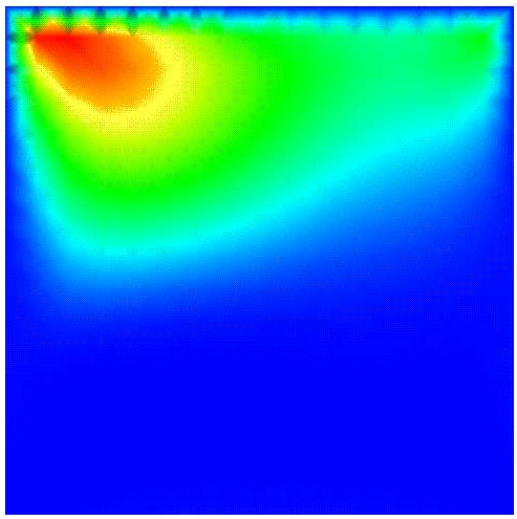

Figure 4.9: Dissipation rate ϵ.

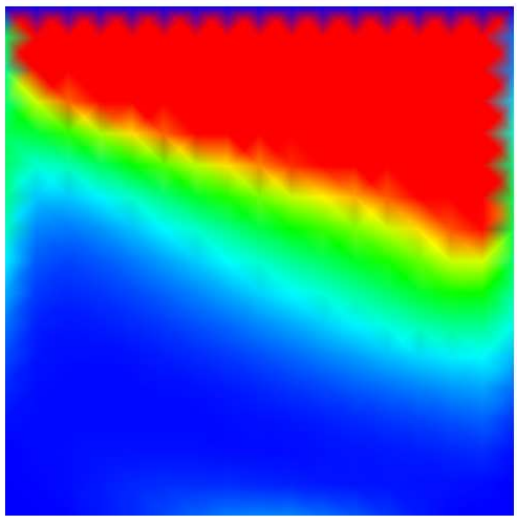

Figure 4.10: Turbulent diffusion ν_T.

Appendix A

Properties of filters

Some of the properties are obvious for some filters, some are by far not trivial or just not fulfilled.

	(a)	(b)	(c)	(d)
1.	X	X	O	O
2.	X	X	O	O
3.	X	X	O	O
4.	X	X	X	O
5.	X	X	X	X

Proof. 1. (a) trivial

(b) We first assume $u \in C^1$. Then

$$\partial_t \langle u \rangle_B = \partial_t \frac{1}{|B|} \int_{B(\mathbf{x},r)} u(\mathbf{y},t)\, \mathrm{d}\mathbf{y} = \frac{1}{|B|} \int_{B(\mathbf{x},r)} \partial_t u(\mathbf{y},t)\, \mathrm{d}\mathbf{y} = \langle \partial_t u \rangle_B$$

and for fixed $x_0 \in \mathcal{G}$

$$\begin{aligned} \nabla \langle u \rangle_B &= \nabla \frac{1}{|B|} \int_{B(\mathbf{x},r)} u(\mathbf{y},t)\, \mathrm{d}\mathbf{y} = \nabla \frac{1}{|B|} \int_{B(\mathbf{x}_0,r)} u(\mathbf{x}-\mathbf{x}_0+\mathbf{y},t)\, \mathrm{d}\mathbf{y} \\ &= \frac{1}{|B|} \int_{B(\mathbf{x}_0,r)} \nabla u(\mathbf{x}-\mathbf{x}_0+\mathbf{y},t)\, \mathrm{d}\mathbf{y} = \frac{1}{|B|} \int_{B(\mathbf{x}_0,r)} \nabla u(\mathbf{y},t)\, \mathrm{d}\mathbf{y}. \end{aligned}$$

(c) Counterexample: $u(\mathbf{x},t) = |\mathbf{x}|^2$. With $|\mathbf{y}|^2 = |\mathbf{x}-\mathbf{y}|^2 - |\mathbf{x}|^2 + 2(\mathbf{x},\mathbf{y})$ and $m_2 := \frac{1}{|B|}\int_{B(0,r)} |\mathbf{y}|^2\, \mathrm{d}\mathbf{y} > 0$

$$\left\langle |x|^2 \right\rangle_B = \frac{1}{|B|} \int_{B(\mathbf{x},r)} |\mathbf{y}|^2\, \mathrm{d}\mathbf{y} = m_2 - |\mathbf{x}|^2 + \underbrace{\frac{2}{|B|} \int_{B(\mathbf{x},r)} (\mathbf{x},\mathbf{y})\, \mathrm{d}\mathbf{y}}_{2|\mathbf{x}|^2} = m_2 + |\mathbf{x}|^2.$$

Hence

$$\left\langle \left\langle |\mathbf{x}|^2 \right\rangle_B \right\rangle_B = m_2 + \left\langle |\mathbf{x}|^2 \right\rangle_B \neq \left\langle |\mathbf{x}|^2 \right\rangle_B .$$

(d) see (c)

2. (a) trivial

(b) Again assume $u \in C^1$. Obviously

$$\partial_t \langle u \rangle_T = \partial_t \frac{1}{T_1} \int_{t-T_1}^{t} u(\mathbf{x}, \tau)\, \mathrm{d}\tau = \frac{1}{T_1} \int_{t-T_1}^{t} \partial_t u(\mathbf{x}, \tau) \mathrm{d}\tau = \langle \partial_t u \rangle_T$$

and (Euler)

$$\nabla \langle u \rangle_T = \nabla \frac{1}{T_1} \int_{t-T_1}^{t} u(\mathbf{x}, \tau)\, \mathrm{d}\tau = \frac{1}{T_1} \int_{t-T_1}^{t} \nabla u(\mathbf{x}, \tau)\, \mathrm{d}\tau = \langle \nabla u \rangle_T .$$

(c) Counterexample: $u(x,t) = 3t^2$.

$$\left\langle 3t^2 \right\rangle_T = \frac{1}{T_1} \int_{t-T_1}^{t} 3\tau^2 \mathrm{d}\, \tau = 3t^2 T_1 - 3tT_1^2 + T_1^3.$$

Hence

$$\left\langle \left\langle 3t^2 \right\rangle_T \right\rangle_T = \left\langle 3t^2 \right\rangle_T - 3tT_1 + T_1^2 \neq \left\langle 3t^2 \right\rangle .$$

(d) see (c)

3. cf. (c) and (d)

4. (a) trivial

(b) For sufficiently smooth $u = \sum_{\mathbf{k}} u_{\mathbf{k}}(t) e^{i(\mathbf{k},\mathbf{x})}$

$$\partial_t \langle u \rangle_F = \partial_t \sum_{|\mathbf{k}| \leq N} u_{\mathbf{k}}(t) e^{i(\mathbf{k},\mathbf{x})} = \sum_{|\mathbf{k}| \leq N} \partial_t u_{\mathbf{k}}(t) e^{i(\mathbf{k},\mathbf{x})} = \langle \partial_t u \rangle_F$$

and

$$\nabla \langle u \rangle_F = \nabla \sum_{|\mathbf{k}| \leq N} u_{\mathbf{k}}(t) e^{i(\mathbf{k},\mathbf{x})} = \sum_{|\mathbf{k}| \leq N} u_{\mathbf{k}}(t) i\mathbf{k} e^{i(\mathbf{k},\mathbf{x})} = \langle \nabla u \rangle_F .$$

(c) Clear:

$$\langle \langle u \rangle_F \rangle_F = \left\langle \sum_{|\mathbf{k}| \leq N} u_{\mathbf{k}}(t) e^{i(\mathbf{k},\mathbf{x})} \right\rangle_F = \sum_{|k| \leq N} u_k(t) e^{i(k,x)} = \langle u \rangle_F .$$

(d) For $u = \sum_k u_k e^{i(k,x)}$, $v = \sum_k v_k e^{i(k,x)}$ in general $\langle u \rangle_F \langle v \rangle_F$ contains terms up to $|k| = 2N$, whereas $\langle u \langle v \rangle_F \rangle_F$ only has terms with $|k| \leq N$.

5. Unfortunately we are not able to contribute to this question.

□

Bibliography

[BBF90] Randolph E. Bank, Josef F. Bürgler, and Wolfgang Fichtner. Some upwinding techniques for finite element approximations of convection-diffusion equations. *Numerische Mathematik*, 58:185–202, 1990.

[BF91] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods.* Springer-Verlag, New York, 1991.

[DL76] G. Duvaut and J.L. Lions. *Inequalities in mechanics and physics*, volume 219 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, 1976. Translated from the French by C.W. John.

[Fei93] Miloslav Feistauer. *Mathematical Methods in Fluid Dynamics.* Number 67 in Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman Scientific & Technical, 1993.

[FLB93] Sixin Fan, Budugur Lakshminarayana, and Mark Barnett. Low-Reynolds-number k-ϵ model for unsteady turbulent boundary-layer flows. *AIAA Journal*, 31(10):1777–1784, October 1993.

[FS91] Leopoldo P. Franca and Rolf Stenberg. Error analysis of some Galerkin least squares methods for the elasticity equations. *SIAM J. Numer. Anal.*, 28(6):1680–1697, December 1991.

[GR86] Vivette Girault and Pierre-Arnaud Raviart. *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms.* Springer Series in Computational Mathematics. Springer, Berlin Heidelberg, 1986.

[GR94] Christian Großmann and Hans-Jörg Roos. *Numerik partieller Differentialgleichungen.* Teubner, Stuttgart, 1994.

[Hac85] W. Hackbusch. *Multigrid Methods and Applications.* Springer-Verlag, Berlin, 1985.

[HFB86] T.J.R Hughes, L.P. Franca, and M. Balestra. A new finite element formulation for fluid dynamics: V. cirumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the Stokes problem acommodating equal-order interpolations. *Comput. Methods Appl. Mech. Engrg.*, 59:85–99, 1986.

[HRT94] John G. Heywood, Rolf Rannacher, and Stefan Turek. Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. Technical Report SFB 359, Interdisziplinäres Zentrum für Wissenschaftliches Rechnen der Universität Heidelberg, 1994.

[LS72] B. E. Launder and D. B. Spalding. *Mathematical models of turbulence.* Academic press, 1972.

[LZK99] Ulrich Langer, Walter Zulehner, and Ferdinand Kickinger. FFF-Projekt Autogen. HTML-Document, 1999. Location in world-wide-web: `www.numa.uni-linz.ac.at/Research/projects.html` → `Project AUTOGEN`.

[MP94] B. Mohammadi and O. Pironneau. *Analysis of the K-Epsilon TURBULENCE MODEL.* John Wiley & Sons, Chichester, 1994.

[Par92] Carlos Parés. Existence, uniqueness and regularity of solution of the equations of a turbulence model for incompressible fluids. *Applicable Analysis*, 43:245–296, 1992.

[Pir89] Olivier Pironneau. *Finite Element Methods for Fluids.* John Wiley & Sons, 1989.

[RS87] J. W. Ruge and K. Stüben. Algebraic multigrid (AMG). In S. F. McCormick, editor, *Multigrid Methods*, volume 3 of *Frontiers in Applied Mathematics*, pages 73–130. SIAM, Philadelphia, 1987.

[SG97] Hermann Schlichting and Klaus Gersten. *Grenzschicht-Theorie.* Springer, Berlin, Heidelberg, 9$^{\text{th}}$ edition, 1997.

[Tru80] Erich Truckenbrodt. *Fluidmechanik. Bd. 1, Grundlagen und elementare Strömungsvorgänge dichtebeständiger Fluide.* Springer Verlag, 1980.

[Vos93] Heinrich Voss. Iterative methods for linear systems of equations. Textbook of the 3$^{\text{rd}}$ International Summerschool in Jyväskylä, Finland, Institute of Applied Mathematics, University of Hamburg, 1993.

[Zei77] E. Zeidler. *Vorlesungen über nichtlineare Funktionalanalysis II: Monotone Operatoren*, volume 9 of *Teubner-Texte zur Mathematik.* B. G. Teubner, Leipzig, 1977.