Polygonal and polyhedral meshes

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Polygonal and polyhedral meshes

- $\Omega \subset \mathbb{R}^d, d = 2, 3 \cdots$ bounded polygonal/polyhedral domain
- $\Gamma = \Gamma_D \cup \Gamma_N \cdots$ boundary consisting of Dirichlet/Neumann components
- **WANT** discretization $\mathcal{K}_h$ of Ω
  - allow meshes with general elements
  - varying numbers of (possibly hanging) nodes

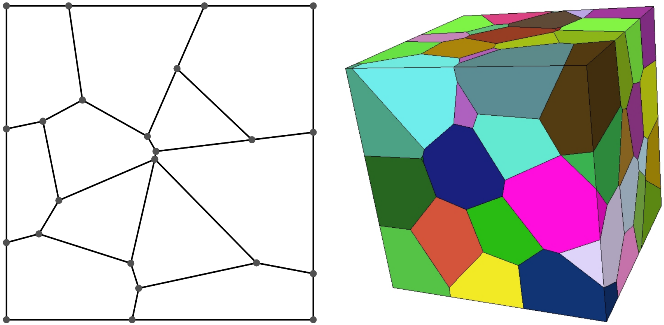


Figure: Two examples for meshes with polygonal and polyhedral elements.

- Elements $K \in \mathcal{K}_h$ are non-overlapping open sets

$$\overline{\Omega} = \bigcup_{K \in \mathcal{K}_h} \overline{K}$$

- Elements consist of **nodes**, **edges** (and **faces** in 3D)
- Edge $E = \overline{\boldsymbol{z}_b \boldsymbol{z}_e}$ located between nodes $\boldsymbol{z}_b$ (beginning) and $\boldsymbol{z}_e$ (end). **Ordering** fixed once per edge. No other nodes allowed on E.

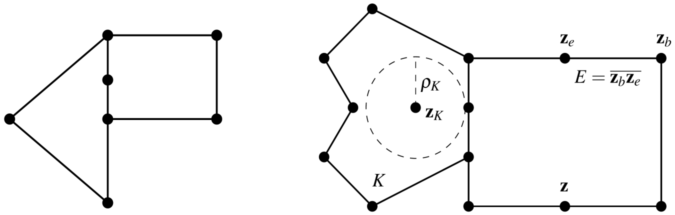


Figure: Two examples of neighbouring elements with additional nodes on the straight boundary.

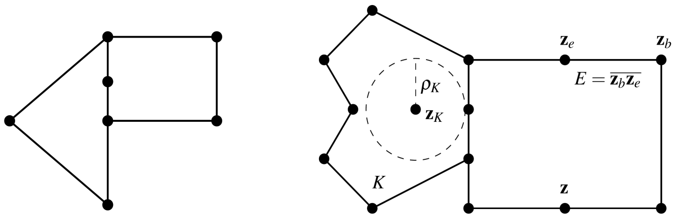


Figure: Two examples of neighbouring elements with additional nodes on the straight boundary.

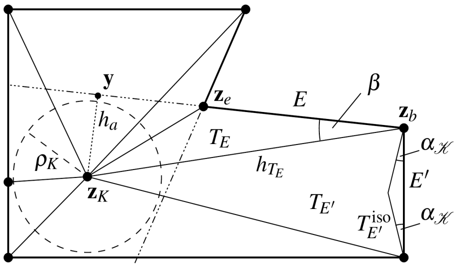
- In 2D there could also be some hanging **nodes on straight lines** of the polygonal boundary ∂K.
- Additional nodes **enrich the approximation space** in FEM.
- *Triangular/Quadrilateral meshes for classical FEM*: Hanging nodes do not influence the accuracy of the approximation.

- In 3D, hanging **nodes appear on edges of elements** and one may have **hanging edges on faces**.

Useful notation

$\mathcal{N}_h$	set of all nodes in mesh $\mathcal{K}_h$
$\mathcal{N}_h = \mathcal{N}_{h,\Omega} \cup \mathcal{N}_{h,D} \cup \mathcal{N}_{h,N}$	set of interiour, Dirichlet (with Dirichlet/Neumann transition nodes) and Neumann nodes
$\mathcal{E}_h$	edges of the mesh
$\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,D} \cup \mathcal{E}_{h,N}$	set of interior, Dirichlet and Neumann edges
$\mathcal{F}_h = \mathcal{F}_{h,\Omega} \cup \mathcal{F}_{h,D} \cup \mathcal{F}_{h,N}$	sets of interiour, Dirichlet and Neumann faces
$\mathcal{N}(K), \mathcal{N}(E), \mathcal{N}(F)$	sets of nodes belonging to element K, edge E or face F
$\mathcal{E}(K), \mathcal{E}(F)$	sets of edges of element K, face F
$\mathcal{F}(K)$	set of faces of element K

Mesh regularity and properties in 2D

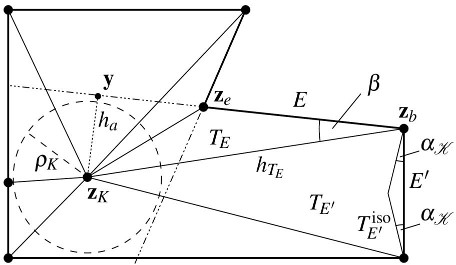


- $h_E \cdots$ length of edge E
- $h_K = diam(K) \cdots$ diameter of element K (wlog $h_K < 1$)

Definition (**REGULARITY**)

The family of meshes $\mathcal{K} := (\mathcal{K}_h)_h$ is called regular iff

1. Each $K \in \mathcal{K}_h$ is a **star-shaped polygon** with respect to a circle of radius ρ_K and midpoint $\boldsymbol{z}_K$.
2. The **aspect ratio is uniformly bounded from above** by $\sigma_{\mathcal{K}}$, ie. $h_k/\rho_K < \sigma_{\mathcal{K}}$ for all $K \in \mathcal{K}_h$.

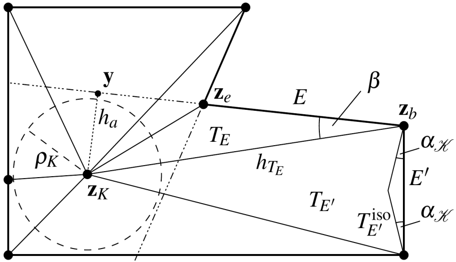


- $h := \max\{h_K : K \in \mathcal{K}_h\} \cdots$ mesh size
- Diameter of element comparable to the length of its shortest edge?

Definition (**STABILITY**)

The family of meshes $\mathcal{K}$ is called stable iff

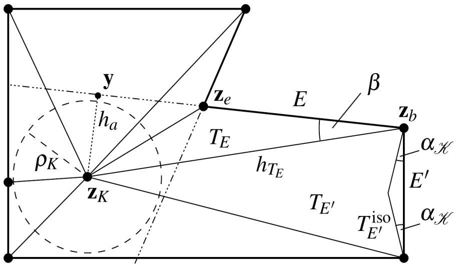
$$\exists c_{\mathcal{K}} > 0 \forall K \in \mathcal{K}_h : \qquad E \in \mathcal{E}(K) \implies h_K \leq c_{\mathcal{K}} h_E$$



- Introduce **auxiliary triangulation** $\mathcal{T}_h(K)$ of element K by connecting $\mathcal{N}(K)$ with $\boldsymbol{z}_K$.
- $\mathcal{T}_h(K)$ consists of triangles T_E for $E = \overline{\boldsymbol{z}_b \boldsymbol{z}_e} \in \mathcal{E}(K)$
- $\mathcal{T} := (\mathcal{T}_h(\mathcal{K}_h))_h, \mathcal{T}_h(\mathcal{K}_h) := \cup_{K \in \mathcal{K}_h} \mathcal{T}_h(K)$

Lemma

Let K be a polygonal element of a **regular and stable mesh** $\mathcal{K}_h$*. Then the auxiliary triangulation* $\mathcal{T}_h(K)$ *is* **shape-regular** *and the aspect ratio of each triangle is uniformly bounded by some constant* $\sigma_{\mathcal{T}}$*, which only depends on* $\sigma_{\mathcal{K}}$ *and* $c_{\mathcal{K}}$*.*



Lemma

Let $\mathcal{K}_h$ *be a* **regular polygonal mesh**. *Then one has*

$$\exists \alpha_{\mathcal{K}} \in (0, \pi/3] \forall K \in \mathcal{K}_h : E \in \mathcal{E}(K) \implies T_E^{iso} \subset T_E \in \mathcal{T}_h(K)$$

- Isosceles triangle (= gleichschenkliges Dreieck) T_E^{iso} with longest side E and two interior angles $\alpha_{\mathcal{K}}$
- The isosceles triangles and the auxiliary triangulation play an important role in the **analysis of error estimates**.

Corollary

Let $\mathcal{K}_h$ be a **regular mesh***. Then*
every node belongs to a uniformly bounded number of elements.

$$\exists c = c(\sigma_{\mathcal{K}}) > 0 \forall \boldsymbol{z} \in \mathcal{N}_{\boldsymbol{h}} : |\{K \in \mathcal{K}_h : \boldsymbol{z} \in \mathcal{N}(K)\}| \leq c$$

Lemma

Let $\mathcal{K}_h$ be **regular and stable mesh***. Then*
every element contains a uniformly bounded number of nodes and edges.

$$\exists c = c(\sigma_{\mathcal{K}}, c_{\mathcal{K}}) > 0 \forall K \in \mathcal{K}_h : |\mathcal{N}(K)| = |\mathcal{E}(K)| \leq c$$

Corollary

Let $\mathcal{K}_h$ be a **regular mesh** *and let $K \in \mathcal{K}_h$. Then the auxiliary triangulations $\mathcal{T}_h(K), \mathcal{T}_h(\mathcal{K}_h)$ fulfil a* **maximum angle condition**, *ie. every angle is uniformly bounded from above by a constant which is trictly less than π. The maximum angle only depends on $\sigma_{\mathcal{K}}$.*

- Therefore, several **approximation properties of finite element interpolation** for linear as well as for higher order basis functions are valid on this discretization. See next Seminar 03.

Mesh regularity and properties in 3D

Definition (REGULAR FACES)

A family of faces $\mathcal{F} := (\mathcal{F}_h)_h$ is called regular iff **all faces are flat <u>regular</u> polygons** with inscribed circle of radius ρ_F, midpoint $\boldsymbol{z}_F$ and with **uniformly bounded regularity parameter** $\sigma_{\mathcal{F}}$.

Definition (REGULARITY)

The family of meshes $\mathcal{K} := (\mathcal{K}_h)_h$ is called regular iff

1. The associated set of faces $\mathcal{F}$ is regular.
2. Each $K \in \mathcal{K}_h$ is a **star-shaped polyhedron** with respect to a ball of radius ρ_K and midpoint $\boldsymbol{z}_K$.
3. The **aspect ratio is uniformly bounded from above** by $\sigma_{\mathcal{K}}$, ie. $h_k/\rho_K < \sigma_{\mathcal{K}}$ for all $K \in \mathcal{K}_h$.

- $h := \max\{h_K : K \in \mathcal{K}_h\} \cdots$ mesh size

Definition (STABILITY)

The family of meshes $\mathcal{K}$ is called stable iff

$$\exists c_{\mathcal{K}} > 0 \forall K \in \mathcal{K}_h : \qquad E \in \mathcal{E}(K) \implies h_K \leq c_{\mathcal{K}} h_E$$

- One hase **intercomparability** for $K \in \mathcal{K}_h, F \in \mathcal{F}(K), E \in \mathcal{E}(K)$

$$h_E \leq h_F \leq h_K \leq c_{\mathcal{K}} h_E \leq c_{\mathcal{K}} h_F.$$

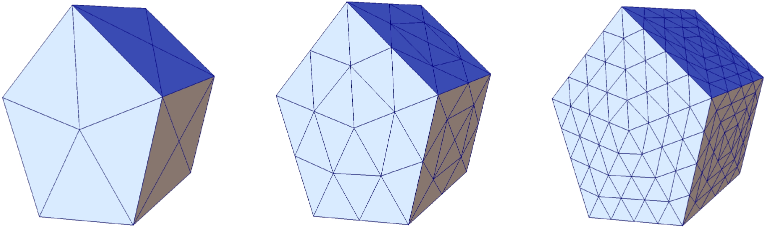


Figure: Polyhedral element with surface triangulations of level $l = 0, 1, 2$.

- Introduce **auxiliary triangulation** $\mathcal{T}_0(F)$ of face $F \in \mathcal{F}_h$
- Introduce family $(\mathcal{T}_l(F))_{l\in\mathbb{N}_0}$ of triangulations, where meshes of level $l \geq 1$ are defined **recursively by splitting each triangle of the previous level into four similar triangles**.
- The set of nodes in the $\mathcal{T}_l(F)$ is denoted by $\mathcal{M}_l(F)$

- Introduce **conforming surface mesh** of element K

$$\mathcal{T}_l(\partial K) = \bigcup_{F \in \mathcal{F}(K)} \mathcal{T}_l(F) \qquad \mathcal{M}_l(\partial K) = \bigcup_{F \in \mathcal{F}(K)} \mathcal{M}_h(F).$$

- Construct **auxiliary tetrahedral mesh** $\mathcal{T}_l(K)$ by connecting $\mathcal{M}_l(\partial K)$ with $\boldsymbol{z}_K$.
- Can **mesh regularity/stability of polyhedral mesh** be controlled?

Lemma

Let K be a polyhedral element of a **regular and stable mesh** $\mathcal{K}_h$*. Then the auxiliary discretizations* $\mathcal{T}_l(\mathcal{K}_h) := \cup_{K \in \mathcal{K}_h} \mathcal{T}_l(K), \mathcal{T} := (\mathcal{T}_l(\mathcal{K}_h))_{h,l}$ *are* **shape-regular** *and the aspect ratio of each tetrahedra is uniformly bounded by some constant* $\sigma_{\mathcal{T}}$*, which only depends on* $\sigma_{\mathcal{K}}, \sigma_{\mathcal{F}}, c_{\mathcal{K}}$ *and the mesh level* l*.*

Corollary

Let $\mathcal{K}_h$ *be a* **regular AND STABLE mesh**. *Then every node belongs to a uniformly bounded number of elements.*

$$\exists c = c(\sigma_{\mathcal{K}}, \sigma_{\mathcal{F}}, c_{\mathcal{K}}) > 0 \forall \boldsymbol{z} \in \mathcal{N}_h : |\{K \in \mathcal{K}_h : \boldsymbol{z} \in \mathcal{N}(K)\}| \leq c$$

Lemma

Let $\mathcal{K}_h$ *be* **regular and stable mesh**. *Then every element contains a uniformly bounded number of nodes, faces and edges.*

$$\exists c = c(\sigma_{\mathcal{K}}, \sigma_{\mathcal{F}}, c_{\mathcal{K}}) > 0 \forall K \in \mathcal{K}_h : |\mathcal{N}(K)|, |\mathcal{E}(K)|, |\mathcal{F}(K)| \leq c$$

Definition (**WEAK STABILITY**)

The family of meshes $\mathcal{K}_h$ is called weakly stable iff

$$\exists c_{\mathcal{F}} > 0 \forall F \in \mathcal{F}_h : E \in \mathcal{E}(F) \implies h_F \leq c_{\mathcal{F}} h_E.$$

Corollary

Let $\mathcal{K}_h$ be a **regular <u>AND weakly stable</u> mesh** *and let $K \in \mathcal{K}_h$. Then the auxiliary triangulations $\mathcal{T}_l(K), \mathcal{T}_l(\mathcal{K}_h)$ fulfil a* **maximum angle condition**, *ie. all dihedral angles between faces and all angles within a triangular face are uniformly bounded from above by a constant which is strictly less than π. The maximum angle only depends on $\sigma_{\mathcal{K}}, \sigma_{\mathcal{F}}, c_{\mathcal{F}}$ and the mesh level l.*

Mesh refinement

- uniform refinement· · · all elements are refined
- adaptive refinement· · · some elements are refined (criterion)

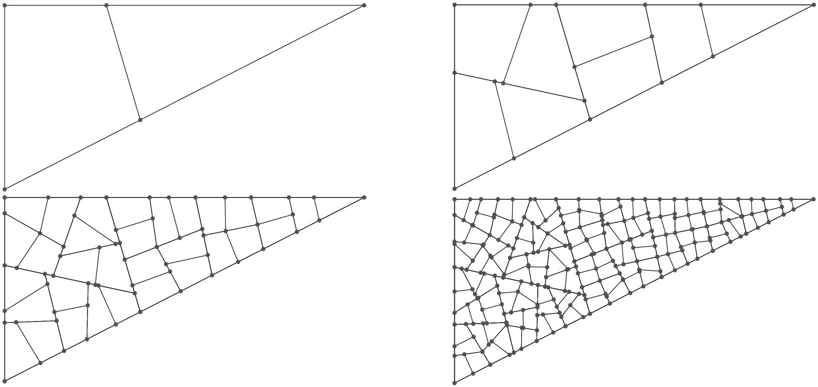


Figure: Uniform refinement of a triangle after one, three, five and seven refinement steps.

How to refine a polygonal/polyhedral element?

- For the refinement process, we choose the **bisection of elements**.
- This works for any dimension.
- **ASSUMPTION for the analysis**: elements K are convex.
- Non-convex/star-shaped elements "only yield technical difficulties".
- Introduce **covariance matrix**

$$M_{Cov}(K) = \frac{1}{|K|} \int_K (\boldsymbol{x} - \overline{\boldsymbol{x}}_K)(\boldsymbol{x} - \overline{\boldsymbol{x}}_K)^\top d\boldsymbol{x}, \qquad \overline{\boldsymbol{x}}_K = \frac{1}{|K|} \int_K \boldsymbol{x} d\boldsymbol{x}$$

- $M_{Cov}(K) \in \mathbb{R}^{d \times d}$ is **symmetric and positive definite**.

- The square root of the eigenvalues give the standard deviation in the direction of the corresponding eigenvector.
- Thus, the eigenvector which belongs to the **biggest eigenvalue** points into the direction of the **longest extend** of the element K.
- Consequently, we **split the element orthogonal to the biggest eigenvector** through the barycenter $\overline{\boldsymbol{x}}_K$.

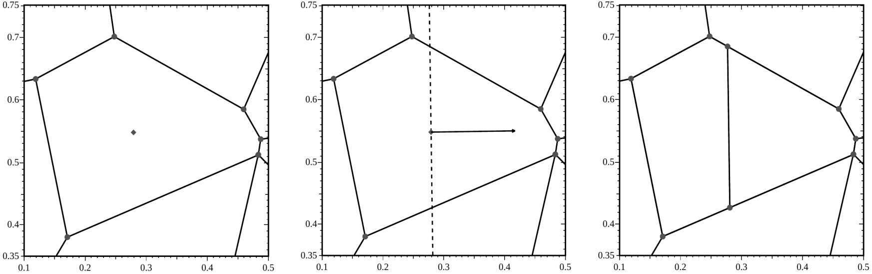


Figure: Refinement of an element: element with center $\overline{\boldsymbol{x}}_K$ (left), element with eigenvector (middle), two new elements (right).

- **OBSERVE:** Even a refinement of a triangle results in an unstructured polygonal mesh.
- For the **bisection** it is possible to perform **local refinements within a few elements**.
- Bisection preserves uniformly bounded aspect ratio for convex elements
- Bisection preserves regularity for convex elements in 2D
- Stability usually lost (small edges/thin faces might occur)

- Classical mesh refinement techniques: **local refinement propagates into neighbouring regions** (to keep admissibility)

Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

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Table

Treatments	Response 1	Response 2
Treatment 1	0.0003262	0.562
Treatment 2	0.0015681	0.910
Treatment 3	0.0009271	0.296

Table: Table caption