Exercise sheet 2

1. To apply the Browder–Minty theorem, it remains to check that A is monotone. We obtain the strong monotonicity by the coercivity and linearity of A as it holds

$$\langle Au - Av, u - v\rangle_X = \langle A(u - v), u - v\rangle_X \geq C\|u - v\|_X^2 \geq 0.$$

2. To appear.

3. (a) For all $t_0 \in [a, b[$ and $h > 0$ but sufficiently small, we have

$$f(t_0 + h) - f(t_0) = \int_{t_0}^{t_0+h} g(s)ds = \int_a^b g(s)1_{[t_0,t_0+h]}(s)ds \underset{h\to 0}{\longrightarrow} 0$$

from Lebesgue's dominated convergence theorem. A similar argument with $h < 0$ shows the continuity of f. We now need to verify that the derivative of f in the sense of distributions is the function g. Let $\varphi \in \mathcal{D}(]a, b[)$; we have

$$-\int_a^b f(t)\varphi'(t)dt = -C\int_a^b \varphi'(t)dt - \int_a^b \left(\int_a^t g(s)\varphi'(t)ds\right) dt$$

The first term is zero because $\varphi(a) = \varphi(b) = 0$ and we apply Fubini's theorem to the second term (the function $(t, s) \mapsto \varphi'(t)g(s)$ is integrable with respect to the two variables). It follows that

$$\begin{aligned}
-\int_a^b f(t)\varphi'(t)dt &= -\int_a^b \left(\int_a^b 1_{[a\leq s\leq t]}g(s)\varphi'(t)ds\right) dt \\
&= -\int_a^b \left(\int_a^b 1_{[a\leq s\leq t]}g(s)\varphi'(t)dt\right) ds \\
&= -\int_a^b \left(g(s)\int_s^b \varphi'(t)dt\right) ds = -\int_a^b (g(s)(\varphi(b) - \varphi(s)))ds \\
&= \int_a^b g(s)\varphi(s)ds
\end{aligned}$$

This indeed proves that $g = f'$ in the sense of distributions and therefore $f \in W^{1,1}(]a, b[)$.

(b) Let $f \in W^{1,1}(]a, b[)$. We introduce

$$g(t) = \int_a^t f'(s)ds$$

According to (a), the function g is continuous in $W^{1,1}(]a, b[)$ and its derivative in the sense of distributions is f'. Hence, $f - g$ is a function for which the distribution derivative is zero. We then know that there exists a real number C such that $f - g = C$ almost everywhere. If we define $\tilde{f} = C + g$, we have shown that f coincides with the continuous function $\tilde{f}$, almost everywhere. From the definition of $\tilde{f}$, it is clear that for all $x, y \in [a, b]$, we have

$$\tilde{f}(y) - \tilde{f}(x) = \left(C + \int_a^y f'(s)ds\right) - \left(C + \int_a^x f'(s)ds\right) = \int_x^y f'(s)ds.$$