# Newest vertex bisection method

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# Introduction

- Adaptive FEM for elliptic equations $\Rightarrow$ Convergence shown by Dörfler and Morin, Nochetto, Siebert
- No rates of convergence in terms of degrees of freedom or number of computations

**Goal**

Give AFEM and prove convergence rates

- Algorithm similar to existing adaptive methods based on chasing of a-posteriori error estimators
- Main difference: Coarsening strategy
- Analysis rely non-linear approximation by piecewise polynomials

- Complications of AFEM analysis:
    - Need of graded meshes
    - Hanging nodes
    - Analysis of a-posteriori error estimator

**Poisson problem**

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega$ is polygonal domain in $\mathbb{R}^2$

# Newest vertex bisection and completion

- Consider only approximations of $u$ by piecewise linear elements using newest vertex bisection

## Notations

$$\Omega = \bigcup_{\Delta \in P} \Delta$$

$$\text{For any } \Delta, \Delta' \in P, \ \text{meas}(\Delta \cap \Delta') = 0$$

$$\mathcal{S}_P = \{S \in C(\Omega) : \ S|_\Delta \in P_1(\Delta) \text{ and } S|_{\partial\Omega} = 0\} \subset H_0^1(\Omega)$$

## Notations

$\mathcal{E}_P$ … set of edges of $P$

$\dot{\mathcal{E}}_P$ … interior edges

$\mathcal{V}_P$ … set of vertices of $P$

$\dot{\mathcal{V}}_P$ … interior vertices

- 2 conditions on $P$:
  - Minimal angle condition for all $\Delta \in P$ for some $a_0 > 0$
  - $P$ is conforming

## Definition

A family of partitions $\mathcal{P}$ is called admissible, if all elements are conforming which satisfy the minimal angle condition with common constant $a_0 > 0$.

- Minimal angle condition implies shape regularity,

$$\exists \hat{C} = \hat{C}(\mathcal{P}) : 1 \leq \frac{\mathrm{diam}(\Delta)^2}{|\Delta|} \leq \hat{C} \quad \text{for all } \Delta \in P, P \in \mathcal{P}$$

- Moreover,

$$\exists G_0 = G_0(\mathcal{P}) : \mathrm{diam}(\Delta) \leq G_0 \mathrm{diam}(\Delta')$$

for all $\Delta, \Delta' \in P$ with $\Delta \cap \Delta' \neq \emptyset$

- Typical AFEM generates sequence of partitions $P_0, P_1, \ldots, P_n$ by subdividing triangles
- Given $P_k$:
    - Mark certain cells $\Delta \in P_k$ for subdivision, $\mathcal{M}_k$
    - Subdivide marked cells which can create hanging nodes
    - Mark additional cells $\mathcal{M}'_k$ for subdividing such that $P_{k+1}$ is admissible

# Newest vertex bisection

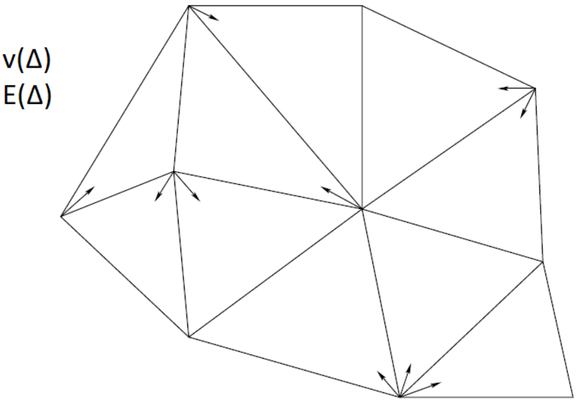


**Fig. 1.** Assignment of newest vertices in $P_0$

- The partitions satisfy the uniform minimal angle condition
- If $P$ is created by Newest vertex bisection and has no hanging nodes $\Rightarrow$ $P$ is conforming $\Rightarrow$ $P$ is admissible
- Completion step provides partition without hanging nodes
- Bound number of additional subdivisions necessary to remove hanging nodes

- Present newest vertex bisection subdivision by an infinite binary tree $T_\star$
- $T_\star$ consists off all triangular cells which can be obtained by sequence of subdivisions
- Roots: triangular cells of $P_0$
- Important fact: Newest vertex of cells are unique and only depend on initial assignment in $P_0$
- This means, children does not depend on preceding sequence of subdivisions

# Completion of subdivision

- Begin with $P_0$ and mark cells for subdivision $\Rightarrow P_1'$
- $P_1'$ not necessarily admissible, therefore make additional subdivision to get admissible $P_1$
- We will see that
  - Completion process does not inflate the number of triangular cells in $P_n$
  - Number of created triangles through completion proportional to number of cells marked in refinement, i.e.

$$\#(P_n) - \#(P_0) \leq C_2(\#(\mathcal{M}_0) + \ldots + \#(\mathcal{M}_{n-1}))$$

## Definition

Suppose $P$ is admissible with $\#(P) > 2$. For $\Delta \in P$ define

$$F(\Delta) = \begin{cases} \varnothing & \text{if } E(\Delta) \text{ is a boundary edge,} \\ \Delta' & \text{if } E(\Delta) \text{ is a edge of } \Delta'. \end{cases}$$

A chain $C(\Delta)$ (with starting cell $\Delta$) in $P$ is a sequence $\Delta, F(\Delta), \ldots, F^m(\Delta)$ with no repetition and with $F^{m+1}(\Delta) = F^k(\Delta)$ for $k = 0, \ldots, m-1$ or $F^{m+1}(\Delta) = \varnothing$.

- Completion of $C(\Delta)$ is collection $\bar{C}(\Delta)$ produced by 2 sets of subdivision
  1) Each cell $\Delta' = F^k(\Delta)$ is subdivided by newest vertex bisection
  2) Subdivide children with hanging nodes
     Hanging nodes occur inside cell $\Delta' = F^k(\Delta)$, if $E(F^{k-1}(\Delta)) \neq E(F^k(\Delta))$
  - Connect midpoint of these two edges

# Labelling

- Use labelling of edges to observe structure of $\bar{C}(\Delta)$
- Label edges in $P_0, P_1, \ldots, P_n$ by non-negative integers
- Given $\Delta \in P_k$, sides are labelled by $(i+1, i+1, i)$ where $i = g(\Delta)$ and lowest labelled side is $E(\Delta)$
- For admissible partitions, labelling is independent of triangles provided we start with suitable labelling of $P_0$

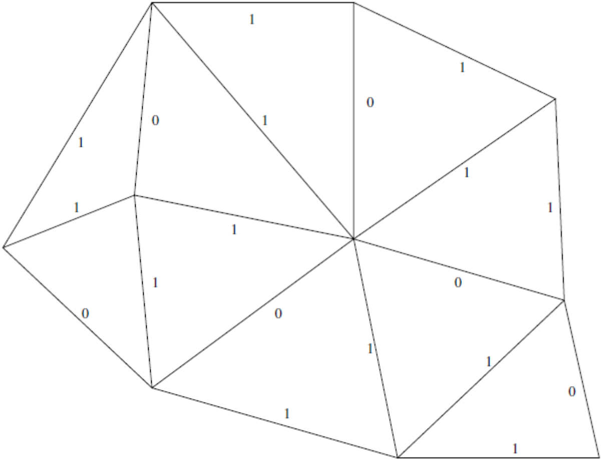


**Fig. 2.** Assignment of newest vertices in $P_0$

## Theorem 1

*For any $P_0$ there is a labelling of the edges in $P_0$ such that each edge is given a label of either 0 or 1 and whenever a triangle $\Delta \in P_0$ then exactly two of its edges are labelled with 1 and the other is labelled with 0.*

- Existence is guaranteed by Petersen's Theorem
- Construction of labelling scheme as in Theorem 1: Suppose, we can find $Q \subset P_0$ such that
  - (i) All triangles in $P_0 \backslash Q$ have at least one edge on $\partial\Omega$
  - (ii) $\Omega_Q$ can be decomposed into an essentially disjoint union of quadrilaterals formed by pairs of adjacent triangles

- For each pair of triangles in $Q$ whose union forms one quadrilateral, assign common edge to 0 and others to 1
- By (i) we missed at most edges on $\partial\Omega$
  - If $E \in \partial\Omega$, and the other sides are interior, then label by 0
  - If $E \in \partial\Omega$ with another boundary edge, label one by 0 and other by 1
- Label problem reduces to:
  Find $Q \subset P_0$ such that (i) and (ii) are satisfied

- Construction of $Q$: Subdivide each $\Delta \in P_0$ into 4 triangles such that new partition $P_0' = Q$ satisfies (i) and (ii)

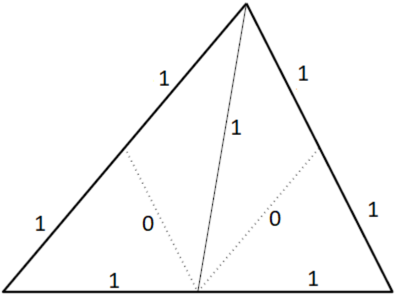


**Fig. 3.** Refinement for labelling

- Do this $\forall \Delta \in P_0$, resulting $P_0'$ has no hanging nodes

- Given the labelling of $P_0$ by Theorem 1, define newest vertex of $\Delta \in P_0$ to be opposite of side with label 0
- Assume, initial labelling of $P_0$ according Theorem 1
- Any chain in $P_0$ has at most 2 cells
- Subdivision of cells gives admissible partition

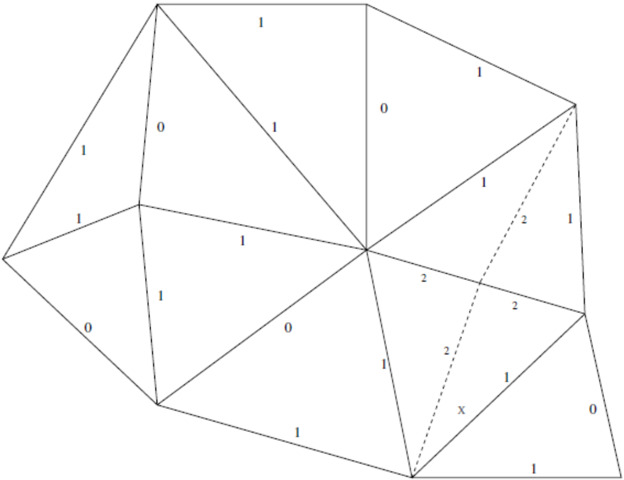


Fig. 4. Labelling of new edges

- Labelling for edges that arise from subdivision completion (2 properties)
    i. Each cell have sides labelled by $(i, i, i-1)$ for $i \in \mathbb{N}$
    ii. Newest vertex is opposite lowest label
- If a cell has label $(i+1, i+1, i)$, then it is of generation $i$, i.e. so many subdivisions are needed to get the cell

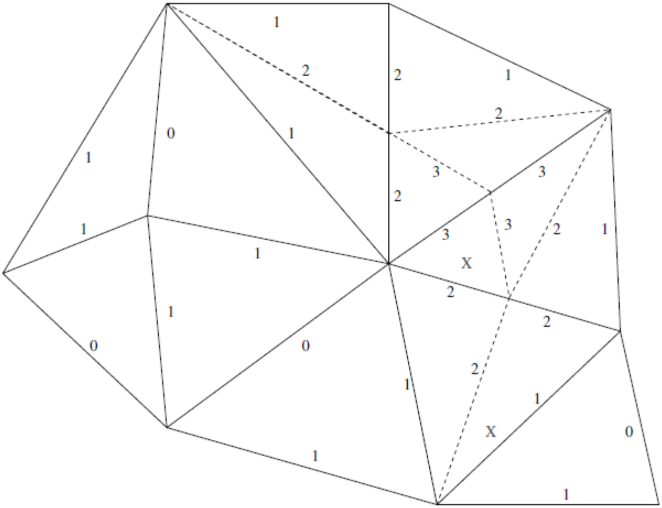


**Fig. 5.** Labelling in the completion process

## Lemma 1

Suppose $P_0$ is an arbitrary partition and its edges and newest vertices are labelled in accordance with Theorem 1. Suppose that $P_1, \dots P_n$ are partitions generated by newest vertex bisection from $P_0$. Then there holds for each $k = 0, 1, \dots, n$:

(i) each edge in $P_k$ has a unique label independent of the two triangles which share this edge.

(ii) If $\Delta \in P_k$ of generation $g(\Delta) = i$, i.e. the edge with label i is the side shared by $\Delta$ and $F(\Delta)$, then $g(F(\Delta)) \in \{i, i-1\}$.
If $g(F(\Delta)) = i$, the flow ends at $F(\Delta)$.

## Lemma 1 cont.

(iii) For any $\Delta \in P_k$ of generation $g(\Delta) = i$, the cells in its chain $C(\Delta) = \{\Delta, F(\Delta), \ldots, F^m(\Delta)\}$ have the property that $g(F^j(\Delta)) = i - j, j = 0, \ldots, m-1$ and $F^m(\Delta)$ for this chain is either of generation $i - m + 1$ or it is a boundary cell with lowest labelled edge an edge of the boundary.

- All admissible partitions, generated by newest vertex bisection with $P_0$ according to Theorem 1, are graded

## Theorem 2

*Let $P_0, \ldots, P_n$ be a sequence of partitions generated as described above. Then, there is a constant $C_2 > 0$ depending only on $P_0$, such that*

$$\#(P_n) \leq \#(P_0) + C_2(\#(\mathcal{M}_0) + \ldots + \#(\mathcal{M}_{n-1})). \tag{2}$$

- Given $P$ and $P'$ partition of $P$, then $m(P|P')$ is number of needed markings
- Thus, (9) can be rewritten as

$$\#(P_n) \leq \#(P_0) + C_2 \sum_{k=1}^{n} m(P_k|P_{k-1}) \qquad (3)$$

# Galerkin approximations

- Weak formulation of Poisson problem (1):
  Find $u \in H_0^1(\Omega)$ such that

$$a(u, w) = (f, w), \quad w \in H_0^1(\Omega)$$

- Notation:

$$|||w|||^2 = a(w, w) = \|\nabla w\|_{L_2(\Omega)}^2$$

- Given admissible $P$, denote Galerkin solution by $u_P$
- $u_P$ is unique element in $\mathcal{S}_P \subset H_0^1(\Omega)$ which satisfies

$$a(u_P, w) = (f, w), \quad w \in \mathcal{S}_P$$

- Replace $f$ by an approximation
- $u_P$ is best approximation to $u$ from $\mathcal{S}_P$ in energy norm

$$|||u - u_P||| = \inf_{S \in \mathcal{S}_P} |||u - S|||$$

- Can't calculate $u_P$ exactly, therefore use numerical scheme

## GAL

- Input: admissible partition $P$, error tolerance $\mu$, initial approximation $\bar{u}_P \in \mathcal{S}_P$ to $u_P$, that satisfies

$$|||u - \bar{u}_P||| \leq A'\mu, \quad A' = const \tag{4}$$

- Apply a preconditioned conjugate gradient scheme with initial guess $\bar{u}_P$ to obtain approximation $\hat{u}_P = \text{GAL}(P, \mu, \bar{u}_P)$ to $u_P$ that satisfies

$$|||u_P - \hat{u}_P||| \leq \delta\mu, \quad \delta \in (0,1) \tag{5}$$

**Remark**

- Needed number of iterations of CG-scheme to achieve (13) depends only on $A'/\delta$
- Since each iteration requires at most $C\#(P)$ computations, it follows that number of computations

$$N(\mathrm{GAL}, P, \mu, \bar{u}_P) \leq C_3(\frac{A'}{\delta})\#(P) \tag{6}$$

where $C_3 : t \to C_3(t)$ increases as a function of $t$

# Adaptive approximation

- Discuss adaptive approximation of a known function $w$ for which local polynomial approximations are computable
- $w$ is the best we can expect in terms of approximating $u$
- Use methods which approximate $f$ and execute coarsening step
- Use adaptive method based on newest vertex subdivision rule starting with $P_0$ and labelling as in Theorem 1

# Adaptive approximation in $H^1(\Omega)$-norm

- Given $w \in H_0^1(\Omega)$ and $P$, we define

$$E(w, \mathcal{S}_P)_{H^1(\Omega)} = \inf_{S \in \mathcal{S}_P} \|w - S\|_{H^1(\Omega)} \tag{7}$$

- Define error

$$\sigma_n(w) = \inf_{P \in \mathcal{P}_n} E(w, \mathcal{S}_P)_{H^1(\Omega)} \tag{8}$$

  of best adaptive approximation

- Cannot expect, that any adaptive algorithm perform exactly the same as $\sigma_n(w)$

- We may expect same asymptotic behavior
- We know (from NumEP) that

$$\|u - v_h\|_{H^1(\Omega)} \leq ch|u|_{H^2(\Omega)} = Mh = Mn^{-1/2}$$

- Introduce class $\mathcal{A}^{1/2} = \mathcal{A}^{1/2}(H_0^1(\Omega))$ of functions $w \in H_0^1(\Omega)$

$$\sigma_n(w) \leq Mn^{-1/2}, \quad n = 1, 2, \dots \tag{9}$$

- In general, for any $s > 0$ the class $\mathcal{A}^s = \mathcal{A}^s(H_0^1(\Omega))$ of functions $w \in H_0^1(\Omega)$, such that

$$\sigma_n(w) \leq Mn^{-s}, \quad n = 1, 2, \ldots \tag{10}$$

- Smallest $M$ for which (14) is satisfied is the norm in $\mathcal{A}^s$

$$\|w\|_{\mathcal{A}^s} = \sup_{n\geq 1} n^s \sigma_n(w) \tag{11}$$

# Adaptive approximation in $H^{-1}(\Omega)$-norm

- Need approximation by piece-wise constants for $f$
- Approximation in $H^{-1}(\Omega)$ with

$$\|g\|_{H^{-1}(\Omega)} = \sup_{\phi \in H^{-1}(\Omega)} \frac{\langle g, \phi \rangle}{|||\phi|||}$$

- Given $P$, we write $\mathcal{S}_P^0$ as class of piecewise constants subordinate to $P$

- For $f \in H^{-1}(\Omega)$, define

$$E(f, \mathcal{S}_P^0)_{H^{-1}(\Omega)} = \inf_{S \in \mathcal{S}_P^0} \|f - S\|_{H^{-1}(\Omega)}$$

which is, best error of approximation for $f$

- Analogous, error of best non-linear approximation

$$\sigma_n(f)_{H^{-1}(\Omega)} = \inf_{P \in \mathcal{P}_n} E(f, \mathcal{S}_P^0)_{H^{-1}(\Omega)}$$

- Introduce for $s > 0$, $\mathcal{A}^s(H^{-1}(\Omega))$ approximation class as before
  - Except: Use $\sigma_n(g)_{H^{-1}(\Omega)}$ in place of $\sigma_n(w)$

- Suppose $g \in L_2(\Omega)$, then $g \in H^{-1}(\Omega)$
- If $P$ is any partition of $\Omega$ and $\Delta \in P$, define

$$g_\Delta = \frac{1}{|\Delta|} \int_\Delta g$$

- $g_\Delta$ best approximation in $L_2(\Delta)$ by constant functions
- Best $L_2(\Omega)$ approximation by piecewise constants subordinate to $P$

$$S_P^0(g) = \sum_{\Delta \in P} g_\Delta \chi_\Delta$$

- For admissible $P$, we have bound for approximation error

$$E(g, \mathcal{S}_P^0)_{H^{-1}(\Omega)}^2 \leq \|g - S_P^0(g)\|_{H^{-1}(\Omega)}^2 \leq C_0 \bar{E}(g, P) \quad (12)$$

where

$$\bar{E}(g, P) = \sum_{\Delta \in P} |\Delta| \, \|g - g_\Delta\|_{L_2(\Omega)}^2$$

- Define another non-linear approximation error

$$\bar{\sigma}_n^2(g) = \inf_{P \in \mathcal{P}_n} \bar{E}(g, P)$$

**Theorem 3 (Petersen's Theorem)**

*Any bridgeless cubic graph has a perfect matching.*

Back