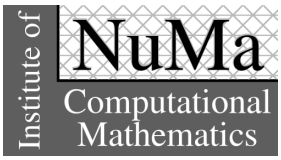

JOHANNES KEPLER UNIVERSITY LINZ

Institute of Computational Mathematics

Scaling Up through Domain Decomposition

Clemens Pechstein

Institute of Comptutational Mathematics, University of Linz,
Altenberger Str. 69, 4040 Linz, Austria

Robert Scheichl
Dept. Mathematical Sciences, University of Bath,
Claverton Down, Bath BA2 7AY, UK.

NuMa-Report No. 2009-01 January 2009

A–4040 LINZ, Altenbergerstraße 69, Austria

Technical Reports before 1998:

1995

95-1	Hedwig Brandstetter *Was ist neu in Fortran 90?*	March 1995
95-2	G. Haase, B. Heise, M. Kuhn, U. Langer *Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations.*	August 1995
95-3	Joachim Schöberl *An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions.*	August 1995

1996

96-1	Ferdinand Kickinger *Automatic Mesh Generation for 3D Objects.*	February 1996
96-2	Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn *Preprocessing in BE/FE Domain Decomposition Methods.*	February 1996
96-3	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	February 1996
96-4	Bodo Heise und Michael Jung *Robust Parallel Newton-Multilevel Methods.*	February 1996
96-5	Ferdinand Kickinger *Algebraic Multigrid for Discrete Elliptic Second Order Problems.*	February 1996
96-6	Bodo Heise *A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation.*	May 1996
96-7	Michael Kuhn *Benchmarking for Boundary Element Methods.*	June 1996

1997

97-1	Bodo Heise, Michael Kuhn and Ulrich Langer *A Mixed Variational Formulation for 3D Magnetostatics in the Space $H(rot) \cap H(div)$*	February 1997
97-2	Joachim Schöberl *Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case.*	June 1997
97-3	Ferdinand Kickinger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl *Numerical Estimates of Inequalities in $H^{\frac{1}{2}}$.*	August 1997
97-4	Joachim Schöberl *Programmbeschreibung NAOMI 2D und Algebraic Multigrid.*	September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see
`http://www.sfb013.uni-linz.ac.at/index.php?id=reports`
From 2004 on reports were also published by RICAM. Please see
`http://www.ricam.oeaw.ac.at/publications/list/`
For a complete list of NuMa reports see
`http://www.numa.uni-linz.ac.at/Publications/List/`

Scaling Up through Domain Decomposition

Clemens Pechstein[a,*] and Robert Scheichl[b]

[a] *Inst. Computational Mathematics, University of Linz, Altenberger Str. 69, 4040 Linz, Austria*
[b] *Dept. Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK.*

January 26, 2009

Abstract

In this paper we discuss domain decomposition parallel iterative solvers for highly heterogeneous problems of flow and transport in porous media. We are particularly interested in highly unstructured coefficient variation where standard periodic or stochastic homogenisation theory is not applicable. When the smallest scale at which the coefficient varies is very small it is often necessary to scale up the equation to a coarser grid to make the problem computationally feasible. Standard upscaling or multiscale techniques, require the solution of local problems in each coarse element, leading to a computational complexity that is at least linear in the global number N of unknowns on the subgrid. Moreover, except for the periodic and the isotropic random case, a theoretical analysis of the accuracy of the upscaled solution is not yet available. Multilevel iterative methods for the original problem on the subgrid, such as multigrid or domain decomposition, lead to similar computational complexity (i.e. $\mathcal{O}(N)$) and are therefore a viable alternative. However, previously no theory was available guaranteeing the robustness of these methods to large coefficient variation. We review a sequence of recent papers where simple variants of domain decomposition methods, such as overlapping Schwarz and one-level FETI, are proposed that are robust to strong coefficient variation. Moreover, we extend the results to other substructuring techniques, such as all-floating FETI and FETI-DP.

Keywords: multiscale PDEs; numerical homogenisation; parallel iterative solvers; additive Schwarz; FETI; conditioning analysis

AMS Subject Classification: 65N55; 65F10; 35B27; 74Q15; 76S05

1 Introduction

In this paper we discuss the use of domain decomposition parallel iterative solvers for highly heterogeneous problems of flow and transport in porous media, in both the deterministic and (Monte-Carlo simulated) stochastic cases. We are particularly interested in the case of highly unstructured coefficient variation where standard periodic or stochastic homogenisation theory is not applicable, and where there is no a priori scale separation. We will restrict attention to the following important model elliptic problem

$$-\nabla \cdot (K \nabla u) \;=\; f \;, \tag{1}$$

in a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3 with suitable boundary data on the boundary $\partial\Omega$. The $d \times d$ tensor $K(x)$ is assumed isotropic and symmetric positive

*Supported by the Austrian Science Fund (FWF) under grant P19255.

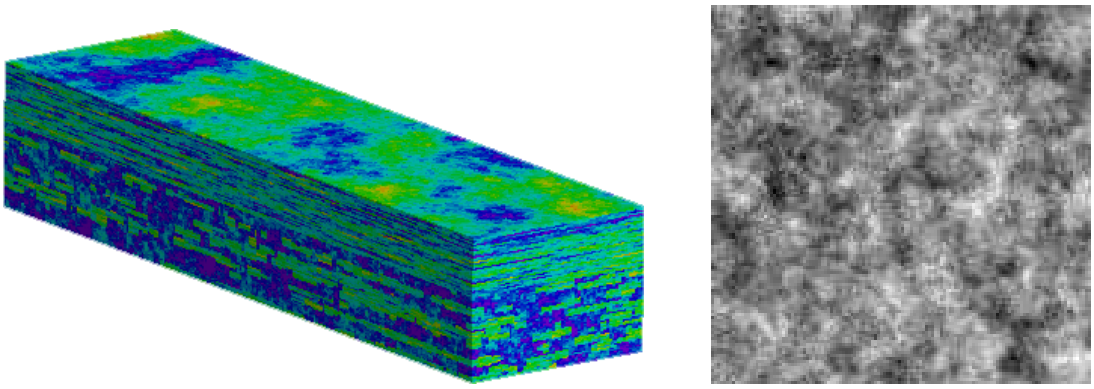

Figure 1: Typical coefficient distributions: benchmark example of the Society of Petroleum Engineer, SPE10 (left), and realisation of a lognormal random field (right).

definite, but may vary over many orders of magnitude in an unstructured way on Ω. Many examples arise in groundwater flow and oil reservoir modelling, e.g. in the context of the SPE10 benchmark problem [4] in Figure 1 (left) or in the Monte Carlo simulated case of stochastic models for strong heteroregeneities [5] in Figure 1 (right).

Let $\mathcal{T}^h$ be a conforming shape-regular simplicial mesh on Ω and let $\mathcal{S}^h(\Omega)$ denote the space of continuous piecewise linear finite elements on $\mathcal{T}^h$. The finite element discretisation of (1) in $\mathcal{V}^h$ (the N-dimensional subspace of functions in $\mathcal{S}^h(\Omega)$ which vanish on essential boundaries), yields the linear system:

$$A\mathbf{u} = \mathbf{f} \ . \tag{2}$$

It is well-known that the size of this system grows like $\mathcal{O}(h^{-d})$, as $\mathcal{T}^h$ is refined, and that the condition number $\kappa(A)$ of A worsens like $\mathcal{O}(h^{-2})$. Moreover the conditioning of A also depends on the heterogeneity (characterised by the range and the variability of K). It is of interest to find solvers for (2) which are robust to changes in the mesh width h as well as to the heterogeneity in K.

When the smallest scale ε, at which the coefficient tensor $K(x)$ varies, is very small it may not be feasible to solve (1) on a mesh of size $h = \mathcal{O}(\varepsilon)$ with standard solvers, and it may be necessary to scale up the equation to a coarser computational grid of size $H \gg \varepsilon$. A large number of computational methods have been suggested over the years in the engineering literature on how to derive such an upscaled equation numerically (see e.g. the reviews [35, 29, 11]). More recently this area has also started to attract the attention of numerical analysts, who have started to try to analyse the approximation properties of such upscaling or multiscale techniques theoretically. Among the methods that have been suggested and analysed are the Variational Multiscale Method [15], the Multiscale Finite Element Method [14], the Heterogeneous Multiscale Method [8], and the Multiscale Finite Volume Method [16]. However, the existing theory is restricted to periodic fine scale variation or to certain isotropic random variation. No theory is yet available that gives a comprehensive analysis of the dependency of the accuracy of the upscaled solution on the coefficient variation in the general case, and it is well known that the approximation can be arbitrarily bad.

Moreover, if the coefficient varies arbitrarily throughout Ω and there is no scale separation into a fine $\mathcal{O}(\varepsilon)$ scale variation and a coarse $\mathcal{O}(H)$ scale variation, then all these methods

require the solution of local "cell" problems, of size $\mathcal{O}((H/\varepsilon)^d)$, in each cell or element of the coarse mesh, i.e. $\mathcal{O}(H^{-d})$ problems. Thus, even if we assume that the local problems can be solved with optimal (linear) complexity, the total computational cost of the method is $\mathcal{O}(\varepsilon^{-d})$. In practice the complexity may actually be worse. A huge advantage is of course the fact that the cell problems are all completely independent from each other. This means that they can be solved very efficiently on a modern multiprocessor machine. This makes this method so attractive to scale up a physical problem,[1] especially if the upscaled matrix can be used for several right hand sides or for several time steps in a time-dependent simulation.

A viable alternative is the use of parallel multilevel iterative solvers, such as multigrid or domain decomposition, for the original fine scale problem (2) on the "subgrid" $\mathcal{T}^h$ where $h = \mathcal{O}(\varepsilon)$. These are known to lead to a similar overall computational complexity $\mathcal{O}(\varepsilon^{-d})$ and, especially in the case of domain decomposition, are designed to scale optimally on modern multiprocessor machines. However, previously no theory was available that guarantees the robustness of these multilevel iterative solvers to heterogeneities in the coefficient, and indeed most of these methods are not robust when used unmodified. The most successful, completely robust method for (2) is algebraic multigrid (AMG), originally introduced in [2, 31]. Many different versions of AMG have emerged since, but unfortunately no theory exists that proves the (observed) robustness of either of these methods to arbitrary spatial variation of $K(x)$. The robustness of geometric multigrid for "layered media" in which discontinuities in K are simple interfaces that can be resolved by the coarsest mesh has recently been proved in [36]. Some ideas towards a theory for more general coefficients can be found in [1]. AMG and the related BoxMG [6] have also recently been used in the context of numerical homogenisation in [24, 20, 23], but this is not the topic of this paper.

The situation is different for domain decomposition methods. There are many papers (with rigorous theory) which solve (2) for "layered media" in which discontinuities in K are simple interfaces that can be resolved by the subdomain partitioning and the coarse mesh (see e.g. [3, 33] and the references therein). However, until recently there was no rigorously justified method for general heterogeneous media. We present here a summary of some recent papers [12, 13, 32, 27, 28, 34] where a new analysis of domain decomposition methods for (2) (which have inherent robustness with respect to h) was presented. This analysis indicates explicitly how subdomains and coarse solves should be designed in order to achieve robustness also with respect to heterogeneities. In addition, we will give here for the first time an extension of the theoretical results for one-level FETI methods in [27], to other substructuring techniques, like all-floating FETI and FETI-DP. The analysis does not require periodicity and does not appeal to homogenisation theory.

For the remainder of the paper let us assume that we have a finite nonoverlapping partitioning of Ω into (open) subdomains $\{\Omega_i : i = 1, \ldots, s\}$, with each $\overline{\Omega}_i$ assumed to consist of a union of elements from $\mathcal{T}^h$. Let us also define the so-called *boundary layer* D_η of width $\eta > 0$ for a subdomain $D \subset \Omega$, such that $\overline{D}_\eta$ consists again of a union of elements from $\mathcal{T}^h$ and

$$\operatorname{dist}(x, \partial D) < \eta, \qquad \text{for all } x \in D_\eta \,.$$

For the purposes of exposition we will only describe the theory for scalar $K = \alpha\, I$ and for homogeneous Dirichlet boundary conditions. For most of the paper we will restrict to the simpler, but still important case of coefficients α that are constant (or varying only mildly)

[1] In this paper we use the term "scaling up" in the sense of solving larger and larger physical problems, which is equivalent to letting $\varepsilon \to 0$ on a fixed size domain.

in the boundary layer $\Omega_{i,\eta}$, for all $i = 1, \ldots, s$ and some $\eta > 0$. We will make some remarks about extensions to more general coefficients in §4. We start by reviewing the analysis for overlapping methods in §2. We will then review the theory for nonoverlapping methods in §3 and extend the results for one-level FETI in [27] to all-floating FETI and FETI-DP.

Throughout the paper, the notation $C \lesssim D$ (for two quantities C, D) means that C/D is bounded from above independently of h and α. Additionally, $C \sim D$ stands for $C \lesssim D$ and $D \lesssim C$.

2 Overlapping Methods

This section is only intended to be a short review of some recent results in a series of papers [12, 13, 32, 34]. Therefore we restrict ourselves to the two-level overlapping additive Schwarz method. The extension of the analysis to other two-level overlapping domain decomposition methods such as multiplicative Schwarz, balancing techniques, or deflation based coarse grid correction follows as usual (see [12, 13] for details). The one-level method is analysed in [12]. For theoretical purposes, we shall assume (in this section) that $\alpha \geq 1$. This is no loss of generality, since problem (2) can be scaled by $(\min_x \alpha(x))^{-1}$ without changing its conditioning.

Given the initial nonoverlapping partitioning $\{\Omega_i : i = 1, \ldots, s\}$ of Ω, we start by extending each subdomain Ω_i to a larger region $\widehat{\Omega}_i$ such that $\widehat{\Omega}_i$ consists again of a union of elements from $\mathcal{T}^h$ and that there exists a $\delta_i > 0$ with

$$x \in \widehat{\Omega}_{i,\delta_i} \quad \Rightarrow \quad x \in \widehat{\Omega}_j\,, \qquad \text{for some } j \neq i,$$

i.e. δ_i is the minimum amount of overlap for subdomain $\widehat{\Omega}_i$. If in addition we are given a coarse space $\mathcal{V}^H \subset \mathcal{V}^h$, then the two-level additive Schwarz preconditioner can be written as

$$M_{AS}^{-1} \;=\; \sum_{i=0}^{s} R_i^T A_i^{-1} R_i\,. \tag{3}$$

Here, for $i = 1, \ldots, s$, R_i denotes the restriction matrix from freedoms in Ω to freedoms in $\widehat{\Omega}_i$. R_0 is the projection onto the coarse space which will be specified later. The matrices A_i are defined via the Galerkin product $A_i := R_i A R_i^T$.

The technical assumptions on the coarse space and on the overlapping subdomains made in the papers [12, 13] and [32, 34] are slightly different. Here we only describe the theory presented in [12, 13] and for that matter we introduce a (shape regular) coarse grid $\mathcal{T}^H$ composed of triangles ($d = 2$) or tetrahedra ($d = 3$). A typical element is the (closed) set K, which again we assume to consist of the union of a set of fine grid elements $\tau \in \mathcal{T}^h$. To simplify the presentation we assume that $\{\Omega_i\} = \mathcal{T}^H$, i.e. the nonoverlapping subdomain partitioning coincides with the coarse grid, and we define a global bound $\delta > 0$ for the (relative) minimum overlap that is defined as $\delta := \min_{i=1,\ldots,s} \frac{H}{H_i}\delta_i$, where $H_i := \operatorname{diam}(\Omega_i)$.

To specify our assumptions on the coarse space we start with a linearly independent set of finite element functions $\{\Phi_j : j = 1, \ldots, N_H\} \subset \mathcal{S}^h(\Omega)$, where each of the Φ_j is associated with a node x_j^H of $\mathcal{T}^H$. A subset of this set will form the basis for our coarse space. The functions have to satsify the following assumptions:

(C1) $\sum_{j=1}^{N_H} \Phi_j(x) = 1$, for all $x \in \bar{\Omega}$;

(C2) $\operatorname{supp}(\Phi_j) \subset \omega_j$ where $\omega_j := \bigcup\{K : x_j^H \in K\}$;

(C3) $\|\Phi_j\|_{L_\infty(\Omega)} \lesssim 1$;

i.e. they form a partition of unity on Ω, and each of the functions Φ_j is bounded and has local support (restricted to the elements K containing coarse node x_j^H). In particular, this implies $\Phi_j(x_k^H) = \delta_{j,k}$, for all $j, k = 1, \ldots, N_H$. If we further assume that the coarse nodes x_j^H are numbered in such a way that $x_j^H \in \Omega$ for all $j \leq N$ and $x_j^H \in \partial\Omega$ for all $j > N$, then we can finally choose the coarse space $\mathcal{V}^H \subset \mathcal{V}^h$ to be $\operatorname{span}\{\Phi_j : j = 1, \ldots, N\}$, i.e. the space spanned by the functions Φ_j that satisfy the Dirichlet boundary conditions on $\partial\Omega$. The restriction matrix R_0 is given by $(R_0)_{j,p} = \Phi_j(x_p^h)$, where x_p^h, $p = 1, \ldots, n$, are the interior nodes of $\mathcal{T}^h$.

Note that the theory below can be generalised to subdomain partitionings that do not coincide with the coarse mesh. It also extends to more general partitions of unity $\{\Phi_j\} \subset \mathcal{S}^h(\Omega)$, not necessarily associated with a simplicial coarse mesh, e.g. aggregation-based coarsening (cf. [32]) or explicit energy minimisation (cf. [34]).

It is well known (see e.g. [33]) that in order to bound $\kappa(M_{AS}^{-1}A)$, we need to assume some upper bounds on $|\Phi_j|^2_{H^1(\Omega)}$ as well. We take a novel approach here and introduce a quantity which also reflects how the coarse space handles the coefficient heterogeneity:

Definition 2.1 (Coarse space robustness indicator)**.**

$$\gamma(\alpha) \;=\; \max_{j=1}^{N_H} \left\{ \operatorname{diam}(\omega_j)^{2-d} \, \|\Phi_j\|_a^2 \right\} \quad \text{where} \quad \|v\|_a^2 := \int_\Omega \alpha |\nabla v|.$$

Note that this robustness indicator is well-behaved if the Φ_j have low energy (independently of any possible variations in α), or in other words, if the Φ_j have small gradient wherever α is large.

The second quantity which we introduce measures (in a certain sense) the ability of the overlapping subdomains $\widehat{\Omega}_i$ to handle the coefficient heterogeneity.

Definition 2.2 (Partitioning robustness indicator)**.**

$$\pi(\alpha) \;=\; \inf_{\{\chi_i\}} \left(\max_{i=1}^{s} \left\{ \delta_i^2 \, \left\| \alpha |\nabla \chi_i|^2 \right\|_{L_\infty(\Omega)} \right\} \right)$$

where the infimum is taken over all partitions of unity $\{\chi_i\} \subset W_\infty^1(\Omega)$ subordinate to the cover $\{\widehat{\Omega}_i\}$.

Roughly speaking, $\pi(\alpha)$ is well-behaved if there is a partition of unity whose members have small gradient wherever α is large.

Using these two robustness indicators and under the assumptions made above we can now state one of the main results from [12, Theorem 3.9]:

Theorem 2.3. *Assume that (C1)–(C3) hold true. Then*

$$\kappa\left(M_{AS}^{-1}A\right) \;\lesssim\; \pi(\alpha)\,\gamma(1)\left(1 + \frac{H}{\delta}\right) \;+\; \gamma(\alpha)\ .$$

$\hat{\alpha}$	$\kappa(M_{AS}^{-1}A)$	$\gamma(\alpha)$
10^0	22.0	3.0
10^2	111	40
10^4	3870	3750
10^6	6000	375000

$\hat{\alpha}$	$\kappa(M_{AS}^{-1}A)$	$\gamma(\alpha)$
10^0	22.0	3.0
10^2	17.7	4.3
10^4	17.6	4.3
10^6	17.6	4.3

Table 1: Two-level additive Schwarz with linear (left) and multiscale (right) coarsening for [12, Example 5.1] with $h = 1/256$, $\delta = 2h$, $H = 8h$.

Thus, provided the overlap δ is sufficiently large w.r.t. the coarse grid size H, i.e. $\delta \sim H$, then the robustness of two-level additive Schwarz can be reduced to bounding the two robustness indicators in Definitions 2.1 and 2.2.

Let us first discuss the partitioning robustness indicator $\pi(\alpha)$. To do this we define for every subdomain Ω_i, $i = 1, \ldots, s$, and for $\eta_i > 0$,

$$\alpha_{i,\eta_i} := \max_{x \in \Omega_{i,\eta_i}} \alpha(x).$$

Then we have the following bound on $\pi(\alpha)$:

Proposition 2.4. $\pi(\alpha) \lesssim \max_{i=1}^{s} \alpha_{i,\eta_i} \left(1 + \frac{\delta_i^2}{\eta_i^2}\right)$

Proof. To prove this we construct a particular partition of unity $\{\chi_i\} \subset W_\infty^1(\Omega)$ subordinate to the cover $\{\widehat{\Omega}_i\}$. Let

$$\widehat{\Omega}_i' := \widehat{\Omega}_i \cap \bigcup\nolimits_{j \neq i} \Omega_{j,\eta_j} \ ,$$

i.e. the subset of $\widehat{\Omega}_i$ that intersects only the boundary layers of its neighbouring domains. Then $\{\widehat{\Omega}_i'\}$ is also a cover of Ω with $\delta_i' = \min\{\delta_i, \eta_i\}$ and $\widehat{\Omega}_i' \subset \widehat{\Omega}_i$. Thus, any partition of unity $\{\chi_i\}$ subordinate to $\{\widehat{\Omega}_i'\}$ is also a partition of unity subordinate to $\{\widehat{\Omega}_i\}$.

Let $\{\chi_i\}$ be the usual partition of unity subordinate to $\{\widehat{\Omega}_i'\}$ based on the distance metric (such as in the proof to [33, Lemma 3.4]). Then $\|\nabla \chi_i\|_{L_\infty(\Omega)} \lesssim 1/\delta_i'$ and so the result follows from the fact that $\nabla \chi_i(x) = 0$ for any $x \notin \bigcup_j \Omega_{j,\eta_j}$. □

Thus $\pi(\alpha) \lesssim 1$, for example, if there exists $\eta_i \sim \delta_i$ such that $\alpha_{i,\eta_i} \sim 1$, for all $i = 1, \ldots, s$, independent of the variation of α in the interior of each of the subdomains Ω_i. However, it is possible to bound $\pi(\alpha)$ independent of α also in the case of strong variation of α near the boundary of any of the Ω_i in certain cases (see §4 for more details).

Now let us investigate the coarse space robustness indicator for different choices of $\{\Phi_j\}$. Since we assumed that $\alpha \geq 1$, we always have $\gamma(1) \leq \gamma(\alpha)$.

Example 2.5 (Linear Finite Element Coarsening)**.** In the classical case, i.e. when $\{\Phi_j\}$ is the standard (nodal) basis for the continuous piecewise linear functions with respect to $\mathcal{T}^H$, we have via standard estimates $\gamma(\alpha) \lesssim \max_{x \in \Omega} \alpha(x)$, and so $\gamma(\alpha) \lesssim 1$ when $\alpha \sim 1$. When $\alpha(x) \to \infty$ for some $x \in \Omega$, on the other hand, then this suggests that linear coarsening may not be robust anymore. The numerical results in Table 1 (left) show that this is indeed the case and that $\gamma(\alpha)$ is a good indicator for the loss of robustness. The results in Table 1 are

for $\Omega = [0,1]^2$ and $\alpha(x) = \hat{\alpha}$ on an "island" in the interior of each coarse element $K \in \mathcal{T}^H$ a distance $\mathcal{O}(H)$ away from ∂K, with $\alpha(x) = 1$ otherwise. (For a precise description of this example see [12, Example 5.1]).

However, our framework leaves open the possibility of choosing $\{\Phi_j\}$ to depend on α in such a way that $\gamma(\alpha)$ is still well-behaved. The next example gives one possible way of constructing such Φ_j.

Example 2.6 (Multiscale Finite Element Coarsening)**.** In this example we use multiscale finite elements on $\mathcal{T}^H$ to define $\mathcal{V}^H$, as proposed in [14].

Let $\mathcal{F}^H$ denote the set of all (closed) faces of elements in $\mathcal{T}^H$ and introduce the *skeleton* $\Gamma_S = \bigcup\{f : f \in \mathcal{F}^H\}$, i.e. the set of all faces of the mesh, including those belonging to the outer boundary $\partial\Omega$. The coarse space basis functions Φ_j are obtained by extending predetermined boundary data into the interior of each element K using a discrete harmonic extension with respect to the original elliptic operator (1). To introduce boundary data for each $j = 1, \ldots, N_H$, we introduce functions $\psi_j : \Gamma_S \to \mathbb{R}$ which are required to be piecewise linear (with respect to the fine mesh $\mathcal{T}^h$ on Γ_S) and to satisfy the following assumptions:

(M1) $\sum_{j=1}^{N_H} \psi_j(x) \;=\; 1\,, \quad \text{for all } \; x \in \Gamma_S;$

(M2) $\psi_j(x_{j'}^H) \;=\; \delta_{j,j'}\,, \quad j, j' = 1, \ldots, N_H;$

(M3) $0 \;\leq\; \psi_j(x) \;\leq\; 1\,, \quad \text{for all } \; x \in \Gamma_S;$

(M4) $\psi_j \;\equiv\; 0 \quad$ on all faces $f \in \mathcal{F}^H$ such that $x_j^H \notin f$.

Using ψ_j as boundary data, the basis function $\Phi_j \in \mathcal{S}^h(\Omega)$ is then defined by discrete $\alpha-$harmonic extension of ψ_j into the interior of each $K \in \mathcal{T}^H$. That is, for each $K \in \mathcal{T}^H$, $\Phi_j|_K \in \{v_h \in \mathcal{S}^h(K) : v_h|_{\partial K} = \psi_j|_{\partial K}\}$ is such that

$$\int_K \alpha \nabla(\Phi_j|_K) \cdot \nabla v_h \;=\; 0 \quad \text{for all } \; v_h \in \mathcal{S}^h(K) \;\; \text{with } \; v_h|_{\partial K} = 0\,, \tag{4}$$

where $\mathcal{S}^h(K)$ is the continuous piecewise linear finite element space with respect to $\mathcal{T}^h$ restricted to K.

The obvious example of boundary data ψ_j satisfying (M1)–(M4) are the standard hat functions on $\mathcal{T}^H$ restricted to the faces of the element K, and these will be sufficient for the results in this section. However, they are not so appropriate if α varies strongly near the boundary ∂K. The "oscillatory" boundary conditions suggested in [14] are more suitable in this case (see §4 and [12] for details).

This recipe specifies $\Phi_j \in S^h(\Omega)$ which can immediately be seen to satisfy the assumptions (C1)–(C3) (see [12] for details). Therefore Theorem 2.3 applies and we have the following bound on $\gamma(\alpha)$ (cf. [12, Theorem 4.3]):

Proposition 2.7. $\gamma(\alpha) \;\lesssim\; \max_{i=1}^{s} \alpha_{i,\eta_i} \dfrac{H_i}{\eta_i}$

Thus again $\gamma(\alpha) \lesssim 1$, for example, if there exists $\eta_i \sim H_i$ such that $\alpha_{i,\eta_i} \sim 1$, for all $i = 1, \ldots, s$ (independent of the variation of α in the interior of each of the subdomains Ω_i).

Again it is also possible to bound $\gamma(\alpha)$ independent of α in the case of strong variation of α near the boundary of any of the Ω_i in certain cases (see §4 for details).

The numerical results in Table 1 (right), obtained for the test problem introduced in Example 2.5 above, show that additive Schwarz with multiscale coarsening can indeed be robust even when the coarse mesh does not resolve discontinuities in α and that our theory accurately predicts this. For more numerical results with multiscale coarsening see [12, 13].

We see already from these results that the choice of the supports ω_j for the coarse basis functions Φ_j is of crucial importance. For highly varying coefficients we will in general require an adaptive choice of the supports taking into account the geometry of the variation of the coefficient α. Ideas on how to do this based on strong and weak connections in A (as in AMG) are given in [32]. In that paper we also study a different type of coarse basis functions based on aggregation. In [34] we look at a method to construct, given a set of supports $\{\omega_j\}$, a coarse basis that explicitly minimises the energy of the coarse basis functions. In both cases, using a slightly different analysis than the one above (involving only one robustness indicator and an additional assumption on the subdomains), we are again able to prove the robustness of our method, if there exists an $\eta_j \sim \text{diam}(\omega_j)$ such that $\alpha \sim 1$ for all $x \in \omega_{j,\eta_j}$ and for all $j = 1, \dots N_H$. The advantage of these two methods is that the supports ω_j can be adapted to the specific coefficient variation of α. Numerical results in [32] show that this leads to a robust method even in the case of random coefficients α.

3 Nonoverlapping Methods/Substructuring Techniques

Finite element tearing and interconnecting (FETI) methods, due to Farhat and Roux [9], and the more recent dual-primal FETI (FETI-DP) methods [10] are one of the most popular type of nonoverlapping domain decomposition methods for FE systems such as (2). They are known to be parallel scalable and quasi-optimal with respect to the number of degrees of freedom (DOF). For a comprehensive presentation and the classical analysis of FETI and FETI-DP we refer to the monograph by Toselli and Widlund [33]. A variant of the classical (or *one-level*) FETI method, is the *all-floating* FETI method, also known as *total FETI* (cf. [7, 25]). For an analysis see [26]. In §3.1–3.2 we review our results from [27] on one-level FETI methods for highly heterogeneous coefficients α and extend them to all-floating FETI. An extension of that theory to FETI-DP is given in §3.3.

3.1 Formulation of one-level and all-floating FETI methods

Recall that $\{\Omega_i : i = 1, \dots, s\}$ is a partitioning of Ω into s non-overlapping subdomains, and denote by $\Gamma := \sum_{i,j}(\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega$ the *interface*. As before $\Gamma_S := \Gamma \cup \partial\Omega$ will be the *skeleton*. Let H_i be again the subdomain diameter, and denote by h_i the mesh size on subdomain Ω_i.

To start with, we introduce on each subdomain separate unknowns $\mathbf{u}_i$ for the solution including the DOFs on the subdomain interfaces. Let u_i denote the function that the coefficient vector $\mathbf{u}_i$ represents. In order to make the solution continuous, constraints of the form

$$u_i(x^h) - u_j(x^h) = 0 \tag{5}$$

are introduced for each finite element node x^h on the interface Γ and for all possible combinations of i and j, even if this leads to redundancies, cf. [33]. This yields the saddle point

problem

$$\begin{pmatrix} A_1 & & 0 & B_1^T \\ & \ddots & & \vdots \\ 0 & & A_s & B_s^T \\ B_1 & \cdots & B_s & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_s \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_s \\ 0 \end{pmatrix}, \tag{6}$$

where the A_i denote the subdomain stiffness matrices, and $\mathbf{f}_i$ are the corresponding load vectors. Solving saddle point system (6) is equivalent to solving the original system (2). The operators B_i are signed Boolean matrices, and each row of the system $\sum_{i=1}^{s} B_i \, \mathbf{u}_i = 0$ corresponds to one of the constraints in (5). The Lagrange multiplier λ plays the role of a continuous flux across Γ.

In the *all-floating* formulation, the Dirichlet boundary conditions are not incorporated in the finite element spaces, but enforced as additional constraints of the form $u_i(x^h) = 0$ for all nodes x^h on Γ_D. These can be easily incorporated in (6) leading to additional Lagrange multipliers.

Introducing a special projection P (see below), the dual problem to (6) can be written in the form

$$P^T F \lambda \;=\; d\,, \tag{7}$$

with $F := \sum_{i=1}^{s} B_i \, A_i^{\dagger} B_i^T$, where the operator $A_i^{\dagger}$ corresponds to the solution of a (possibly) regularised Neumann problem on subdomain Ω_i. For the standard one-level formulation, subdomains with contributions from the Dirichlet boundary require no regularisation since the corresponding subdomain stiffness matrix is regular. For the remaining (*floating*) subdomains, the local Neumann problems are not uniquely solvable, and so we need to employ a standard regularisation of A_i to define the corresponding pseudoinverses $A_i^{\dagger}$. Since in the all-floating formulation, the Dirichlet boundary conditions are only weakly imposed, all the subdomain stiffness matrices are singular and can be treated by the same type of regularisation. In the following, let R_i denote a full-rank matrix that spans $\ker A_i$, and set the projection $P := I - Q\,G(G^T Q\,G)^{-1} G^T$, where $G = [B_1\,R_1 | \ldots | B_s\,R_s]$ and Q is a diagonal scaling matrix.

The FETI method is now a special projected preconditioned conjugate gradient (PCG) method for (7). For each subdomain let S_i denote the Schur complement of A_i eliminating the interior DOFs in Ω_i (which requires the solution of a local Dirichlet problem on Ω_i). The FETI preconditioner is chosen to be

$$M^{-1} := P \sum_{i=1}^{s} D_i \, B_i \, S_i \, B_i^T D_i \,, \tag{8}$$

where D_i is a diagonal scaling matrix. Here and in the following we implicitly assume (for ease of notation) that matrices like S_i in the formula above are extended with zero rows and columns to interior DOFs where necessary. Note that the entries of Q and D_i need to be carefully chosen w.r.t. to the coefficient α, cf. [18, 27].

The projection P involves the solution of a coarse problem that corresponds to a sparse linear system of dimension $\mathcal{O}(s)$. Usually, one selects the subdomain partition in a way that the local subdomain problems and the coarse problem are of a size that can be efficiently handled by sparse direct solvers. The factorisations of the local system matrices can be computed in a preprocessing phase and kept in memory during the whole FETI iteration.

Note that these local, decoupled problems can be parallelised in a straightforward manner, e.g. treating each subdomain on a different processor. Once problem (7) is solved, the actual solution u can easily be determined from the Lagrange multiplier λ. The spectral condition number κ of the preconditioned system can finally be bounded by

$$\kappa \;\leq\; C^*(\alpha)\, \max_{i=1}^{s} \big(1+\log(H_i/h_i)\big)^2 , \tag{9}$$

where the constant $C^*(\alpha)$ is independent of H_i, h_i, and s. In a parallel scheme, the total computational complexity of the FETI-PCG method is $\mathcal{O}\big((\mathcal{D}(s)+\mathcal{D}(N_{loc}))\,\log(TOL^{-1})\,\sqrt{\kappa}\big)$, where $N_{loc} \sim \max_{i=1}^{s}(H_i/h_i)^d$ is the maximum number of DOFs per subdomain, $\mathcal{D}(\cdot)$ is the cost of the direct solver, and $TOL > 0$ is the desired relative error reduction in the energy norm.

If the heterogeneities in the coefficient α are resolved by the subdomain partition, i.e. α is constant on each Ω_i, then, Klawonn & Widlund [18] proved that $C^*(\alpha) \sim 1$. However, in general, using classical proof techniques, we only get

$$C^*(\alpha) \;\lesssim\; \max_{i=1}^{s} \max_{x,y\in\Omega_i} \frac{\alpha(x)}{\alpha(y)} , \tag{10}$$

i.e. the bound is proportional to the maximum local variation of α in the subdomains, which can be arbitrarily large. However, as noticed by several authors (e.g. [30, 21]) this asymptotic bound is in general far too pessimistic, and robustness is observed for many special kinds of coefficient distributions.

3.2 Robustness results for one-level and all-floating FETI methods

For each subdomain Ω_i, let Ω_{i,η_i} denote again the boundary layer of witdh $\eta_i > 0$. For neighbouring subdomains Ω_i, Ω_j, assume that $H_i \sim H_j$, $h_i \sim h_j$, and $\eta_i \sim \eta_j$. Furthermore we agree on the standard (technical) assumptions made in [33, Assumption 4.3] for the partitioning $\{\Omega_i\}$. To be brief, each subdomain needs to be a union of a uniformly bounded number of simplices, which alltogether form a geometrically conforming and shape-regular coarse mesh of Ω. Our additional (but not significantly stronger) regularity assumptions on the subdomain boundary layers Ω_{i,η_i} can be found in [27].

Theorem 3.1 (One-level and all-floating FETI)**.** *The condition numbers of the one-level and the all-floating FETI method (with suitably chosen scaling matrices D_i and Q) satisfy the bound*

$$\kappa \;\lesssim\; \Big\{ \max_{j=1}^{s} \max_{x,y\in\Omega_{j,\eta_j}} \frac{\alpha(x)}{\alpha(y)} \Big\} \max_{i=1}^{s} \Big(\frac{H_i}{\eta_i}\Big)^2 \max_{k=1}^{s} \big(1+\log(H_k/h_k)\big)^2 .$$

If, in addition, $\alpha(x) \gtrsim \min_{y\in\Omega_{i,\eta_i}} \alpha(y)$ for all $x \in \Omega_i$, then the quadratic dependence on H_i/η_i is reduced to a linear one.

Thus again $\kappa \lesssim 1$, for example, if there exists $\eta_i \sim H_i$ such that $\max_{x,y\in\Omega_{j,\eta_j}} \frac{\alpha(x)}{\alpha(y)} \sim 1$, for all $i = 1, \ldots, s$ (independent of the variation of α in the interior of each of the subdomains Ω_i). And again it is also possible to bound κ independent of α in the case of strong variation of α near the boundary of any of the Ω_i in certain cases (see §4 for details).

Sketch of Proof of Theorem 3.1: In the following we sketch how to prove Theorem 3.1 by modifying the theory given in [18]. For details see [27, Theorem 3.3 and Section 4], as well as [26, Chap. 5] for the all-floating case.

Let us denote by W_i the space of discrete α-harmonic functions on Ω_i, and by W the corresponding product space (functions in W are typically discontinuous across subdomain interfaces). Furthermore, we introduce the energy (semi)norm $|w_i|_{S_i} = (\int_{\Omega_i} \alpha |\nabla w_i|^2 \, dx)^{1/2}$ on W_i. Besides many algebraic arguments that do not need to be adjusted significantly, the key ingredient to the FETI condition number bound is the estimate

$$\sum_{i=1}^{s} |(P_D\, w)_i|^2_{S_i} \;\leq\; C^*(\alpha) \max_{j=1}^{s} \big(1 + \log(H_j/h_j)\big)^2 \sum_{i=1}^{s} |w_i|^2_{S_i}\,, \tag{11}$$

where P_D is a projection operator defined by

$$(P_D\, w)_i(x^h) = \sum\nolimits_{j \in \mathcal{N}_{x^h}} \delta_j^\dagger(x^h) \big(w_i(x^h) - w_j(x^h)\big)\,, \tag{12}$$

for each node $x^h \in \Gamma_S$. Here, $\mathcal{N}_{x^h}$ is the index set of subdomains that share the node x^h. For Dirichlet nodes x^h we set $(P_D w)_i(x^h) = w_i(x^h)$. Inequality (11) needs to be shown for all functions $w = [w_1|\dots|w_s]$ in a suitable subspace of W, where on all floating subdomains Ω_i the averages of w_i are "balanced" with respect to the projection P. (To be more precise, $\sum_{i=1}^s B_i\, w_i \in \text{range} P^T$, cf. [33, Sect. 6.3].) The values $\{\delta_j^\dagger(x^h)\}_{j \in \mathcal{N}_{x^h}}$, which are the entries in the diagonal scaling matrices D_i, are chosen as a partition of unity on Γ_S. In [27] we describe suitable choices, among them $\delta_j^\dagger(x^h) = \widehat{\alpha}_j(x^h) / \max_{k \in \mathcal{N}_{x^h}} \widehat{\alpha}_k(x^h)$, where $\widehat{\alpha}_j(x^h)$ is the maximum of α over all elements $\tau \in \mathcal{T}^h \cap \Omega_j$ that contain x^h. This choice is closely related to the suggestions made in [30].

Usually, the estimate (11) is shown in two steps. Let $z_i = |\partial\Omega_i|^{-1} \int_{\partial\Omega_i} w_i \, dx$ denote the subdomain boundary averages of w_i. First, (11) is shown for $w - z$, using a decomposition into subdomain face, edge, and vertex terms, see [33, Sect. 4.6]. This is necessary due to the nature of P_D, so that contributions of neighbouring functions can be estimated separately and then collected subdomain-wise. This decomposition leads to a logarithmic factor and an appropriately scaled full H^1-norm of each of $w_i - z_i$, that can be estimated by the H^1-seminorm using a Poincaré type inequality, or using a discrete Friedrichs/Sobolev type inequality if Dirichlet boundary conditions are available. Secondly, (11) is shown for z directly. Here, only a careful choice of the scaling matrix Q leads to the correct estimate. Previously, the above described decomposition was available only in the H^1-seminorm, not in the energy norm, leading to the suboptimal bound (10).

In [27] we describe how this crude estimate can be circumvented by using cut-off arguments and generalised Poincaré, Friedrichs, and Sobolev type inequalities. Our cut-off argument reads

$$|v|^2_{S_i} \;\lesssim\; \max_{x \in \Omega_{i,\eta_i}} \alpha(x) \left\{ |v|^2_{H^1(\Omega_{i,\eta_i})} + \eta_i^{-1} \|v\|^2_{L_2(\partial\Omega_i)} \right\} \qquad \forall v \in W_i\,, \tag{13}$$

cf. [27, Lemma 4.1], and can be easily shown using a cut-off function χ in Ω_i that is one on $\partial\Omega_i$ and zero on $\Omega_i \setminus \Omega_{i,\eta_i}$, and that satisfies $\|\nabla\chi\|_{L_\infty} \lesssim \eta_i^{-1}$. Using this argument, we can remove the dependence on α in the interior $\Omega_i \setminus \Omega_{i,\eta_i}$ of each subdomain: we apply (13), lift

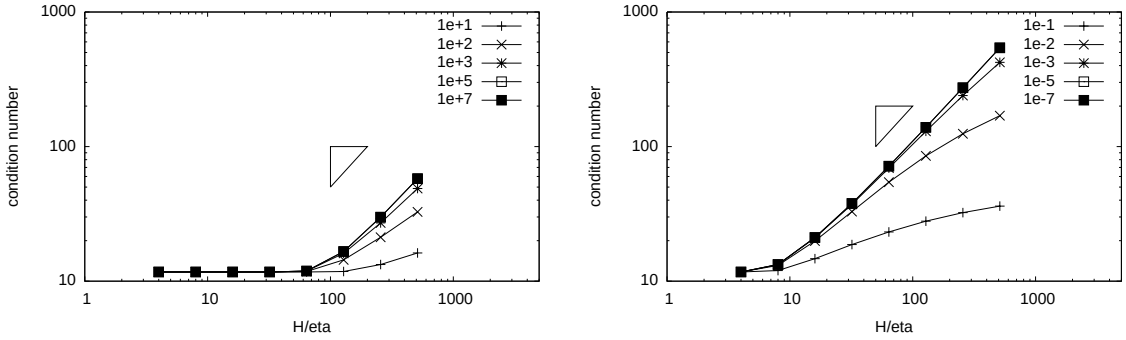

Figure 2: Estimated condition number of one-level FETI for Example 3.2 for different values of H/h and β. Fixed discretisation $H/h = 512$. Left: $\beta \geq 0$. Right: $\beta \leq 0$.

each H^1-seminorm to the whole of Ω_i, perform the usual arguments there, and apply (13) once more, formally for $\alpha \equiv 1$. It then remains to estimate

$$\max_{x \in \Omega_{i,\eta_i}} \alpha(x) \left\{ |w_i|^2_{H^1(\Omega_{i,\eta_i})} + \eta_i^{-1} \|w_i\|^2_{L_2(\partial\Omega_i)} \right\}$$

in terms of $|w_i|^2_{S_i}$. This can be done using a generalised Poincaré inequality

$$\|v\|_{L_2(\partial\Omega_i)} \lesssim \frac{H_i^2}{\eta_i} |v|^2_{H^1(\Omega_{i,\eta_i})}, \tag{14}$$

cf. [27, Lemma 4.3], which holds for all functions $v \in W_i$ whose average over $\partial\Omega_i$ vanishes. For subdomains with Dirichlet boundary conditions we need to use a similar inequality, cf. [27, Lemma 4.5]. In case the coefficient in each subdomain interior is not significantly smaller than in the boundary layer, one can replace the last step by a lifting argument and a standard Poincaré inequality, leading to a better dependence on the ratio H_i/η_i. □

At the end of this subsection we would like to give a relatively simple numerical example that confirms the theory above. For more results see [27] and §4.

Example 3.2 (FETI for an "island" coefficient)**.** In this example ([27, Example 1]), we subdivide the unit square Ω into 25 congruent square-shaped subdomains of width $H = 1/5$. We choose the coefficient α to be 10^β in a square region that is contained in the interior of the central subdomain, and separated by a distance η from its boundary. On the rest of Ω, we choose $\alpha = 1$. In Fig. 2 we display the condition number of the one-level FETI method (estimated by the Lanczos method) for different values of the exponent β from -7 to $+7$, and for different ratios of H/η. The left part of the figure corresponds to the case where our theory predicts a linear dependence on H/η, which is perfectly reproduced by the numerical experiments. The case of a negative exponent β does indeed prove to be the harder case and leads to a worse conditioning, as predicted by our theory. However, the quadratic dependence in Theorem 3.1 seems to be overly pessimistic.

3.3 FETI-DP methods

In contrast to the FETI methods discussed before, in dual-primal methods, one keeps certain DOFs continuous. These DOFs, called *primal* DOFs, form a coarse problem for the FETI-DP method, and they are chosen such that each of the local subdomain problems becomes regular. In two dimensions, it is sufficient to choose individual DOFs of the original problem (2) associated with vertices $x^h \in \Gamma_S$ as primal DOFs, whereas in three dimensions one needs to add at least some (subdomain) edge or face averages in order to get a stable method, cf. [33, Sect. 6.4.2].

Some notation: we reorder the DOFs in each subdomain stiffness matrix A_i and group them into a *primal* block (subscript Π), a *dual* block (with the remaining DOFs on the subdomain boundaries, subscript Δ), and the remaining block of interior DOFs (subscript I). Subassembling the subdomain stiffness matrices only at the primal DOFs (indicated by a tilde) leads to a global matrix $\widetilde{A}$, i.e.

$$A_i = \begin{pmatrix} A_{\Pi\Pi}^{(i)} & A_{\Pi\Delta}^{(i)} & A_{\Pi I}^{(i)} \\ A_{\Delta\Pi}^{(i)} & A_{\Delta\Delta}^{(i)} & A_{\Delta I}^{(i)} \\ A_{I\Pi}^{(i)} & A_{I\Delta}^{(i)} & A_{II}^{(i)} \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} \widetilde{A}_{\Pi\Pi} & \widetilde{A}_{\Pi\Delta} & \widetilde{A}_{\Pi I} \\ \widetilde{A}_{\Delta\Pi} & A_{\Delta\Delta} & A_{\Delta I} \\ \widetilde{A}_{I\Pi} & A_{I\Delta} & A_{II} \end{pmatrix}. \tag{15}$$

With this notation, we can introduce jump operators B_i analogously to §3.1. However, here the B_i only operate on non-primal DOFs. With $B = [B_1|\dots|B_s]$, the resulting saddle point system reads

$$\begin{pmatrix} \widetilde{A} & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{u}} \\ \lambda \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{f}} \\ 0 \end{pmatrix}, \tag{16}$$

where the vector $\widetilde{\mathbf{u}}$ consists of a primal (global) block, and local blocks that correspond to the subdomains and are not coupled. The lower right 2×2 block of $\widetilde{A}$ is block diagonal since the DOFs are separated subdomain-wise. Using the idea of Cholesky factorisation, the action of the inverse $\widetilde{A}^{-1}$ can be performed by solving local problems in the dual and interior DOFs, which are all regular now (due to the Dirichlet conditions imposed at the primal DOFs), and a coarse problem in the primal DOFs, whose system matrix is again sparse, cf. e.g. [33, 22]. The resulting Lagrange multiplier problem reads

$$F_{DP}\,\lambda \;=\; d_{DP}\,, \tag{17}$$

where $F_{DP} := B\,\widetilde{A}^{-1}B^T$. It is solved using conjugate gradients, preconditioned by

$$M_{DP}^{-1} := \sum_{i=1}^{s} D_i\,B_i\,S_{\Delta\Delta}^{(i)}\,B_i^T D_i\,, \tag{18}$$

with the Schur complement $S_{\Delta\Delta}^{(i)} := A_{\Delta\Delta}^{(i)} - A_{\Delta I}^{(i)}[A_{II}^{(i)}]^{-1}A_{I\Delta}^{(i)}$ eliminating the interior DOFs, and with the diagonal scaling matrices D_i chosen as in §3.1, §3.2.

As proved by Klawonn, Widlund, and Dryja [19], the condition number κ_{DP} of the preconditioned FETI-DP system fulfils the estimate

$$\kappa_{DP} \;\le\; C^*(\alpha)\,\max_{i=1}^{s}\big(1+\log(H_i/h_i)\big)^2\,, \tag{19}$$

where $C^*(\alpha) \sim 1$ if α is piecewise constant on the subdomains.

The following theorem extends the results of Theorem 3.1 to FETI-DP methods.

Theorem 3.3 (FETI-DP). *Under the assumptions made in §3.2, the condition number κ_{DP} of the FETI-DP method (with the set of primal DOFs and the D_i suitably chosen) satisfies the bound*

$$\kappa_{DP} \lesssim \left\{ \max_{j=1}^{s} \max_{x,y\in\Omega_{j,\eta_j}} \frac{\alpha(x)}{\alpha(y)} \right\} \max_{i=1}^{s} \left(\frac{H_i}{\eta_i}\right)^2 \max_{k=1}^{s} \left(1+\log(H_k/h_k)\right)^2 .$$

If, in addition, $\alpha(x) \gtrsim \min_{y\in\Omega_{i,\eta_i}} \alpha(y)$ for all $x \in \Omega_i$, then the quadratic dependence on H_i/η_i is reduced to a linear one.

Proof. We give a sketch of the proof in 3D for the case that the set of primal DOFs consists of the subdomain vertices, edge averages, and face averages (cf. [33, Sect. 6.4.2, Algorithm B]). However, all of the other usual choices for the primal DOFs in 3D are also admissable. With the discrete α-harmonic spaces W_i and the product space W defined as in §3.2, let $\widetilde{W}$ be the subspace of W of functions w that are continuous in the primal DOFs. The crucial estimate to show is again

$$\sum_{i=1}^{s} |(P_\Delta\, w)_i|^2_{S_i} \leq C^*(\alpha) \max_{j=1}^{s} \left(1+\log(H_j/h_j)\right)^2 |w_i|^2_{S_i} , \tag{20}$$

for all functions $w \in \widetilde{W}$, where P_Δ is a projection operator, basically identical to P_D from §3.2. In the standard theory, cf. [33, Sect. 6.4.3], the contributions of faces, edges, and vertices are again separated, similarly to the analysis of the one-level methods. In order to estimate the full H^1-norm in terms of seminorms, one uses shift invariances by adding face or edge averages constructively (this can only be done because the corresponding averages are continuous when working in the space $\widetilde{W}$). Then, by a Poincaré type inequality any L_2 contribution can be removed. Applying our cut-off argument (13) from §3.2 to each subdomain, we can again remove the dependence on α in the interiors $\Omega_i \setminus \Omega_{i,\eta_i}$ at the cost of a L_2-term over the boundary $\partial\Omega_i$ multiplied by a factor of η_i^{-1}. Applying the decomposition and the cut-off argument once more, we are left with terms of the form

$$\max_{x\in\Omega_{i,\eta_i}} \alpha(x) \left(1+\log(H_i/h_i)\right)^2 \left\{ |w_i - \overline{w_i}^{\mathcal{F}_{ij}}|^2_{H^1(\Omega_{i,\eta_i})} + \eta_i^{-1} \|w_i - \overline{w_i}^{\mathcal{F}_{ij}}\|^2_{L_2(\partial\Omega_i)} \right\},$$

where $\overline{w_i}^{\mathcal{F}_{ij}}$ denotes the face average. For simplicity we only consider face contributions here. Using our generalised Poincaré inequality [27, Lemma 4.3], we can remove the L_2-term at the cost of a factor $(H_i/\eta_i)^2$ in front of the H^1 norm of w_i. Edge and vertex contributions are treated similarly. Under the stronger assumption on α the last argument can be replaced by a standard Poincaré type inequality again, which leads to the improved factor H_i/η_i. □

4 Interface Variation

As already mentioned several times above, our results are not restricted to the case of coefficients that are benign in a boundary layer of each subdomain/coarse grid element. In [12] we were also able to bound the coarse space robustness indicator for overlapping additive Schwarz in more general situations where the coefficient is allowed to vary strongly along the interface between two coarse elements. We have also recently managed to obtain similar results for FETI methods in certain cases of strong interface variation (see the forthcoming paper [28] for details).

4.1 Schwarz methods

The numerical results in [12, 13, 32, 34] show robustness of overlapping Schwarz methods also for coefficients that vary along the boundary of coarse elements or along subdomain interfaces. To give a flavour of our theoretical results, we focus again on the results in [12] for multiscale finite element coarsening.

With the notation introduced in §2, piecewise linear boundary data ψ_j (as used in Example 2.6 and in particular in Proposition 2.7) are not sufficient anymore when the coefficient α varies strongly along the boundary ∂K of a coarse grid element $K \in \mathcal{T}^H$ (cf. [12, Example 5.3]). The "oscillatory" boundary conditions suggested in [14] are more suitable in this case (see [12] for details). Roughly speaking, to find the boundary data ψ_j in this case, a projection of the PDE onto each face $f \in \mathcal{F}^H$ is solved. It can be shown that the resulting ψ_j satisfy assumptions (M1)–(M4) (cf. [12]). With this choice of ψ_j we were able to prove the following result in [12]:

Let each element $K \in \mathcal{T}^H$ be subdivided into $K = \bigcup_{\ell=0}^{L_K} K_\ell \cup \widehat{K}$, such that $\operatorname{dist}(K_0, \partial K) \gtrsim \eta_K$ and $\operatorname{dist}(K_\ell, K_{\ell'}) \gtrsim \eta_K$ for some $\eta_K > 0$, i.e. K_0 denotes the interior of K and $\widehat{K}$ the remainder (see Figure 3, left, for an example). Moreover, let the boundary islands K_ℓ, $\ell = 1, \ldots, L_K$, be polygonal (poyhedral) with side lengths $\gtrsim \eta_K$. The following proposition is [12, Theorem 4.5].

Proposition 4.1. Suppose that $\alpha \geq 1$ is continuous across the interface between two coarse grid elements, and that on a particular element $K \in \mathcal{T}^H$ we have $\alpha(x) = \alpha_{K,\ell}$ for all $x \in K_\ell$ and $\ell = 1, \ldots, L_K$. Then the coarse space robustness indicator for multiscale finite elements with oscillatory boundary conditions satisfies

$$\gamma(\alpha) \;\lesssim\; \max_{K \in \mathcal{T}^H} \; \widehat{\alpha}_K \left\{ \left(\frac{H_K}{\eta_K} \right)^2 \left(1 + \log \left(\widehat{\alpha}_K \frac{H_K}{\eta_K} \right) \right) \;+\; \widehat{\alpha}_K \frac{H_K}{\eta_K} \right\} ,$$

where $H_K = \operatorname{diam}(K)$ and $\widehat{\alpha}_K := \max_{x \in \widehat{K}} \alpha(x)$.

Thus again $\gamma(\alpha) \lesssim 1$, provided there exists an $\eta_K \sim H_K$ such that $\widehat{\alpha}_K \sim 1$, for all $K \in \mathcal{T}^H$ (independent of the constant values of α on each of the islands $K_1, \ldots, K_{L_K}$ and independent of the variation of α in the interior K_0 of each element). The partitioning robustness indicator $\pi(\alpha)$ can also be bounded independently of α in this case provided the overlap δ is suffciently large. For numerical results confirming this see [12, Example 5.3].

4.2 FETI methods

Besides the numerical results in [30, 17, 21] on FETI and FETI-DP methods, our computations in [27, Sect. 5.3 and 5.4] show that in particular one-level FETI methods with suitably chosen scalings (D_i and Q) can be suprisingly robust even in case of large coefficient variation along subdomain interfaces. In the forthcoming paper [28] we will demonstrate how to prove such robustness for certain kinds of coefficient discributions, where the coefficient along the boundary jumps only a few number of times. The key tool to such an analysis is the following weighted Poincaré inequality,

$$\int_{\Omega_{i,\eta_i}} \alpha |v|^2 \, dx \;\lesssim\; C \int_{\Omega_{i,\eta_i}} \alpha |\nabla v|^2 \, dx \, ,$$

which holds with a constant C independent of α,

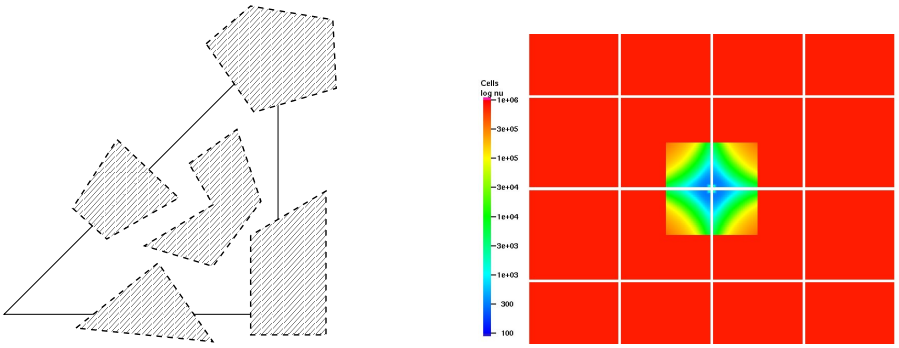

Figure 3: Left: Typical coefficient distribution admissable for Proposition 4.1 (islands K_ℓ in grey; remainder $\widehat{K}$ in white). Right: Coefficient distribution and subdomain partitioning for Example 4.2.

- if α takes only two different values in two connected subregions of Ω_{i,η_i}, and
- if v has a vanishing average over the interface between those subregions.

The constant C depends on the relative sizes and shapes of these subregions.

Example 4.2. We finish with an example from nonlinear mangetostatics ([27, Sect. 5.4]), where the subdomain partition is chosen such that coefficient peaks (that arise due to singularities in the solution) are in the centre of the subdomains, whereas material interfaces are allowed to cut through subdomain interfaces (cf. Fig. 3). Our theory in [28] gives a condition number bound of $\mathcal{O}(10^2)$ for an interface variation of $\mathcal{O}(10^4)$. The estimated condition number of 13.7 is well within this bound, and only 16 FETI-PCG steps are needed for a residual reduction by 10^{-6}.

References

[1] B. Aksoylu, I.G. Graham, H. Klie and R. Scheichl, *Towards a rigorously justified algebraic preconditioner for high-contrast diffusion problems*, Comput. Visual. Sci. 11 (2008), pp. 319–331.

[2] A. Brandt, S. McCormick and J.W. Ruge, *Algebraic multigrid (AMG) for sparse matrix equations*, in *Sparsity and its Applications* (D.J. Evans, ed.), Cambridge University Press, 1984, pp. 257–284.

[3] T.F. Chan and T. Mathew. *Domain decomposition methods*, In *Acta Numerica*, Cambridge University Press, 2004, pp. 61–143.

[4] M.A. Christie and M.J. Blunt, *Tenth SPE Comparative Solution Project: A Comparison of Upscaling Techniques*, SPE Reservoir Engineering 12 (2001), pp. 308–317.

[5] K.A. Cliffe, I.G. Graham, R. Scheichl and L. Stals, *Parallel computation of flow in heterogeneous media modelled by mixed finite elements*, J. Comput. Phys. 164 (2000), pp. 258–282.

[6] J.E. Dendy, *Black-box multigrid*, J. Comput. Phys. 48 (1982), pp. 366–386.

[7] Z. Dostál, D. Horák and R. Kučera, *Total FETI: An easier implementable variant of the FETI method for numerical solution of elliptic PDE*, Commun. Numer. Methods Eng. 22 (2006), pp. 1155–1162.

[8] W. E and B. Engquist, *The heterogeneous multi-scale method*, Comm. Math. Sci. 1 (2003), pp. 87–132.

[9] C. Farhat and F.-X. Roux, *A method of finite element tearing and interconnecting and its parallel solution algorithm*, Int. J. Numer. Meth. Engrg. 32 (1991), pp. 1205–1227.

[10] C. Farhat, M. Lesoinne, P. Le Tallec, K. Pierson, and D. Rixen, *FETI-DP: a dual-primal unified FETI method I: A faster alternative to the two-level FETI method*, Int. J. Numer. Meth. Engrg. 50 (2001), pp. 1523–1544.

[11] C.L. Farmer, *Upscaling: A review*, Int. J. Numer. Meth. Fl. 40 (2002), pp. 63–78.

[12] I.G. Graham, P. Lechner and R. Scheichl. *Domain decomposition for multiscale PDEs*, Numer. Math. 106 (2007), pp. 589–626.

[13] I.G. Graham and R. Scheichl, *Robust Domain Decomposition Algorithms for Multiscale PDEs*, Numer. Meth. Part. Diff. Eqns. 23 (2007), pp. 859–878.

[14] T.Y. Hou and X.H. Wu, *A multiscale finite element method for elliptic problems in composite materials and porous media*, J. Comput. Phys. 134 (1997), pp. 169–189.

[15] T.J.R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods*, Comput. Meth. Appl. Mech. Eng. 127 (1995), pp. 387–401.

[16] P. Jenny, S.H. Lee and H.A. Tchelepi, *Multi-scale finite-volume method for elliptic problems in subsurface flow simulation*, J. Comput. Phys. 187 (2003), pp. 47–67.

[17] A. Klawonn and O. Rheinbach, *Robust FETI-DP methods for heterogeneous three dimensional elasticity problems*, Comput. Methods Appl. Mech. Engrg. 196 (2007), pp. 1400–1414.

[18] A. Klawonn and O.B. Widlund, *FETI and Neumann-Neumann iterative substructuring methods: connections and new results*, Comm. Pure Appl. Math. 54 (2001), pp. 57–90.

[19] A. Klawonn, O.B. Widlund and M. Dryja, *Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients*, SIAM J. Numer. Anal. 40 (2002), pp. 159–179.

[20] S. Knapek, *Matrix-dependent multigrid homogenization for diffusion problems*, SIAM J. Sci. Comput. 20 (1998), pp. 515–533.

[21] U. Langer and C. Pechstein, *Coupled finite and boundary element tearing and interconnecting solvers for nonlinear potential problems*, Z. Angew. Math. Mech. 86 (2006), pp. 915–931.

[22] J. Li and O.B. Widlund, *FETI-DP, BDDC, and block Cholesky methods*, Int. J. Numer. Meth. Engrg. 66 (2006), pp. 250–271.

[23] S.P. MacLachlan and J.D. Moulton, *Multilevel upscaling through variational coarsening*, Water Resour. Res. 42 (2006), W02418.

[24] J.D. Moulton, J.E. Dendy and J.M. Hyman, *The black box multigrid numerical homogenization algorithm*, J. Comput. Phys. 142 (1998), pp. 80–108.

[25] G. Of, *The all-floating BETI method: Numerical results*, in *Domain Decomposition Methods in Science and Engineering XVII* (U. Langer, M. Discacciati et al., Eds.), Lecture Notes in Computational Science and Engineering 60, Springer, Berlin, 2008, pp. 295–302.

[26] C. Pechstein, *Finite and Boundary Element Tearing and Interconnecting Methods for Multiscale Elliptic Partial Differential Equations*, PhD Thesis, Johannes Kepler University Linz, Austria, 2008.

[27] C. Pechstein and R. Scheichl, *Analysis of FETI Methods for Multiscale PDEs*, Numer. Math. 111(2) (2008), pp. 293–333.

[28] C. Pechstein and R. Scheichl, *Analysis of FETI Methods for Multiscale PDEs – Part II: Interface Variation*, in preparation, 2009.

[29] P. Renard and G. de Marsily, *Calculating equivalent permeability: A review*, Adv. Water. Resour. 20 (1997), pp. 253–278.

[30] D. Rixen and C. Farhat, *A simple and efficient extension of a class of substructure based preconditioners to heterogeneous structural mechanics problems*, Int. J. Numer. Meth. Engrg. 44 (1999), pp. 489-516.

[31] J.W. Ruge and K. Stüben. *Algebraic multigrid*, in *Multigrid methods, Frontiers Appl. Math., 3*, SIAM, Philadelphia, 1987, pp. 73–130.

[32] R. Scheichl and E. Vainikko, *Additive Schwarz and aggregation-based coarsening for elliptic problems with highly variable coefficients*, Computing 80(4) (2007), pp. 319–343.

[33] A. Toselli and O. Widlund, *Domain Decomposition Methods Algorithms and Theory*, Springer, New York, 2005.

[34] J. Van lent, R. Scheichl and I.G. Graham, *Energy Minimizing Coarse Spaces for Two-level Schwarz Methods for Multiscale PDEs*, submitted, BICS Preprint 12/08, University of Bath, 2008, pp. 1–25.

[35] X.-H. Wen and J.J. Gomez-Hernandez, *Upscaling hydraulic conductivities in heterogeneous media: An overview*, J. Hydrol. 183 (1996), pp. ix–xxxii.

[36] X.J. Xu and Y. Zhu, *Uniform convergent multigrid methods for elliptic problems with strongly discontinuous coefficients*, Math. Mod. Meth. Appl. Sci. 18(1) (2008), pp. 77–105.

Latest Reports in this series

2009

2009-01 Clemens Pechstein and Robert Scheichl
Scaling Up through Domain Decomposition January 2009

From 1998 to 2008 reports were published by SFB013. Please see
http://www.sfb013.uni-linz.ac.at/index.php?id=reports
From 2004 on reports were also published by RICAM. Please see
http://www.ricam.oeaw.ac.at/publications/list/
For a complete list of NuMa reports see
http://www.numa.uni-linz.ac.at/Publications/List/