JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

On Metric Pseudo-(sub)Regularity of Multifunctions and Optimality Conditions for Degenerated Mathematical Programs

Helmut Gfrerer
Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

NuMa-Report No. 2012-10 December 2012

A–4040 LINZ, Altenbergerstraße 69, Austria

Technical Reports before 1998:

1995

95-1 Hedwig Brandstetter
Was ist neu in Fortran 90? — March 1995

95-2 G. Haase, B. Heise, M. Kuhn, U. Langer
Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations. — August 1995

95-3 Joachim Schöberl
An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions. — August 1995

1996

96-1 Ferdinand Kickinger
Automatic Mesh Generation for 3D Objects. — February 1996

96-2 Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn
Preprocessing in BE/FE Domain Decomposition Methods. — February 1996

96-3 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. — February 1996

96-4 Bodo Heise und Michael Jung
Robust Parallel Newton-Multilevel Methods. — February 1996

96-5 Ferdinand Kickinger
Algebraic Multigrid for Discrete Elliptic Second Order Problems. — February 1996

96-6 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. — May 1996

96-7 Michael Kuhn
Benchmarking for Boundary Element Methods. — June 1996

1997

97-1 Bodo Heise, Michael Kuhn and Ulrich Langer
A Mixed Variational Formulation for 3D Magnetostatics in the Space $H(rot) \cap H(div)$ — February 1997

97-2 Joachim Schöberl
Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case. — June 1997

97-3 Ferdinand Kickinger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl
Numerical Estimates of Inequalities in $H^{\frac{1}{2}}$. — August 1997

97-4 Joachim Schöberl
Programmbeschreibung NAOMI 2D und Algebraic Multigrid. — September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`

On metric pseudo-(sub)regularity of multifunctions and optimality conditions for degenerated mathematical programs

Helmut Gfrerer*

Abstract

This paper is devoted to the analysis of a special kind of regularity of multifunction which we call metric pseudo-(sub)regularity, when the multifunction is not metrically (sub)regular at a given point but is metrically (sub)regular at certain points in a neighborhood with moduli possibly tending to infinity with a certain order. By using advanced techniques of generalized differentiation we derive conditions both necessary and sufficient for this property. As a byproduct we obtain a new sufficient condition for metric subregularity. Then we apply these results to multifunctions composed by a smooth mapping and a generalized polyhedral multifunction and obtain explicit formulas for this case. Finally we show how the theory can be used to obtain necessary optimality conditions when the constraint qualification condition of metric (sub)regularity is violated.

Key words. Metric regularity, subregularity, coderivative, optimality conditions

AMS subject classification. 49J53 49K27 90C48

1 Introduction

Recall that a multifunction $M : X \rightrightarrows Y$ between Banach spaces X and Y is said to be *metrically regular near the point* $(\bar{x},\bar{y}) \in \operatorname{gph} M := \{(x,y) \in X \times Y \,|\, y \in M(x)\}$ with modulus $\kappa > 0$, if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathrm{d}(x, M^{-1}(y)) \le \kappa \mathrm{d}(y, M(x)) \;\forall (x,y) \in U \times V. \tag{1}$$

Here $\mathrm{d}(x,\Omega)$ denotes the usual distance between a point x and a set Ω. Metric regularity plays an important role in many fields of nonlinear analysis and its applications, in particular in optimization, constraint qualification conditions and stability analysis. For a survey on the theory of metric regularity and also on the related notions of *pseudo-Lipschitz continuity*, *Aubin property*, *Lipschitz-like property* and *openness with a linear rate* we refer to [19] and to the monographs [21], [26], [29] and the references therein.

When fixing $y = \bar{y}$ in (1) we obtain the weaker property of *metric subregularity* of M at $(\bar{x},\bar{y})$, i.e. we require the estimate

$$\mathrm{d}(x, M^{-1}(\bar{y})) \le \kappa \mathrm{d}(\bar{y}, M(x)) \quad \forall x \in U \tag{2}$$

with some neighborhood $U$of $\bar{x}$ and a positive real $\kappa > 0$. This property as well as its equivalent *calmness* counterpart for the inverse multifunction and their applications have been studied extensively in recent years, see e.g. [5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 23, 27, 28, 30, 31, 32, 35, 36].

*Institute of Computational Mathematics, Johannes Kepler University Linz, A-4040 Linz, Austria, `helmut.gfrerer@jku.at`

In this paper we consider the case that the multifunction $M : X \rightrightarrows Y$ is not metrically regular near $(\bar{x},\bar{y})$ respectively not metrically subregular at $(\bar{x},\bar{y})$. *Metric regularity of order* $\gamma \geq 1$ has been studied in [8] which is defined as the validity of the estimate (1) with $\mathrm{d}(y,M(x))$ replaced by $\mathrm{d}(y,M(x))^{1/\gamma}$, see also [34]. Similarly, when replacing $\mathrm{d}(\bar{y},M(x))$ in (2) by $\mathrm{d}(\bar{y},M(x))^q$, $0 < q \leq 1$, one obtains the property of *metric (Hölder) q-subregularity* [25]. Here we use a different approach. We will introduce directional versions of the concept of *metric pseudo-(sub)regularity of order* γ, $\gamma \geq 1$, where M is assumed to be regular near certain points $(x,y) \in \mathrm{gph}\,M$ respectively subregular at points $(x,\bar{y}) \in \mathrm{gph}\,M$ near $(\bar{x},\bar{y})$, but the moduli κ behave like $O(1/\|x-\bar{x}\|^{\gamma-1})$ for $x \to \bar{x}$ and also the size of the neighborhoods U and V depend on x and γ. For smooth mappings $F : X \to Y$, i.e. $M(x) = \{F(x)\}$, this concept with order $\gamma = 2$ is closely related to the 2-regularity property of F [1]. Further, metric pseudo-(sub)regularity of order γ implies metric Hölder $1/\gamma$-subregularity. We will also consider the case that the multifunction $M(x) = (M_1(x),\ldots,M_s(x))$ is composed by several multifunctions $M_i : X \rightrightarrows Y_i$ and the components M_i have pseudo-(sub)regular behavior of different orders γ_i.

The rest of the paper is organized as follows. After recalling some preliminaries in section 2, we state in section 3 the definitions as well as the basic properties of metric pseudo-(sub)regularity and present some examples. In section 4 we give an equivalent characterization of metric pseudo-regularity and a sufficient criterion for pseudo-subregularity of arbitrary order by means of limiting objects from generalized differentiation as originated by Mordukhovich. In section 5 we consider multifunctions $M(x) = (M_1(x),M_2(x))$ of the form $M_i(x) = F_i(x)+S_i(x)$, $i = 1,2$, where $F_i : X \to Y_i$ is a sufficiently smooth mapping and $S_i : X \rightrightarrows Y_i$ is a generalized polyhedral multifunction and formulate explicit conditions for pseudo-(sub)regularity of order $\gamma_i = i$, $i = 1,2$. Finally, in the last section we treat optimization problems of the form

$$\min_{x\in X} f(x) \text{ subject to } 0 \in F(x)+S(x),$$

where F is a sufficiently smooth mapping and S is a generalized polyhedral multifunction. We obtain necessary optimality conditions which, when applied to the mathematical programming problem with equality and inequality constraints, contain and even improve the optimality conditions as presented in [2, 3, 9].

Throughout this paper let X, Y, W and Z be Banach spaces equipped with norm $\|\cdot\|$. By X^* we denote the topological dual of X with the canonical pairing $\langle\cdot,\cdot\rangle$ between X and X^*. $\mathscr{B}_X := \{x \in X \mid \|x\| \leq 1\}$ denotes the closed unit ball and $\mathscr{S}_X := \{x \in X \mid \|x\| = 1\}$ denotes the unit sphere. Unless otherwise stated, we assume that the product space $X \times Y$ of two spaces X and Y is equipped with a norm satisfying $\max\{\|x\|,\|y\|\} \leq \|(x,y)\| \leq \|x\|+\|y\|$. Given an element of a product space, e.g. $z \in X \times Y$ respectively $w \in W_1 \times \ldots \times W_m$, we denote by appropriate indices the corresponding components, e.g. $z = (z_x,z_y) \in X \times Y$ respectively $w = (w_1,\ldots,w_m) \in \prod_{i=1}^m W_i$. Given a closed subspace $L \subset X$, we denote by $L^\perp := \{x^* \in X^* \mid \langle x^*,l\rangle = 0 \forall l \in L\}$ the annihilator of L and by X/L the quotient space. Note that $(X/L)^*$ can be identified with $L^\perp$.

2 Preliminaries

At first let us recall some constructions from variational analysis. Let Ω be a nonempty subset of a Banach space X and let $x \in \Omega$. The *contingent cone* to Ω at x, denoted by $T(x;\Omega)$, is given by

$$T(x;\Omega) := \{d \in X \mid \exists (x_k) \in \Omega, (t_k) \downarrow 0 : \frac{x_k - x}{t_k} \to d\}.$$

Given $\varepsilon \geq 0$ we denote by

$$\hat{N}_\varepsilon(x;\Omega) = \{x^* \in X^* \mid \limsup_{x' \xrightarrow{\Omega} x} \frac{\langle x^*, x'-x\rangle}{\|x'-x\|} \leq \varepsilon\} \tag{3}$$

the *set of ε-normals* to Ω. When $\varepsilon = 0$, elements of (3) are called *regular/Fréchet normals* and their collection is denoted by $\hat{N}(x;\Omega)$. Finally, the *limiting/Mordukhovich normal cone* to Ω at x is defined by

$$N(x;\Omega) := \{x^* \mid \exists(\varepsilon_k) \downarrow 0,\ (x_k) \xrightarrow{\Omega} x,\ (x_k^*) \xrightarrow{w^*} x^* : x_k^* \in \hat{N}_{\varepsilon_k}(x_k;\Omega) \forall k\}.$$

If $x \notin \Omega$ we put $T(x;\Omega) = \emptyset$, $N(x;\Omega) = \emptyset$ and $\hat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \geq 0$.

The Mordukhovich normal cone is generally nonconvex whereas the Fréchet normal cone is always convex. In the case of a convex set Ω, both the Fréchet normal cone and the Mordukhovich normal cone coincide with the standard normal cone from convex analysis and moreover, the contingent cone is equal to the tangent cone in the sense of convex analysis. Further, for convex sets Ω there holds

$$\hat{N}_\varepsilon(x;\Omega) = \{x^* \in X^* \mid \mathrm{d}(x^*, \hat{N}(x;\Omega)) \leq \varepsilon\}.$$

Given a multifunction $M : X \rightrightarrows Y$ and a point $(\bar{x},\bar{y}) \in \operatorname{gph} M$, the *contingent derivative* of M at $(\bar{x},\bar{y})$ is defined as the set-valued mapping $CM(\bar{x},\bar{y}) : X \rightrightarrows Y$ with the values $CM(\bar{x},\bar{y})(u) := \{v \in Y \mid (u,v) \in T((\bar{x},\bar{y}); \operatorname{gph} M)\}$, i.e. $CM(\bar{x},\bar{y})(u)$ is the collection of all $v \in Y$ such that there are sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ with $(\bar{x}+t_k u_k, \bar{y}+t_k v_k) \in \operatorname{gph} M$.

The *normal coderivative* of M at $(\bar{x},\bar{y})$ is a multifunction $D_N^* M(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$, where $D_N^* M(\bar{x},\bar{y})(y^*)$ is the collection of all $x^* \in X^*$ for which there are sequences $(\varepsilon_k) \downarrow 0$, $(x_k,y_k) \to (\bar{x},\bar{y})$ and $(x_k^*,y_k^*) \xrightarrow{w^*} (x^*,y^*)$ with $(x_k^*,-y_k^*) \in \hat{N}_{\varepsilon_k}((x_k,y_k); \operatorname{gph} M)$.

The *reversed mixed coderivative* of M is the version of the normal coderivative with strong convergence in X^* and weak* convergence in Y^* and is denoted by $\tilde{D}_M^* M(\bar{x},\bar{y})$. For more details on the Mordukhovich normal cone and coderivatives we refer the reader to the monograph of Mordukhovich [26].

The following directional version of these limiting constructions were introduced in [11]. Given a direction $u \in X$, the Mordukhovich normal cone to a subset $\Omega \subset X$ in direction u at $x \in \Omega$ is defined by

$$N(x;\Omega;u) := \{x^* \mid \exists(\varepsilon_k) \downarrow 0,\ (t_k) \downarrow 0,\ (u_k) \to u,\ (x_k^*) \xrightarrow{w^*} x^* : x_k^* \in \hat{N}_{\varepsilon_k}(x+t_k u_k;\Omega) \forall k\}.$$

For a multifunction $M : X \rightrightarrows Y$ and a direction $(u,v) \in X \times Y$, the normal coderivative (respectively reversed mixed coderivative) of M in direction (u,v) at $(\bar{x},\bar{y}) \in \operatorname{gph} M$ is defined as the multifunction $D_N^* M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$ (respectively $\tilde{D}_M^* M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$), where $D_N^* M((\bar{x},\bar{y});(u,v))(y^*)$ (respectively $\tilde{D}_M^* M((\bar{x},\bar{y});(u,v))(y^*)$) is the collection of all $x^* \in X^*$ for which there exist sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ and $(x_k^*,y_k^*) \xrightarrow{w^*} (x^*,y^*)$ (respectively $x_k^* \to x^*$, $y_k^* \xrightarrow{w^*} y^*$) with $(x_k^*,-y_k^*) \in \hat{N}_{\varepsilon_k}((\bar{x}+t_k u_k, \bar{y}+t_k v_k); \operatorname{gph} M)$.

Note that $N(x;\Omega) = N(x;\Omega;0)$, $D_N^* M(\bar{x},\bar{y}) = D_N^* M((\bar{x},\bar{y});(0,0))$ and $\tilde{D}_M^* M(\bar{x},\bar{y}) = \tilde{D}_M^* M((\bar{x},\bar{y});(0,0))$.

In [12],the *strong coderivative of M in direction* (u,v) at $(\bar{x},\bar{y})$ is defined as the set-valued mapping $D_S^* M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$, where $D_S^* M((\bar{x},\bar{y});(u,v))(y^*)$ is the collection of all $x^* \in X^*$ for which there exist sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ and $(x_k^*,y_k^*) \to (x^*,y^*)$ with $(x_k^*,-y_k^*) \in \hat{N}((\bar{x}+t_k u_k, \bar{y}+t_k v_k); \operatorname{gph} M)$. Further we set $D_S^* M(\bar{x},\bar{y}) := D_S^* M((\bar{x},\bar{y});(0,0))$.

To study the directional behavior of multifunctions, it is convenient to introduce the following neighborhoods of directions: Given a Banach space X, a direction $u \in X$ and positive numbers $\varepsilon, \delta > 0$, the set $V_{\varepsilon,\delta}(u)$, is defined by

$$V_{\varepsilon,\delta}(u) := \{x \in \varepsilon\mathscr{B}_X \mid \big\| \|u\|x - \|x\|u \big\| \le \delta \|x\|\ \|u\|\}. \tag{4}$$

This can be written also in the form

$$V_{\varepsilon,\delta}(u) = \begin{cases} \{0\} \cup \{x \in \varepsilon\mathscr{B}_X \setminus \{0\} \mid \big|\frac{x}{\|x\|} - \frac{u}{\|u\|}\big| \le \delta\} & \text{if } u \neq 0, \\ \varepsilon\mathscr{B}_X & \text{if } u = 0. \end{cases}$$

Note that $V_{\varepsilon,\delta}(u) = V_{\varepsilon,\delta}(\alpha u)$, $\forall \alpha > 0$ and that, given $\bar{x} \in Z$ and a sequence $(x_k) \to \bar{x}$, there exist sequences $(t_k) \downarrow 0$, $(u_k) \to u$ with $x_k = \bar{x} + t_k u_k$ if and only if for every $\varepsilon > 0$, $\delta > 0$ there is some index $k_{\varepsilon,\delta}$ such that $x_k \in \bar{z} + \operatorname{int} V_{\varepsilon,\delta}(u)$, $\forall k \ge k_{\varepsilon,\delta}$. Further, for all $0 \neq u, u'$ we have

$$\|\frac{u}{\|u\|} - \frac{u'}{\|u'\|}\| \le 2\frac{\|u-u'\|}{\max\{\|u\|, \|u'\|\}}.$$

In the sequel we will also consider generalized polyhedral multifunctions. A *convex generalized polyhedral set* is the intersection of a closed affine subspace and finitely many closed halfspaces. A set-valued mapping $S : X \rightrightarrows Y$ is called a *generalized polyhedral multifunction*, if its graph is the union of finitely many convex generalized polyhedral sets, i.e.

$$\operatorname{gph} S = \bigcup_{i=1}^{p} P_i \subset X \times Y, \tag{5}$$

$$P_i = \{(x,y) \in (x_i', y_i') + L_i \mid \langle x_{ij}^*, x\rangle + \langle y_{ij}^*, y\rangle \le \zeta_{ij}, j = 1, \ldots, m_i\},\ i = 1, \ldots, p, \tag{6}$$

where for each $i = 1, \ldots, p$, $(x_i', y_i') \in X \times Y$, $L_i \subset X \times Y$ denotes a closed subspace, $x_{ij}^* \in X^*$, $y_{ij}^* \in Y^*$, $j)1, \ldots, m_i$ are continuous linear functionals and $\zeta_{ij} \in \mathbb{R}$.

It was shown in [12, Lemma 3.8] that for a generalized polyhedral multifunction $S : X \rightrightarrows Y$ the strong coderivative and the normal one coincide for all directions (u, v):

$$D_N^* S((\bar{x},\bar{y}); (u,v)) = D_S^* S((\bar{x},\bar{y}); (u,v)) =: D^* S((\bar{x},\bar{y}); (u,v)).$$

Finally, given an extended-real-valued function $f : X \to \bar{R}$, the *regular/Fréchet subdifferential* at $x \in \operatorname{dom} f$ is given by

$$\hat{\partial} f(x) := \{x^* \in X^* \mid \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \ge 0\}$$

and $\hat{\partial} f(x) := \emptyset$ for $x \notin \operatorname{dom} f$.

3 Metric pseudo-(sub)regularity of multifunctions

Throughout this section let $s \ge 1$ be an integer and let $M_i : X \rightrightarrows Y_i$, $i = 1, \ldots, s$ be closed multifunctions between the Banach spaces X and Y_i. Further we define the multifunction

$$M : X \rightrightarrows Y := Y_1 \times \ldots \times Y_s, \quad M(x) = (M_1(x), \ldots, M_s(x)). \tag{7}$$

For given $y \in Y$ we denote by $y_i \in Y_i$, $i = 1, \ldots, s$ the components of y

Definition 1. *Let M be given by* (7), $(\bar{x},\bar{y}) \in \mathrm{gph} M$, *and let the real numbers $\gamma_i \geq 1$, $i = 1,\dots,s$ be given.*

1. *Given a direction $(u,v) \in X \times Y$, we say that M is* metrically pseudo-regular of order $(\gamma_1,\dots,\gamma_s)$ *in direction (u,v) near $(\bar{x},\bar{y})$, if there are positive real numbers ε, δ and κ, such that*
$$\mathrm{d}(x, M^{-1}(y)) \leq \sum_{i=1}^{s} \frac{\kappa}{\|x-\bar{x}\|^{\gamma_i - 1}} \mathrm{d}(y_i, M_i(x)) \tag{8}$$
holds for all $(x,y) \in (\bar{x},\bar{y}) + V_{\varepsilon,\delta}(u,v)$ with $x \neq \bar{x}$ and $\mathrm{d}(y_i, M_i(x)) \leq \delta \|x-\bar{x}\|^{\gamma_i}$, $i = 1,\dots,s$.
We say that M is metrically pseudo-regular of order $(\gamma_1,\dots,\gamma_s)$ near $(\bar{x},\bar{y})$, if this definition holds for $(u,v) = (0,0)$.

2. *Given a direction $u \in X$, we say that M is* metrically pseudo-subregular of order $(\gamma_1,\dots,\gamma_s)$ *in direction u at $(\bar{x},\bar{y})$, if there are positive real numbers ε, δ and κ, such that*
$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \sum_{i=1}^{s} \frac{\kappa}{\|x-\bar{x}\|^{\gamma_i - 1}} \mathrm{d}(\bar{y}_i, M_i(x)) \tag{9}$$
holds for all $x \in \bar{x} + V_{\varepsilon,\delta}(u)$ with $x \neq \bar{x}$.
In case that this definition is fulfilled for $u = 0$ we say that M is metrically pseudo-subregular of order $(\gamma_1,\dots,\gamma_s)$ at $(\bar{x},\bar{y})$.

By the definition it follows that M is metrically pseudo-regular of every order in direction (u,v) near $(\bar{x},\bar{y})$, if $v \notin CM(\bar{x},\bar{y})(u)$. Further, metric pseudo-subregularity of order 1 in direction u is the same property as metric subregularity in direction u, which is defined by the validity of (2) for $x \in \bar{x} + V_{\varepsilon,\delta}(u)$ (cf.[11]).

Example 1.

1. *For every $\gamma \geq 1$ the mapping $M : \mathbb{R} \rightrightarrows \mathbb{R}$, $M(x) := \{|x|^\gamma\}$ is metrically pseudo-regular of order γ near $(0,0)$. The case $\gamma = 1$ provides an example of a multifunction, which is metrically pseudo-regular of order* 1, *but not metrically regular.*

2. *Given real numbers $\gamma_i \geq 1$, $i = 1,\dots,s$, the multifunction $M : \mathbb{R}^s \rightrightarrows \mathbb{R}$, $M(x) := \{|x_1|^{\gamma_1},\dots,|x_s|^{\gamma_s}\}$ is metrically pseudo-regular of order $(\gamma_1,\dots,\gamma_s)$ near $(0,0)$ in every direction (u,v) satisfying $\prod_{i=1}^{s} u_i \neq 0$.*

Lemma 1. *Let $M : X \rightrightarrows Y$ be metrically pseudo-regular of order $\gamma \geq 1$ in direction (u,v) near $(\bar{x},\bar{y}) \in \mathrm{gph} M$. Then there are positive reals $\kappa', \varepsilon', \delta', \rho > 0$ such that for every $(\tilde{x},\tilde{y}) \in ((\bar{x},\bar{y}) + V_{\varepsilon',\delta'}(u,v)) \cap \mathrm{gph} M$ with $\tilde{x} \neq \bar{x}$ one has*
$$\mathrm{d}(x, M^{-1}(y)) \leq \frac{\kappa'}{\|\tilde{x}-\bar{x}\|^{\gamma-1}} \mathrm{d}(y, M(x))$$
for all $(x,y) \in (\tilde{x} + \rho\|\tilde{x}-\bar{x}\|\mathscr{B}_X) \times (\tilde{y} + \rho\|\tilde{x}-\bar{x}\|^\gamma \mathscr{B}_Y)$.

Proof. Let the reals $\varepsilon, \kappa, \delta$ be chosen according to the definition of metric pseudo-regularity of order γ and choose $\varepsilon' = \min\{\varepsilon/2, 1\}$, $\delta' = \delta/2$. Then for arbitrary $\kappa' > \kappa$ we choose $\rho < 1/2$ small enough such that $4\rho < \delta/2$, $\kappa/(1-\rho)^{\gamma-1} \leq \kappa'$ and $\rho(\kappa+1)/(\delta(1-\rho)^\gamma) \leq \kappa'$. Now let $(\tilde{x},\tilde{y}) \in ((\bar{x},\bar{y}) + V_{\varepsilon',\delta'}(u,v)) \cap \mathrm{gph} M$ with $\tilde{x} \neq \bar{x}$ be arbitrarily fixed and consider $(x,y) \in (\tilde{x} + \rho\|\tilde{x}-\bar{x}\|\mathscr{B}_X) \times (\tilde{y} + \rho\|\tilde{x}-\bar{x}\|^\gamma \mathscr{B}_Y)$. Since $\|\tilde{x}-\bar{x}\| \leq \varepsilon' \leq 1$ and $\gamma \geq 1$ we have $\|\tilde{x}-\bar{x}\|^\gamma \leq \|\tilde{x}-\bar{x}\|$ and therefore $\|(x,y)-(\tilde{x},\tilde{y})\| \leq 2\rho\|\tilde{x}-\bar{x}\| \leq 2\rho\|(\tilde{x},\tilde{y})-(\bar{x},\bar{y})\|$ Hence $\|(x,y)-(\bar{x},\bar{y})\| \leq (1+2\rho)\varepsilon' < \varepsilon$ and
$$\left\| \frac{(x,y)-(\bar{x},\bar{y})}{\|(x,y)-(\bar{x},\bar{y})\|} - \frac{(\tilde{x},\tilde{y})-(\bar{x},\bar{y})}{\|(\tilde{x},\tilde{y})-(\bar{x},\bar{y})\|} \right\| \leq 4\rho$$

showing $(x,y) \in (\bar{x},\bar{y}) + V_{\varepsilon,\delta}(u,v)$. If $\mathrm{d}(y,M(x)) \le \delta\|x-\bar{x}\|^\gamma$ then

$$\mathrm{d}(x,M^{-1}(y)) \le \frac{\kappa}{\|x-\bar{x}\|^{\gamma-1}}\mathrm{d}(y,M(x)) \le \frac{\kappa}{((1-\rho)\|\tilde{x}-\bar{x}\|)^{\gamma-1}}\mathrm{d}(y,M(x)) \le \frac{\kappa'}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(y,M(x)).$$

On the other hand, if $\mathrm{d}(y,M(x)) \ge \delta\|x-\bar{x}\|^\gamma$, we obtain because of $\mathrm{d}(y,M(\tilde{x})) \le \rho\|\tilde{x}-\bar{x}\|^\gamma \le \delta\|\tilde{x}-\bar{x}\|^\gamma$

$$\begin{aligned}
\mathrm{d}(x,M^{-1}(y)) &\le \|x-\tilde{x}\| + \mathrm{d}(\tilde{x},M^{-1}(y)) \le \rho\|\tilde{x}-\bar{x}\| + \frac{\kappa}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(y,M(\tilde{x})) \\
&\le \rho\|\tilde{x}-\bar{x}\| + \frac{\kappa}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\rho\|\tilde{x}-\bar{x}\|^\gamma = \frac{\rho(1+\kappa)}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\|\tilde{x}-\bar{x}\|^\gamma \\
&\le \frac{\rho(1+\kappa)}{\|\tilde{x}-\bar{x}\|^{\gamma-1}(1-\rho)^\gamma}\|x-\bar{x}\|^\gamma \le \frac{\rho(1+\kappa)}{\delta\|\tilde{x}-\bar{x}\|^{\gamma-1}(1-\rho)^\gamma}\mathrm{d}(y,M(x)) \\
&\le \frac{\kappa'}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(y,M(x)).
\end{aligned}$$

□

Lemma 2. *Let $M : X \rightrightarrows Y$ be metrically pseudo-subregular of order $\gamma \ge 1$ in direction u at $(\bar{x},\bar{y}) \in \operatorname{gph} M$. Then there are positive reals $\kappa',\varepsilon',\delta',\rho > 0$ such that for every $\tilde{x} \in (\bar{x} + V_{\varepsilon',\delta'}(u)) \cap M^{-1}(\bar{y})$ with $\tilde{x} \ne \bar{x}$ one has*

$$\mathrm{d}(x,M^{-1}(\bar{y})) \le \frac{\kappa'}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x))$$

for all $x \in \tilde{x} + \rho\|\tilde{x}-\bar{x}\|\mathscr{B}_X$.

Proof. Let the reals $\varepsilon,\kappa,\delta$ be chosen according to the definition of metric pseudo-regularity of order γ and choose $\varepsilon' = \varepsilon/2$, $\delta' = \delta/2$. Then we choose some $\kappa' > \kappa$ and $\rho < 1$ small enough such that $2\rho < \delta/2$, $\kappa/(1-\rho)^{\gamma-1} \le \kappa'$. Now let $\tilde{x} \in (\bar{x} + V_{\varepsilon',\delta'}(u)) \cap M^{-1}(\bar{y})$ with $\tilde{x} \ne \bar{x}$ be arbitrarily fixed and consider $x \in \tilde{x} + \rho\|\tilde{x}-\bar{x}\|\mathscr{B}_X$. Then

$$\left\|\frac{x-\bar{x}}{\|x-\bar{x}\|} - \frac{\tilde{x}-\bar{x}}{\|\tilde{x}-\bar{x}\|}\right\| \le 2\rho < \frac{\delta}{2}$$

and together with $\|x-\bar{x}\| \le \|x-\tilde{x}\| + \|\tilde{x}-\bar{x}\| \le (1+\rho)\|\tilde{x}-\bar{x}\| < 2\varepsilon' = \varepsilon$ we conclude $x \in \bar{x} + V_{\varepsilon,\delta}(u)$. Hence

$$\mathrm{d}(x,M^{-1}(\bar{y})) \le \frac{\kappa}{\|x-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x)) \le \frac{\kappa}{(1-\rho)^{\gamma-1}\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x)) \le \frac{\kappa'}{\|\tilde{x}-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x))$$

□

Lemma 3. *Let $M : X \rightrightarrows Y$ be metrically pseudo-regular of order $\gamma \ge 1$ in direction $(u,0)$ near $(\bar{x},\bar{y}) \in \operatorname{gph} M$. Then M is also metrically pseudo-subregular of order γ in direction u at $(\bar{x},\bar{y})$.*

Proof. Let the reals $\varepsilon,\kappa,\delta$ be chosen according to the definition of metric pseudo-regularity of order γ. Then, if $x \in \bar{x} + V_{\varepsilon,\delta}(u)$ and $\mathrm{d}(\bar{y},M(x)) > \delta\|x-\bar{x}\|^\gamma$ we obtain $\mathrm{d}(x,M^{-1}(\bar{y})) \le \|x-\bar{x}\| \le \frac{1}{\delta\|x-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x))$ and hence

$$\mathrm{d}(x,M^{-1}(\bar{y})) \le \|x-\bar{x}\| \le \max\{\kappa,\frac{1}{\delta}\}\frac{1}{\|x-\bar{x}\|^{\gamma-1}}\mathrm{d}(\bar{y},M(x)) \ \forall x \in \bar{x} + V_{\varepsilon,\delta}(u), x \ne \bar{x}$$

follows. □

We now show that pseudo-subregularity implies Hölder subregularity as considered in [25]. The following lemma states a directional version.

Lemma 4. *Let $M : X \rightrightarrows Y$ be metrically pseudo-subregular of order $\gamma \geq 1$ in direction u at $(\bar{x}, \bar{y}) \in$ $\mathrm{gph}M$. Then it is also metrically Hölder subregular of order $\frac{1}{\gamma}$ in direction u at $(\bar{x}, \bar{y})$, i.e. there exist positive real numbers $\varepsilon', \delta', \kappa' > 0$ such that*

$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \kappa' \mathrm{d}(\bar{y}, M(x))^{\frac{1}{\gamma}} \ \forall x \in \bar{x} + V_{\varepsilon', \delta'}(u)$$

Proof. Let the reals $\varepsilon, \kappa, \delta$ be chosen according to the definition of metric pseudo-subregularity of order γ and set $\varepsilon' = \varepsilon$, $\delta' = \delta$, $\kappa' = \kappa^{\frac{1}{\gamma}}$. Consider $x \in \bar{x} + V_{\varepsilon', \delta'}(u)$. If $\mathrm{d}(\bar{y}, M(x)) \geq \frac{1}{\kappa} \|x - \bar{x}\|^\gamma$ then we obviously have $\mathrm{d}(x, M^{-1}(\bar{y})) \leq \|x - \bar{x}\| \leq \kappa' \mathrm{d}(\bar{y}, M(x))^{\frac{1}{\gamma}}$. On the other hand, if $\mathrm{d}(\bar{y}, M(x)) < \frac{1}{\kappa} \|x - \bar{x}\|^\gamma$ then

$$\mathrm{d}(x, M^{-1}(\bar{y})) \leq \frac{\kappa}{\|x - \bar{x}\|^{\gamma - 1}} \mathrm{d}(\bar{y}, M(x)) < \frac{\kappa}{(\kappa \mathrm{d}(\bar{y}, M(x)))^{\frac{\gamma - 1}{\gamma}}} \mathrm{d}(\bar{y}, M(x)) = \kappa' \mathrm{d}(\bar{y}, M(x))^{\frac{1}{\gamma}}.$$

□

The following example demonstrates that the order of pseudo-subregularity can be less than that of pseudo-regularity.

Example 2. *Consider the multifunction $M : \mathbb{R} \rightrightarrows \mathbb{R}^2$, $M(x) := (x, 1 - x^2) - \mathscr{B}_{\mathbb{R}^2}$ at $(\bar{x}, \bar{y}) = (0, (0, 0))$, where $\mathbb{R}^2$ is equipped with the Euclidean norm. Then $M^{-1}(0, 0) = [-1, 1]$ and therefore M is metrically subregular (and hence metrically pseudo-subregular of order 1) at $(\bar{x}, \bar{y})$. However, it is easy to see that M is not metrically pseudo-regular of order 1 in any direction $(u, (0, 0))$. Now let us verify that M is metrically pseudo-regular of order 2 in every direction $(u, (0, 0))$, $0 \neq u \in \mathbb{R}$. Consider arbitrary $(x, (y_1, y_2)) \in (\bar{x}, \bar{y}) + V_{\varepsilon, \delta}(u, (0, 0))$ with $x \neq 0$, $0 < \mathrm{d}((y_1, y_2), M(x)) \leq \delta |x|^2$ with $\varepsilon = \delta = 0.01$. Then*

$$\frac{\|(y_1, y_2)\|}{\|(x, (y_1, y_2))\|} \leq \|\frac{(x, (y_1, y_2))}{\|(x, (y_1, y_2))\|} - (\frac{u}{|u|}, (0, 0))\| \leq \delta$$

and $\|(y_1, y_2)\| \leq \frac{\delta}{1 - \delta} |x| \leq 0.011 |x|$ follows. Further

$$\begin{aligned} \mathrm{d}((y_1, y_2), M(x)) &= \sqrt{(x - y_1)^2 + (1 - x^2 - y_2)^2} - 1 = \zeta((x - y_1)^2 + (1 - x^2 - y_2)^2 - 1) \\ &= \zeta((x - y_1)^2 - (x^2 + y_2)(2 - x^2 - y_2)) \end{aligned}$$

where $\zeta = 1/(\sqrt{(x - y_1)^2 + (1 - x^2 - y_2)^2} + 1) \in [0.4999, 0.5001]$.

Now we will show that $(y_1, y_2) \in M((1 + f)x)$, where $f = 2\mathrm{d}((y_1, y_2), M(x))/x^2$. We have $(y_1, y_2) \in$ $M((1 + f)x)$ if and only if $\|((1 + f)x - y_1, 1 - (1 + f)^2 x^2 - y_2)\|^2 \leq 1$ and this holds true because of

$$\begin{aligned} &\|((1 + f)x - y_1, 1 - (1 + f)^2 x^2 - y_2)\|^2 - 1 \leq \|((1 + f)x - y_1, 1 - (1 + 2f)x^2 - y_2)\|^2 - 1 \\ &= (x - y_1)^2 - (x^2 + y_2)(2 - x^2 - y_2) + 2fx^2(1 - \frac{y_1}{x} + \frac{f}{2}) - 2fx^2(2 - 2fx^2 - 2(x^2 + y_2))) \\ &= (x - y_1)^2 - (x^2 + y_2)(2 - x^2 - y_2) - 2fx^2(1 + \frac{y_1}{x} - \frac{f}{2} - 2fx^2 - 2(x^2 + y_2)) \\ &= \mathrm{d}((y_1, y_2), M(x))(\frac{1}{\zeta} - 4(1 + \frac{y_1}{x} - \frac{f}{2} - 2fx^2 - 2(x^2 + y_2))) < 0. \end{aligned}$$

by taking into account $|y_1| \leq 0.011 |x|$, $f \leq 2\delta = 0.02$, $x^2 \leq 0.0001$ and $|y_2| \leq 0.00011$.

Hence $\mathrm{d}(x, M^{-1}(y_1,y_2)) \leq f|x| = 2\mathrm{d}((y_1,y_2),M(x))/|x-\bar{x}|$ *showing that* M *is metrically pseudo-regular of order* 2 *in direction* u.

Note that the mapping $x \to (x, 1-x^2)$ *is not 2-regular at* 0 *with respect to* $\mathscr{B}_{\mathbb{R}^2}$ *in any direction* u *in the sense of [4, Definition 1].*

4 Pointbased characterizations of metric pseudo-(sub)regularity

Our sufficient conditions for metric pseudo-(sub)regularity are only valid under the following assumption:

(A1) We say that the multifunction $M : X \rightrightarrows Y$ between Banach spaces fulfills assumption (A1) at $\bar{x} \in X$ if either

1. both X and Y are Asplund spaces, or
2. Y is Fréchet smooth, or
3. M is of the form $M(x) = F(x) + S(x)$, where the mapping $F : X \to Y$ is continuously differentiable in a neighborhood of $\bar{x}$ and $S : X \rightrightarrows Y$ is either a generalized polyhedral multifunction or a closed convex multifunction.

Theorem 1. *Let the closed multifunction* $M : X \rightrightarrows Y$ *be given by* (7), *let* $(\bar{x},\bar{y}) \in \operatorname{gph} M$, *let* $(u,v) \in X \times Y$ *and let* $\gamma_i \geq 1$, $i = 1,\ldots,s$.

1. *Assume that assumption (A1) is fulfilled and that there do not exist sequences* $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \in \mathscr{S}_{X\times Y}, (x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*, \ldots, y_{ks}^*) \in \mathscr{S}_{Y^*}$ *with* $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $x_k^* \to 0$ *and*
$$(x_k^*, (-\|x_k' - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|x_k' - \bar{x}\|^{1-\gamma_s} y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k', y_k'); \operatorname{gph} M),$$
where $x_k' := \bar{x} + t_k u_k$, $y_k' := \bar{y} + t_k v_k$. *Then* M *is metrically pseudo-regular of order* $(\gamma_1,\ldots,\gamma_s)$ *in direction* (u,v) *near* $(\bar{x},\bar{y})$.

2. *If assumption (A1) is fulfilled and there do not exist sequences* $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \in \mathscr{S}_{X\times Y}, (x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*, \ldots, y_{ks}^*) \in \mathscr{S}_{Y^*}$, $(x_k', y_k') \in X \times Y$ *with* $\|u_k\| \to 1$, $\|u\|u_k \to u$, $t_k^{1-\gamma_i} v_{ki} \to 0$, $i = 1,\ldots,s$, $x_k^* \to 0$, $(x_k', y_k') = (\bar{x},\bar{y}) + t_k(u_k,v_k)$, $x_k' \neq \bar{x}$, $y_k' \neq \bar{y}$,
$$(x_k^*, (-\|x_k' - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|x_k' - \bar{x}\|^{1-\gamma_s} y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k', y_k'); \operatorname{gph} M) \tag{10}$$
and
$$\lim_{k\to\infty} \frac{\sum_{i=1}^s \langle y_{ki}^*, \|x_k' - \bar{x}\|^{1-\gamma_i}(y_{ki}' - \bar{y}_i)\rangle}{\|(\|x_k' - \bar{x}\|^{1-\gamma_1}(y_{k1}' - \bar{y}_1), \ldots, \|x_k' - \bar{x}\|^{1-\gamma_s}(y_{ks}' - \bar{y}_s))\|} = 1, \tag{11}$$
then M *is metrically pseudo-subregular of order* $(\gamma_1,\ldots,\gamma_s)$ *in direction* u *at* $(\bar{x},\bar{y})$.

3. *Assume that there exist sequences* $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \in \mathscr{S}_{X\times Y}, (x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*, \ldots, y_{ks}^*) \in \mathscr{S}_{Y^*}$ *with* $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $x_k^* \to 0$ *and*
$$(x_k^*, (-\|x_k - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|x_k - \bar{x}\|^{1-\gamma_s} y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k, y_k); \operatorname{gph} M),$$
where $x_k := \bar{x} + t_k u_k$, $y_k := \bar{y} + t_k v_k$. *Then* M *is not metrically pseudo-regular of order* $(\gamma_1,\ldots,\gamma_s)$ *in direction* (u,v) *near* $(\bar{x},\bar{y})$.

Proof. We proof the first part by contraposition. Assuming on the contrary that M is not pseudo-regular of order $(\gamma_1,\ldots,\gamma_s)$ in direction (u,v), we can find for each k elements $(x_k,y_k)\in(\bar{x},\bar{y})+V_{\frac{1}{k},\frac{1}{k}}(u,v)$, $y_k=(y_{k1},\ldots,y_{ks})$ such that $x_k\neq\bar{x}$, $\mathrm{d}(y_{ki},M_i(x_k))\leq\frac{1}{k}\|x_k-\bar{x}\|^{\gamma_i},\ i=1,\ldots,s$ and

$$\mathrm{d}(x_k,M^{-1}(y_k))>k\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\mathrm{d}(y_{ki},M_i(x_k)).$$

Then we can choose for each k some $\check{y}_k=(\check{y}_{k1},\ldots,\check{y}_{ks})\in M(x_k)$, such that $\|y_{ki}-\check{y}_{ki}\|\leq 2\mathrm{d}(y_{ki},M_i(x_k))$ and

$$\mathrm{d}(x_k,M^{-1}(y_k))>k\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\|y_{ki}-\check{y}_{ki}\|.$$

Define $\Psi_k:Y\to\mathbb{R}$ by

$$\Psi_k(y_1,\ldots,y_s)=\|(\|x_k-\bar{x}\|^{1-\gamma_1}y_1,\ldots,\|x_k-\bar{x}\|^{1-\gamma_s}y_s)\|.$$

Then $\varepsilon:=\Psi_k(\check{y}_k-y_k)\leq\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\|y_{ki}-\check{y}_{ki}\|\leq\frac{2s}{k}\|x_k-\bar{x}\|$ and by Ekeland's variational principle there are $(\tilde{x}_k,\tilde{y}_k)\in\mathrm{gph}M$ such that $\Psi_k(\tilde{y}_k-y_k)\leq\Psi_k(\check{y}_k-y_k)$, $\|(\tilde{x}_k,\tilde{y}_k)-(x_k,\check{y}_k)\|\leq\frac{\sqrt{k}}{2s}\varepsilon\leq\frac{1}{\sqrt{k}}\|x_k-\bar{x}\|$ and

$$\Psi_k(\tilde{y}_k-y_k)\leq\Psi_k(y-y_k)+\frac{2s}{\sqrt{k}}\|(x,y)-(\tilde{x}_k,\tilde{y}_k)\|,\ \forall(x,y)\in\mathrm{gph}M. \tag{12}$$

Note that

$$0<(1-\frac{1}{\sqrt{k}})\|x_k-\bar{x}\|\leq\|\tilde{x}_k-\bar{x}\|\leq(1+\frac{1}{\sqrt{k}})\|x_k-\bar{x}\|\ \forall k\geq 2. \tag{13}$$

Further we have

$$\mathrm{d}(x_k,M^{-1}(y_k))>k\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\|y_{ki}-\check{y}_{ki}\|\geq k\Psi_k(\check{y}_k-y_k)=k\varepsilon$$

and therefore

$$\mathrm{d}(\tilde{x}_k,M^{-1}(y_k))>k\varepsilon-\|\tilde{x}_k-x_k\|\geq(k-\frac{\sqrt{k}}{2s})\varepsilon, \tag{14}$$

showing $y_k\notin M(\tilde{x}_k)$ and $y_k\neq\tilde{y}_k$.

Since $\|\check{y}_k-y_k\|\leq\sum_{i=1}^{s}\|\check{y}_{ki}-y_{ki}\|\leq\frac{2}{k}\sum_{i=1}^{s}\|x_k-\bar{x}\|^{\gamma_i}\leq\frac{2s}{k}\|x_k-\bar{x}\|$ we obtain

$$\begin{aligned}\|(\tilde{x}_k,\tilde{y}_k)-(x_k,y_k)\| &\leq \|(\tilde{x}_k,\tilde{y}_k)-(x_k,\check{y}_k)\|+\|(x_k,\check{y}_k)-(x_k,y_k)\|\\ &\leq (\frac{1}{\sqrt{k}}+\frac{2s}{k})\|x_k-\bar{x}\|\leq(\frac{1}{\sqrt{k}}+\frac{2s}{k})\|(x_k,y_k)-(\bar{x},\bar{y})\|,\end{aligned}$$

implying

$$\lim_{k\to\infty}\frac{(\tilde{x}_k,\tilde{y}_k)-(\bar{x},\bar{y})}{\|(\tilde{x}_k,\tilde{y}_k)-(\bar{x},\bar{y})\|}-\frac{(x_k,y_k)-(\bar{x},\bar{y})}{\|(x_k,y_k)-(\bar{x},\bar{y})\|}=0. \tag{15}$$

In case that both X and Y are Asplund spaces, by using the fuzzy (semi-Lipschitzian) sum rule (see for example [26, Theorem 2.33])), for arbitrary $\delta_k>0$ we can find points $(x'_k,y'_k),(x''_k,y''_k)$ in $(\tilde{x}_k,\tilde{y}_k)+\delta_k\mathscr{B}_{X\times Y}$ and linear functionals $(\hat{x}^*_k,\hat{y}^*_k)\in\hat{N}((x'_k,y'_k),\mathrm{gph}M)$ and $(\acute{x}^*_k,\acute{y}^*_k)\in\hat{\partial}(\Psi_k(y''_k-y_k)+\frac{2s}{\sqrt{k}}\|(x''_k,y''_k)-(\tilde{x}_k,\tilde{y}_k)\|)$ such that $\|(\hat{x}^*_k,\hat{y}^*_k)+(\acute{x}^*_k,\acute{y}^*_k)\|\leq\delta_k$. Taking $\delta_k=\frac{1}{k}\min\{\|\tilde{x}_k-\bar{x}\|,\|\tilde{y}_k-y_k\|\}$ we have $y''_k \neq y_k$ and therefore, by convex analysis there is some linear functional $\check{y}^*_k = (\check{y}^*_{k1}, \ldots, \check{y}^*_{ks}) \in \mathscr{S}_{Y^*}$ such that

$$\tilde{\check{y}}^*_k := (\|x_k - \bar{x}\|^{1-\gamma_1} \check{y}^*_{k1}, \ldots, \|x_k - \bar{x}\|^{1-\gamma_s} \check{y}^*_{ks}) \in \hat{\partial}\Psi_k(y''_k - y_k)$$

and $\|\hat{y}^*_k + \tilde{\check{y}}^*_k\| \leq \delta_k + \frac{2s}{\sqrt{k}}$. Further we have $\|\hat{x}^*_k\| \leq \delta_k + \frac{2s}{\sqrt{k}}$. Defining $\tilde{y}^*_{ki} := -\hat{y}^*_{ki}\|x'_k - \bar{x}\|^{\gamma_i - 1}$, $i = 1, \ldots, s$ we conclude $\|\tilde{y}^*_{ki}\left(\frac{\|x_k - \bar{x}\|}{\|x'_k - \bar{x}\|}\right)^{\gamma_i - 1} - \check{y}^*_{ki}\| \leq (\delta_k + \frac{2s}{\sqrt{k}})\|x_k - \bar{x}\|^{\gamma_i - 1} \to 0$, $i = 1, \ldots, s$ and therefore $\|\tilde{y}^*_k\| \to 1$ by taking into account (13) and consequently

$$\begin{aligned} 0 &< (1 - \frac{1}{k})(1 - \frac{1}{\sqrt{k}})\|x_k - \bar{x}\| \leq (1 - \frac{1}{k})\|\tilde{x}_k - \bar{x}\| \leq \|x'_k - \bar{x}\| \\ &\leq (1 + \frac{1}{k})\|\tilde{x}_k - \bar{x}\| \leq (1 + \frac{1}{k})(1 + \frac{1}{\sqrt{k}})\|x_k - \bar{x}\| \ \forall k \geq 2. \end{aligned}$$

Setting $t_k := \|(x'_k, y'_k) - (\bar{x}, \bar{y})\|$, $(u_k, v_k) := ((x'_k, y'_k) - (\bar{x}, \bar{y}))/t_k$, $x^*_k := \hat{x}^*_k / \|\tilde{y}^*_k\|$, $y^*_k := \tilde{y}^*_k / \|\tilde{y}^*_k\|$ we obtain the contradiction

$$(x^*_k, (-\|x'_k - \bar{x}\|^{1-\gamma_1} y^*_{k1}, \ldots, -\|(x'_k - \bar{x}\|^{1-\gamma_s} y^*_{ks})) \in \hat{N}(x'_k, y'_k), \mathrm{gph} M),$$

since $(x'_k, y'_k) = (\bar{x}, \bar{y}) + t_k(u_k, v_k)$, $u_k \neq 0$, $x^*_k \to 0$ and $\|(u, v)\|(u_k, v_k) \to (u, v)$ follows from (15) and $\|(x'_k, y'_k) - (\tilde{x}_k, \tilde{y}_k)\| \leq \delta_k \leq \frac{1}{k}\|(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y})\|$.

In case that Y admits a Fréchet smooth renorm we can assume without loss of generality that the original norm $\|\cdot\|$ on Y is Fréchet smooth. Because of $\tilde{y}_k \neq y_k$ the function Ψ_k is Fréchet differentiable at $\tilde{y}_k - y_k$ and there exists a functional $\tilde{y}^*_k = (\tilde{y}^*_{k1}, \ldots, \tilde{y}^*_{ks}) \in \mathscr{S}_{Y^*}$ with

$$\nabla\Psi_k(\tilde{y}_k - y_k) = (\|x_k - \bar{x}\|^{1-\gamma_1} \tilde{y}^*_{k1}, \ldots, \|x_k - \bar{x}\|^{1-\gamma_s} \tilde{y}^*_{ks})$$

By the definition of Fréchet differentiability there exists a positive radius ρ_k such that $\Psi_k(y - y_k) - \Psi_k(\tilde{y}_k - y_k) \leq \langle \nabla\Psi_k(\tilde{y}_k - y_k), y - \tilde{y}_k \rangle + \frac{s}{\sqrt{k}}\|y - \tilde{y}_k\|$, $\forall y \in \tilde{y}_k + \rho_k \mathscr{B}_Y$, implying

$$0 \leq \nabla\Psi(\tilde{y}_k - y_k)(y - \tilde{y}_k) + \frac{3s}{\sqrt{k}}\|(x, y) - (\tilde{x}_k, \tilde{y}_k)\|$$

for all $(x, y) \in \mathrm{gph} M \cap ((\tilde{x}_k, \tilde{y}_k) + \rho_k \mathscr{B}_{X \times Y})$ and consequently $(0, -\nabla\Psi(\tilde{y}_k - y_k)) \in \hat{N}_{\frac{3s}{\sqrt{k}}}((\tilde{x}_k, \tilde{y}_k); \mathrm{gph} M)$. Because of (13) we have $\tilde{u}_k \neq 0$ and the functionals $\hat{y}^*_k = (\hat{y}^*_{k1}, \ldots, \hat{y}^*_{ks})$ with

$$\hat{y}^*_{ki} := \left(\frac{\|x_k - \bar{x}\|}{\|\tilde{x}_k - \bar{x}\|}\right)^{1-\gamma_i} \tilde{y}^*_{ki}, \ i = 1, \ldots, s$$

satisfy $\lim_{k\to\infty} \|\hat{y}^*_k\| = 1$. Taking $t_k := \|(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y})\|$, $(u_k, v_k) := (\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y}))/t_k$, $x^*_k = 0$, $y^*_k = \hat{y}^*_k / \|\hat{y}^*_k\|$, $\varepsilon_k = \frac{3s}{\sqrt{k}\|\hat{y}^*_k\|}$, we obtain the contradiction

$$(x^*_k, (-\|\tilde{x}_k - \bar{x}\|^{1-\gamma_1} y^*_{k1}, \ldots, -\|(\tilde{x}_k - \bar{x}\|^{1-\gamma_s} y^*_{ks})) \in \hat{N}_{\varepsilon_k}((\tilde{x}_k, \tilde{y}_k), \mathrm{gph} M).$$

Next consider the case when $M(x) = F(x) + S(x)$ where the mapping $F : X \to Y$ is continuously differentiable near $\bar{x}$ and S is either a generalized polyhedral multifunction or a closed convex mapping. Let $F_i : X \to Y_i$, $i = 1, \ldots, s$ denote the components of F. Then we can find a radius $0 < \rho_k < \frac{1}{k}$ such that

$$\|F_i(x) - F_i(\tilde{x}_k) - \nabla F_i(\tilde{x}_k)(x - \tilde{x}_k)\| \leq \frac{s}{\sqrt{k}}\|x_k - \bar{x}\|^{\gamma_i - 1}\|x - \tilde{x}_k\| \ \forall x \in \tilde{x}_k + \rho_k \mathscr{B}_X, \ i = 1, \ldots, s$$

We can assume without loss of generality that for all k the ball $x_k+\rho_k\mathscr{B}_X$ is contained in some neighborhood of $\bar{x}$ where F is Lipschitz continuous with modulus L. Since

$$|\Psi(y)-\Psi(y')|\leq\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\|y_i-y_i'\|\ \forall y,y'\in Y,$$

we deduce from (12) that

$$\Psi(F(\tilde{x}_k)+\tilde{p}_k-y_k)\leq\Psi(F(\tilde{x}_k)+\nabla F(\tilde{x}_k)(x-\tilde{x}_k)+p-y_k)+\frac{3s}{\sqrt{k}}(1+L)\|(x-\tilde{x}_k,p-\tilde{p}_k)\|=:\tilde{\Psi}(x,p)$$

holds for all $(x,p)\in\operatorname{gph}S\cap(\tilde{x}_k+\rho_k\mathscr{B}_X)\times Y$, where $\tilde{p}_k:=\tilde{y}_k-F(\tilde{x}_k)$. In case that S is a generalized polyhedral multifunction we obtain by using [12, Lemma 4.2] that $0\in\hat{\partial}\tilde{\Psi}(\tilde{x}_k,\tilde{p}_k)+N((\tilde{x}_k,\tilde{p}_k);\operatorname{gph}S)$. Hence there is some linear functional linear $(\hat{x}_k^*,\hat{p}_k^*)\in N((\tilde{x}_k,\tilde{p}_k),\operatorname{gph}S)\cap(-\hat{\partial}\tilde{\Psi}(\tilde{x}_k,\tilde{p}_k))$. From [12, Lemma 3.4] it follows that for arbitrary $\delta_k>0$ there is some $(x_k',p_k')\in(\tilde{x}_k,\tilde{p}_k)+\delta_k\mathscr{B}_{X\times Y}$ with $(\hat{x}_k^*,\hat{p}_k^*)\in\hat{N}((x_k',p_k');\operatorname{gph}S)$ and by convex analysis it follows that there is some linear functional $\check{y}_k^*=(\check{y}_{k1}^*,\ldots,\check{y}_{ks}^*)\in\mathscr{S}_{Y^*}$ with

$$\|(\sum_{i=1}^{s}\|x_k-\bar{x}\|^{1-\gamma_i}\nabla F_i(\tilde{x}_k)^*\check{y}_{ki}^*+\hat{x}_k^*,(\|x_k-\bar{x}\|^{1-\gamma_1}\check{y}_{k1}^*,\ldots,\|x_k-\bar{x}\|^{1-\gamma_s}\check{y}_{ks}^*)+\hat{p}_k^*)\|\leq\frac{3s}{\sqrt{k}}(1+L).$$

Setting $\tilde{y}_{ki}^*:=-\hat{p}_{ki}^*\|x_k'-\bar{x}\|^{\gamma_i-1}$, $\tilde{x}_k^*:=\hat{x}_k^*-\nabla F(x_k')^*\hat{p}_k^*$ we have

$$\begin{aligned}&\langle\tilde{x}_k^*,x-x_k'\rangle+\langle-(\|x_k'-\bar{x}\|^{1-\gamma_1}\tilde{y}_{k1}^*,\ldots,\|x_k'-\bar{x}\|^{1-\gamma_s}\tilde{y}_{ks}^*),F(x)+p-F(x_k')-p_k'\rangle\\ =\ &\langle\hat{x}_k^*,x-x_k'\rangle+\langle\hat{p}_k^*,p-p_k'\rangle+\langle\hat{p}_k^*,F(x)-F(x_k')-\nabla F(x_k')(x-x_k')\rangle\end{aligned}$$

for all $(x,p)\in X\times Y$ and therefore

$$(\tilde{x}_k^*,-(\|x_k'-\bar{x}\|^{1-\gamma_1}\tilde{y}_{k1}^*,\ldots,\|x_k'-\bar{x}\|^{1-\gamma_s}\tilde{y}_{ks}^*))\in\hat{N}((x_k',F(x_k')+p_k');\operatorname{gph}M).$$

Taking $\delta_k\leq\frac{1}{k}\min\{\|\tilde{x}_k-\bar{x}\|,\|\tilde{y}_k-y_k\|\}$ small enough such that $\|\nabla F_i(\tilde{x}_k)^*-\nabla F_i(x_k')^*\|\leq\frac{1}{k}\|x_k-\bar{x}\|^{\gamma_i-1}$, $i=1,\ldots,s$ and taking into account $\|\hat{x}_k^*-\nabla F(\tilde{x}_k)^*\hat{p}_k^*\|\leq(1+\|\nabla F(\tilde{x}_k)^*\|)\frac{3s}{\sqrt{k}}(1+L)$ we obtain

$$\begin{aligned}\|\tilde{x}_k^*\|&\leq\|\hat{x}_k^*-\nabla F(\tilde{x}_k)^*\hat{p}_k^*\|+\|(\nabla F(\tilde{x}_k)^*-\nabla F(x_k')^*)\hat{p}_k^*\|\\&\leq\frac{3s}{\sqrt{k}}(1+\|\nabla F(\tilde{x}_k)^*\|)(1+L)+\sum_{i=1}^{s}\|(\nabla F_i(\tilde{x}_k)^*-\nabla F_i(x_k')^*)\hat{p}_{ki}^*\|\\&\leq\frac{3s}{\sqrt{k}}(1+\|\nabla F(\tilde{x}_k)^*\|+\sum_{i=1}^{s}\|\nabla F_i(\tilde{x}_k)^*-\nabla F_i(x_k')^*\|)(1+L)\\&\qquad+\sum_{i=1}^{s}\|(\nabla F_i(\tilde{x}_k)^*-\nabla F_i(x_k')^*)\|x_k-\bar{x}\|^{1-\gamma_i}\check{y}_{ki}^*\|\\&\leq\frac{3s}{\sqrt{k}}(1+\|\nabla F(\tilde{x}_k)^*\|+\sum_{i=1}^{s}\|\nabla F_i(\tilde{x}_k)^*-\nabla F_i(x_k')^*\|)(1+L)+\frac{s}{k}\end{aligned}$$

showing $\tilde{x}_k^*\to 0$ as $k\to\infty$. Further we have for each $i=1,\ldots,s$

$$\|\check{y}_{ki}^*-\tilde{y}_{ki}^*\|\leq\|x_k-\bar{x}\|^{\gamma_i-1}\left\|\|x_k-\bar{x}\|^{1-\gamma_i}\check{y}_{ki}^*+\hat{p}_{ki}^*\right\|\leq\frac{3s}{\sqrt{k}}(1+L)\|x_k-\bar{x}\|^{\gamma_i-1}$$

implying $\|\tilde{y}_k^*\| \to 1$. Setting $y_k' = F(x_k') + p_k'$, $t_k = \|(x_k', y_k') - (\bar{x}, \bar{y})\|$, $(u_k, v_k) := ((x_k', y_k') - (\bar{x}, \bar{y}))/t_k$, $x_k^* := \tilde{x}_k^*/\|\tilde{y}_k^*\|$, $y_k^* := \tilde{y}_k^*/\|\tilde{y}_k^*\|$ we obtain the contradiction

$$(x_k^*, (-\|x_k' - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|(x_k' - \bar{x}\|^{1-\gamma_s} y_{ks}^*)) \in \hat{N}(x_k', y_k'), \mathrm{gph} M).$$

In case that S is a closed convex multifunction we have $0 \in \hat{\partial}\tilde{\Psi}(\tilde{x}_k, \tilde{p}_k) + \hat{N}((\tilde{x}_k, \tilde{p}_k); \mathrm{gph} S)$ by convex analysis and similar arguments as before yield the contradiction.

Next we show the assertion about pseudo-subregularity by contraposition. Assuming that M is not pseudo-subregular of order $(\gamma_1, \ldots, \gamma_s)$ in direction u, we can find for each k an element $x_k \in \bar{x} + V_{\frac{1}{k}, \frac{1}{k}}(u)$ such that

$$\mathrm{d}(x_k, M^{-1}(\bar{y})) > k \sum_{i=1}^{s} \|x_k - \bar{x}\|^{1-\gamma_i} \mathrm{d}(\bar{y}_i, M_i(x_k)).$$

This implies $x_k \neq \bar{x}$ and for each i

$$\|x_k - \bar{x}\|^{1-\gamma_i} \mathrm{d}(\bar{y}_i, M_i(x_k)) < \frac{1}{k} \mathrm{d}(x_k, M^{-1}(\bar{y})) \leq \frac{1}{k} \|x_k - \bar{x}\|$$

and therefore

$$\mathrm{d}(\bar{y}_i, M_i(x_k)) \leq \frac{1}{k} \|x_k - \bar{x}\|^{\gamma_i}.$$

Then we can proceed as before with $y_k = \bar{y}$, $v = 0$ to find the sequence $(\tilde{x}_k, \tilde{y}_k)$. Since $\Psi_k(\tilde{y}_k - y_k) \leq \Psi_k(\check{y}_k - y_k) = \varepsilon_k \leq \frac{2s}{k} \|x_k - \bar{x}\|$ we conclude $\|\tilde{y}_{ki} - \bar{y}_i\| \leq \frac{2s}{k} \|x_k - \bar{x}\|^{\gamma_i}$, $i = 1, \ldots, s$. Hence, if we define the sequences $\tilde{t}_k := \|(\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y})\|$, $(\tilde{u}_k, \tilde{v}_k) = ((\tilde{x}_k, \tilde{y}_k) - (\bar{x}, \bar{y}))/\tilde{t}_k$ we obtain $(\tilde{u}_k, \tilde{v}_k) \in \mathscr{S}_{X \times Y}$ and, together with (13), $\|\tilde{u}_k\| \to 1$, $\|u\| \tilde{u}_k \to u$, $\tilde{t}_k^{1-\gamma_i} \tilde{v}_{ki} \to 0$, $i = 1, \ldots, s$. Now, if we choose δ_k small enough such that $\delta_k \leq \frac{1}{k} \|x_k - \bar{x}\|^{\max_i \gamma_i}$ also holds, then we can also conclude $\|u_k\| \to 1$, $\|u\| u_k \to u$, $t_k^{1-\gamma_i} v_{ki} \to 0$, $i = 1, \ldots, s$. There remains to show (11). In case that both X and Y are Asplund or in case that M is of the form $M(X) = F(x) + S(x)$, this follows from

$$\sum_{i=1}^{s} \langle \check{y}_{ki}^*, \|x_k - \bar{x}\|^{1-\gamma_i} (y_{ki}'' - \bar{y}_i) \rangle = \|(\|x_k - \bar{x}\|^{1-\gamma_1} (y_{k1}'' - \bar{y}_1), \ldots, \|x_k - \bar{x}\|^{1-\gamma_s} (y_{ks}'' - \bar{y}_s))\|,$$

where $y_k'' := \tilde{y}_k$ in case of $M(x) = F(x) + S(x)$, together with (13), our choice of δ_k and $y_k^* - \check{y}_k^* \to 0$. If Y is Fréchet smooth then we can use the relations

$$\sum_{i=1}^{s} \langle \tilde{y}_{ki}^*, \|x_k - \bar{x}\|^{1-\gamma_i} (\tilde{y}_{ki} - \bar{y}_i) \rangle = \|(\|x_k - \bar{x}\|^{1-\gamma_1} (\tilde{y}_{k1} - \bar{y}_1), \ldots, \|x_k - \bar{x}\|^{1-\gamma_s} (\tilde{y}_{ks} - \bar{y}_s))\|,$$

$y_k^* - \tilde{y}_k^* \to 0$ and (13) in order to show (11).

Now let us show the last part of the theorem. Assume that there exist sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k, v_k) \in \mathscr{S}_{X \times Y}$, $(x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*, \ldots, y_{ks}^*) \in \mathscr{S}_{Y^*}$ with $u_k \neq 0$, $\|(u, v)\| (u_k, v_k) \to (u, v)$, $x_k^* \to 0$ and

$$(x_k^*, (-\|x_k - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|x_k - \bar{x}\|^{1-\gamma_s} y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k, y_k), \mathrm{gph} M),$$

where $x_k := \bar{x} + t_k u_k$, $y_k := \bar{y} + t_k v_k$. By passing to a subsequence we can assume that $\|x_k - \bar{x}\| \leq 1$ and $\|x_k^*\| + \varepsilon_k < \frac{1}{8sk}$ holds for all k. Then for each k we can find some positive radius $\rho_k > 0$ such that

$$\langle x_k^*, x - x_k \rangle + \langle \hat{y}_k^*, y - y_k \rangle \leq (\varepsilon_k + \frac{1}{8sk}) \|(x - x_k, y - y_k)\|$$

holds for every $(x, y) \in \mathrm{gph} M \cap ((x_k, y_k) + \rho_k \mathscr{B}_{X \times Y})$, where $\hat{y}_k^* := (-\|x_k - \bar{x}\|^{1-\gamma_1} y_{k1}^*, \ldots, -\|x_k - \bar{x}\|^{1-\gamma_s} y_{ks}^*)$. Further we can find elements $z_{ki} \in \mathscr{S}_{Y_i}$ with $\langle y_{ki}^*, z_{ki} \rangle \geq \frac{1}{2} \|y_{ki}^*\|$, $i = 1, \ldots, s$. Now define

$\hat{y}_k = (\hat{y}_{k1},\dots,\hat{y}_{ks})$ by $\hat{y}_{ki} := y_{ki} - \alpha_k\|x_k-\bar{x}\|^{\gamma_i} z_{ki}$, $i = 1,\dots,s$, where $\alpha_k := \min\{\rho_k/(2sk\|x_k-\bar{x}\|), 1/k\}$. We obtain

$$\|\hat{y}_k - y_k\| \le \sum_{i=1}^s \alpha_k\|x_k-\bar{x}\|^{\gamma_i} \le s\alpha_k\|x_k-\bar{x}\|$$

and

$$\langle \hat{y}_k^*, \hat{y}_k - y_k\rangle = \sum_{i=1}^s \alpha_k\|x_k-\bar{x}\|\langle y_{ki}^*, z_{ki}\rangle \ge \frac{\alpha_k}{2}\|x_k-\bar{x}\|\sum_{i=1}^s \|y_{ki}^*\| \ge \frac{\alpha_k}{2}\|x_k-\bar{x}\| \tag{16}$$

For each $i \in \{1,\dots,s\}$ we have $\mathrm{d}(\hat{y}_{ki}, M_i(x_k)) \le \alpha_k\|x_k-\bar{x}\|^{\gamma_i}$ and thus $\sum_{i=1}^s \|x_k-\bar{x}\|^{1-\gamma_i}\mathrm{d}(\hat{y}_{ki}, M_i(x_k)) \le s\alpha_k\|x_k-\bar{x}\|$. On the other hand we have $\mathrm{d}(x_k, M^{-1}(\hat{y}_k)) \ge ks\alpha_k\|x_k-\bar{x}\|$ since if there would be some $\hat{x}_k \in x_k + ks\alpha_k\|x_k-\bar{x}\|\mathscr{B}_X$ with $(\hat{x}_k,\hat{y}_k) \in \mathrm{gph}\,M$ we would obtain

$$\begin{aligned}\|(\hat{x}_k,\hat{y}_k) - (x_k,y_k)\| &\le \|\hat{x}_k - x_k\| + \|\hat{y}_k - y_k\| \\ &\le (k+1)s\alpha_k\|x_k-\bar{x}\| \le \rho_k\frac{k+1}{2k} \le \rho_k\end{aligned}$$

and hence

$$\begin{aligned}\langle \hat{y}_k^*, \hat{y}_k - y_k\rangle &\le (\varepsilon_k + \frac{1}{8ks} + \|x_k^*\|)\|(\hat{x}_k - x_k, \hat{y}_k - y_k)\| \\ &\le (\varepsilon_k + \frac{1}{8ks} + \|x_k^*\|)(k+1)s\alpha_k\|x_k-\bar{x}\| \\ &< \frac{k+1}{4k}\alpha_k\|x_k-\bar{x}\|\end{aligned}$$

contradicting (16) because of $\frac{k+1}{4k} \le \frac{1}{2}$. Thus

$$\mathrm{d}(x_k, M^{-1}(\hat{y}_k)) \ge k\sum_{i=1}^s \|x_k-\bar{x}\|^{1-\gamma_i}\mathrm{d}(\hat{y}_{ki}, M_i(x_k))$$

and since $\mathrm{d}(\hat{y}_{ki}, M_i(x_k)) \le \alpha_k\|x_k-\bar{x}\|^{\frac{1}{q_i}}$, $i = 1,\dots,s$, $\alpha_k \to 0$, $t_k^{-1}((x_k,\hat{y}_k) - (\bar{x},\bar{y})) = (u_k, v_k + t_k^{-1}(\hat{y}_k - y_k))$ and

$$0 \le \lim_{k\to\infty}\|\frac{\hat{y}_k - y_k}{t_k}\| \le \lim_{k\to\infty} s\alpha_k\frac{\|x_k-\bar{x}\|}{t_k} = \lim_{k\to\infty} s\alpha_k\|u_k\| = 0,$$

M is not metrically pseudo-regular of order $(\gamma_1,\dots,\gamma_s)$ in direction (u,v) at $(\bar{x},\bar{y})$. □

Remark 1. *It follows from the proof that the sufficient conditions for metric pseudo-regularity respectively metric pseudo-subregularity remain valid for $\varepsilon_k \equiv 0$ in case that both X and Y are Asplund spaces or $M(x)$ is of the form $M(x) = F(x) + S(x)$ with F continuously differentiable at $\bar{x}$ and S is either a generalized polyhedral multifunction or a closed convex multifunction.*

Applying the second part Theorem 1 with $s = 1$ and $\gamma_1 = 1$ we obtain the following new characterization of (directional) metric subregularity, which improves the ones of [10], [11], [12] by the additional condition (17).

Corollary 1. *Given the multifunction $M : X \rightrightarrows Y$, a point $(\bar{x},\bar{y}) \in \mathrm{gph}\,M$ and a direction $u \in X$, assume that M fulfills asuumption (A1) at $\bar{x}$ and that there do not exist sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(x_k^*) \in X^*$, $(y_k^*) \in \mathscr{S}_{Y^*}$ with $\|u_k\| \to 1$, $\|u\|u_k \to u$, $v_k \to 0$, $x_k^* \to 0$,*

$$(x_k^*, -y_k^*) \in \hat{N}_{\varepsilon_k}((x_k', y_k'); \mathrm{gph}\,M)$$

and

$$\lim_{k\to\infty}\frac{\langle y_k^*, y_k'-\bar y\rangle}{\|y_k'-\bar y\|}=1, \tag{17}$$

where $x_k' := \bar x + t_k u_k \neq \bar x$, $y_k' := \bar y + t_k v_k \neq \bar y$. *Then* M *is metrically subregular in direction* u *at* $(\bar x, \bar y)$.

Remark 2. *By taking into account* (14) *and the fact, that in case of proving pseudo-subregularity we have* $y_k = \bar y$, *the sufficient conditions for metric pseudo-subregularity of Theorem 1(2.) could be strengthened by the additional requirement* $x_k' \notin M^{-1}(\bar y)$. *However, we do not use this condition, because on the one hand the required information on the set* $M^{-1}(\bar y)$ *is in general not available and on the other hand it makes it more challenging or even impossible to develop comprehensive calculus rules, see e.g. [10, Example 1.2]. We also refer to [25, Remark 3.5] for more discussion on this topic.*

Example 3. *Let us apply Theorem 1 and Corollary 1 to the multifunction* $M:\mathbb{R}\rightrightarrows\mathbb{R}^2$, $M(x) := F(x) - \mathscr{B}_{\mathbb{R}^2}$ *of Example 2 at* $(\bar x,\bar y) = (0,(0,0))$, *where* $F(x) := (x, 1-x^2)$. *Given* $x\in\mathbb{R}$ *and* $p=(p_1,p_2)\in\mathbb{R}^2$ *we obtain*

$$\hat N((x,F(x)-p);\operatorname{gph} M) = \begin{cases} \emptyset & \text{if } p \neq \mathscr{B}_{\mathbb{R}^2},\\ \{(0,(0,0))\} & \text{if } p\in \operatorname{int}\mathscr{B}_{\mathbb{R}^2},\\ \mathbb{R}_+\{(p_1-2xp_2,(-p_1,-p_2))\} & \text{if } p\in\mathscr{S}_{R^2}\end{cases}$$

We first show that M *is metrically subregular at* $(\bar x,\bar y)$. *Consider sequences* $(u_k,v_k)\in\mathscr{S}_{\mathbb{R}\times\mathbb{R}^2}$, $(x_k^*)\in\mathbb{R}$, $(y_k^*)\in\mathscr{S}_{\mathbb{R}_2}$ *with* $\|u_k\|\to 1$, $v_k\to(0,0)$, $x_k^*\to 0$ *such that* $(x_k^*,-y_k^*)\in\hat N((x_k',y_k');\operatorname{gph} M)$, *where* $(x_k',y_k') = (t_ku_k,t_kv_k)$ *and according to Remark 1 we can take* $\varepsilon_k = 0$. *Using* $y_k^*\in\mathscr{S}_{\mathbb{R}^2}$ *we obtain* $t_kv_k = F(x_k') - p_k$ *with* $p_k\in\mathscr{S}_{\mathbb{R}^2}$, $y_k^* = p_k$ *and* $x_k^* = p_{k1} - 2x_k'p_{k2}\to 0$. *Since* $\|F(x_k')\|^2 = 1 - x_k'^2 + x_k'^4 < 1$ *for all* k *sufficiently large we have*

$$\langle y_k^*, y_k'-\bar y\rangle = \langle p_k, F(x_k') - p_k\rangle \le \|p_k\|\|F(x_k')\| - \|p_k\|^2 = \|F(x_k')\| - 1 < 0$$

and thus (17) *cannot be fulfilled, showing that* M *is metrically subregular at* $(\bar x,\bar y)$.

Next we show by contradiction that M *is metrically pseudo-regular of order* 2 *in direction* $(u,(0,0))$ *for every* $u\neq 0$. *Assume there are sequences* $(u_k,v_k)\in\mathscr{S}_{\mathbb{R}\times\mathbb{R}^2}$, $(x_k^*)\in\mathbb{R}$, $(y_k^*)\in\mathscr{S}_{\mathbb{R}_2}$ *with* $\|u_k\|\to 1$, $v_k\to(0,0)$, $x_k^*\to 0$ *satisfying* $(x_k^*, -\|x_k'-\bar x\|^{-1}y_k^*)\in\hat N((x_k',y_k');\operatorname{gph} M)$ *and* $\|(u,(0,0))\|(u_k,v_k)\to(u,(0,0))$, *where* $(x_k',y_k') = (t_ku_k,t_kv_k)$. *As before we obtain from* $y_k^*\in\mathscr{S}_{\mathbb{R}^2}$ *that* $t_kv_k = F(x_k) - p_k$ *with* $p_k\in\mathscr{S}_{\mathbb{R}^2}$ *and consequently* $y_k^* = p_k$ *and* $x_k^* = \|x_k'-\bar x\|^{-1}(p_{k1} - 2x_k'p_{k2})$. *From* $v_k = t_k^{-1}(F(x_k') - p_k) = u_kx_k'^{-1}(x_k' - p_{k1}, 1 - x_k'^2 - p_{k2})\to(0,0)$ *and* $|u_k|\to 1$ *we conclude* $p_{k1}/|x_k'|\to\operatorname{sign} u$ *and* $p_{k2}\to 1$, *showing* $x_k^* = p_{k1}/|x_k'| - 2p_{k2}\operatorname{sign} u_k\to -\operatorname{sign} u\neq 0$, *a contradiction.*

Finally we show that for every $u\neq 0$ *and every* $\gamma\in[1,2)$ M *is not metrically pseudo-regular of order* γ *in direction* $(u,(0,0))$. *Let* $u\neq 0$ *and* $\gamma\in[1,2)$ *be arbitrarily fixed and let* $x_k' := \frac{1}{k}\operatorname{sign} u$, $p_k := (x_k', \sqrt{1-x_k'^2})\in\mathscr{S}_{\mathbb{R}^2}$, $y_k' := F(x_k') - p_k = (0, 1-\sqrt{1-1/k^2})$, $y_k^* := p_k$, $x_k^* := |x_k'|^{1-\gamma}(p_{k1} - 2x_k'p_{k2}) = k^{\gamma-2}\operatorname{sign} u(1-2\sqrt{1-1/k^2})$, $(u_k,v_k) := (x_k',y_k')/\|(x_k',y_k')\|$, $t_k := \|(x_k',y_k')\|$. *Then obviously* $x_k^*\to 0$ *and since* $\|y_k'\| = 1-\sqrt{1-1/k^2} = \xi_k/k^2$ *with* $\xi_k\to 1/2$ *we obtain* $\lim kt_k = \lim\|(kx_k',ky_k')\| = 1$ *and thus* $t_k\to 0$, $\lim v_k = \lim ky_k' = (0,0)$. *Together with the fact that* $(x_k^*, -|x_k'|^{1-\gamma}y_k^*)\in\hat N((x_k',y_k');\operatorname{gph} M)$ *we conclude from Theorem 1(3.), that* M *is not metrically pseudo-regular of order* γ *in direction* $(u,(0,0))$.

We consider now the following coderivative-like construction:

Definition 2. *Let* $M : X\rightrightarrows Y$ *be of the form* (7), $(\bar x,\bar y)\in\operatorname{gph} M$, $(u,v)\in X\times Y$ *and* $\gamma_i\ge 1$, $i = 1,\ldots,s$.

1. *The* reversed mixed pseudo-coderivative of order $\gamma := (\gamma_1,\dots,\gamma_s)$ in direction (u,v) *is the multifunction* $\tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(u,v)) : Y^* \rightrightarrows X^*$ *whose values* $\tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(u,v))(y^*)$ *consist of such* $x^* \in X^*$ *for which there are sequences* $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}, (x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*,\dots,y_{ks}^*) \in Y^*$ *with* $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $y_k^* \xrightarrow{w^*} y^*$, $x_k^* \to x^*$ *and*
$$(x_k^*,(-\|x_k'-\bar{x}\|^{1-\gamma_1}y_{k1}^*,\dots,-\|x_k'-\bar{x}\|^{1-\gamma_s}y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k',y_k');\operatorname{gph}M),$$
where $x_k' := \bar{x}+t_ku_k$, $y_k' := \bar{y}+t_kv_k$. *In case* $(u,v) = (0,0)$ *we call* $\tilde{D}^*_{M,\gamma}M(\bar{x},\bar{y}) := \tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(0,0))$ *the* reversed mixed pseudo-coderivative of order $\gamma := (\gamma_1,\dots,\gamma_s)$.

2. *The mapping* M *is called* partially sequentially normally compact of order $\gamma := (\gamma_1,\dots,\gamma_s)$ (PSNC(γ)) in direction (u,v) *at* $(\bar{x},\bar{y})$ *with respect to* Y, *if for all sequences* $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*,\dots,y_{ks}^*) \in Y^*$, (x_k',y_k') *with* $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $y_k^* \xrightarrow{w^*} 0$, $x_k^* \to 0$, $(x_k',y_k') = (\bar{x},\bar{y}) + t_k(u_k,v_k)$ *and*
$$(x_k^*,(-\|x_k'-\bar{x}\|^{1-\gamma_1}y_{k1}^*,\dots,-\|x_k'-\bar{x}\|^{1-\gamma_s}y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k',y_k');\operatorname{gph}M)$$
one has $y_k^* \to 0$ *as* $k \to \infty$.

If $s = 1$ and $\gamma_1 = 1$, the reversed mixed pseudo-coderivative of order 1 in direction (u,v) coincides with the reversed mixed coderivative of M in direction (u,v) as defined in [11], provided $u \neq 0$. Similarly, if $u \neq 0$ the properties PSNC(1) in direction (u,v) and PSNC in direction (u,v) (cf. [11]) are the same.

Corollary 2. *Let* $M : X \rightrightarrows Y$ *be a closed-graph multifunction of the form* (7), $(\bar{x},\bar{y}) \in \operatorname{gph}M$, $(u,v) \in X\times Y$ *and let* $\gamma_i \geq 1$, $i = 1,\dots,s$. *Assume that assumption (A1) is fulfilled and assume that every bounded sequence in* Y^* *has a weak** *convergent subsequence. Then the following statements are equivalent:*

(a) M *is metrically pseudo-regular of order* $\gamma := (\gamma_1,\dots,\gamma_s)$ *in direction* (u,v) *at* $(\bar{x},\bar{y})$.

(b) M *is PSNC(γ) in direction* (u,v) *at* $(\bar{x},\bar{y})$ *with respect to* Y *and* $\ker\tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(u,v)) = \{0\}$.

Proof. We prove $(a) \Rightarrow (b)$ by contradiction. If M is not PSNC(γ) in direction (u,v) at $(\bar{x},\bar{y})$ with respect to Y or $0 \neq y^* \in \ker\tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(u,v))$, then there are sequences $(\tilde{\varepsilon}_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(\tilde{x}_k^*) \in X^*$, $(\tilde{y}_k^*) = (\tilde{y}_{k1}^*,\dots,\tilde{y}_{ks}^*) \in Y^*$, (x_k',y_k') with $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $\tilde{x}_k^* \to 0$, $(x_k',y_k') = (\bar{x},\bar{y}) + t_k(u_k,v_k)$, $(\tilde{x}_k^*,(-\|x_k'-\bar{x}\|^{1-\gamma_1}\tilde{y}_{k1}^*,\dots,-\|x_k'-\bar{x}\|^{1-\gamma_s}\tilde{y}_{ks}^*)) \in \hat{N}_{\tilde{\varepsilon}_k}((x_k',y_k');\operatorname{gph}M)$ and $0 < \liminf_{k\to\infty}\|\tilde{y}_k^*\|$. Hence, by taking $(\varepsilon_k,x_k^*,y_k^*) := (\tilde{\varepsilon}_k,\tilde{x}_k^*,\tilde{y}_k^*)/\|\tilde{y}_k^*\|$ we have $\varepsilon_k \downarrow 0$, $x_k^* \to 0$ and $y_k^* \in \mathscr{S}_{Y^*}$ and $(x_k^*,(-\|x_k'-\bar{x}\|^{1-\gamma_1}y_{k1}^*,\dots,-\|x_k'-\bar{x}\|^{1-\gamma_s}y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k',y_k');\operatorname{gph}M)$. Now it follows from Theorem 1(3.) that M is not metrically pseudo-regular of order γ in direction (u,v). The reverse implication $(b) \Rightarrow (a)$ is also shown by contradiction. Let us assume that M is not metrically pseudo-regular of order γ in direction (u,v). Then, by Theorem 1(3.) we can find sequences $(\varepsilon_k) \downarrow 0$, $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(x_k^*) \in X^*$, $(y_k^*) = (y_{k1}^*,\dots,y_{ks}^*) \in \mathscr{S}_{Y^*}$, (x_k',y_k') with $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $x_k^* \to 0$, $(x_k',y_k') = (\bar{x},\bar{y}) + t_k(u_k,v_k)$ and $(x_k^*,(-\|x_k'-\bar{x}\|^{1-\gamma_1}y_{k1}^*,\dots,-\|x_k'-\bar{x}\|^{1-\gamma_s}y_{ks}^*)) \in \hat{N}_{\varepsilon_k}((x_k',y_k');\operatorname{gph}M)$. By our assumption the sequence (y_k^*) has a weak* convergent subsequence and we may assume that the sequence (y_k^*) weakly* converges to some $y^* \in Y^*$. If $y^* = 0$ then M is not PSNC(γ) in direction (u,v) since $y_k^* \not\to 0$ because of $y_k^* \in \mathscr{S}_{Y^*}$. On the other hand, if $y^* \neq 0$ we obtain the contradiction $0 \neq y^* \in \ker\tilde{D}^*_{M,\gamma}M((\bar{x},\bar{y});(u,v))$. □

Remark 3. *Corollary 2 is an analogue to the famous Mordukhovich criterion for metric regularity of multifunctions in Asplund spaces, cf. [26, Theorem 4.18]. Note that the weak* sequential compactness assumption of bounded sequences in Y^* is automatically fulfilled in Asplund spaces and hence it is only needed in the last case of assumption (A1) when $M(x) = F(x) + S(x)$. However, we conjecture that in the case, when S is generalized polyhedral, by using similar arguments as in [12, Theorem 4.5], both this weak* sequential compactness assumption and the PSNC(γ) property can be replaced by the requirement that certain linear operators have closed range.*

Remark 4. *In a similar way we could also find a sufficient coderivative-like criterion for metric pseudo-subregularity, we omit working out details. We want only discuss some relations with the reverse outer mixed q-coderivative as defined by Li and Mordukhovich [25]. In case that $s = 1$ and both X and Y are Asplund spaces, we can take $\varepsilon_k = 0$ and hence* (10) *holds iff*

$$(\hat{x}_k^*, -y_k^*) \in \hat{N}((x_k', y_k'); \operatorname{gph} M)$$

where $\hat{x}_k^* := \|x_k' - \bar{x}\|^{\gamma-1} x_k^*$. Hence, in this situation a sufficient coderivative-like criterion based on Theorem 1(2.), together with the condition $x_k' \notin M^{-1}(\bar{y})$, but without the condition (11)*, yields the sufficient conditions of [25, Theorem 3.6] with $q = 1/\gamma$ and the reverse outer mixed q-coderivative modified as indicated in [25, Remark 3.5(iii)]*

5 Characterizations of pseudo-(sub)regularity of order $(1,2)$ for a class of multifunctions

In this section we study multifunctions of the form

$$M := (M_1, M_2) : X \rightrightarrows Y := Y_1 \times Y_2, \quad M_l(x) = F_l(x) + S_l(x),\ l = 1,2 \tag{18}$$

between Banach spaces, where $F_l : X \to Y_l$, $l = 1,2$ are sufficiently smooth mappings and $S_l : X \rightrightarrows Y_l$, $l = 1,2$ are generalized polyhedral multifunctions. In the sequel we denote by $F := (F_1, F_2)$ and $S(x) := S_1(x) \times S_2(x)$ the composite mappings. Then by [12, Lemma 3.9], S is again a generalized polyhedral multifunction, and we assume that the graph of S has the representation (5),(6).

In what follows we denote for $(x,s) \in \operatorname{gph} S$ by $\mathscr{P}(x,s) := \{i \in \{1,\ldots,p\} \mid (x,s) \in P_i\}$ the index set of the convex generalized polyhedral sets containing (x,s) and by $\mathscr{A}_i(x,s) = \{j \in \{1,\ldots,m_i\} \mid \langle x_{ij}^*, x\rangle + \langle y_{ij}^*, y\rangle = \zeta_{ij}\}$, $i \in \mathscr{P}(x,s)$ the index sets of active inequality constraints.

Given $(\bar{x},\bar{s}) \in \operatorname{gph} S$ and a direction $(u,v) \in X \times Y$ we denote by $\mathscr{I}((\bar{x},\bar{s});(u,v))$ the collection of index sets $\mathscr{P} \subset \{1,\ldots,p\}$ such that there exist sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ with $(\bar{x}+t_k u_k, \bar{s}+t_k v_k) \in \operatorname{gph} S$ and $\mathscr{P} = \mathscr{P}(\bar{x}+t_k u_k, \bar{s}+t_k v_k)$. Given in addition functionals $x^* \in X^*$, $y^* \in Y^*$ we denote by $\mathscr{K}((\bar{x},\bar{s});(u,v);(x^*,y^*))$ the collection of all index sets $\mathscr{P}$ such that there are sequences $(t_k) \downarrow 0$, $(u_k,v_k) \to (u,v)$ with $(x^*,-y^*) \in \hat{N}((\bar{x}+t_k u_k, \bar{s}+t_k v_k); \operatorname{gph} S)$ and $\mathscr{P} = \mathscr{P}(\bar{x}+t_k u_k, \bar{s}+t_k v_k)$. It follows from [12, Lemma 3.8] that $\mathscr{K}((\bar{x},\bar{s});(u,v);(x^*,y^*)) \neq \emptyset$ if and only if $x^* \in D^*S((\bar{x},\bar{s});(u,v))(y^*)$.

For every nonempty index set $\emptyset \neq \mathscr{P} \subset \{1,\ldots,p\}$ we define the operator $\check{A}_{\mathscr{P}}^* : Y_2^* \times \prod_{i\in\mathscr{P}} L_i^\perp \to \prod_{i\in\mathscr{P}} (X^* \times Y^*)$ by

$$\check{A}_{\mathscr{P}}^*(y_2^*, (l_i^*)_{i\in\mathscr{P}}) := ((\nabla F_2(\bar{x})^* y_2^*, (0, y_2^*)) + l_i^*)_{i\in\mathscr{P}}$$

and for given $u \neq 0$ we define the operator $\hat{A}_{\mathscr{P}}^*(u) : Y_2^* \times \prod_{i\in\mathscr{P}} L_i^\perp \times Y^* \times \prod_{i\in\mathscr{P}} L_i^\perp \to \prod_{i\in\mathscr{P}} (X^* \times Y^* \times X^* \times Y^*)$ by

$$\hat{A}_{\mathscr{P}}^*(u)(y_2^*, (l_i^*)_{i\in\mathscr{P}}, z^*, (w_i^*)_{i\in\mathscr{P}}) := \begin{pmatrix} (\nabla F_2(\bar{x})^* y_2^*, (0, y_2^*)) + l_i^* \\ (\nabla F(\bar{x})^* z^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* y_2^*, z^*) + w_i^* \end{pmatrix}_{i\in\mathscr{P}}$$

Theorem 2. *Let the multifunction M be given by* (18)*, let $(\bar{x},\bar{y}) \in \operatorname{gph} M$ and let $(u,v) \in X \times Y$ with $u \neq 0$. Further assume that F is continuously differentiable at $\bar{x}$ and that F_2 is twice Fréchet differentiable at $\bar{x}$ and assume that for every nonempty subset $\emptyset \neq \mathscr{P} \in \mathscr{I}((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))$ the operators $\check{A}^*_{\mathscr{P}}$ and $\hat{A}^*_{\mathscr{P}}(u)$ have closed range.*

1. *M is metrically pseudo-regular of order $(1,2)$ in direction (u,v) at $(\bar{x},\bar{y})$ if and only if there are not multipliers $\bar{y}^* = (0,\bar{y}^*_2) \in Y_1^* \times Y_2^*$ and $\bar{z}^* \in Y^*$ such that $(\bar{z}^*_1,\bar{y}^*_2) \neq 0$,*
$$-\nabla F(\bar{x})^*\bar{y}^* \in D^*S((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))(\bar{y}^*),$$
$$-\nabla F(\bar{x})^*\bar{z}^* - (\nabla^2 F_2(\bar{x})\frac{u}{\|u\|})^*\bar{y}^*_2 \in CD^*S((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))(\bar{y}^*,-\nabla F(\bar{x})^*\bar{y}^*)(\bar{z}^*),$$
where $\bar{s} := \bar{y} - F(\bar{x})$.

2. *Assume that $v = 0$ and that there are not multipliers $\bar{y}^* = (0,\bar{y}^*_2) \in Y_1^* \times Y_2^*$, $\bar{z}^* \in Y^*$ and an index set $\emptyset \neq \mathscr{P} \in \mathscr{K}((\bar{x},\bar{s});(u,-\nabla F(\bar{x})u);(-\nabla F(\bar{x})^*\bar{y}^*,\bar{y}^*))$ such that $(\bar{z}^*_1,\bar{y}^*_2) \neq 0$,*
$$-\nabla F(\bar{x})^*\bar{y}^* \in D^*S((\bar{x},\bar{s});(u,-\nabla F(\bar{x})u))(\bar{y}^*), \tag{19}$$
$$-\nabla F(\bar{x})^*\bar{z}^* - (\nabla^2 F_2(\bar{x})\frac{u}{\|u\|})^*\bar{y}^*_2 \in CD^*S((\bar{x},\bar{s});(u,-\nabla F(\bar{x})u))(\bar{y}^*,-\nabla F(\bar{x})^*\bar{y}^*)(\bar{z}^*), \tag{20}$$
and
$$\langle \bar{y}^*_2, \nabla^2 F_2(\bar{x})(u,u)\rangle = \sup\{\langle y^*_2, \nabla^2 F_2(\bar{x})(u,u)\rangle \mid (-\nabla F_2(\bar{x})^* y^*_2,(0,-y^*_2)) \in N((\bar{x},\bar{s});P_i)\}$$
for every $i \in \mathscr{P}$. Then M is metrically pseudo-subregular of order $(1,2)$ in direction u at $(\bar{x},\bar{y})$.

Proof. We proof the if-part of the first assertion by contraposition. Assuming that M is not pseudo-regular of order $(1,2)$ in direction (u,v), then by Theorem 1(1.) and taking into account Remark 1 there are sequences $(t_k) \downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(x^*_k) \in X^*$, $(y^*_k) \in \mathscr{S}_{Y^*}$ with $u_k \neq 0$, $\|(u,v)\|(u_k,v_k) \to (u,v)$, $x^*_k \to 0$ and
$$(x^*_k,(-y^*_{k1},-\|x'_k-\bar{x}\|^{-1}y^*_{k2})) \in \hat{N}((x'_k,y'_k),\operatorname{gph} M),$$
where $x'_k := \bar{x}+t_k u_k$, $y'_k := \bar{y}+t_k v_k$. Then for each k we have $(x'_k,s'_k) \in \operatorname{gph} S$, where $s'_k := y'_k - F(x'_k) = \bar{s}+t_k(v_k-\nabla F(\bar{x})u_k+r_k)$ with $r_k \to 0$. By passing to a subsequence if necessary we can assume that there are index sets $\mathscr{P} \in \mathscr{I}((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))$ and $\mathscr{A}_i \subset \{1,\ldots,m_i\}$, $i \in \mathscr{P}$, such that for all k we have $\mathscr{P} = \mathscr{P}(x'_k,s'_k)$ and $\mathscr{A}_i(x'_k,s'_k) = \mathscr{A}_i$, $i \in \mathscr{P}$. By denoting $\hat{y}^*_k := (y^*_{k1},\|x'_k-\bar{x}\|^{-1}y^*_{k2})$ we obtain $(x^*_k-\nabla F(x'_k)^*\hat{y}^*_k,-\hat{y}^*_k) \in \hat{N}((x'_k,s'_k),\operatorname{gph} S)$ as a consequence of [26, Theorem 1.62] and, because $\hat{N}((x'_k,s'_k),\operatorname{gph} S)$ is a cone, we also have $\|x'_k-\bar{x}\|(x^*_k-\nabla F(x'_k)^*\hat{y}^*_k,-\hat{y}^*_k) \in \hat{N}((x'_k,s'_k),\operatorname{gph} S)$. Using the relation $\hat{N}((x'_k,s'_k),\operatorname{gph} S) = \bigcap_{i\in\mathscr{P}} \hat{N}((x'_k,s'_k);P_i)$ and the generalized Farkas Lemma (see, e.g. [12]) we can find for each $i \in \mathscr{P}$ some linear functionals $l^*_{ki} \in L_i^\perp$ and nonnegative numbers $\lambda_{kij} \geq 0$, $j \in \mathscr{A}_i$ such that
$$\|(\nabla F_2(\bar{x})^*y^*_{k2},(0,y^*_{k2})) + l^*_{ki} + \sum_{j\in\mathscr{A}_i} \lambda_{kij}(x^*_{ij},y^*_{ij})\| \leq \|x'_k-\bar{x}\|\tau_k$$
where $\tau_k := \|(x^*_k-\nabla F_1(x'_k)^*y^*_{k1},(-y^*_{k1},0))\| + \|x'_k-\bar{x}\|^{-1}\|(\nabla F_2(x'_k)-\nabla F_2(\bar{x}))^*y^*_{k2}\|$ is bounded by some constant τ. Next consider the operator which assigns to every $y^*_2 \in Y^*_2$, $(l^*_i)_{i\in\mathscr{P}} \in \prod_{i\in\mathscr{P}} L_i^\perp$ and $(\lambda_{ij})_{i\in\mathscr{P},j\in\mathscr{A}_i}$ the value
$$((\nabla F_2(\bar{x})^*y^*_2,(0,y^*_2)) + l^*_i + \sum_{j\in\mathscr{A}_i} \lambda_{ij}(x^*_{ij},y^*_{ij}))_{i\in\mathscr{P}}.$$

The range of this operator is the sum of range the of $\mathrm{Im}\check{A}^*_{\mathscr{P}}$ and a finite dimensional subspace and therefore closed by our assumption. Hence we can invoke Hoffman's Lemma [18, Theorem 3] to find $\tilde{y}^*_{k2} \in Y_2^*$, $\tilde{l}^*_{ki} \in L_i^\perp$ and nonnegative numbers $\tilde{\lambda}_{kij} \geq 0$, $i \in \mathscr{P}$, $j \in \mathscr{A}_i$ such that

$$\|y^*_{k2} - \tilde{y}^*_{k2}\| + \sum_{i\in\mathscr{P}} (\|l^*_{ki} - \tilde{l}^*_{ki}\| + \sum_{j\in\mathscr{A}_i} |\lambda_{kij} - \tilde{\lambda}_{kij}|) \leq \beta\tau\|x'_k - \bar{x}\|,$$

where the constant $\beta > 0$ does not depend on k, and

$$(\nabla F_2(\bar{x})^*\tilde{y}^*_{k2}, (0, \tilde{y}^*_{k2})) + \tilde{l}^*_{ki} + \sum_{j\in\mathscr{A}_i} \tilde{\lambda}_{kij}(x^*_{ij}, y^*_{ij}) = 0,\ i \in \mathscr{P}.$$

By eventually passing once more to a subsequence we can find index sets $\bar{\mathscr{A}_i} \subset \mathscr{A}_i$, $i \in \mathscr{P}$ and a constant $\bar{\delta} > 0$ such that

$$\tilde{\lambda}_{kij} \geq 3\bar{\delta}\ \forall k, i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}$$

and

$$\delta_k := \sum_{i\in\mathscr{P}} \sum_{j\in\mathscr{A}_i\setminus\bar{\mathscr{A}_i}} \tilde{\lambda}_{kij} \to 0$$

Using Hoffman's Lemma once more we find $\check{y}^*_{k2} \in Y_2^*$, $\check{l}^*_{ki} \in L_i^\perp$ and nonnegative numbers $\check{\lambda}_{kij} \geq 0$, $i \in \mathscr{P}$, $j \in \bar{\mathscr{A}_i}$ such that

$$\|\check{y}^*_{k2} - \tilde{y}^*_{k2}\| + \sum_{i\in\mathscr{P}} (\|\check{l}^*_{ki} - \tilde{l}^*_{ki}\| + \sum_{j\in\bar{\mathscr{A}_i}} |\check{\lambda}_{kij} - \tilde{\lambda}_{kij}|) \leq \check{\beta}\delta_k,$$

where the constant $\check{\beta} > 0$ does not depend on k, and

$$(\nabla F_2(\bar{x})^*\check{y}^*_{k2}, (0, \check{y}^*_{k2})) + \check{l}^*_{ki} + \sum_{j\in\bar{\mathscr{A}_i}} \check{\lambda}_{kij}(x^*_{ij}, y^*_{ij}) = 0,\ i \in \mathscr{P}.$$

Since $\lim_k \delta_k = 0$ we can assume without loss of generality that $\check{\lambda}_{kij} \geq 2\bar{\delta}$, $\forall k, i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}$. Now consider for each $i \in \mathscr{P}$ the convex cone

$$K_i := \{(x,s) \in L_i \,|\, \langle x^*_{ij}, x\rangle + \langle y^*_{ij}, s\rangle = 0, j \in \bar{\mathscr{A}_i};\ \langle x^*_{ij}, x\rangle + \langle y^*_{ij}, s\rangle \leq 0,\ j \in \mathscr{A}_i \setminus \bar{\mathscr{A}_i}\}$$

Then we have

$$\langle \nabla F_2(\bar{x})^*\check{y}^*_{k2}, x\rangle + \langle (0, \check{y}^*_{k2}), s\rangle = 0,\ \forall (x,s) \in K_i$$

and, since K_i is a subset of the tangent cone to P_i at (x'_k, s'_k), we obtain for all $(x,s) \in K_i$

$$\begin{aligned}
0 &\geq \langle x^*_k - \nabla F(x'_k)^*\hat{y}^*_k, x\rangle + \langle -\hat{y}^*_k, s\rangle \\
&= \langle x^*_k - \nabla F(x'_k)^*\hat{y}^*_k, x\rangle + \langle -\hat{y}^*_k, s\rangle + \|x'_k - \bar{x}\|^{-1}(\langle \nabla F_2(\bar{x})^*\check{y}^*_{k2}, x\rangle + \langle (0, \check{y}^*_{k2}), s\rangle) \\
&\geq \langle -\nabla F_1(\bar{x})^* y^*_{k1} - (\nabla^2 F_2(\bar{x})\frac{u}{\|u\|})^*\check{y}^*_{k2} - \nabla F_2(\bar{x})^*\frac{y^*_{k2} - \check{y}^*_{k2}}{\|x'_k - \bar{x}\|}, x\rangle + \langle (-y^*_{k1}, -\frac{y^*_{k2} - \check{y}^*_{k2}}{\|x'_k - \bar{x}\|}), s\rangle \\
&\qquad -\varepsilon_k\|(x,s)\|,
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_k &:= \|x^*_k\| + \|(\nabla F_1(x'_k) - \nabla F_1(\bar{x}))^* y^*_{k1}\| + \|\left(\frac{\nabla F_2(x'_k) - \nabla F_2(\bar{x})}{\|x'_k - \bar{x}\|} - \nabla^2 F_2(\bar{x})\frac{u}{\|u\|}\right)^* y^*_{k2}\| \\
&\qquad + \|(\nabla^2 F_2(\bar{x})\frac{u}{\|u\|})^*(y^*_{k2} - \check{y}^*_{k2})\|.
\end{aligned}$$

Denoting $z_k^* := (y_{k1}^*, \frac{y_{k2}^* - \check{y}_{k2}^*}{\|x_k' - \bar{x}\|})$ we obtain

$$-(\nabla F(\bar{x})^* z_k^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \check{y}_{k2}^*, z_k^*) \in \hat{N}_{\varepsilon_k}((0,0); K_i)$$

and, since $\hat{N}((0,0), K_i)$ is the polar cone of K_i, by the generalized Farkas Lemma there are functionals $w_{ki}^* \in L_i^\perp$ and real numbers μ_{kij}, $j \in \mathscr{A}_i$ with $\mu_{kij} \geq 0$, $j \in \mathscr{A}_i \setminus \bar{\mathscr{A}_i}$ such that

$$\|(\nabla F(\bar{x})^* z_k^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \check{y}_{k2}^*, z_k^*) + w_{ki}^* + \sum_{j \in \mathscr{A}_i} \mu_{kij} (x_{ij}^*, y_{ij}^*)\| \leq \varepsilon_k.$$

Then, since the range of the operator which assigns to every $y_2^* \in Y_2^*$, $(l_i^*)_{i \in \mathscr{P}} \in \prod_{i \in \mathscr{P}} L_i^\perp$, $(\lambda_{ij})_{i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}}$, $z^* \in Y^*$, $(w_i^*)_{i \in \mathscr{P}} \in \prod_{i \in \mathscr{P}} L_i^\perp$ and $(\mu_{ij})_{i \in \mathscr{P}, j \in \mathscr{A}_i}$ the value

$$\begin{pmatrix} (\nabla F_2(\bar{x})^* y_2^*, (0, y_2^*)) + l_i^* + \sum_{j \in \bar{\mathscr{A}_i}} \lambda_{ij} (x_{ij}^*, y_{ij}^*) \\ (\nabla F(\bar{x})^* z^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* y_2^*, z^*) + w_i^* + \sum_{j \in \mathscr{A}_i} \mu_{ij} (x_{ij}^*, y_{ij}^*) \end{pmatrix}_{i \in \mathscr{P}},$$

is the sum of $\mathrm{Im} \hat{A}_{\mathscr{P}}^*(u)$ and a finite dimensional subspace and therefore closed by our assumption, we can again apply Hoffman's lemma to find $\bar{y}_{k2}^* \in Y_2^*$, $(\bar{l}_{ki}^*)_{i \in \mathscr{P}} \in \prod_{i \in \mathscr{P}} L_i^\perp$, $(\bar{\lambda}_{kij})_{i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}}$, $\bar{z}_k^* \in Y^*$, $(\bar{w}_{ki}^*)_{i \in \mathscr{P}} \in \prod_{i \in \mathscr{P}} L_i^\perp$ and $(\bar{\mu}_{kij})_{i \in \mathscr{P}, j \in \mathscr{A}_i}$ with $\bar{\lambda}_{kij} \geq 0$, $j \in \bar{\mathscr{A}_i}$, $i \in \mathscr{P}$, $\bar{\mu}_{kij} \geq 0$, $j \in \mathscr{A}_i \setminus \bar{\mathscr{A}_i}$ such that

$$\|\bar{y}_{k2}^* - \check{y}_{k2}^*\| + \|\bar{z}_k^* - z_k^*\| + \sum_{i \in \mathscr{P}} (\|\bar{l}_{ki}^* - \check{l}_{ki}^*\| + \|\bar{w}_{ki}^* - w_{ki}^*\| + \sum_{j \in \bar{\mathscr{A}_i}} |\bar{\lambda}_{kij} - \check{\lambda}_{kij}| + \sum_{j \in \mathscr{A}_i} |\bar{\mu}_{kij} - \mu_{kij}|) \leq \bar{\beta} \varepsilon_k$$

for some constant $\bar{\beta}$ independent of k, and for each $i \in \mathscr{P}$

$$(\nabla F_2(\bar{x})^* \bar{y}_{k2}^*, (0, \bar{y}_{k2}^*)) + \bar{l}_{ki}^* + \sum_{j \in \bar{\mathscr{A}_i}} \bar{\lambda}_{kij} (x_{ij}^*, y_{ij}^*) = 0,$$
$$(\nabla F(\bar{x})^* \bar{z}_k^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_{k2}^*, \bar{z}_k^*) + \bar{w}_{ki}^* + \sum_{j \in \mathscr{A}_i} \bar{\mu}_{kij} (x_{ij}^*, y_{ij}^*) = 0.$$

Because of $\varepsilon_k \to 0$ we can assume without loss of generality that $\bar{\lambda}_{kij} \geq \bar{\delta} > 0, \forall k, i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}$. Hence for every k there is some positive real $\bar{\alpha}_k > 0$ such that $\bar{\lambda}_{kij} + \alpha \bar{\mu}_{kij} \geq 0$, $\forall \alpha \in [0, \bar{\alpha}_k], i \in \mathscr{P}, j \in \bar{\mathscr{A}_i}$ and therefore we have

$$\begin{aligned} &-(\nabla F(\bar{x})^* \bar{y}_k^*, \bar{y}_k^*) - \alpha (\nabla F(\bar{x})^* \bar{z}_k^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_{k2}^*, \bar{z}_k^*) \\ &= \bar{l}_{ki}^* + \alpha \bar{w}_{ki}^* + \sum_{j \in \bar{\mathscr{A}_i}} (\bar{\lambda}_{kij} + \alpha \bar{\mu}_{kij})(x_{ij}^*, y_{ij}^*) + \sum_{j \in \mathscr{A}_i \setminus \bar{\mathscr{A}_i}} \alpha \bar{\mu}_{kij} (x_{ij}^*, y_{ij}^*) \in \hat{N}((x_k', s_k'); P_i) \end{aligned} \tag{21}$$

for all $i \in \mathscr{P}$ and for all $\alpha \in [0, \bar{\alpha}_k]$, where $\bar{y}_k^* := (0, \bar{y}_{k2}^*)$. Using [12, Lemma 3.8] we conclude that

$$-\nabla F(\bar{x})^* \bar{z}_k^* - (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_{k2}^* \in CD^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x}) u))(\bar{y}_k^*, -\nabla F(\bar{x})^* \bar{y}_k^*)(\bar{z}_k^*).$$

Since $\|\bar{z}_{k1}^* - y_{k1}^*\| + \|\bar{y}_{k2}^* - y_{k2}^*\| \to 0$ we have $(\bar{z}_{k1}^*, \bar{y}_{k2}^*) \neq 0$ for all k sufficiently large and taking into account that $-\nabla F(\bar{x})^* \bar{y}_k^* \in D^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x}) u))(\bar{y}_k^*)$, we obtain a contradiction and the if-part is proved.

To show the only-if-part, assume that there is some $\bar{z}^* \in Y^*$, $\bar{y}^* = (0, \bar{y}_2^*) \in Y_1^* \times Y_2^*$ such that $(\bar{z}_1^*, \bar{y}_2^*) \neq 0$, $-\nabla F(\bar{x})^* \bar{y}^* \in D^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x})u))(\bar{y}^*)$ and

$$-\nabla F(\bar{x})^* \bar{z}^* - (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_2^* \in CD^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x})u))(\bar{y}^*, -\nabla F(\bar{x})^* \bar{y}^*)(\bar{z}^*).$$

The Mordukhovich normal cone of the union of finitely many convex generalized polyhedral sets can be written as the union of finitely many intersections of finitely many normal cones of convex generalized polyhedral sets, see [12, Lemma 3.4]. Since the normal cone of a convex generalized polyhedral set is again generalized polyhedral (see e.g. [12, Example 2]) and the intersection of two generalized polyhedral sets is generalized polyhedral, we conclude that $N((\bar{x}, \bar{s}); \operatorname{gph} S; (u, v - \nabla F(\bar{x})u))$ and hence also $\operatorname{gph} D^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x})u))$ is generalized polyhedral. Hence the contingent cone of $\operatorname{gph} D^* S((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x})u))$ is the union of finitely many tangent cones to convex generalized polyhedral sets and thus there is some $\bar{\alpha} > 0$ such that

$$\begin{aligned} w^*(\alpha) &:= -(\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*) - \alpha(\nabla F(\bar{x})^* \bar{z}^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_2^*, \bar{z}^*) \\ &\in N((\bar{x}, \bar{s}); \operatorname{gph} S; (u, v - \nabla F(\bar{x})u)) \end{aligned}$$

holds for all $\alpha \in [0, \bar{\alpha}]$. Taking an arbitrary sequence $(\alpha_k) \downarrow 0$, we conclude from [12, Lemma 3.4], that there are sequences $(x_k', s_k') \to (\bar{x}; \bar{s})$ such that

$$\lim_{k \to \infty} \frac{(x_k', s_k') - (\bar{x}, \bar{s})}{\|(x_k', s_k') - (\bar{x}, \bar{s})\|} = \frac{(u, v - \nabla F(\bar{x})u)}{\|(u, v - \nabla F(\bar{x})u)\|} \tag{22}$$

and $w^*(\alpha_k) \in \hat{N}((x_k', s_k'); \operatorname{gph} S)$. After eventually passing to a subsequence we can assume that there are index sets $\mathscr{P} \in \mathscr{I}((\bar{x}, \bar{s}); (u, v - \nabla F(\bar{x})u))$, $\mathscr{A}_i$, $i \in \mathscr{P}$ such that for all k we have $\mathscr{P} = \mathscr{P}(x_k', s_k')$, $\mathscr{A}_i = \mathscr{A}_i(x_k', s_k')$, $i \in \mathscr{P}$, implying

$$\hat{N}((x_k', s_k'); \operatorname{gph} S) = \bigcap_{i \in \mathscr{P}} (L_i^\perp + \{\sum_{j \in \mathscr{A}_i} \lambda_{ij} (x_{ij}^*, y_{ij}^*) \,|\, \lambda_{ij} \geq 0\}).$$

Since the set on the right hand side does not depend on k and is the intersection of convex sets, we can conclude that $w^*(\alpha) \in \hat{N}((x_k', s_k'); \operatorname{gph} S)$ for all k and all $\alpha \in (0, \alpha_1)$. Because of (22) and $u \neq 0$ we can assume that $x_k' \neq \bar{x}\ \forall k$ and $(x_k' - \bar{x})/\|x_k' - \bar{x}\| \to u/\|u\|$. Then, setting $\tilde{y}_{k1}^* := \bar{z}_1^*$, $\tilde{y}_{k2}^* := \bar{y}_2^* + \|x_k' - \bar{x}\| \bar{z}_2^*$,

$$\begin{aligned} \tilde{x}_k^* &:= \left(\frac{\nabla F_2(x_k') - \nabla F_2(\bar{x}) - \nabla^2 F_2(\bar{x})(x_k' - \bar{x})}{\|x_k' - \bar{x}\|} + \nabla^2 F_2(\bar{x}) (\frac{x_k' - \bar{x}}{\|x_k' - \bar{x}\|} - \frac{u}{\|u\|}) \right)^* \bar{y}_2^* \\ &\quad + (\nabla F(x_k') - \nabla F(\bar{x}))^* \bar{z}^* \end{aligned}$$

we obtain

$$\begin{aligned} w^*(\|x_k' - \bar{x}\|) &= (\|x_k' - \bar{x}\| (\tilde{x}_k^* - \nabla F_1(x_k') \tilde{y}_{k1}^*) - \nabla F_2(x_k')^* \tilde{y}_{k2}^*, (-\|x_k' - \bar{x}\| \tilde{y}_{k1}^*, -\tilde{y}_{k2}^*)) \\ &\in \hat{N}((x_k', s_k'); \operatorname{gph} S). \end{aligned}$$

Since $\hat{N}((x_k', s_k'); \operatorname{gph} S)$ is a cone, we also have $w^*(\|x_k' - \bar{x}\|)/(\|x_k' - \bar{x}\| \|\tilde{y}_k^*\|) \in \hat{N}((x_k', s_k'); \operatorname{gph} S)$. Then we obtain from [26, Theorem 1.62] $(x_k^*, -(y_{k1}^*, \|x_k' - \bar{x}\|^{-1} y_{k2}^*)) \in \hat{N}((x_k', F(x_k') + s_k'); \operatorname{gph} M)$, where $(x_k^*, y_k^*) := (\tilde{x}_k^*, \tilde{y}_k^*)/\|\tilde{y}_k^*\|$. Because of $\|\tilde{y}_k^*\| \to \|(\bar{z}_1^*, \bar{y}_2^*)\| \neq 0$ we have $x_k^* \to 0$. Now define $t_k' := \|(x_k', s_k') - (\bar{x}, \bar{s})\| / \|(u, v - \nabla F(\bar{x})u)\|$, $u_k' := (x_k' - \bar{x})/t_k'$, $\sigma_k' := (s_k' - \bar{s})/t_k'$. Then $t_k' \to 0$ and $u_k' \to u$, $\sigma_k' \to v - \nabla F(\bar{x})u$ by (22) and, by the definition of Fréchet differentiability, we have $v_k' :=$

$\sigma'_k + (F(\bar{x}+t'_k u'_k) - F(\bar{x}))/t'_k \to v$. Defining $(u_k,v_k) := (u'_k,v'_k)/\|(u'_k,v'_k)\|$, $t_k := t'_k\|(u'_k,v'_k)\|$, $y'_k := F(\bar{x})+\bar{s}+t_k v_k = F(x'_k)+s'_k$ we have $(x'_k,y'_k) = (\bar{x},\bar{y})+t_k(u_k,v_k)$, $(u_k,v_k) \to (u,v)/\|(u,v)\|$ and by Theorem 1(3.) it follows that M is not metrically pseudo-regular of order $(1,2)$ in direction (u,v) at $(\bar{x},\bar{y})$.

We prove the assertion about metric pseudo-subregularity also by contraposition. Assuming that M is not pseudo-subregular of order $(1,2)$ in direction u, by Theorem 1(2.) there are sequences $(t_k)\downarrow 0$, $(u_k,v_k) \in \mathscr{S}_{X\times Y}$, $(x^*_k) \in X^*$, $(y^*_k) \in \mathscr{S}_{Y^*}$ with $u_k \neq 0$, $\|u_k\| \to 1$, $\|u\|u_k \to u$, $v_{k1} \to 0$, $t_k^{-1}v_{k2} \to 0$, $x^*_k \to 0$ and

$$(x^*_k,(-y^*_{k1},-\|x'_k-\bar{x}\|^{-1}y^*_{k2})) \in \hat{N}((x'_k,y'_k),\mathrm{gph}M),$$

where $x'_k := \bar{x}+t_k u_k$, $y'_k := \bar{y}+t_k v_k$. Then we can proceed as before to find the sequences (s'_k), $(\bar{y}^*_{k2})$ and $(\bar{z}^*_k)$. Fixing some index $\bar{k}$ sufficiently large and setting $\bar{y}^* := (0,\bar{y}^*_2) := (0,\bar{y}^*_{\bar{k}2})$, $\bar{z}^* = \bar{z}^*_{\bar{k}}$ we have $(\bar{z}^*_1,\bar{y}^*_2) \neq 0$, conditions (19) and (20) are fulfilled, $-(\nabla F(\bar{x})^*\bar{y}^*,\bar{y}^*)) \in \hat{N}((x'_k,s'_k);\mathrm{gph}S)$, $\forall k$ and $\langle \nabla F(\bar{x})^*\bar{y}^*, x'_k-\bar{x}\rangle + \langle \bar{y}^*, s'_k-\bar{s}\rangle = \langle \nabla F_2(\bar{x})^*\bar{y}^*_2, x'_k-\bar{x}\rangle + \langle \bar{y}^*_2, s'_{k2}-\bar{s}_2\rangle = 0$, $\forall k$ since $\mathscr{P} \subset \mathscr{P}(\bar{x},\bar{s})$, $\mathscr{A}_i \subset \mathscr{A}_i(\bar{x},\bar{s})$, $i \in \mathscr{P}$. Taking into account $\|x'_k-\bar{x}\|/t_k = \|u_k\| \to 1$ we obtain

$$\begin{aligned}
0 &= \lim_{k\to\infty} t_k^{-1}\langle \bar{y}^*_2, v_{k2}\rangle = \lim_{k\to\infty} t_k^{-2}\langle \bar{y}^*_2, y'_{k2}-\bar{y}_2\rangle = \lim_{k\to\infty} t_k^{-2}\langle \bar{y}^*_2, F_2(x'_k)+s'_{k2}-F_2(\bar{x})-\bar{s}_2\rangle \\
&= \lim_{k\to\infty} t_k^{-2}\langle \bar{y}^*_2, \nabla F_2(\bar{x})(x'_k-\bar{x})+s'_{k2}-\bar{s}_2+\frac{1}{2}\nabla^2 F_2(\bar{x})(x'_k-\bar{x},x'_k-\bar{x})\rangle \\
&= \langle \bar{y}^*_2, \frac{1}{2}\nabla^2 F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle.
\end{aligned}$$

By construction we have $\mathscr{P} \in \mathscr{K}((\bar{x},\bar{s});(u,-\nabla F(\bar{x})u);(-\nabla F(\bar{x})^*\bar{y}^*,\bar{y}^*)$. Now let $i \in \mathscr{P}$ and $y^*_2 \in Y^*_2$ with $(-\nabla F_2(\bar{x})^*y^*_2,(0,-y^*_2)) \in N((\bar{x},\bar{s});P_i)$ be arbitrarily fixed. Then

$$0 \geq \langle -\nabla F_2(\bar{x})^*y^*_2, x'_k-\bar{x}\rangle + \langle (0,-y^*_2), s'_k-\bar{s}\rangle = -\langle y^*_2, \nabla F_2(\bar{x})(x'_k-\bar{x})+s'_{k2}-\bar{s}_2\rangle$$

and we obtain

$$\begin{aligned}
0 &= \langle \bar{y}^*_2, \frac{1}{2}\nabla^2 F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle \\
&= \lim_{k\to\infty} t_k^{-1}\langle y^*_2, v_{k2}\rangle = \lim_{k\to\infty} t_k^{-2}\langle y^*_2, y'_{k2}-\bar{y}_2\rangle = \lim_{k\to\infty} t_k^{-2}\langle y^*_2, F_2(x'_k)+s'_{k2}-F_2(\bar{x})-\bar{s}_2\rangle \\
&= \lim_{k\to\infty} t_k^{-2}\langle y^*_2, \nabla F_2(\bar{x})(x'_k-\bar{x})+s'_{k2}-\bar{s}_2+\frac{1}{2}\nabla^2 F_2(\bar{x})(x'_k-\bar{x},x'_k-\bar{x})\rangle \\
&\geq \lim_{k\to\infty} t_k^{-2}\langle y^*_2, \frac{1}{2}\nabla^2 F_2(\bar{x})(x'_k-\bar{x},x'_k-\bar{x})\rangle \\
&= \langle y^*_2, \frac{1}{2}\nabla^2 F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle,
\end{aligned}$$

a contradiction. □

Remark 5. *Let us consider the special case* $S(x) \equiv \{0\}$. *Then we have* $\mathrm{gph}S = X\times\{0\} = P_1 = L_1$, $L_1^{\perp} = \{0\}\times Y^*$

$$D^*S((\bar{x},0);(u,v))(y^*) = \begin{cases} \emptyset & \text{if } v \neq 0 \\ \{0\} & \text{if } v = 0 \end{cases},$$

$$CD^*S((\bar{x},0);(u,0))(y^*,x^*)(v^*) = \begin{cases} \emptyset & \text{if } x^* \neq 0 \\ \{0\} & \text{if } x^* = 0 \end{cases}$$

and

$$\mathscr{I}((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u)) = \begin{cases} \{\{1\}\} & \text{if } v = \nabla F(\bar{x})u \\ \emptyset & \text{else} \end{cases}$$

Now let $u \neq 0$. If $v = \nabla F(\bar{x})u$, then the only nonempty subset $\emptyset \neq \mathscr{P} \in \mathscr{I}((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))$ is $\mathscr{P} = \{1\}$ and it follows that the operators $\check{A}^*_{\mathscr{P}}$ and $\hat{A}^*_{\mathscr{P}}(u)$ have closed range, if and only if the operators $Y_2^* \ni y_2^* \to \nabla F_2(\bar{x})^* y_2^* \in X^*$ and $Y_2^* \times Y^* \ni (y_2^*, z^*) \to (\nabla F_2(\bar{x})^* y_2^*, \nabla F(\bar{x})^* z^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* y_2^*) \in X^* \times X^*$ have closed range. It turns out that this is equivalent to the condition that the operators $\nabla F_2(\bar{x})^* : Y_2^* \to X^*$ and $G_u^* : Y^* \times \ker \nabla F_2(\bar{x})^* \to X^*$, $G_u^*(z^*, y_2^*) := \nabla F(\bar{x})^* z^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* y_2^*$ have closed range. By virtue of the Closed Range Theorem this holds true if and only if the operators $\nabla F_2(\bar{x})$ and $G_u : X \to Y \times (Y_2/\operatorname{Im} \nabla F_2(\bar{x}))$, $G_u(h) = (\nabla F(\bar{x})h, [\nabla^2 F_2(\bar{x})(\frac{u}{\|u\|}, h)])$ have closed range, where $[y_2]$ denotes the equivalence class of an element $y_2 \in Y_2$ defined by the equivalence relation $y_2 \sim y_2' \Leftrightarrow y_2 - y_2' \in \operatorname{Im} \nabla F_2(\bar{x})$ and we have taken into account that $\ker \nabla F_2(\bar{x})^* = (\operatorname{Im} \nabla F_2(\bar{x}))^\perp$ is the dual space of $Y_2/\operatorname{Im} \nabla F_2(\bar{x})$.

Hence, if $\nabla F_2(\bar{x})$ and G_u have closed range, then the multifunction $M(x) = \{F(x)\}$ is metrically pseudo-regular of order $(1,2)$ in direction $(u, \nabla F(\bar{x})u)$ at $(\bar{x}, 0)$, if and only if there does not exist multipliers $\bar{y}_2^* \in Y_2^*$ and $z^* \in Y^*$ such that $(\bar{y}_2^*, \bar{z}_1^*) \neq 0$ and

$$\nabla F_2(\bar{x})^* \bar{y}_2^* = 0,\ \nabla F(\bar{x})^* \bar{z}^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* \bar{y}_2^* = 0,$$

which is equivalent to $(\operatorname{Im} G_u)^\perp = \ker G_u^* = (\{0\} \times \ker \nabla F_2(\bar{x})^*) \times \{0\}$ respectively $\operatorname{Im} G_u = (Y_1 \times \operatorname{Im} \nabla F_2(\bar{x})) \times (Y_2/\operatorname{Im} \nabla F_2(\bar{x}))$.

If $F = F_2$, $Y = Y_2$ and the direction u satisfies $\nabla F(\bar{x})u = 0$, $\nabla^2 F(\bar{x})(u,u) \in \operatorname{Im} \nabla F(\bar{x})$, the condition $\operatorname{Im} G_u = \operatorname{Im} \nabla F(\bar{x}) \times (Y/\operatorname{Im} \nabla F(\bar{x}))$ was already used by Avakov [1], the mapping F is then usually called 2-regular *in direction u. We see that for single-valued mappings the concepts of 2-regularity and metric pseudo-regularity of order 2 are equivalent for such directions u. However, as already pointed out in Example 2, this is not true for more general multifunctions.*

The following proposition states a result which is closely related to directional metric pseudo-subregulaity of order $(1,2)$. We denote by $\Xi(u)$ the set of all multipliers $\bar{y}^* = (0, \bar{y}_2^*) \in Y_1^* \times Y_2^*$, $\bar{z}^* \in Y^*$ such that the sufficient condition for metric pseudo-subregularity of order $(1,2)$ in direction u of Theorem 2 are violated, i.e. the set of all $\bar{y}^* = (0, \bar{y}_2^*) \in y_1^* \times Y_2^*$, $\bar{z}^* \in Y^*$ fulfilling $(\bar{z}_1^*, \bar{y}_2^*) \neq 0$, (19), (20) and there is an index set $\mathscr{P} \in \mathscr{K}((\bar{x},\bar{s});(u, -\nabla F(\bar{x})u);(-\nabla F(\bar{x})^* \bar{y}^*, \bar{y}^*))$ such that

$$\langle \bar{y}_2^*, \nabla^2 F_2(\bar{x})(u,u) \rangle = \sup\{\langle y_2^*, \nabla^2 F_2(\bar{x})(u,u) \rangle \,|\, (-\nabla F_2(\bar{x})^* y_2^*, (0, -y_2^*)) \in N((\bar{x},\bar{s}); P_i)\}\ \forall i \in \mathscr{P}.$$

Note that $\langle \bar{y}_2^*, \nabla^2 F_2(\bar{x})(u,u) \rangle = 0\ \forall \bar{y}_2^* \in \Xi(u)$ follows, because the set $\{y_2^* \,|\, (-\nabla F_2(\bar{x})^* y_2^*, (0, -y_2^*)) \in N((\bar{x},\bar{s}); P_i)\}$ is a nonempty cone for all $i \in \mathscr{P}$.

Proposition 1. *Let the multifunction M be given by* (18), *let $(\bar{x},\bar{y}) \in \operatorname{gph} M$ and let $0 \neq u \in X$. Further assume that F is twice Fréchet differentiable at $\bar{x}$ and that F_2 is 3 times Fréchet differentiable at $\bar{x}$ and assume that for every nonempty subset $\emptyset \neq \mathscr{P} \in \mathscr{I}((\bar{x},\bar{s});(u,v-\nabla F(\bar{x})u))$ the operators $\check{A}^*_{\mathscr{P}}$ and $\hat{A}^*_{\mathscr{P}}(u)$ have closed range. Then, if*

$$\langle z^*, \nabla^2 F(\bar{x})(\frac{u}{\|u\|}, \frac{u}{\|u\|}) \rangle + \frac{1}{3} \langle y_2^*, \nabla^3 F_2(\bar{x})(\frac{u}{\|u\|}, \frac{u}{\|u\|}, \frac{u}{\|u\|}) \rangle < 0$$

holds for all $((0, y_2^*), z^*) \in \Xi(u)$, then there are constants $\delta > 0$ and $\kappa > 0$ such that for all $x \in \bar{x} + \delta B_X$ with $x \neq \bar{x}$, $\left\| x - \bar{x} - \|x - \bar{x}\| \frac{u}{\|u\|} \right\| \leq \delta \|x - \bar{x}\|^{3/2}$ and

$$\mathrm{d}(\bar{y}_i, M_i(x)) \leq \delta \|x - \bar{x}\|^{i+1/2},\ i = 1,2$$

one has

$$\mathrm{d}(x,M^{-1}(\bar{y})) \le \kappa(\mathrm{d}(\bar{y}_1,M_1(x)) + \|x-\bar{x}\|^{-1}\mathrm{d}(\bar{y}_2,M_2(x))).$$

Proof. Assume on the contrary that there is a sequence $(x_k)\to\bar{x}$ such that for every k we have $x_k\neq\bar{x}$, $\left\|x_k-\bar{x}-\|x_k-\bar{x}\|\frac{u}{\|u\|}\right\|\le\frac{1}{k}\|x_k-\bar{x}\|^{3/2}$, $\mathrm{d}(\bar{y}_i,M_i(x_k))\le\frac{1}{k}\|x_k-\bar{x}\|^{i+1/2},\ i=1,2$ and

$$\mathrm{d}(x_k,M^{-1}(\bar{y})) > k(\mathrm{d}(\bar{y}_1,M_1(x_k)) + \|x_k-\bar{x}\|^{-1}\mathrm{d}(\bar{y}_2,M_2(x_k))).$$

Then we can proceed as in the proof of Theorem 1(2.) with $s=2$, $(\gamma_1,\gamma_2)=(1,2)$ to find the sequence $(\tilde{x}_k,\tilde{y}_k)$. By our assumption we have $\varepsilon=\Psi_k(\check{y}_k-y_k)\le\frac{4}{k}\|x_k-\bar{x}\|^{3/2}$ and therefore $\|\tilde{x}_k-x_k\|\le\frac{\sqrt{k}}{4}\varepsilon\le\frac{1}{\sqrt{k}}\|x_k-\bar{x}\|^{3/2}$. Then choosing $\delta_k<\frac{1}{k}\|x_k-\bar{x}\|^{3/2}$ we can conclude that

$$\lim_{k\to\infty}\frac{\left\|x'_k-\bar{x}-\|x'_k-\bar{x}\|\frac{u}{\|u\|}\right\|}{\|x'_k-\bar{x}\|^{3/2}}=0. \tag{23}$$

Then we can proceed as in the proof of Theorem 2 to find the sequences $(\bar{y}^*_k)$ and $(\bar{z}^*_k)$, such that $(\bar{y}^*_k,\bar{z}^*_k)\in\Xi(u)$ holds for all k sufficiently large and $\langle\bar{z}^*_k,\nabla^2F(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle+\frac{1}{3}\langle\bar{y}^*_{k2},\nabla^3F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle<0$ for all k sufficiently large follows. By eventually passing to a subsequence,

$$\limsup_{k\to\infty}\langle\bar{z}^*_k,\nabla^2F(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle+\frac{1}{3}\langle\bar{y}^*_{k2},\nabla^3F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle=:-2\sigma<0$$

follows, since otherwise we could construct $\bar{y}^*_{k2},\bar{z}^*_k$ fulfilling also the equation

$$\langle\bar{z}^*,\nabla^2F(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle+\frac{1}{3}\langle\bar{y}^*_2,\nabla^3F_2(\bar{x})(\frac{u}{\|u\|},\frac{u}{\|u\|},\frac{u}{\|u\|})\rangle=0, \tag{24}$$

because adding the single equation (24) to the system defining $\bar{y}^*_{k2},\bar{z}^*_k$ does not affect the applicability of Hoffman's lemma. Now taking into account that $\|\bar{z}^*_{k1}-y^*_{k1}\|+\|\bar{y}^*_{k2}-y^*_{k2}\|\to0$, $\|x'_k-\bar{x}\|\bar{z}^*_{k2}\to0$ and $\langle y^*_{k1},y'_{k1}-\bar{y}_1\rangle+\|x'_k-\bar{x}\|^{-1}\langle y^*_{k2},y'_{k2}-\bar{y}_1\rangle\ge\frac{1}{2}\|(y'_{k1}-\bar{y}_1,\|x'_k-\bar{x}\|^{-1}(y'_{k2}-\bar{y}_2))\|$ by (11), we obtain for all k sufficiently large

$$\langle\bar{z}^*_{k1},y'_{k1}-\bar{y}_1\rangle+\|x'_k-\bar{x}\|^{-1}\langle\bar{y}^*_{k2}+\|x'_k-\bar{x}\|\bar{z}^*_{k2},y'_{k2}-\bar{y}_1\rangle\ge\frac{1}{4}\|(y'_{k1}-\bar{y}_1,\|x'_k-\bar{x}\|^{-1}(y'_{k2}-\bar{y}_2))\|$$

implying

$$\begin{aligned}
0 &\le \|x'_k-\bar{x}\|^{-3}\langle\bar{y}^*_k+\|x'_k-\bar{x}\|\bar{z}^*_k,y'_k-\bar{y}\rangle\\
&= \|x'_k-\bar{x}\|^{-3}\langle(\bar{y}^*_k+\|x'_k-\bar{x}\|\bar{z}^*_k,F(x'_k)+s'_k-F(\bar{x})-\bar{s}\rangle\\
&= \|x'_k-\bar{x}\|^{-3}(\langle(0,\bar{y}^*_{k2})+\|x'_k-\bar{x}\|\bar{z}^*_k,\nabla F(\bar{x})(x'_k-\bar{x})+s'_k-\bar{s}+\frac{1}{2}\nabla^2F(\bar{x})(x'_k-\bar{x},x'_k-\bar{x})\rangle\\
&\quad+\frac{1}{6}\langle\bar{y}^*_{k2},\nabla^3F_2(\bar{x})(x'_k-\bar{x},x'_k-\bar{x},x'_k-\bar{x})\rangle)+\eta_k,
\end{aligned}$$

where $\eta_k\to0$. Since $\mathscr{P}\subset\mathscr{P}(\bar{x},\bar{s})$ and $\mathscr{A}_i\subset\mathscr{A}_i(\bar{x},\bar{s})\ \forall i\in\mathscr{P}$ we deduce from (21) that

$$\langle(\nabla F(\bar{x})^*\bar{y}^*_k,\bar{y}^*_k)+\|x'_k-\bar{x}\|(\nabla F(\bar{x})^*\bar{z}^*_k+(\nabla^2F_2(\bar{x})\frac{u}{\|u\|})^*\bar{y}^*_{k2},\bar{z}^*_k),(x'_k-\bar{x},s'_k-\bar{s})\rangle=0$$

and since $\langle \bar{y}^*_{k2}, \nabla^2 F_2(\bar{x})(u,u)\rangle = 0$ we have

$$\begin{aligned}
&\langle (0,\bar{y}^*_{k2}) + \|x'_k - \bar{x}\|\bar{z}^*_k, \nabla F(\bar{x})(x'_k - \bar{x}) + s'_k - \bar{s} + \frac{1}{2}\nabla F^2(\bar{x})(x'_k - \bar{x}, x'_k - \bar{x})\rangle \\
= \;& \langle (\nabla F(\bar{x})^* \bar{y}^*_k, \bar{y}^*_k) + \|x'_k - \bar{x}\|(\nabla F(\bar{x})^* \bar{z}^*_k + (\nabla^2 F_2(\bar{x})\frac{u}{\|u\|})^* \bar{y}^*_{k2}, \bar{z}^*_k), (x'_k - \bar{x}, s'_k - \bar{s})\rangle \\
&+ \frac{1}{2}\langle \bar{y}^*_{k2}, \nabla^2 F_2(\bar{x})(x'_k - \bar{x} - \|x'_k - \bar{x}\|\frac{u}{\|u\|}, x'_k - \bar{x} - \|x'_k - \bar{x}\|\frac{u}{\|u\|})\rangle \\
&- \frac{1}{2}\|x'_k - \bar{x}\|^2 \langle \bar{y}^*_{k2}, \nabla^2 F_2(\bar{x})(\frac{u}{\|u\|}, \frac{u}{\|u\|})\rangle + \frac{1}{2}\|x'_k - \bar{x}\|\langle \bar{z}^*_k, \nabla^2 F(\bar{x})(x'_k - \bar{x}, x'_k - \bar{x})\rangle \\
= \;& \frac{1}{2}\langle \bar{y}^*_{k2}, \nabla^2 F_2(\bar{x})(x'_k - \bar{x} - \|x'_k - \bar{x}\|\frac{u}{\|u\|}, x'_k - \bar{x} - \|x'_k - \bar{x}\|\frac{u}{\|u\|})\rangle \\
&+ \frac{1}{2}\|x'_k - \bar{x}\|\langle \bar{z}^*_k, \nabla^2 F(\bar{x})(x'_k - \bar{x}, x'_k - \bar{x})\rangle.
\end{aligned}$$

Hence, using (23) we obtain the contradiction

$$\begin{aligned}
0 \;\leq\;& \limsup_{k\to\infty} \|x'_k - \bar{x}\|^{-3}\langle (0,\bar{y}^*_{k2}) + \|x'_k - \bar{x}\|\bar{z}^*_k, y'_k - \bar{y}\rangle \\
=\;& \limsup_{k\to\infty} \frac{1}{2}\langle \bar{z}^*_k, \nabla^2 F(\bar{x})(\frac{x'_k - \bar{x}}{\|x'_k - \bar{x}\|}, \frac{x'_k - \bar{x}}{\|x'_k - \bar{x}\|})\rangle + \frac{1}{6}\langle \bar{y}^*_{k2}, \nabla^3 F_2(\bar{x})(\frac{x'_k - \bar{x}}{\|x'_k - \bar{x}\|}, \frac{x'_k - \bar{x}}{\|x'_k - \bar{x}\|}, \frac{x'_k - \bar{x}}{\|x'_k - \bar{x}\|})\rangle \\
=\;& \limsup_{k\to\infty} \frac{1}{2}\langle \bar{z}^*_k, \nabla^2 F(\bar{x})(\frac{u}{\|u\|}, \frac{u}{\|u\|})\rangle + \frac{1}{6}\langle \bar{y}^*_{k2}, \nabla^3 F_2(\bar{x})(\frac{u}{\|u\|}, \frac{u}{\|u\|}, \frac{u}{\|u\|})\rangle = -\sigma < 0
\end{aligned}$$

and the proposition is proved. □

6 Optimality conditions

Consider the optimization problem

$$\min f(x) \text{ subject to } 0 \in M_i(x),\; i = 1,\dots,s \tag{25}$$

where the objective function $f : X \to \mathbb{R}$ is defined on the Banach space X and the multifunctions $M_i : X \rightrightarrows Y_i$, $i = 1,\dots,s$ have values in another Banach space Y_i. Given a point $\bar{x} \in X$ we define the multifunction $M_0 : X \rightrightarrows \mathbb{R}$ by $M_0(x) := f(x) - f(\bar{x}) + \mathbb{R}_+$ respectively for arbitrary $x^* \in X^*$ and $\gamma \geq 1$ we define the multifunction $M_0^{x^*,\gamma} : X \rightrightarrows \mathbb{R}$ by $M_0^{x^*,\gamma}(x) := f(x) - f(\bar{x}) + |\langle x^*, x - \bar{x}\rangle|^{\gamma+2} + \mathbb{R}^+$.

The base of deriving necessary optimality conditions from sufficient conditions for metric pseudo-(sub)regularity is given by the following observation:

Proposition 2. *Let $\bar{x}$ be a local minimizer for the problem* (25) *and let $u \in X$. Assume that there are real numbers $\gamma_0, \gamma_1 \dots, \gamma_s \geq 1$ and a sequence $(x_k) \to \bar{x}$ such that $x_k \neq \bar{x}$, $\|u\|\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \to u$ and*

$$\lim_{k\to\infty} \frac{\mathrm{d}(0, M_i(x_k))}{\|x_k - \bar{x}\|^{\gamma_i}} = 0,\; i = 0,\dots,s$$

Then $(M_0, M_1, \dots, M_s)$ is not metrically pseudo-regular of order $(\gamma_0, \dots, \gamma_s)$ in direction $(u, 0)$ at $(\bar{x}, 0)$. In addition, if $u \neq 0$ then there is a functional $x^* \in X^*$ such that $(M_0^{x^*,\gamma_0}, M_1, \dots, M_s)$ is not metrically pseudo-subregular of order $(\gamma_0, \dots, \gamma_s)$ in direction u at $(\bar{x}, 0)$.

Proof. To show the first assertion, assume on the contrary that $\tilde{M} := (M_0, M_1, \ldots, M_s)$ is metrically pseudo-regular of order $(\gamma_0, \ldots, \gamma_s)$ in direction $(u, 0)$ near $(\bar{x}, 0)$ with modulus κ. By passing to a subsequence we can assume that $\mathrm{d}(0, M_i(x_k)) \le \frac{\|x_k - \bar{x}\|^{\gamma_i}}{k}$, $i = 0, \ldots, s$ holds for all k. Taking $y_k := (-\frac{\|x_k - \bar{x}\|^{\gamma_0}}{k}, 0, \ldots, 0)$ we obtain for all k sufficiently large that

$$\mathrm{d}(x_k, \tilde{M}^{-1}(y_k)) \le \kappa \sum_{i=0}^{s} \frac{\mathrm{d}(y_{ki}, M_i(x_k))}{\|x_k - \bar{x}\|^{\gamma_i - 1}} \le \frac{\kappa(s+2)\|x_k - \bar{x}\|}{k}.$$

Thus there is some $\tilde{x}_k \in x_k + \kappa(s+3)\frac{\|x_k - \bar{x}\|}{k}$ such that $y_k \in \tilde{M}(\tilde{x}_k)$, i.e. $f(\tilde{x}_k) - f(\bar{x}) \le -\frac{\|\tilde{x}_k - \bar{x}\|^{\gamma_0}}{k} < 0$, $0 \in M_i(\tilde{x}_k)$, $i = 1, \ldots, s$, contradicting the optimality of $\bar{x}$.

To show the second assertion choose $x^* \in \mathscr{S}_{X^*}$ such that $\langle x^*, u \rangle \ne 0$. Now assuming that $\tilde{M} := (M_0^{x^*, \gamma_0}, M_1, \ldots, M_s)$ is metrically pseudo-subregular of order $(\gamma_0, \ldots, \gamma_s)$ in direction u at $(\bar{x}, 0)$ with modulus κ and, by passing to a subsequence, that $\mathrm{d}(0, M_i(x_k)) \le \frac{\|x_k - \bar{x}\|^{\gamma_i}}{k}$, $i = 0, \ldots, s$ holds for all k, we obtain for all k sufficiently large

$$\mathrm{d}(x_k, \tilde{M}^{-1}(0, \ldots, 0)) \le \kappa\Big(\frac{\mathrm{d}(0, M_0^{x^*, \gamma_0}(x_k))}{\|x_k - \bar{x}\|^{\gamma_0 - 1}} + \sum_{i=1}^{s} \frac{\mathrm{d}(0, M_i(x_k))}{\|x_k - \bar{x}\|^{\gamma_i - 1}}\Big) \le \frac{\kappa(s+1)\|x_k - \bar{x}\|}{k} + \kappa\|x_k - \bar{x}\|^3.$$

Thus for all k sufficiently large there is some $\tilde{x}_k \in x_k + \kappa(s+2)\frac{\|x_k - \bar{x}\|}{k}$ such that $0 \in \tilde{M}(\tilde{x}_k)$, i.e. $f(\tilde{x}_k) - f(\bar{x}) \le -|\langle x^*, \tilde{x}_k - \bar{x} \rangle|^{\gamma_0 + 2}$, $0 \in M_i(\tilde{x}_k)$, $i = 1, \ldots, s$. Since

$$\lim_{k \to \infty} \frac{\tilde{x}_k - \bar{x}}{\|\tilde{x}_k - \bar{x}\|} = \lim_{k \to \infty} \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} = \frac{u}{\|u\|}$$

we obtain $\langle x^*, \tilde{x}_k - \bar{x} \rangle \ne 0$ and consequently the contradiction $f(\tilde{x}_k) - f(\bar{x}) < 0$. □

Remark 6. *If $\bar{x}$ is a local minimizer for the problem* (25)*, $u \ne 0$ and the sequence (x_k) of Proposition 2 fulfills*

$$\lim_{k \to \infty} \frac{\frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} - \frac{u}{\|u\|}}{\|x_k - \bar{x}\|^{\frac{1}{2}}} = 0,\ \lim_{k \to \infty} \frac{\mathrm{d}(0, M_i(x_k))}{\|x_k - \bar{x}\|^{\gamma_i + \frac{1}{2}}} = 0,\ i = 0, \ldots, s, \tag{26}$$

then one can show by using similar arguments as before that we can find some functional $x^* \in X^*$ in such a way that there cannot be real numbers $\delta > 0$, $\kappa > 0$ such that for all $x \in \bar{x} + \delta\mathscr{B}_X$ with $x \ne \bar{x}$, $\|x - \bar{x} - \|x - \bar{x}\|\frac{u}{\|u\|}\| \le \delta\|x - \bar{x}\|^{\frac{3}{2}}$ and $\mathrm{d}(0, M_i(x)) \le \delta\|x - \bar{x}\|^{\gamma_i + \frac{1}{2}}$, $i = 0, \ldots, s$ one has

$$\mathrm{d}(x, (M_0^{x^*, \gamma_0}, M_1, \ldots, M_s)^{-1}(0, \ldots, 0)) \le \kappa\Big(\frac{\mathrm{d}(0, M_0^{x^*, \gamma_0})}{\|x - \bar{x}\|^{\gamma_0 - 1}} + \sum_{i=1}^{s} \frac{\mathrm{d}(0, M_i)}{\|x - \bar{x}\|^{\gamma_i - 1}}\Big).$$

We will now explicitly formulate the necessary optimality conditions for the problem

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & 0 \in M_i(x) := F_i(x) + S_i(x),\ i = 1, 2 \end{aligned} \tag{27}$$

with $\gamma_i = i$, $i = 1, 2$ and $\gamma_0 \in \{1, 2\}$, where $M := (M_1, M_2)$ is of the form (18) as we studied in the previous section. We associate with the problem (27) the Lagrangian

$$\mathscr{L} : X \times \mathbb{R} \times Y^* \to \mathbb{R},\ \mathscr{L}(x, y_0^*, y^*) := y_0^* f(x) + \langle y^*, F(x) \rangle.$$

Further we denote by

$$\mathscr{C}(\bar{x}) := \{u \in X \mid \langle \nabla f(\bar{x}), u \rangle \le 0, -\nabla F(\bar{x})u \in CS(\bar{x}, -F(\bar{x}))(u))\}$$

the *cone of critical directions*.

Theorem 3. *Let $\bar{x}$ be a local minimizer of the problem* (27) *and assume that f and F are continuously differentiable at $\bar{x}$ and F_2 is twice Fréchet differentiable at $\bar{x}$. Let $0 \neq u \in \mathscr{C}(\bar{x})$ be any nonzero critical direction such that there exist some $w \in X$ with*

$$(w, -\nabla F_2(\bar{x})w - \frac{1}{2}\nabla^2 F_2(\bar{x})(u,u)) \in T((u, -\nabla F_2(\bar{x})u); T((\bar{x}, -F_2(\bar{x})); \operatorname{gph} S_2)) \tag{28}$$

and for every nonempty subset $\emptyset \neq \mathscr{P} \in \mathscr{I}((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))$ the operators $\check{A}^*_{\mathscr{P}}$ and $\hat{A}^*_{\mathscr{P}}(u)$ have closed range. Then the following hold:

1. *($\gamma_0 = 1$): There exist multipliers $\bar{y}_0^* = 0$, $\bar{y}^* = (0, \bar{y}_2^*) \in Y_1^* \times Y_2^*$, $\bar{z}_0^* \in \mathbb{R}_+$, $\bar{z}^* = (\bar{z}_1^*, \bar{z}_2^*) \in Y_1^* \times Y_2^*$ with $(\bar{z}_0^*, \bar{z}_1^*, \bar{y}_2^*) \neq 0$, $\bar{z}_0^* \langle \nabla f(\bar{x}), u \rangle = 0$,*

$$0 \in \nabla_x \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*) + D^* S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(\bar{y}^*), \tag{29}$$

$$0 \in \nabla_x \mathscr{L}(\bar{x}, \bar{z}_0^*, \bar{z}^*) + \nabla_x^2 \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*)\frac{u}{\|u\|} + CD^* S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(\bar{y}^*, -\nabla_x \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*))(\bar{z}^*), \tag{30}$$

   and some index set $\mathscr{P} \in \mathscr{K}((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u); (-\nabla_x \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*), \bar{y}^*))$ with

$$\nabla_x^2 \mathscr{L}(\bar{x}, 0, \bar{y}^*)(u,u) = \sup\{\nabla_x^2 \mathscr{L}(\bar{x}, 0, (0, y_2^*))(u,u) \mid (-\nabla F_2(\bar{x})^* y_2^*, (0, -y_2^*)) \in N((\bar{x}, -F(\bar{x})); P_i)\},\ \forall i \in \mathscr{P}.$$

   If in addition f and F are twice Fréchet differentiable at $\bar{x}$ and F_2 is three times Fréchet differentiable at $\bar{x}$, then one also has

$$\nabla_x^2 \mathscr{L}(\bar{x}, \bar{z}_0^*, \bar{z}^*)(\frac{u}{\|u\|}, \frac{u}{\|u\|}) + \frac{1}{3}\nabla_x^3 \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*)(\frac{u}{\|u\|}, \frac{u}{\|u\|}, \frac{u}{\|u\|}) \geq 0. \tag{31}$$

2. *($\gamma_0 = 2$): If f is twice Fréchet differentiable at $\bar{x}$ and*

$$-\langle \nabla f(\bar{x}), w \rangle - \frac{1}{2}\nabla^2 f(\bar{x})(u,u) \in T(-\langle \nabla f(\bar{x}), u \rangle; \mathbb{R}_+), \tag{32}$$

   then there exist multipliers $\bar{y}_0^* \in \mathbb{R}_+$, $\bar{y}^* = (0, \bar{y}_2^*) \in Y_1^* \times Y_2^*$, $\bar{z}_0^* \in T(\bar{y}_0^*; \mathbb{R}_+)$, $\bar{z}^* = (\bar{z}_1^*, \bar{z}_2^*) \in Y_1^* \times Y_2^*$ such that $(\bar{y}_0^*, \bar{z}_1^*, \bar{y}_2^*) \neq 0$, $\bar{y}_0^* \langle \nabla f(\bar{x}), u \rangle = \bar{z}_0^* \langle \nabla f(\bar{x}), u \rangle = 0$, conditions (29) *and* (30) *are fulfilled and there is some index set $\mathscr{P} \in \mathscr{K}((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u); (-\nabla_x \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*), \bar{y}^*))$ with*

$$\begin{aligned} \nabla_x^2 \mathscr{L}(\bar{x}, \bar{y}_0^*, \bar{y}^*)(u,u) \quad = \quad & \sup\{\nabla_x^2 \mathscr{L}(\bar{x}, y_0^*, (0, y_2^*))(u,u) \mid (-y_0^* \nabla f(\bar{x}) - \nabla F_2(\bar{x})^* y_2^*, (0, -y_2^*)) \\ & \in N((\bar{x}, -F(\bar{x})); P_i), y_0^* \geq 0, y_0^* \langle \nabla f(\bar{x}), u \rangle = 0\},\ \forall i \in \mathscr{P}. \end{aligned} \tag{33}$$

   If in addition F is twice Fréchet differentiable at $\bar{x}$ and f, F_2 are three times Fréchet differentiable at $\bar{x}$, then one also condition (31) *holds true.*

Proof. $T((u, -\nabla F_2(\bar{x})u); T((\bar{x}, -F_2(\bar{x})); \operatorname{gph} S_2))$ is the union of finitely many convex generalized polyhedral cones because the contingent cone to the union of finitely many convex generalized polyhedral sets at some point is the union of finitely many convex generalized polyhedral cones. Hence

$$(u + tw, -\nabla F_2 u - t(\nabla F_2(\bar{x})w + \frac{1}{2}\nabla^2 F_2(\bar{x})(u,u))) \in T((\bar{x}, -F_2(\bar{x})); \operatorname{gph} S_2)\ \forall t \geq 0$$

follows and consequently there is some $\bar{t} \geq 0$ such that

$$(\bar{x}, -F_2(\bar{x})) + t(u + tw, -\nabla F_2(\bar{x})u - t(\nabla F_2(\bar{x})w + \frac{1}{2}\nabla^2 F_2(\bar{x})(u,u))) \in \operatorname{gph} S_2 \forall t \in [0, \bar{t}].$$

Since u is a critical direction we can also assume

$$(\bar{x}, -F(\bar{x})) + t(u, -\nabla F(\bar{x})u) \in \operatorname{gph} S(\bar{x}, -F(\bar{x}))\ \forall t \in [0, \bar{t}].$$

To show the first assertion, choose any sequence $(t_k) \downarrow 0$ and define the sequence $x_k := \bar{x} + t_k u + t_k^2 w$. Then $\lim_{k\to\infty} \|x_k - \bar{x}\|/t_k = \|u\|$,

$$\lim_{k\to\infty} \frac{\mathrm{d}(0, M_0(x_k))}{\|x - x_k\|} \leq \lim_{k\to\infty} \frac{|f(x_k) - f(\bar{x}) - t_k \langle \nabla f(\bar{x}), u\rangle|}{t_k \|u\|} = 0,$$

$$\lim_{k\to\infty} \frac{\mathrm{d}(0, M_1(x_k))}{\|x - x_k\|} \leq \lim_{k\to\infty} \frac{\|F_1(x_k) - F_1(\bar{x}) - t_k \nabla F_1(\bar{x})u\|}{t_k \|u\|} = 0,$$

$$\lim_{k\to\infty} \frac{\mathrm{d}(0, M_2(x_k))}{\|x - x_k\|^2} \leq \lim_{k\to\infty} \frac{\|F_2(x_k) - F_2(\bar{x}) - t_k(\nabla F_2(\bar{x})u + t_k(\nabla F_2(\bar{x})w + \frac{1}{2}\nabla^2 F_2(\bar{x})(u,u)))\|}{t_k^2 \|u\|^2} = 0$$

and by Proposition 2 there is some $x^* \in X^*$ such that the multifunction $\tilde{M} := (\tilde{M}_1, \tilde{M}_2) : X \rightrightarrows (\mathbb{R} \times Y_1) \times Y_2$

$$\tilde{M}_1(x) := \tilde{F}_1(x) + \tilde{S}_1(x) := (f(x) - f(\bar{x}) + |\langle x^*, x - \bar{x}\rangle|^3, F_1(x)) + \mathbb{R}_+ \times S_1(x),\ \tilde{M}_2(x) := F_2(x) + S_2(x)$$

is not metrically pseudo-regular of order $(1,2)$ in direction u at $(\bar{x}, 0)$. Now consider the multifunction $\tilde{S}(x) := \mathbb{R}_+ \times S(x)$ and let $(x, (\varphi, s)) \in \operatorname{gph} \tilde{S}$, i.e. $\varphi \geq 0$, $s \in S(x)$. Then we obviously have

$$\hat{N}((x, (\varphi, s)); \operatorname{gph} \tilde{S}) = \{(x^*, (y_0^*, y^*)) \,|\, (x^*, y^*) \in \hat{N}((x, s); \operatorname{gph} S), y_0^* \in \hat{N}(\varphi, \mathbb{R}_+)\}$$

and

$$\begin{aligned} & D^*\tilde{S}((\bar{x}, (0, -F(\bar{x}))); (u, -(\langle \nabla f(\bar{x}), u\rangle, \nabla F(\bar{x})u)))(y_0^*, y^*) \\ = & \begin{cases} D^*S((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))(y^*) & \text{if } y_0^* \geq 0,\ y_0^* \langle \nabla f(\bar{x}), u\rangle = 0, \\ \emptyset & \text{else} \end{cases} \end{aligned}$$

follows. Further, $\operatorname{gph} \tilde{S} = \bigcup_{i=1}^p \tilde{P}_i$, where $\tilde{P}_i = \{(x, \phi, s) \,|\, (x, s) \in P_i, \varphi \geq 0\}$. Hence the subspaces $\tilde{L}_i$ associated with the convex generalized polyhedral sets $\tilde{P}_i$ are $\tilde{L}_i = \{(x, \phi, s) \,|\, (x, s) \in L_i, \phi \in \mathbb{R}\}$ and thus $\tilde{L}_i^\perp = \{(x^*, 0, s^*) \,|\, (x^*, s^*) \in L_i^\perp\}$. Moreover, if $\varphi \geq 0$, then $\{i \,|\, (x, \varphi, s) \in \tilde{P}_i\} = \{i \,|\, (x, s) \in P_i\}$ Hence, denoting $\tilde{F} := (\tilde{F}_1, F_2)$ and given some index set $\mathscr{P} \in \mathscr{I}((\bar{x}, -\tilde{F}(\bar{x})); (u, -\nabla \tilde{F}(\bar{x})u))$ associated with $(\tilde{M}_1, \tilde{M}_2)$, we also have $\mathscr{P} \in \mathscr{I}((\bar{x}, -F(\bar{x})); (u, -\nabla F(\bar{x})u))$ associated with (M_1, M_2) and the range of the operator $\check{A}^*_{\mathscr{P}}$ associated with $(\tilde{M}_1, \tilde{M}_2)$ can be identified with the range of the operator $\check{A}^*_{\mathscr{P}}$ associated with (M_1, M_2) and is therefore closed. Further the range of the operator $\hat{A}^*_{\mathscr{P}}(u)$ associated with $(\tilde{M}_1, \tilde{M}_2)$ is the collection of all

$$\begin{pmatrix} (\nabla F_2(\bar{x})^* y_2^*, (0, y_2^*)) + \tilde{l}_i^* \\ (\nabla f(\bar{x})\varphi^* + \nabla F(\bar{x})^* z^* + (\nabla^2 F_2(\bar{x}) \frac{u}{\|u\|})^* y_2^*, (\varphi^*, z^*)) + \tilde{w}_i^* \end{pmatrix}_{i \in \mathscr{P}},$$

where $y_2^* \in Y_2^*$, $(\tilde{l}_i^*)_{i\in\mathscr{P}}, (\tilde{w}_i^*)_{i\in\mathscr{P}} \in \prod_{i\in\mathscr{P}} \tilde{L}_i^\perp$, $z^* \in Y^*$, $\varphi^* \in \mathbb{R}$. Thus it can be identified by the sum of the range of the operator $\hat{A}^*_{\mathscr{P}}(u)$ associated with (M_1, M_2) and a finite dimensional subspace and is therefore closed by the assumption of the Theorem.

Now straightforward application of Theorem 2 gives the stated result. To show (31) we take into account that f, F_1 are twice Fréchet differentiable and F_2 is three times Fréchet differentiable and therefore

$$\limsup_{k\to\infty}\frac{\mathrm{d}(0,M_0(x_k))}{\|x-x_k\|^2}\le\lim_{k\to\infty}\frac{|f(x_k)-f(\bar{x})-t_k\langle\nabla f(\bar{x}),u\rangle|}{t_k^2\|u\|^2}=\frac{|\nabla f(\bar{x})w+\frac{1}{2}\nabla^2 f(\bar{x})(u,u)|}{\|u\|^2}<\infty,$$

$$\limsup_{k\to\infty}\frac{\mathrm{d}(0,M_1(x_k))}{\|x-x_k\|^2}\le\lim_{k\to\infty}\frac{\|F_1(x_k)-F_1(\bar{x})-t_k\nabla F_1(\bar{x})u\|}{t_k^2\|u\|^2}=\frac{\|\nabla F_1(\bar{x})w+\frac{1}{2}\nabla^2 F_2(\bar{x})(u,u)\|}{\|u\|^2}<\infty,$$

$$\begin{aligned}\limsup_{k\to\infty}\frac{\mathrm{d}(0,M_2(x_k))}{\|x-x_k\|^3}&\le\lim_{k\to\infty}\frac{\|F_2(x_k)-F_2(\bar{x})-t_k(\nabla F_2(\bar{x})u+t_k(\nabla F_2(\bar{x})w+\frac{1}{2}\nabla^2 F_2(\bar{x})(u,u)))\|}{t_k^3\|u\|^3}\\&=\frac{\|\nabla^2 F_2(\bar{x})(u,w)+\frac{1}{6}\nabla^3 F_2(\bar{x})(u,u,u)\|}{\|u\|^3}<\infty.\end{aligned}$$

This implies that (26) is fulfilled and (31) follows from Remark 6 and Proposition 1.

The second assertion follows by using similar arguments with $\gamma_0=2$. We omit the details. □

We now apply these results to the mathematical programming problem with equality and inequality constraints

$$\begin{aligned}&\min f_0(x)\\ \text{subject to}\quad &G(x)=0,\\ &f_i(x)\le 0,\ i=1,\dots m.\end{aligned}\tag{34}$$

where $G:X\to V$ carries the Banach space X on another Banach space V and $f_i:X\to\mathbb{R}$, $i=0,\dots,m$. Associated with problem (34) is the Lagrangian

$$\mathscr{L}:X\times\mathbb{R}\times\mathbb{R}^m\times V^*\to\mathbb{R},\mathscr{L}(x,\lambda_0,\lambda,v^*):=\sum_{i=0}^{m}\lambda_i f_i(x)+\langle v^*,G(x)\rangle.$$

and the cone of critical directions at some feasible point $\bar{x}$ is

$$\mathscr{C}(\bar{x})=\{u\in X\,|\,\langle\nabla f_i(\bar{x}),u\rangle\le 0,i\in\{0\}\cup I(\bar{x}),\nabla G(\bar{x})u=0\}$$

where $I(\bar{x}):=\{i\in\{1,\dots,m\}\,|\,f_i(\bar{x})=0\}$ denotes the index set of active inequality constraints. We further denote by

$$\Lambda(\bar{x}):=\{(\lambda_0,\lambda,v^*)\in\mathbb{R}_+\times\mathbb{R}_+^m\times V^*\,|\,\nabla_x\mathscr{L}(\bar{x},\lambda_0,\lambda,\bar{v}^*)=0,\lambda_i=0,\ i\in\{1,\dots,m\}\setminus I(\bar{x})\}$$

the set of multipliers fulfilling the first-order optimality condition of Fritz-John-type.

Corollary 3. *Let $\bar{x}$ be a local minimizer of the problem* (34)*, assume that f_i, $i=0,\dots,m$ and G are three times Fréchet differentiable at $\bar{x}$ and that the range of $\nabla G(\bar{x})$ is closed. Then for any nonzero critical direction $0\neq u\in\mathscr{C}(\bar{x})$ there is some nonzero multiplier $(0,0,0)\neq(\bar{\lambda}_0,\bar{\lambda},\bar{v}^*)\in\Lambda(\bar{x})$ such that*

$$\nabla_x^2\mathscr{L}(\bar{x},\bar{\lambda}_0,\bar{\lambda},\bar{v}^*)(u,u)\ge 0.\tag{35}$$

If

$$0=\nabla_x^2\mathscr{L}(\bar{x},\bar{\lambda}_0,\bar{\lambda},\bar{v}^*)(u,u)=\sup\{\nabla_x^2\mathscr{L}(\bar{x},\lambda_0,\lambda,v^*)(u,u)\,|\,(\lambda_0,\lambda,v^*)\in\Lambda(\bar{x})\}\tag{36}$$

and the operator $A(u) : X \times X \to V \times V$*,*

$$A(u)(x^1,x^2) := (\nabla G(\bar{x})x^1 + \nabla^2 G(\bar{x})(\frac{u}{\|u\|},x^2), \nabla G(\bar{x})x^2) \tag{37}$$

has closed range, then there also exist multipliers $(\bar{\mu}_0,\bar{\mu},\bar{w}^*) \in \mathbb{R}\times\mathbb{R}^m\times V^*$ *such that* $(\bar{\mu}_0,\bar{\mu}) \in T((\bar{\lambda}_0,\bar{\lambda});\mathbb{R}_+\times\mathbb{R}^m_+)$*,* $\bar{\mu}_i = 0$*,* $i\in\{1,\dots,m\}\setminus I(\bar{x})$*,*

$$\nabla_x\mathscr{L}(\bar{x},\bar{\mu}_0,\bar{\mu},\bar{w}^*) + \nabla^2_x\mathscr{L}(\bar{x},\bar{\lambda}_0,\bar{\lambda},\bar{v}^*)\frac{u}{\|u\|} = 0 \tag{38}$$

and

$$\nabla^2_x\mathscr{L}(\bar{x},\bar{\mu}_0,\bar{\mu},\bar{w}^*)(\frac{u}{\|u\|},\frac{u}{\|u\|}) + \frac{1}{3}\nabla^3_x\mathscr{L}(\bar{x},\bar{\lambda}_0,\bar{\lambda},\bar{v}^*)(\frac{u}{\|u\|},\frac{u}{\|u\|},\frac{u}{\|u\|}) \geq 0. \tag{39}$$

Proof. First note that (35) is the standard second-order optimality condition due to Levitin, Miljutin and Osmolovskii [24], if $\nabla G(\bar{x})$ is surjective. If $\nabla G(\bar{x})$ is not surjective, there exists $0 \neq v^* \in V^*$ with $\nabla G(\bar{x})^*v^* = 0$ and (35) follows with $(\bar{\lambda}_0,\bar{\lambda},\bar{v}^*) = (0,0,\pm v^*)$. Now assume that condition (36) is fulfilled. To ease the notation we assume $I(\bar{x}) = \{1,\dots,m\}$. We apply Theorem 3 ($\gamma_0 = 2$) to problem (34) with $Y_1 := \{0\}$, $F_1(x) := 0$, $S_1(x) := \{0\}$ and $Y_2 := \mathbb{R}^m\times V$, $F_2 = (f_1,\dots,f_m,G)$, $S_2(x) = \mathbb{R}^m_+\times\{0_V\}$. Hence $\operatorname{gph} S$ is formed by one convex generalized polyhedral set $P_1 = X\times\mathbb{R}^m_+\times\{0_V\}$. For every $v^*\in\ker\nabla G(\bar{x})^*$ we have $(0,0,\pm v^*)\in\Lambda$ and we deduce from condition (36) that $\nabla^2_x\mathscr{L}(\bar{x},0,0,\pm v^*)(u,u) = \langle\pm v^*,\nabla^2 G(\bar{x})(u,u)\rangle \leq 0$. Thus $\nabla G^2(\bar{x})(u,u)\in(\ker\nabla G(\bar{x})^*)^\perp$ and by using the Closed Range Theorem we obtain $\nabla G^2(\bar{x})(u,u)\in\operatorname{Im}\nabla G(\bar{x})$. Hence there is some element $w_0\in X$ satisfying $\nabla G(\bar{x})w_0 + \nabla^2 G(\bar{x})(u,u) = 0$. Now we choose some finite dimensional subspace $E\subset\ker\nabla g(\bar{x})$ such that $(\nabla f_0(\bar{x}),\dots,\nabla f_m(\bar{x}))(E) = (\nabla f_0(\bar{x}),\dots,\nabla f_m(\bar{x}))(\ker\nabla G(\bar{x}))$ and let denote a_i^* respectively $\tilde{a}_i^*$ denote the restriction of $\nabla f_i(\bar{x})$ to $\ker\nabla G(\bar{x})$ respectively E. Then for every $\lambda_0,\dots,\lambda_m\geq 0$ satisfying $\sum_{i=0}^m\lambda_i\tilde{a}_i^* = 0$ we also have $\sum_{i=0}^m\lambda_i a_i^* = 0$ and therefore $\sum_{i=1}^m\lambda_i\nabla f_i(\bar{x})\in(\ker\nabla G(\bar{x}))^\perp$. By utilizing the Closed Range Theorem once more we can find some $v^*\in V^*$ such that $\sum_{i=0}^m\lambda_i\nabla f_i(\bar{x}) + \nabla G(\bar{x})^*v^* = 0$ and therefore $(\lambda_0,\lambda,v^*)\in\Lambda(\bar{x})$. Hence

$$0 \geq \nabla^2_x\mathscr{L}(\bar{x},\lambda_0,\lambda,v^*)(u,u) = \sum_{i=0}^m\lambda_i\nabla^2 f_i(\bar{x})(u,u) - \langle\nabla G(\bar{x})^*v^*,w_0\rangle = \sum_{i=0}^m\lambda_i(\nabla^2 f_i(\bar{x})(u,u) + \langle\nabla f_i(\bar{x}),w_0\rangle)$$

and due to the Farkas Lemma we can find some element $w_1\in E\subset\ker\nabla G(\bar{x})$ such that

$$\langle\tilde{a}_i^*,w_1\rangle + (\nabla^2 f_i(\bar{x})(u,u) + \langle\nabla f_i(\bar{x}),w_0\rangle) = \langle\nabla f_i(\bar{x}),w_0+w_1\rangle + \nabla^2 f_i(\bar{x})(u,u) \leq 0, i = 0,\dots,m.$$

It follows that $w := \frac{1}{2}(w_0+w_1)$ fulfills conditions (28) and (32). There remains to show that the operators $\check{A}^*_{\mathscr{P}}$ and $\hat{A}^*_{\mathscr{P}}(u)$, $\mathscr{P} = \{1\}$, have closed range. The linear subspace L_1 associated with P_1 is $X\times\mathbb{R}^m\times\{0_V\}$. Hence $L_1^\perp = \{0_X\}\times\{0_{\mathbb{R}^m}\}\times V^*$ is the dual space of the quotient space $X\times\mathbb{R}^m\times\{0_V\}/L_1$ which can be identified with V. Hence $\check{A}^*_{\mathscr{P}}$ is the adjoint operator of $\check{A}_{\mathscr{P}} : X\times(\mathbb{R}^m\times V)\to(\mathbb{R}^m\times V)\times V$,

$$\check{A}_{\mathscr{P}}(x,(r_1,\dots,r_m),v) = (((\langle\nabla f_i(\bar{x}),x\rangle + r_i)_{i=1,\dots,m}, \nabla G(\bar{x})x + v, v)$$

and due to the closed range of $\nabla G(\bar{x})$, the operator $\check{A}_{\mathscr{P}}$ and its adjoint operator $\check{A}^*_{\mathscr{P}}$ have closed range. Further $\hat{A}^*_{\mathscr{P}}(u)$ is the adjoint operator of $\hat{A}_{\mathscr{P}}(u) : X\times(\mathbb{R}^m\times V)\times X\times(\mathbb{R}^m\times V)\to(\mathbb{R}^m\times V)\times V\times(\mathbb{R}^m\times V)\times V$,

$$\hat{A}_{\mathscr{P}}(u)(x^1,(r^1_1,\dots,r^1_m),v^1,x^2,(r^2_1,\dots,r^2_m),v^2) = (((\langle\nabla f_i(\bar{x}),x^1\rangle + r^1_i + \nabla^2 f_i(\bar{x})(\frac{u}{\|u\|},x^2))_{i=1,\dots,m},$$
$$\nabla G(\bar{x})x^1 + v^1 + \nabla^2 G(\bar{x})(\frac{u}{\|u\|},x^2), v^1, (\langle\nabla f_i(\bar{x}),x^2\rangle + r^2_i)_{i=1,\dots,m}, \nabla G(\bar{x})x^2 + v^2, v^2).$$

Since $A(u)$ is assumed to have closed range, it follows that also $\hat{A}_{\mathscr{P}}(u)$ and consequently $\hat{A}^*_{\mathscr{P}}(u)$ have closed range. Hence the assertion follows from Theorem 3 ($\gamma_0 = 2$). □

Condition (38) can be already found in [9, Corollary 5.1] but the necessary condition (39) is new. Let us mention that in a similar way the necessary optimality conditions of [3, Theorem 4] for the problem (34) follow from Theorem 3 ($\gamma_0 = 1$). Further, the necessary optimality conditions of [2, Theorem 2] also follow from Theorem 3 ($\gamma_0 = 1$) with $F_1(x) := (f_1(x), \ldots, f_m(x))$, $S_1(x) := \mathbb{R}^+_m$, $F_2(x) := G(x)$, $S_2(x) := \{0_V\}$.

Example 4. *Consider the problem*

$$\min x^3 \text{ subject to } x^3 \leq 0.$$

Then at $\bar{x} = 0$ *the necessary optimality conditions of [3, Theorem 4] and [2, Theorem2] are fulfilled. However, the necessary conditions of Corollary 3 are violated for* $u = -1$*, verifying that* $\bar{x} = 0$ *is not a local minimizer.*

References

[1] E. R.AVAKOV, *Extremum conditions for smooth problems with equality-type constraints*, USSR Comput. Math. and Math. Phys., 25 (1985), pp. 24–32.

[2] E. R.AVAKOV, *Necessary extremum conditions for smooth abnormal problems with equality and inequality-type constraints*, Mathem. Notes, 45 (1989), pp. 431–437.

[3] E. R. AVAKOV, A. V. ARUTYUNOV , A. F. IZMAILOV, *Necessary Conditions for an Extremum in a Mathematical Programming Problem*, Proc. Steklov Inst. Math., 256 (2007), pp. 2-25.

[4] E. R. AVAKOV, A. V. ARUTYUNOV , A. F. IZMAILOV, *Necessary optimality conditions for constrained optimization problems under relaxed constraint qualifications*, Math.Program., Ser. A, 114 (2008), pp. 37–68.

[5] J. V. BURKE, *Calmness and exact penalization*, SIAM J. Control Optim., 29 (1991), pp. 493–497.

[6] A. L. DONTCHEV, R. T. ROCKAFELLAR, *Regulartity and conditioning of solution mappings in variatonal anlysis*, Set-Valued Anal., 12 (2004), pp. 79–109.

[7] M. J. FABIAN, R. HENRION, A. Y. KRUGER, J. V. OUTRATA, *Error bounds: necessary and sufficient conditions*, Set-Valued Anal., 18 (2010), pp. 121-149.

[8] H. FRANKOWSKA, M. QUINCAMPOIX, *Hölder metric regularity of set-valued maps*, Math.Program., Ser. A, 132 (2012), pp. 333354.

[9] H. GFRERER, *Second-order necessary conditions for nonlinear optimization problems with abstract constraints: the degenerate case*, SIAM J. Optim., 18 (2007), pp. 589-612.

[10] H. GFRERER, *First order and second order characterizations of metric subregularity and calmness of constraint set mappings*, SIAM J. Optim., 21 (2011), pp. 1439–1474.

[11] H. GFRERER, *On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs*, Set-Valued Var. Anal., (2012), DOI: 10.1007/s11228-012-0220-5

[12] H. GFRERER, *On directional metric subregularity and second-order optimality conditions for a class of nonsmooth mathematical programs*, preprint (submitted), (2012), pp. 1–35.

[13] R. HENRION, A. JOURANI, *Subdifferential conditions for calmness of convex constraints*, SIAM J. Optim., 13(2002), pp. 520–534.

[14] R. HENRION, A. JOURANI, J. V. OUTRATA, *On the calmness of a class of multifunctions*, SIAM J. Optim., 13 (2002), pp. 603–618.

[15] R. HENRION, J. V. OUTRATA, *A subdifferential condition for calmness of multifunctions*, J. Math. Anal. Appl., 258 (2001), pp. 110–130.

[16] R. HENRION, J. V. OUTRATA, *Calmness of constraint systems with applications*, Math. Program., Ser. B, 104 (2005), pp. 437–464.

[17] A. D. IOFFE, *Necessary and sufficient conditions for a local minimum 1: A reduction theorem and first order conditions*, SIAM J. Control Optim., 17 (1979), pp. 245–250.

[18] A. D. IOFFE, *Regular points of Lipschitz functions*, Trans. Amer. Math. Soc., 251 (1979), pp. 61–69.

[19] A. D. IOFFE, *Metric regularity and subdifferential calculus*, Russian Math. Surveys, 55 (2000), pp. 501–558.

[20] A. D. IOFFE, J. V. OUTRATA, *On metric and calmness qualification conditions in subdifferential calculus*, Set-Valued Anal., 16 (2008), pp. 199–227.

[21] D. KLATTE, B. KUMMER, *Nonsmooth equations in optimization. Regularity, calculus, methods and applications*, Nonconvex Optimization and its Applications 60, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.

[22] D. KLATTE, B. KUMMER, *Optimization methods and stability of inclusions in Banach spaces*, Math. Program., Ser. B, 117 (2009), pp. 305–330.

[23] B. KUMMER, *Inclusions in general spaces: Hoelder stability, solution schemes and Ekeland's principle*, J. Math. Anal. Appl., 358 (2009), pp. 327–344.

[24] E. S. LEVITIN, A. A.MILYUTIN AND N. P. OSMOLOVSKII, *Conditions of high order for a local minimum in problems with constraints*, Russian Math. Surveys, 33 (1978), pp. 97–168.

[25] G. LI, B. S. MORDUKHOVICH, *Hölder metric subregularity with applications to proximal point method*, to appear in SIAM J. Control Optim.

[26] B. S. MORDUKHOVICH, *Variational analysis and generalized differentiation, I: Basic theory, II: Applications*, Springer, Berlin, Heidelberg, 2006.

[27] J.-P. PENOT, *Error bounds, calmness and their applications in nonsmooth analysis*, in Nonlinear Analysis and Optimization II: Optimization, A. Leizarowitz, B. S. Mordukhovich, I. Shafrir, and A. J. Zaslavski, eds., vol. 514 of Contemp. Math., AMS, Providence, RI, 2010, pp. 225–248.

[28] S. M. ROBINSON, *Some continuity properties of polyhedral multifunctions*, Math. Program. Studies, 14 (1981), pp. 206–214.

[29] R. T. ROCKAFELLAR, R. J-B. WETS, *Variational analysis*, Springer, Berlin, 1998.

[30] Z. WU, J. J. YE, *Sufficient conditions for error bounds*, SIAM J. Optim., 12 (2001), pp. 421–435.

[31] Z. WU, J. J. YE, *First-order and second-order conditions for error bounds*, SIAM J. Optim., 14 (2003), pp. 621–645.

[32] J. J. YE, *Constraint qualifications and necessary optimality conditions for optimization problems with variational inequality constraints*, SIAM J. Optim., 10 (2000), pp. 943–962.

[33] J. J. YE, X. Y. YE, *Necessary optimality conditions for optimization problems with variational inequality constraints*, Math. Oper. Res., 22 (1997), pp. 977997.

[34] N. D. YEN, J. C. YAO, B. T. KIEN, *Covering properties at positive-order rates of multifunctions and some related topics*, J. Math. Anal. Appl., 338 (2008), pp. 467–478.

[35] X. Y. ZHENG, K. F. NG, *Metric subregularity and constraint qualifications for convex generalized equations in Banach spaces*, SIAM J. Optim., 18 (2007), pp. 437–460.

[36] X. Y. ZHENG, K. F. NG, *Metric subregularity and calmness for nonconvex generalized equations in Banach spaces*, SIAM J. Optim. 20 (2010), 21192136.

Latest Reports in this series

2009 - 2010

[..]

2011

[..]			
2011-08	Michael Kolmbauer *A Robust FEM-BEM MinRes Solver for Distributed Multiharmonic Eddy Current Optimal Control Problems in Unbounded Domains*	November 2011
2011-09	Michael Kolmbauer *Efficient Solvers for Multiharmonic Eddy Current Optimal Control Problems with Various Constraints*	November 2011
2011-10	Markus Kollmann, Michael Kolmbauer, Ulrich Langer, Monika Wolfmayr and Walter Zulehner *A Finite Element Solver for a Multiharmonic Parabolic Optimal Control Problem*	December 2011

2012

2012-01	Markus Kollmann and Walter Zulehner *A Robust Preconditioner for Distributed Optimal Control for Stokes Flow with Control Constraints*	January 2012
2012-02	Michael Kolmbauer and Ulrich Langer *A Robust Preconditioned MinRes-Solver for Time-Periodic Eddy Current Problems*	January 2012
2012-03	Wolfgang Krendl, Valeria Simoncini and Walter Zulehner *Stability Estimates and Structural Spectral Properties of Saddle Point Problems*	February 2012
2012-04	Helmut Gfrerer *On Directional Metric Regularity, Subregularity and Optimality Conditions for Nonsmooth Mathematical Programs*	February 2012
2012-05	Veronika Pillwein and Stefan Takacs *A Multigrid Fourier Analysis of a Multigrid Method using Symbolic Computation*	April 2012
2012-06	Stefan Takacs and Walter Zulehner *Convergence Analysis of All-at-once Multigrid Methods for Elliptic Control Problems Under Partial Elliptic Regularity*	June 2012
2012-07	Helmut Gfrerer *On Directional Metric Subregularity and Second-Order Optimality Conditions for a Class of Nonsmooth Mathematical Programs*	August 2012
2012-08	Michael Kolmbauer and Ulrich Langer *Efficient Solvers for Some Classes of Time-Periodic Eddy Current Optimal Control Problems*	November 2012
2012-09	Clemens Hofreither, Ulrich Langer and Clemens Pechstein *FETI Solvers for Non-Standard Finite Element Equations Based on Boundary Integral Operators*	November 2012
2012-10	Helmut Gfrerer *On Metric Pseudo-(sub)Regularity of Multifunctions and Optimality Conditions for Degenerated Mathematical Programs*	December 2012

From 1998 to 2008 reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`