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Numerical Simulation of Contact Problems with Coulomb Friction

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Abstract

Contact problems with friction appear in many applications. This thesis is concerned with modeling, analyzing and finally solving the elastic two-body contact problem with Coulomb friction. The temper rolling process in steel industry is used as motivation.

Starting from the time dependent classical contact problem with Coulomb friction, a primal and a mixed variational formulation are derived. Given that the coefficient of friction is small enough, existence of a solution can be shown for both formulations.

The problem is discretized using the Finite Element Method. Error estimates for the approximations are provided.

For the solution of the discrete problem, two methods are proposed. Both were implemented and successfully tested.

Zusammenfassung

In vielen Anwendungen treten Kontaktprobleme mit Reibung auf. Deshalb beschäftigt sich diese Diplomarbeit mit der Modellierung, Analysis und Auflösung von elastischen Kontaktproblemen mit Coulomb-Reibung. Als Motivation wird der Dressier-Walzvorgang in der Stahlerzeugung verwendet.

Beginnend beim klassischen, zeitabhängigen Zwei-Körper-Kontaktproblem mit Coulomb-Reibung, wird eine primale und eine gemischte variationelle Formulierung hergeleitet. Unter der Annahme, dass der Reibungskoeffizient hinreichend klein ist, kann die Existenz einer Lösung garantiert werden.

Die Finite-Elemente-Methode wird verwendet, um das variationelle Problem zu diskretisieren. Es werden Fehlerabschätzungen für die Näherungslösung gezeigt.

Weiters werden zwei Algorithmen zur Auflösung des diskretisierten Problems vorgestellt. Beide wurden implementiert und erfolgreich getestet.

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Contents

Chapter 1

Introduction

The problem we consider is inspired by the process of temper rolling in steel industry. Temper rolling represents the final step in forming the strip of steel. The main goals of this process are to harden the steel and improve its surface qualities. Thereby also the thickness of the strip is reduced slightly. A more detailed description of the process can be found in [5]. To optimize the quality of the product, one is interested in finding the deformation which occurs in the strip of steel. It is also necessary to compute the contact pressures which arise between strip and work roll.

We consider a simplified geometry of the process, which consists of a single roll and a strip of steel. The roll is pressed against the strip, therefore both bodies are deformed. According to [5], it is not possible to neglect the deformation of the roll when considering the process of temper rolling. Therefore, we need to compute the displacement in both bodies. Simultaneously we find the distribution of the arising contact stress.

Inside the strip, there are high temperatures and pressures. Therefore, there are elastic as well as plastic deformations taking place. In dry temper rolling, there is no liquid in between the roll and the steel. This way there is a considerable amount of friction acting on the contact surfaces. This friction leads to contact stress which influences the surface texture of the strip. One is interested in finding the arising stress between roll and strip.

In this approach, we focus on the friction along the contact surfaces. Therefore, we will not concentrate on modeling elastoplastic materials. We will rather consider an elastic contact problem with Coulomb friction. We study the behaviour of two general interacting bodies, having in mind the problem concerning the temper rolling process. We adopt the simplified geometry for our model problem, although it does not fully correspond to the industrial process. For the sake of simplicity, we shall consider the plane problem only. Of course, all results can be transferred directly to three dimensions.

Task of this thesis

In this thesis, we deal with the following problems:

- For the two-body contact problem with Coulomb friction, the deformations and the occurring stresses on the contact surfaces are related in a nonlinear way. Therefore, we have to rewrite the problem as a fixed point problem, for which we can show existence of a solution for small coefficients of friction.
- The linearized problem consists of a mixed variational inequality. We show existence and uniqueness of a solution.
- We use the Finite Element Method for discretization. For discrete problems, we see that the fixed point formulation is contractive, and therefore the solution can be computed by the method of successive approximations.
- For the solution of the discretized system, we propose two methods: Uzawa's method and a proportioning algorithm, which uses the cg method. Both algorithms are analyzed and finally implemented.

Organization of this thesis

- Chapter 2
  A mathematical model for the elastic contact problem with Coulomb friction is derived. Existence of a solution is shown in detail, and stability estimates are given.
- Chapter 3
  In this chapter, the problem is discretized using the Finite Element Method. Error estimates for the approximation of the solution are provided.
- Chapter 4
  Two algorithms for the solution of the system of equations and inequalities are presented: the Uzawa method, and a proportioning algorithm, which makes use of the cg method. Both methods are analyzed, and convergence is shown.
- Chapter 5
  The problem is implemented, and both algorithms discussed in Chapter 4 are tested. Numerical results are given.

Chapter 2

Mathematical Model

2.1 Contact problem without friction

2.1.1 Elastic problem

In this section, we derive the elastic two-body contact problem without friction. Further details can be found in [6], also the notation we use is close to this reference.

We want to find a model that describes the deformations of two bodies which get in contact of each other. For our model problem, the temper rolling process in steel industry, these two bodies are the roll and the strip of steel. Before the deformation, the bodies are represented by the closure of two non-intersecting domains Ω_R^a and Ω_R^b in $\mathbb{R}^2$. Let $\Omega_R = \Omega_R^a \cup \Omega_R^b$ be the union of these two sets. We will call Ω_R the referential configuration of the system. We want to find the deformation

$$\phi : \Omega_R \times [0, T] \to \mathbb{R}^2$$

of the bodies when they are pressed against each other in time $[0, T]$. Each particle $X \in \Omega_R$ is transformed to the spatial point $x = \phi(X, t)$. We will assume that this deformation is sufficiently smooth, and keeps the orientation of the system,

$$\det(\nabla\phi(X, t)) > 0 \quad X \in \Omega_R, t \in [0, T].$$

We can define the displacement of a particle $X \in \Omega_R$ by

$$u(X, t) = \phi(X, t) - X.$$

For given data that we will specify later, we want to find u as a function of X and t.

Let σ be the Cauchy stress tensor. Then the traction $\mathbf{t}$ acting on a surface with normal vector n in some spatial point x at time t is given by

$$\mathbf{t}(x, n, t) = \sigma(x, t)n.$$

After the deformation, the bodies have to satisfy a balance equation. We assume that there are given volume forces $f \in L^2(\Omega_R)$ acting in Ω_R. For each sufficiently smooth subset $\omega \subset \Omega_R$, surface tractions on $\partial\omega$ and body forces in ω have to equal inertia forces induced by acceleration:

$$\int_{\phi(\omega)} \rho\ddot{u}(x,t)\ dx = \int_{\phi(\partial\omega)} \sigma(x,t)n\ ds + \int_{\phi(\omega)} f\ dx.$$

Applying Green's formula we get

$$\int_{\phi(\omega)} \rho\ddot{u}(x,t)\ dx = \int_{\phi(\omega)} \operatorname{div}(\sigma(x,t)) + f(x)\ dx,$$

where div denotes the divergence

$$\operatorname{div}(\sigma)_i = \sum_{j=1}^{2} \frac{\partial\sigma_{ij}}{\partial x_j}, \qquad i = 1, 2.$$

As this relation has to hold true for all domains $\omega \subseteq \Omega_R$, we get the balance equation for $x \in \phi(\Omega_R), t \in [0,T]$

$$\rho\ddot{u}(x,t) - \operatorname{div}(\sigma(x,t)) = f(x,t). \tag{2.1}$$

Referring to material coordinates X, we have for $X \in \Omega_R, t \in [0,T]$

$$\rho\ddot{u}(X,t) - \operatorname{div}(\nabla\phi(X,t)\Sigma(X,t)) = f(X,t),$$

where $\Sigma = \det(\nabla\phi)\nabla\phi^{-1}\sigma\nabla\phi^{-T}$ denotes the second Piola-Kirchhoff stress tensor. We consider bodies made from elastic materials. This means, that the stress tensor Σ only depends on the Green-St. Vernant strain tensor

$$E(u) = \frac{1}{2}(\nabla\phi^T\nabla\phi - I) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T\nabla u). \tag{2.2}$$

Hooke's law states that the stress tensor depends linearly on $E(u)$

$$\Sigma(u) = CE(u). \tag{2.3}$$

The coefficient tensor $C = C_{ijkl}$ has to be symmetric, positive definite and bounded, i. e. there have to exist constants $C_1, C_2 > 0$ such that

$$C_1 E : E \leq (CE) : E \leq C_2 E : E \tag{2.4}$$

for all symmetric matrices E. By : we denote the matrix inner product

$$A : B := \sum_{i,j=1}^{2} A_{ij}B_{ij}.$$

We see that, although the relation (2.3) is linear in E, it is not linear in u. As a linearization is only justified for small deformations, we consider a small interval of time $[t_0, t_0 + \Delta t]$. We assume that u is known for all times $t \leq t_0$. Let $\phi_0 = \phi(., t_0)$ denote the deformation of the bodies at time t_0. We refer to the position of a particle X at time t_0 by $Y = \phi_0(X) = X + u(X, t_0)$. Then, the configuration of the bodies is given by $\Omega_a = \phi_0(\Omega_R^a), \Omega_b = \phi_0(\Omega_R^b)$. We want to find the increment $\bar{u}$ as a function of $Y \in \Omega = \phi_0(\Omega_R)$ and t such that

$$\bar{u}(Y, t) = u(X, t) - u(X, t_0), \quad t \in [t_0, t_0 + \Delta t].$$

Analogously, we define the additional deformation $\bar{\phi}$ such that

$$\phi(X, t) = \bar{\phi}(\phi_0(X), t).$$

For all times $t \in [t_0, t_0 + \Delta t]$ there holds the relation

$$\begin{aligned} E(\phi(t)) - E(\phi_0) &= \frac{1}{2}(\nabla\phi_0^T \, \nabla\bar{\phi}^T \nabla\bar{\phi} \nabla\phi_0 \, - \nabla\phi_0^T \, \nabla\phi_0 \,) \\ &= \nabla\phi_0^T \, E(\bar{\phi}) \nabla\phi_0 \\ &\approx \nabla\phi_0^T \, \varepsilon(\bar{u}) \nabla\phi_0 \end{aligned}$$

where ε is the linearized strain tensor

$$\varepsilon(v) = \frac{1}{2}(\nabla v + \nabla v^T).$$

Setting $J_0 := |\det(\nabla\phi_0\,)|$, there holds

$$\begin{aligned} \operatorname{div}_X(\nabla\phi\Sigma) &= \operatorname{div}_Y(\nabla\phi \, \Sigma \, \nabla\phi_0^T \, J_0^{-1}) \\ &\approx \operatorname{div}_Y(\nabla\phi_0 \, \Sigma \nabla\phi_0^T \, J_0^{-1}) \\ &\approx \operatorname{div}_Y(\sigma(Y, t_0)). \end{aligned}$$

Note that σ is affine linear in $\bar{u}$,

$$\sigma = \sigma_0 + \bar{\sigma}(\bar{u})$$

where

$$\begin{aligned} \sigma_0 &= J_0^{-1} \nabla\phi_0 \, C E(\phi_0) \nabla\phi_0^T \\ \bar{\sigma}(\bar{u}) &= J_0^{-1} \nabla\phi_0 \, C \nabla\phi_0^T \, \varepsilon(\bar{u}) \nabla\phi_0 \, \nabla\phi_0^T \, . \end{aligned}$$

We want to find $\bar{u} : \Omega \times [t_0, t_0 + \Delta t] \to \mathbb{R}^2$, which satisfies the linearized equations

$$\left.\begin{aligned} \ddot{\bar{u}}(Y, t) - \operatorname{div}_Y(\sigma(Y, t)) &= f(Y, t) \\ \sigma(Y, t) &= J_0^{-1} \nabla\phi_0 \, C E(Y, t) \, \nabla\phi_0^T \\ E(Y, t) &= E(\phi_0) + \nabla\phi_0^T \, \varepsilon(\bar{u}) \nabla\phi_0 \, |_{(Y,t)} \end{aligned}\right\} \text{in } \Omega \tag{2.5}$$

with initial conditions

$$\bar{u}(Y, t_0) = 0, \quad \dot{\bar{u}}(Y, t_0) = \dot{u}(X, t_0) \qquad Y \in \Omega.$$

Throughout the following, we only consider the interval $[t_0, t_0 + \Delta t]$. In order to obtain a more standard notation, we write X for Y and u for $\bar{u}$ as long as not specified otherwise.

Boundary conditions

We still need to specify conditions on the body surfaces. Therefore, we define the normal and tangential component of a vector $v \in \mathbb{R}^2$ with respect to the normal vector n

$$v_n := v \cdot n, \quad v_T := v - v_n n.$$

In Figure 2.1, one can find a sketch of the geometry of the problem. There is an axis of symmetry dividing the strip horizontally. On this line there is no displacement in direction normal to the surface, and no stress in the tangential direction. Denoting this part of the boundary by Γ_S, we get the condition

$$\left.\begin{array}{r} u_n = u_2 = 0 \\ \sigma_T = (\sigma n)_1 = 0 \end{array}\right\} \text{on } \Gamma_S, \tag{2.6}$$

where we used that in our problem Γ_S is parallel to the X_1-axis.

On those parts Γ_N of the surface, where the bodies do not touch, there shall be no forces acting

$$\sigma n = 0 \qquad \text{on } \Gamma_N. \tag{2.7}$$

Around the center of the roll, we prescribe the rotation. We consider a small circle Γ_D^a around the center X_M. For this inner boundary, there has to hold

$$u = u_g^a \qquad \text{on } \Gamma_D^a. \tag{2.8}$$

The given displacement consists on one hand of the rotation of the roll, and of some movement downwards on the other. Assuming that the roll is shifted by $u_d = (0, -d)^T$, we can specify the boundary condition on Γ_D^a by

$$u_g(X, t) = (Q(t - t_0) - I)(X - X_M) + u_d, \quad Q(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

We also fix the displacement that occurs at the end of the strip. On this part Γ_D^b we set

$$u = u_g^b \qquad \text{on } \Gamma_D^b. \tag{2.9}$$

By $\Gamma_D = \Gamma_D^a \cup \Gamma_D^b$ we denote the union of these boundary parts, whereas $u_g = u_g^z$ on $\Gamma_D^z, z = a, b$ shall be the respective given displacement.

For the part Γ_C, where the surfaces can, but need not meet, we will now specify conditions.

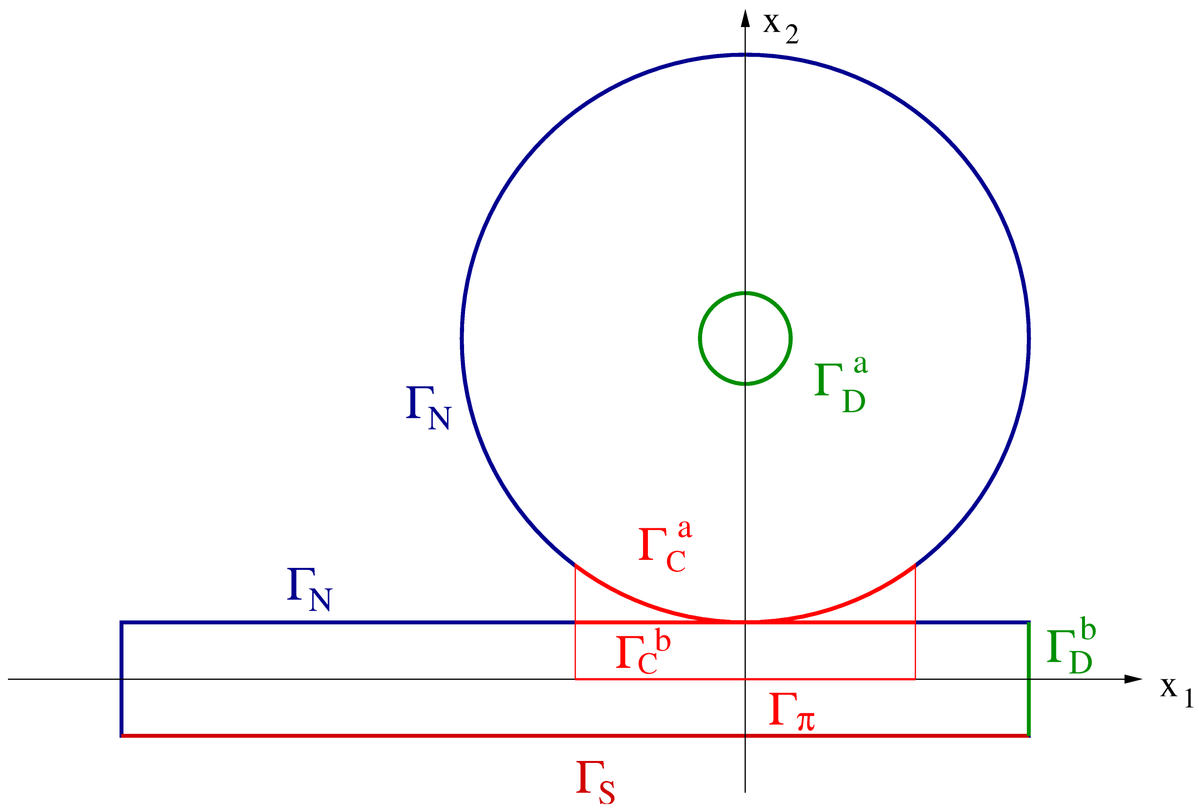

Figure 2.1: Boundary conditions

2.1.2 Contact condition

In our configuration, Ω_a is located above Ω_b. There is a part of the boundary $\Gamma_C = \Gamma_C^a \cup \Gamma_C^b \subset \Gamma$ where the two bodies can, but need not touch. By Γ_C^a and Γ_C^b we denote the respective boundary parts of Ω_a and Ω_b. We now want to find conditions on the displacement $u(X,t)$ for $t \in [t_0, t_0 + \Delta t]$ fixed.

We have chosen the spatial system of coordinates such that the X_1 axis is almost parallel to the contact surfaces. By Γ_π, we denote the segment which we get from projecting Γ_C onto the X_1 axis. Then the contact surfaces can be parametrized by

$$\begin{aligned}\Gamma_C^a &= \{X | X_2 = \gamma_a(X_1), X_1 \in \Gamma_\pi\} \\ \Gamma_C^b &= \{X | X_2 = \gamma_b(X_1), X_1 \in \Gamma_\pi\}\end{aligned}$$

with smooth functions γ_a and γ_b.

Let X^a be a particle on the contact boundary of the roll, where $X^a = (X_1^a, \gamma_a(X_1^a))$. There exists a unique particle X^b on Γ_C^b that lies above X^a at time $t \in [t_0, t_0 + \Delta t]$, this means

$$X_1^a + u_1(X^a, t) = X_1^b + u_1(X^b, t).$$

As the bodies can not penetrate each other, we get the condition

$$\gamma_a(X_1^a) + u_2(X^a, t) \geq \gamma_b(X_1^b) + u_2(X^b, t).$$

We may assume that $X_1^a \approx X_1^b$ and therefore we have $u_1((X_1^b, \gamma_a(X_1^b), t) \approx u_1((X_1^a, \gamma_a(X_1^a)), t)$. To simplify notation, we shall write

$$u^z(X_1, t) = u((X_1, \gamma_z(X_1)), t), \quad z = a, b.$$

As the boundaries are smooth, there holds

$$\gamma_b(X_1^b) \approx \gamma_b(X_1^a) + \gamma_b'(X_1^a)^T (X_1^b - X_1^a).$$

So the condition for non-penetration reduces to

$$\begin{aligned}\gamma_a(X_1^a) + u_2^a(X_1^a, t) &\geq \gamma_b(X_1^a) + \gamma_b'(X_1^a)^T (X_1^b - X_1^a) + u_2^a(X_1^a, t) \\ &\approx \gamma_b(X_1^a) + \gamma_b'(X_1^a)^T (u_1^b(X_1^a, t) - u_1^a(X_1^a, t)) + u_2^b(X_1^a, t).\end{aligned}$$

Rewriting this condition, we finally get

$$u_n^R := u^R \cdot n \leq g \quad \text{on } \Gamma_\pi \times [t_0, t_0 + \Delta t]$$

where

$$u^R(X_1, t) := u^a(X_1, t) - u^b(X_1, t), \;\; X_1 \in \Gamma_\pi, t \in [t_0, t_0 + \Delta t]$$

denotes the relative displacement of the two boundary segments Γ_C^a and Γ_C^b. The unit vector outward normal on Γ_C^a is almost parallel to the one on Γ_C^b, so we may set

$$n = -n_b = \frac{(\gamma_b', -1)^T}{\sqrt{(\gamma_b')^2 + 1}} \approx n_a.$$

The right hand side is a given "gap function"

$$g = \frac{(\gamma_a - \gamma_b)}{\sqrt{(\gamma_a')^2 + 1}}.$$

We can now formulate the contact condition on the boundary segment Γ_C. For $X_1 \in \Gamma_\pi, t \in [t_0, t_0 + \Delta t]$, the condition "the roll Ω_a lies above the strip of steel Ω_b" is given by

$$u_n^R(X_1, t) \leq g(X_1, t).$$

On Γ_C, we consider the arising normal stress σ_n as well as the tangential stress σ_T. They are given by

$$\sigma_n := n^T \sigma n, \quad \sigma_T := \sigma n - \sigma_n n.$$

At the points of contact, the two bodies are pressed against each other. The normal stress σ_n appearing on these surfaces has to be compressive, i. e.

$$\sigma_n \leq 0.$$

We assume that there are no tractions applied to the boundary, so as long as no contact occurs, the normal stress is zero. Furthermore, the two bodies are balanced, so the stress occurring in a point of contact has to be the same on both boundary segments

$$\sigma(X_1, \gamma_a(X_1), t)n = \sigma(X_1, \gamma_b(X_1), t)n$$

Therefore, $\sigma_n|_{\Gamma_C}$ is a function of X_1 and t only.

Together we get for $X_1 \in \Gamma_\pi, t \in [t_0, t_0 + \Delta t]$

$$\begin{aligned} u^R(X_1, t) \cdot n \leq g(X_1), & \qquad \sigma_n(X_1, t) \leq 0 \\ u^R(X_1, t) \cdot n < g(X_1) & \;\Rightarrow\; \sigma_n(X_1, t) = 0 \\ \sigma_n(X_1, t) < 0 & \;\Rightarrow\; u^R(X_1, t) \cdot n = g(X_1). \end{aligned}$$

We can reformulate the so-called complementarity condition above: We see that either there is no stress in normal direction, $\sigma_n = 0$, or the bodies touch, $u^R(X_1, t) \cdot n - g(X_1) = 0$. Therefore, their product $\sigma_n(X_1, t)(u^R(X_1, t) \cdot n - g(X_1))$ has to vanish on Γ_π. Using this, we obtain for $X_1 \in \Gamma_\pi, t \in [t_0, t_0 + \Delta t]$

$$\begin{aligned} u^R(X_1, t) \cdot n &\leq g(X_1) \\ \sigma_n(X_1, t) &\leq 0 \\ \sigma_n(X_1, t)(u^R(X_1, t) \cdot n - g(X_1)) &= 0 \end{aligned}$$

As a first approach to the problem, we will assume that there occurs no friction between the roll and the strip of steel. Although this is not true in our application, it will help us understand the concept of contact between two bodies. If the contact is frictionless, the tangential component of stress σ_T has to be zero. Thus, we must have for $x_1 \in \Gamma_\pi, t \in [t_0, t_0 + \Delta t]$

$$\sigma_T(x_1, t) = 0$$

2.1.3 Classical formulation

Collecting all the results from the previous subsections, we can now formulate the elastic two body contact problem without friction. We want to find the displacement field $u : \Omega \times [t_0, t_0 + \Delta t] \to \mathbb{R}^2$, that describes the deformation of the two bodies Ω_a and Ω_b. They are made of an elastic material obeying Hooke's law. There are no surface tractions acting on the bodies. We assume that we may neglect inertia forces. We have boundary conditions arising from symmetry on a part Γ_S of the boundary, and the deformation is given on another part Γ_D. We are given the initial gap g between the two surfaces Γ_C^a and Γ_C^b. This leaves us with the following system of partial differential equations for all times $t \in [t_0, t_0 + \Delta t]$

$$\left.\begin{aligned} -\operatorname{div}(\sigma) &= f \\ \sigma &= J_0^{-1} \nabla\phi_0 \, CE \, \nabla\phi_0^T \\ E &= E(\phi_0) + \nabla\phi_0^T \, \varepsilon(u) \nabla\phi_0 \end{aligned}\right\} \quad \text{in } \Omega \tag{2.10}$$

$$u = u_g \quad \text{on } \Gamma_D \tag{2.11}$$

$$\left.\begin{aligned} u_2 &= 0 \\ (\sigma n)_1 &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_S \tag{2.12}$$

$$\sigma_n = 0 \quad \text{on } \Gamma_N \tag{2.13}$$

$$\left.\begin{aligned} u^R \cdot n &\le g \\ \sigma_n &\le 0 \\ \sigma_n(u^R \cdot n - g) &= 0 \\ \sigma_T = \sigma n - \sigma_n n &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_\pi \tag{2.14}$$

2.1.4 Variational Formulation

We will now find an alternative formulation of the contact problem. From this variational formulation we will be able to prove existence and uniqueness of a solution, and find stability estimates. Throughout the following the parentheses $(.,.)$ denote the L^2 scalar product on Ω. The scalar product and the norm in the Sobolev space $H^k(\Omega)$ are denoted by $(.,.)_k$ and $\|.\|_k$, respectively.

Let V be a real Hilbert space equipped with scalar product $(.,.)_V$ and norm $\|.\|_V$. Let V^* denote its dual and $\langle .,. \rangle$ the duality pairing. For $u, v \in V$ we require that the virtual work $(\sigma(u) : \varepsilon(v))$ is well defined and that the trace of v on the boundary exists. We will see later, that $V = H^1(\Omega)$ is an appropriate choice for our problem

We define the linear manifold V_g of functions in V, that satisfy given essential boundary conditions on Γ_D and Γ_S, whereas V_0 shall be the subspace of V where the respective homogeneous conditions are fulfilled:

$$\begin{aligned} V_0 &:= \{v \in V|\ v = 0 \text{ on } \Gamma_D, v_2 = 0 \text{ on } \Gamma_S\} \\ V_g &:= \{v \in V|\ v = u_g \text{ on } \Gamma_D, v_2 = 0 \text{ on } \Gamma_S\} \end{aligned}$$

The contact condition $u_n^R \leq g$ on Γ_π adds a further constraint to the set of admissible solutions K, which is defined by

$$K := \{v \in V_g|\ v_n^R \leq g \text{ on } \Gamma_\pi\}. \tag{2.15}$$

We note that, as long as Γ_D and Γ_S do not touch, K is a nonempty, closed, convex cone in V. Otherwise we needed to have compatibility of the respective boundary conditions on those parts.

Let $u \in K$ be a solution to the contact problem (2.10) - (2.14). For an arbitrary $v \in K$, $v - u \in V_0$ is a virtual displacement satisfying the homogeneous essential boundary conditions. Multiplying the balance equation (2.10) with $(v - u)$ and integrating over Ω yields

$$-\int_\Omega \operatorname{div}(\sigma(u)) \cdot (v-u)\ dX = \int_\Omega f \cdot (v-u)\ dX \qquad \forall v \in K.$$

As the solution u, the test function $v - u$ and the boundary Γ are smooth enough, we can apply Green's formula to obtain

$$\int_\Omega \sigma(u) : \nabla(v-u)\ dX - \int_\Gamma (\sigma(u)n) \cdot (v-u)\ ds_X = \int_\Omega f \cdot (v-u)\ dX \qquad \forall v \in K.$$

Using the fact that the stress tensor σ is symmetric, one can see that there holds

$$\int_\Omega \sigma(u) : \nabla(v-u)\ dX = \int_\Omega \sigma(u) : \varepsilon(v-u)\ dX \qquad \forall v \in K.$$

Next, we include essential and natural boundary conditions on Γ_D, Γ_S and Γ_N

$$\begin{aligned} \int_\Gamma (\sigma(u)n) \cdot (v-u)\ ds_X &= \int_{\Gamma_D} (\sigma(u)n) \cdot \underbrace{(v-u)}_{=0}\ ds_X + \int_{\Gamma_S} \underbrace{(\sigma(u)n)}_{(\sigma(u)n)_1=0} \cdot \underbrace{(v-u)}_{(v-u)_2=0}\ ds_X \\ &\quad + \int_{\Gamma_N} \underbrace{(\sigma(u)n)}_{=0} \cdot (v-u)\ ds_X + \int_{\Gamma_C} (\sigma(u)n) \cdot (v-u)\ ds_X \\ &= \int_{\Gamma_C} (\sigma(u)n) \cdot (v-u)\ ds_X. \end{aligned}$$

Using this relation, we obtain

$$\int_\Omega \sigma(u) : \varepsilon(v-u)\; dX - \int_{\Gamma_C} (\sigma(u)n)\cdot(v-u)\; ds_X = \int_\Omega f\cdot(v-u)\; dX \qquad \forall v\in K.$$

Up to now, this was more or less a standard procedure for finding a variational formulation for elasticity problems. Next, we need to include the contact conditions (2.14) into the equation. By bounding the integral over Γ_C we can find a weak formulation of the classical conditions. Let therefore be $v \in K$, then we can estimate

$$\begin{aligned}
\int_{\Gamma_C} (\sigma(u)n)\cdot(v-u)\; ds &= \int_{\Gamma_\pi} (\sigma(u)n)\cdot(v^R-u^R)\; dX_1 \\
&= \int_{\Gamma_\pi} (\sigma_n(u)n + \underbrace{\sigma_T(u)}_{=0})\cdot(v^R-u^R)\; dX_1 \\
&= \int_{\Gamma_\pi} \sigma_n(u)(v_n^R-u_n^R)\; dX_1 \\
&= \int_{\Gamma_\pi} \underbrace{\sigma_n(u)}_{\le 0}\underbrace{(v_n^R-g)}_{\le 0} - \underbrace{\sigma_n(u)(u_n^R-g)}_{=0}\; dX_1 \\
&\ge 0.
\end{aligned}$$

Thus, the solution u satisfies the inequality

$$\int_\Omega \sigma(u) : \varepsilon(v-u)\; dX \ge \int_\Omega f\cdot(v-u)\; dX \qquad \forall v\in K.$$

We call it a variational inequality of the first kind. We have seen that σ is affine linear in u, there holds $\sigma(u) = \sigma_0 + \bar\sigma(u)$. Inserting this into the inequality above, we get

$$\int_\Omega \bar\sigma(u) : \varepsilon(v-u)\; dX \ge \int_\Omega f\cdot(v-u)\; dX - \int_\Omega \sigma_0 : \varepsilon(v-u)\; dX \qquad \forall v\in K.$$

To obtain a more abstract variational formulation of our problem, we define the bilinear form $a : V\times V \to \mathbb{R}$ and the continuous linear operator $F : V \to \mathbb{R}$

$$\begin{aligned}
a(u,v) &:= \int_\Omega \bar\sigma(u) : \varepsilon(v)\; dX && (2.16)\\
&= \int_\Omega J_0^{-1}(\nabla\phi_0\, C \nabla\phi_0{}^T \varepsilon(u)\nabla\phi_0\,) : (\varepsilon(v)\nabla\phi_0\,)\; dX \qquad \forall u,v\in V \\
\langle F, v\rangle &:= \int_\Omega f\cdot v\; dX - \int_\Omega \sigma_0 : \varepsilon(v)\; dX && (2.17)\\
&= \int_\Omega f\cdot v\; dX - \int_\Omega (J_0^{-1}\nabla\phi_0\, CE(\phi_0)\nabla\phi_0^T\,) : \varepsilon(v)\; dX \qquad \forall v\in V
\end{aligned}$$

Using these definitions, we can pose a variational problem of the first kind: Find $u \in K = \{v \in V : v_n^R \leq g\}$ such that

$$a(u, v-u) \geq \langle F, v-u \rangle \qquad \forall v \in K. \tag{2.18}$$

One can directly see that a is symmetric:

$$a(u, v) = a(v, u) \qquad \forall u, v \in V$$

We will see that the bilinear form a is bounded and coercive on the subspace V_0. This will help providing a result on existence and uniqueness of a solution to the two-body contact problem.

2.2 Contact problem with friction

2.2.1 Coulomb friction

So far, we have assumed that there occurs no friction along the contact surfaces. This implies that the tangential stress σ_T vanishes. For the two-body contact problem with Coulomb friction, we get different conditions on the stress acting on the boundary segment where the surfaces may touch each other. Still, the bodies cannot penetrate, so we have that the relative normal displacement is smaller or equal to some given gap function g,

$$u_n^R \leq g \qquad \text{on } \Gamma_\pi.$$

Also, the normal stress is compressive, i. e.

$$\sigma_n \leq 0 \qquad \text{on } \Gamma_\pi.$$

And again, there holds the complementarity condition, which states that if the bodies do not touch, the stress in normal direction vanishes.

For the tangential stress Coulomb's law states that there holds the condition

$$\left.\begin{array}{lcr} |\sigma_T(u)| \leq \mathcal{F}|\sigma_n(u)| & & \\ |\sigma_T(u)| < \mathcal{F}|\sigma_n(u)| & \Rightarrow & \dot{u}_T^R = 0 \\ |\sigma_T(u)| = \mathcal{F}|\sigma_n(u)| & \Rightarrow & \exists \alpha \leq 0 : \ \dot{u}_T^R = \alpha \sigma_T(u) \end{array}\right\} \text{on } \Gamma_\pi \tag{2.19}$$

where the dot denotes the time derivative and $\mathcal{F} \geq 0 \in L^\infty(\Gamma_\pi)$ is the coefficient of friction. This means that the normal stress bounds the tangential one in some way. The tangential component σ_T cannot exceed the normal stress σ_n times the coefficient of friction $\mathcal{F}$. If we have $|\sigma_T| < \mathcal{F}|\sigma_n|$, there is no relative displacement between the two surfaces, the bodies stick to each other. Sliding only occurs if the absolute value of the tangential stress reaches the critical value of $|\mathcal{F}\sigma_n|$. The tangential velocity of the relative movement of the surfaces is then given by $\dot{u}_T^R$. It is proportional to the tangential stress. This implies that the sliding takes the same direction as the stress tangential to the surfaces does. Note that, for $\mathcal{F} = 0$, we have contact without friction. Therefore the contact problem without friction can be treated as a special case of the more general problem with Coulomb friction.

We get the following system of partial differential equations and boundary conditions, which describes the elastic two body contact problem with friction:

$$\left.\begin{aligned} -\mathrm{div}(\sigma) &= f \\ \sigma &= J_0^{-1} \nabla\phi_0 \, CE \, \nabla\phi_0^T \\ E &= E(\phi_0) + \nabla\phi_0^T \, \varepsilon(u) \nabla\phi_0 \end{aligned}\right\} \quad \text{in } \Omega \tag{2.20}$$

$$u = u_g \quad \text{on } \Gamma_D \tag{2.21}$$

$$\left.\begin{aligned} u_2 &= 0 \\ (\sigma n)_1 &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_S \tag{2.22}$$

$$\sigma_n = 0 \quad \text{on } \Gamma_N \tag{2.23}$$

$$\left.\begin{aligned} u^R \cdot n &\leq g \\ \sigma_n &\leq 0 \\ \sigma_n(u^R \cdot n - g) &= 0 \\ |\sigma_T(u)| &\leq \mathcal{F}|\sigma_n(u)| \\ \dot{u}_T^R &= \alpha\sigma_T(u), \ \alpha \leq 0 \\ |\dot{u}_T^R|(|\sigma_T(u)| - \mathcal{F}|\sigma_n(u)|) &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_C \tag{2.24}$$

Again, we want to derive a variational formulation of the contact problem with Coulomb friction. Let again denote $V = H^1(\Omega)$, and its subsets V_0, V_g and K be defined as before. Assume that $u \in K$ is a solution to the Coulomb problem with friction. We choose an arbitrary test function $\dot{v} \in V_0$, which corresponds to a velocity. We multiply the partial differential equation (2.20) by $\dot{v}$ and integrate over Ω. We apply Green's formula and insert the given boundary conditions on Γ_D, Γ_S and Γ_N. As before, we get the equation

$$\int_\Omega \sigma(u) : \varepsilon(\dot{v}) \, dX - \int_{\Gamma_C} (\sigma(u)n) \cdot \dot{v} \, ds = \int_\Omega f \cdot \dot{v} \, dX \qquad \forall \dot{v} \in V_0.$$

Now we have to consider the Coulomb conditions (2.24). We include them into the variational formulation by bounding the integral over Γ_C. For some test function $\dot{v} \in V_0$, we have

$$\begin{aligned} \int_{\Gamma_C} (\sigma(u)n) \cdot \dot{v} \, ds &= \int_{\Gamma_C} (\sigma_n(u)n + \sigma_T(u)) \cdot \dot{v} \, ds \\ &= \int_{\Gamma_C} \sigma_n(u)\dot{v}_n \, ds + \int_{\Gamma_C} \sigma_T(u) \cdot \dot{v}_T \, ds \\ &= -\int_{\Gamma_\pi} \lambda_1 \dot{v}_n^R \, dX_1 - \int_{\Gamma_\pi} \lambda_2 \cdot \dot{v}_T^R \, dX_1, \end{aligned}$$

where we set

$$\lambda_1 := -\sigma_n, \quad \lambda_2 := -\sigma_T \qquad \text{on } \Gamma_\pi.$$

Later we will see that the so called "dual variables" λ_1, λ_2 can be interpreted as Lagrange multipliers. To determine the spaces where to find λ_1, λ_2, we consider the following: As we have $u \in V = H^1(\Omega)^2$, the restriction of u to the boundary lies in $H^{1/2}(\Gamma)^2$, which can be defined via

$$H^{1/2}(\Gamma) = \{g \in L^2(\Gamma) : \exists v \in H^1(\Omega) \text{ s. t. } v|_\Gamma = g\}.$$

By $H^{1/2}(\Gamma_\pi)$ we denote the restriction of $H^{1/2}(\Gamma)$ onto Γ_C and projection onto Γ_π. Let $H^{-1/2}(\Gamma_\pi)$ denote its dual,

$$H^{-1/2}(\Gamma_\pi) = H^{1/2}(\Gamma_\pi)^*.$$

Then for the stress along the zone of contact, we have $\lambda_1 n + \lambda_2 = -\sigma(u)n \in H^{-1/2}(\Gamma_\pi)^2$. The Lagrange multipliers λ_1, λ_2 represent the normal and tangential component of the vector-valued function $-\sigma(u)n$. As λ_2 is orthogonal to n, there holds $\lambda_2 \cdot n = 0$, and further

$$\begin{aligned}
\lambda_1 \in Q_1 &:= H^{-1/2}(\Gamma_\pi), \\
\lambda_2 \in Q_2 &:= \{\mu_2 \in H^{-1/2}(\Gamma_\pi)^2 | \mu_2 \cdot n = 0\}.
\end{aligned}$$

The respective norms in these Hilbert spaces we denote by $\|.\|_{Q_i}, i = 1, 2$. Let $\langle ., . \rangle_{Q_i} : Q_i^* \times Q_i \to \mathbb{R}$ be the respective duality pairings. By $Q = Q_1 \times Q_2$ we denote the product space, which is isomorphic to the original space $H^{-1/2}(\Gamma_\pi)^2$ for $\sigma(u)n$.

The conditions of Coulomb friction do not only imply that $\lambda_i \in Q_i, i = 1, 2$. For the normal stress, we have that it is compressive. For λ_1 this implies

$$\lambda_1 \in \Lambda_1 := \{\mu_1 \in Q_1 | \ \mu_1 \geq 0\}.$$

We can find a weak formulation of the non-penetration condition. For $\mu_1 \in \Lambda_1$ there holds

$$\begin{aligned}
\int_{\Gamma_\pi} (u_n^R - g)(\mu_1 - \lambda_1) \, dX_1 &= \int_{\Gamma_\pi} (u_n^R - g)\mu_1 \ - \underbrace{(u_n^R - g)\lambda_1}_{=0} dX_1 \\
&= \int_{\Gamma_\pi} \underbrace{(u_n^R - g)}_{\leq 0} \mu_1 \, dX_1 \leq 0 \quad \forall \mu_1 \in \Lambda_1.
\end{aligned}$$

The size of the tangential stress $|\lambda_2|$ is bounded by $\mathcal{F}\lambda_1$. Therefore we have

$$\lambda_2 \in \Lambda_2(\lambda_1) := \{\mu_2 \in Q_2 | \ |\mu_2| \leq \mathcal{F}|\lambda_1|\}.$$

Again, we can deduce a weak formulation of the condition for contact obeying Coulomb's law of friction: Because $\dot{u}_T^R$ and λ_2 are parallel and have the same direction, there holds

$$\dot{u}_T^R \cdot \lambda_2 = |\dot{u}_T^R||\lambda_2|.$$

Using this and the condition

$$|\dot{u}_T^R|(|\lambda_2| - \mathcal{F}\lambda_1) = |\dot{u}_T^R|(|\sigma_T(u)| - \mathcal{F}|\sigma_n(u)|) = 0;$$

we obtain

$$\begin{aligned}\int_{\Gamma_\pi} \dot{u}_T^R \cdot \lambda_2 \, dX_1 &= \int_{\Gamma_\pi} |\dot{u}_T^R||\lambda_2| \, dX_1 \\ &= \int_{\Gamma_\pi} |\dot{u}_T^R|\mathcal{F}\lambda_1 \, dX_1 \\ &\geq \int_{\Gamma_\pi} \dot{u}_T^R \cdot \mu_2 \, dX_1 \quad \forall \mu_2 \in \Lambda_2(\lambda_1).\end{aligned}$$

Collecting our results, we get a variational inequality of the second kind for $u \in H^1([t_0, t_0 + \Delta t], V_g)$, $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2(\lambda_1)$

$$\begin{aligned} a(u, \dot{v}) + \langle \dot{v}_n^R, \lambda_1 \rangle_{Q_1} + \langle \dot{v}_T^R, \lambda_2 \rangle_{Q_2} &= \langle F, \dot{v} \rangle \quad \forall \dot{v} \in V_0 \\ \langle u_n^R, \mu_1 - \lambda_1 \rangle_{Q_1} + \langle \dot{u}_T^R, \mu_2 - \lambda_2 \rangle_{Q_2} &\leq \langle g, \mu_1 - \lambda_1 \rangle_{Q_1} \quad \forall \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2(\lambda_1).\end{aligned}$$

To find a more abstract formulation, we introduce bilinear forms $b_1 : V \times Q_1 \to \mathbb{R}$ and $b_2 : V \times Q_2 \to \mathbb{R}$ such that

$$\begin{aligned} b_1(v, \mu_1) &= \langle v_n^R, \mu_1 \rangle_{Q_1} \qquad \forall v \in V, \mu_1 \in Q_1 \\ b_2(v, \mu_2) &= \langle v_T^R, \mu_2 \rangle_{Q_2} \qquad \forall v \in V, \mu_2 \in Q_2 \end{aligned}$$

Then we define the bilinear form $b : V \times Q \to \mathbb{R}$,

$$b(v, \mu) = b_1(v, \mu_1) + b_2(v, \mu_2) \quad \forall v \in V, \mu \in Q$$

and the bounded linear operator $G : Q \to \mathbb{R}$,

$$\langle G, \mu \rangle_Q := \langle g, \mu_1 \rangle_{Q_1} \quad \forall \mu \in Q.$$

We can rewrite the problem:
Find $u \in H^1([t_0, t_0 + \Delta t], V_g)$, $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2(\lambda_1)$

$$\begin{aligned} a(u, \dot{v}) + b(\dot{v}, \lambda) &= \langle F, \dot{v} \rangle \quad \forall \dot{v} \in V_0 \\ b_1(u, \mu_1 - \lambda_1) + b_2(\dot{u}, \mu_2 - \lambda_2) &\leq \langle G, \mu \rangle_Q \quad \forall \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2(\lambda_1).\end{aligned}$$

We can eliminate the Lagrange multipliers λ_1 and λ_2 from the system above. First, the conditions on the normal component of u and the normal stress $-\lambda_1 = \sigma_n(u)$

$$\sigma_n(u) \leq 0,\ u_n^R \leq g,\ \sigma_n(u)(u_n^R - g) = 0$$

are equivalent to

$$u \in K : \quad b_1(z - u, \sigma_n(u)) \geq 0 \quad \forall z \in K.$$

We choose the test function $\dot{v} - \dot{u} \in V_0$, and obtain the variational inequality

$$\begin{aligned} a(u, \dot{v} - \dot{u}) - b(\dot{v} - \dot{u}, \sigma_n(u)) &= \langle F, \dot{v} - \dot{u} \rangle \quad \forall \dot{v} \in V_0 \\ b_2(\dot{u}, \mu_2 - \lambda_2) &\leq 0 \quad \forall \mu_2 \in \Lambda_2(\sigma_n(u)) \\ b_1(z - u, \sigma_n(u)) &\geq 0 \quad \forall z \in K \end{aligned}$$

We can bound the duality product involving λ_2:

$$\begin{aligned} b_2(\dot{v} - \dot{u}, \lambda_2) &= -\int_{\Gamma_\pi} \sigma_T(u) \cdot (\dot{v} - \dot{u}) \, dX_1 \\ &\leq \int_{\Gamma_\pi} |\sigma_T(u)||\dot{v}_T^R| + \underbrace{\sigma_T(u) \cdot \dot{u}_T^R}_{=-|\sigma_T(u))||\dot{u}_T^R|} \, dX_1 \\ &= \int_{\Gamma_\pi} |\sigma_T(u)|(|\dot{v}_T^R| - |\dot{u}_T^R|) \, dX_1 \\ &\leq \int_{\Gamma_\pi} \mathcal{F}|\sigma_n(u)|(|\dot{v}_T^R| - |\dot{u}_T^R|) \, dX_1. \end{aligned}$$

This motivates the following definition of the nonlinear functional $j : V \times V \to \mathbb{R}$, such that

$$j(u, v) := \int_{\Gamma_\pi} \mathcal{F}|\sigma_n(u)||v_T^R| \, dX_1. \tag{2.25}$$

We can now pose a variational inequality for the solution $u \in H^1([t_0, t_0 + \Delta t], K)$

$$\begin{aligned} a(u, \dot{v} - \dot{u}) + j(u, \dot{v}) - j(u, \dot{u}) - b_1(\dot{v} - \dot{u}, \sigma_n) &\geq \langle F, \dot{v} - \dot{u} \rangle \ \forall \dot{v} \in V_0, \\ b_1(z - u, \sigma_n(u)) &\geq 0 \qquad\qquad \forall z \in K. \end{aligned} \tag{2.26}$$

We have shown that a solution u to the classical problem (2.20) - (2.24), which is in $H^1(\Omega)$, fulfills the variational inequality (2.26). One can also show, that each sufficiently smooth solution to the variational problem (2.26) is a classical solution in the sense of (2.20) - (2.24).

We now discretize the problem in time, using an implicit scheme, as was done in [1]. In each time step , u denotes the increment of the deformation taking place in t_0 to $t_0 + \Delta t$. We can approximate the time derivative by $\dot{u}(t) \approx u(t+\Delta t)/\Delta t$. We get the variational inequality for $u = u(., t_0 + \Delta t) \in K$

$$\begin{aligned} a(u, \dot{v} - \frac{u}{\Delta t}) + j(u, \dot{v}) - j(u, \frac{u}{\Delta t}) - b_1(\dot{v} - \frac{u}{\Delta t}, \sigma_n(u)) \geq \langle F, \dot{v} - \frac{u}{\Delta t} \rangle \qquad & \forall \dot{v} \in V_0, \\ b_1(z - u, \sigma_n(u)) \geq 0 \qquad & \forall z \in K. \end{aligned}$$

We multiply the inequality above by Δt and set $v = \Delta t\ \dot{v}$. This test function v has to satisfy the same essential boundary conditions as u, this means $v \in V_g$. Then $v - u$ lies in V_0. We get the equivalent problem of finding $u \in K$ such that

$$\begin{aligned} a(u, v-u) + j(u, v) - j(u, u) - b_1(v - u, \sigma_n(u)) \ &\geq \ \langle F, v - u \rangle \quad \forall v \in V_g \\ b_1(z - u, \sigma_n(u)) \ &\geq \ 0 \qquad\qquad \forall z \in K. \end{aligned}$$

Setting $v = z \in K$ and adding the two inequalities above, we see that any $u \in K$ is a solution, if and only if

$$a(u, v-u) + j(u, v) - j(u, u) \geq \langle F, v - u \rangle \quad \forall v \in K. \tag{2.27}$$

This is the problem we have to solve in each time step, therefore we will analyze it in the following. However, this variational inequality is nonlinear, due to the nonlinear functional j. Additionally, the term $j(u, u)$ is not convex and not differentiable. This causes severe problems in proving the existence of a solution. In fact, the question of existence of a solution to the general problem (2.20) - (2.24) remains open.

2.2.2 Contact problem with given friction

In order to get existence of a solution, we will rewrite the problem as a fixed point equation. From the definition of j, we see that $j(u, v)$ depends on $\sigma_n(u)$ and v only. We will now replace the unknown negative normal stress $-\sigma_n(u) = \lambda_1$ by a given contact pressure $g_n \geq 0$. Assuming that $u_{g_n} \in K$ is some deformation for which there holds $-\sigma_n(u_{g_n}) = g_n$, we will consider the problem

$$u \in K: \quad a(u, v-u) + j(u_{g_n}, v) - j(u_{g_n}, u) \geq \langle F, v - u \rangle \qquad \forall v \in K.$$

We will be able to prove existence of a solution $u(g_n)$ to this problem. Thus, we can define a mapping Φ, which maps the contact pressure g_n to the resulting negative normal stress $-\sigma_n(u(g_n))$ on the contact boundary,

$$\Phi : g_n \mapsto \Phi(g_n) = -\sigma_n(u(g_n)).$$

We will show that Φ has a fixed point for small coefficients of friction $\mathcal{F}$, i. e. there exists some g_n such that $g_n = \sigma_n(u(g_n))$. The deformation we get for this special contact pressure is then a solution to the general contact problem.

Reduced contact condition

In our approach, we replace the unknown absolute value of the normal stress $|\sigma_n(u)|$ in the Coulomb friction condition by a given compressive slip stress $g_n \geq 0$. Our task is now to find a displacement $u \in K$ such that the balance equation (2.20) and the boundary conditions (2.21) - (2.23) are satisfied. On the contact boundary Γ_C we get the conditions

$$\left.\begin{aligned} u^R \cdot n &\leq g \\ \sigma_n &\leq 0 \\ \sigma_n(u^R \cdot n - g) &= 0 \\ |\sigma_T(u)| &\leq \mathcal{F} g_n \\ \dot{u}_T^R = \alpha \sigma_T(u),\ \alpha &\leq 0 \\ |\dot{u}_T^R|(|\sigma_T(u)| - \mathcal{F} g_n) &= 0 \end{aligned}\right\} \text{ on } \Gamma_C. \tag{2.28}$$

Variational formulation

Let V, V_0, V_g, K, the bilinear form $a(.,.)$ and the functional F be defined as in section 2.1. The variational formulation of the incremental formulation reduces to

$$u \in K \text{ s.t. } \quad a(u, v - u) + j(v) - j(u) \geq \langle F, v - u \rangle \qquad \forall v \in K \tag{2.29}$$

where for $v \in V$

$$j(v) := \int_{\Gamma_\pi} \mathcal{F} g_n |v_T^R| \, dX_1. \tag{2.30}$$

This functional $j(.)$ is still nonlinear, but it is continuous and convex on K.

We can again give a mixed formulation of the problem. Therefore, we choose the Hilbert spaces Q_1 and Q_2 and the convex cone Λ_1 as before. Note that $\Lambda_2 := \Lambda_2(g_n)$ does not depend on λ_1 anymore. Using the same time discrete scheme as before, we can derive a mixed variational inequality for $u \in V_g, \lambda \in \Lambda$

$$\begin{array}{lcll} a(u, v) + b(v, \lambda) & = & \langle F, v \rangle & \forall v \in V_0 \\ b(u, \mu - \lambda) & \leq & \langle G, \mu - \lambda \rangle_Q & \forall \mu \in \Lambda \end{array} \tag{2.31}$$

with the nonempty, convex, closed set

$$\Lambda := \{\mu = (\mu_1, \mu_2) : \mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2\}. \tag{2.32}$$

If there exists a Lagrange parameter $\lambda \in \Lambda$, the primal variational formulation is equivalent to the mixed one.

2.3 Existence of a solution

As we have seen in the previous section, the contact problem with Coulomb friction is described by a variational inequality. There the displacement u and the normal stress σ_n on the contact boundary are related in a nonlinear way, and one of the resulting functionals was non-convex and not differentiable. So we cannot show existence of a solution directly. However, if we rewrite the relation to get a fixed point formulation for the normal stress σ_n, we can show existence of a solution u at least for small coefficients of friction $\mathcal{F}$. So our first step is analyzing the contact problem with given friction. As it can be described by an abstract variational inequality of the type

$$u \in K: \; a(u, v-u) + j(v) - j(u) \geq \langle F, v \rangle \qquad \forall v \in K, \tag{2.33}$$

we will now collect some results on the theory of such inequalities.

2.3.1 Existence and uniqueness for the primal problem

If not specified otherwise, we will assume throughout this section that V is a Hilbert space with dual V^*. Additionally, we need V_0 to be a subspace of V. Then $V_g := V_0 + u_g$ denotes the linear manifold containing all functions satisfying essential boundary conditions. The set of admissible solutions $K \subseteq V_g$ shall be nonempty, closed and convex. The bilinear form $a : V \times V \to \mathbb{R}$ is symmetric, bounded and elliptic on V_0

$$|a(u,v)| \leq \alpha_2 \|u\|_V \|v\|_V \qquad \forall u, v \in V \tag{2.34}$$

$$a(v,v) \geq \alpha_1 \|v\|_V^2 \qquad \forall v \in V_0. \tag{2.35}$$

Furthermore, we consider $j : V \to \mathbb{R}$ to be weakly lower semi-continuous. That means for each sequence (v_n) converging weakly to some $v \in V$, $v_n \rightharpoonup v$, there has to hold that

$$j(v) \leq \liminf j(v_n).$$

Then there holds the following theorem.

Theorem 1 *Let V be Hilbert space and K a nonempty, closed and convex subset of V_g. Let a be symmetric and satisfy (2.34) and (2.35) Let $j : V \to \mathbb{R}$ be nonnegative, convex and weakly lower semi-continuous and $F \in V^*$. Then the variational inequality*

$$u \in K: \; a(u, v-u) + j(v) - j(u) \geq \langle F, v-u \rangle \quad \forall v \in K \tag{2.36}$$

has a unique solution $u \in K$. Furthermore, for solutions u_1, u_2 to (2.36) with right hand sides $F_1, F_2 \in V^*$ there holds the stability estimate

$$\|u_1 - u_2\| \leq \frac{1}{\alpha_1} \|F_1 - F_2\|_{V^*}. \tag{2.37}$$

Proof.

As a is symmetric, the variational inequality is equivalent to a minimization problem

$$u \in K : \; J(u) = \min_{v \in K} J(v) \tag{2.38}$$

for the energy functional

$$J(v) := \frac{1}{2} a(v, v) + j(v) - \langle F, v \rangle \quad \forall v \in V. \tag{2.39}$$

We note that J is coercive and weakly lower semi-continuous. To see the equivalence of the formulations, assume u to be a solution to the minimization problem (2.38). This holds true if and only if for all $v \in K$, $t \in (0, 1)$

$$\begin{aligned} J(u) &\leq J(u + t(v - u)) \\ &= J(u) + t(a(u, v - u) + \langle F, v - u \rangle) + \frac{t^2}{2} a(v - u, v - u) \\ &\quad -j(u) + j(u + t(v - u)) \\ &\leq J(u) + t\,(a(u, v - u) + j(v) - j(u) - \langle F, v - u \rangle) + \frac{t^2}{2} a(v - u, v - u) \\ &\leq J(u) + t\,(a(u, v - u) + j(v) - j(u) - \langle F, v - u \rangle) \end{aligned}$$

which is equivalent to

$$a(u, v - u) + j(v) - j(u) - \langle F, v - u \rangle \geq 0 \quad \forall v \in K.$$

Thus the variational inequality (2.36) is satisfied.

The energy functional J is bounded from below, since

$$J(v) \geq \frac{\alpha_1}{2} \|v\|^2 - \|F\| \|v\| \geq -\frac{1}{2} \|F\|^2.$$

So the infimum of J over K is finite, and there exists a sequence (u_n) in K such that

$$J(u_n) \to \inf_{v \in K} J(v) > -\infty.$$

This sequence is bounded, as J is coercive on K,

$$\lim_{\|v\| \to \infty} J(v) = \infty.$$

Hence, there exists a weakly convergent subsequence, we assume $u_n \rightharpoonup u \in K$. As J is weakly lower semi-continuous, there holds

$$J(u) \leq \liminf J(u_n) = \inf_{v \in K} J(v).$$

So u minimizes J over K, and is a solution to (2.36).

For solutions u_1, u_2 to the variational problem (2.36) with different right hand sides $F_1, F_2 \in V^*$ we have

$$\begin{aligned} a(u_1, u_2 - u_1) + j(u_2) - j(u_1) &\geq \langle F_1, u_2 - u_1 \rangle \\ a(u_2, u_1 - u_2) + j(u_1) - j(u_2) &\geq \langle F_2, u_1 - u_2 \rangle. \end{aligned}$$

By adding these inequalities and using the coercivity of a we see that

$$\alpha_1 \|u_1 - u_2\|^2 \leq \langle F_1 - F_2, u_2 - u_1 \rangle$$

and thus there holds the stability estimate (2.37). For $F_1 = F_2 = F$ we can also see uniqueness of a solution. □

2.3.2 Existence and uniqueness for the mixed problem

In addition to the assumptions on V, V_0, V_g and a we made in the previous section, let Q be a Hilbert space with norm $\|.\|_Q$. Let Q^* denote its dual and $\langle .,. \rangle_Q$ be the duality pairing. The bilinear form $b : V \times Q \to \mathbb{R}$ shall be bounded and satisfy an inf-sup condition, there exist positive constants $\beta_1, \beta_2 > 0$ such that

$$|b(v, \mu)| \leq \beta_2 \|v\|_V \|\mu\|_Q \qquad \forall v \in V, \mu \in Q \tag{2.40}$$

$$\inf_{\mu \in \Lambda} \sup_{v \in V_0} \frac{b(v, \mu)}{\|v\|_V \|\mu\|_Q} \geq \beta_1 > 0 \tag{2.41}$$

The problem of finding $u \in V_g, \lambda \in \Lambda$ such that

$$\begin{array}{lcll} a(u, v) + b(v, \mu) & = & \langle F, v \rangle & \forall v \in V_0 \\ b(u, \mu - \lambda) & \leq & \langle G, \mu - \lambda \rangle_Q & \forall \mu \in \Lambda, \end{array} \tag{2.42}$$

is equivalent to finding a saddle point $(u, \lambda) \in V_g \times \Lambda$ of the Lagrangian $\mathcal{L}$ on $V_g \times \Lambda$:

$$\mathcal{L}(u, \lambda) = \inf_{\mu \in \Lambda} \sup_{v \in V_g} \mathcal{L}(v, \mu) \tag{2.43}$$

$$\mathcal{L}(v, \mu) = \frac{1}{2} a(v, v) + b(v, \mu) - \langle F, v \rangle - \langle G, \mu \rangle_Q \tag{2.44}$$

We shall prove this by the following argument:

Assume (u, λ) to be a solution to the saddle point problem. The optimality condition with respect to u holds true if and only if for any $v \in V_0, s > 0$

$$\begin{aligned} 0 &\leq \mathcal{L}(u + sv, \lambda) - \mathcal{L}(u, \lambda) \\ &= \frac{s^2}{2} a(v, v) + s a(u, v) + s b(v, \lambda) - s \langle F, v \rangle. \end{aligned}$$

Dividing the inequality by $s > 0$, sending s to 0 and choosing test functions $v, -v$ yields

$$0 = a(u, v) + b(v, \lambda) - \langle F, v \rangle \quad \forall v \in V_0.$$

Due to the non-negativity of $a(v, v)$ also the reverse implication holds true.

The inequality in (2.42) is equivalent to the maximality condition for λ

$$\mathcal{L}(u, \lambda) \geq \mathcal{L}(u, \mu) \quad \forall \mu \in \Lambda.$$

As shown in [4], there holds the following theorem.

Theorem 2 *Let A and B be two non-empty, closed convex subsets of V and Q, respectively. Let the Lagrangian $\mathcal{L} : V \times Q \to \mathbb{R}$ satisfy*

1. *for all $\mu \in B, v \mapsto \mathcal{L}(v, \mu)$ is convex and weakly lower semi-continuous*
2. *for all $v \in A, \mu \mapsto \mathcal{L}(v, \mu)$ is concave and weakly upper semi-continuous*
3. *there exists $\mu_0 \in B$ such that*
$$\lim_{\|v\| \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty$$
4. *there holds*
$$\lim_{\|\mu\| \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v, \mu) = -\infty$$

Then there exists a saddle point $(u, \lambda) \in A \times B$ of $\mathcal{L}$ on $A \times B$:

$$\mathcal{L}(u, \lambda) = \inf_{v \in A} \sup_{\mu \in B} \mathcal{L}(v, \mu)$$

2.3.3 Contact problem with Coulomb friction

So far we have seen, that, for given contact pressure g_n, we can compute a displacement field u and the normal stress $\sigma_n(u)$ along the contact boundary by solving the primal variational problem (2.46). Using the notation introduced in the previous section, we define the mapping

$$\Phi : \Lambda_1 \to \Lambda_1, \; g_n \mapsto \sigma_n(u). \tag{2.45}$$

Then every fixed point of Φ is a solution to the problem obeying Coulomb's law of friction. One can show that the mapping $\Psi : \Lambda_1 \to V, g_n \mapsto u(g_n)$ is $\frac{1}{2}$-Hölderian, i. e. for $g_1, g_2 \in Q$ and corresponding solutions u_1, u_2 there holds

$$\|u_2 - u_1\|_V \leq c \|g_1 - g_2\|_Q^{1/2}.$$

Using this relation, one can prove weak continuity of the mapping $L^2(\Gamma_\pi) \to L^2(\Gamma_\pi), \; g_n \mapsto \sigma_n(u(g_n))$. Then a weak version of Schauder's theorem implies the existence of a fixed point, provided the coefficient of friction is small enough. We will give a proof for existence of a solution in Section 3.2 after analyzing discrete approximations to the infinite dimensional problem.

2.3.4 Application to the contact problem

We have seen general theorems on variational inequalities, that assure existence and uniqueness of a solution, provided all conditions are satisfied. So our next aim is to prove that the functionals we defined in sections 2.1 and 2.2 satisfy the required conditions.

Primal variational formulation

We deduced the following formulation of the contact problem with given contact pressure:

Find $u \in K = \{v \in H^1(\Omega) : v_2 = u_g \text{ on } \Gamma_S, v_n^R \leq g \text{ on } \Gamma_\pi\}$ such that

$$a(u, v-u) + j(v) - j(u) \geq \langle F, v-u \rangle \qquad \forall v \in K \tag{2.46}$$

where for $u, v \in V$

$$\begin{aligned} a(u,v) &= \int_\Omega J_0^{-1}(\nabla\phi_0\, C\nabla\phi_0{}^T\varepsilon(u)\nabla\phi_0\,) : (\varepsilon(v)\nabla\phi_0\,)\; dX \\ j(v) &= \int_{\Gamma_\pi} \mathcal{F} g_n |v_T^R|\; dX_1 \\ \langle F, v\rangle &= \int_\Omega f \cdot v\; dX \end{aligned}$$

We see immediately, that K is a nonempty, convex, closed subset of V_g.

Clearly, the bilinear form $a : V \times V \to \mathbb{R}$ is symmetric. It is also bounded and coercive on V_0

$$\begin{aligned} |a(u,v)| &\leq \alpha_2 \|u\|_V \|v\|_V \qquad && \forall u, v \in V \\ a(v,v) &\geq \alpha_1 \|v\|_V^2 \qquad && \forall v \in V_0. \end{aligned}$$

Boundedness follows directly from Cauchy's inequality, there exists a constant $\alpha_2 > 0$ such that

$$\begin{aligned} |a(u,v)| &= \int_\Omega J_0^{-1}(\nabla\phi_0\, C\nabla\phi_0{}^T\varepsilon(u)\nabla\phi_0\,) : (\varepsilon(v)\nabla\phi_0\,)\; dX \\ &\leq \max(J_0^{-1})\lambda_{max}(\nabla\phi_0\, C\nabla\phi_0{}^T) \int_\Omega (\varepsilon(u)\nabla\phi_0\,) : (\varepsilon(v)\nabla\phi_0\,)\; dX \\ &\leq \max(J_0^{-1})\lambda_{max}(\nabla\phi_0\, C\nabla\phi_0{}^T) \|\varepsilon(u)\nabla\phi_0\,\|_0 \|\varepsilon(v)\nabla\phi_0\,\|_0 \\ &\leq \alpha_2 \|u\|_V \|v\|_V \end{aligned}$$

One can prove coercivity of a by using Korn's inequality

$$\begin{aligned} a(v,v) &= \int_\Omega J_0^{-1}(\nabla\phi_0\, C\nabla\phi_0{}^T\varepsilon(v)\nabla\phi_0\,):(\varepsilon(v)\nabla\phi_0\,)\;dXdX \\ &\geq J_0^{-1}\lambda_{min}(\nabla\phi_0\, C\nabla\phi_0{}^T)\int_\Omega(\varepsilon(v)\nabla\phi_0\,):(\varepsilon(v)\nabla\phi_0\,)\;dX \\ &\geq J_0^{-1}\lambda_{min}(\nabla\phi_0\, C\nabla\phi_0{}^T)c_k\|v\|_V^2 \\ &= \alpha_1\|v\|_V^2. \end{aligned}$$

Note that it is necessary to have one part of the boundary Γ_D fixed, otherwise the solution would only be unique up to a rigid body motion v, for which there holds $\varepsilon(v)=0$.

For given $g_n \in H^{-1/2}(\Gamma_\pi)$ and $\mathcal{F} \in L^\infty(\Gamma_\pi)$, satisfying $g_n, \mathcal{F} \geq 0$, the functional j is nonnegative, convex and lower semi-continuous. Clearly F is bounded and linear. Therefore we get existence and uniqueness of a solution $u \in K$ by Theorem 1.

Mixed formulation

Let V, V_0, V_g and Q be defined as in sections 2.1 and 2.2. We consider the following mixed problem:

Find $u \in V_g, \lambda \in \Lambda$ such that

$$\begin{aligned} a(u,v)+b(v,\lambda) &= \langle F,v\rangle && \forall v \in V_0 \\ b(u,\mu-\lambda) &\leq \langle G,\mu-\lambda\rangle_Q && \forall \mu \in \Lambda \end{aligned} \tag{2.47}$$

where, in addition to the definitions of a and F above, we have

$$\begin{aligned} b(v,\mu) &= \langle v_n^R,\mu_1\rangle_{Q_1}+\langle v_T^R,\mu_2\rangle_{Q_2} && \forall v \in V, \mu \in Q \\ \langle G,\mu\rangle_Q &= \langle g,\mu_1\rangle_{Q_1} && \forall \mu \in Q \end{aligned} \tag{2.48}$$

Before we prove the existence of a solution to the mixed formulation of the contact problem with given friction, we will analyze the bilinear form b. It is bounded on $V \times Q$. To prove this, we need a trace theorem, which states that the trace operator γ from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous:

$$\|\gamma(v)\|_{1/2} \leq c_\gamma\|v\|_1 \quad \forall v \in H^1(\Omega)$$

Using this, we see that for $v \in V$ and $\mu \in Q$ there holds

$$|b(v,\mu)| \leq \|v_n^R\|_{-1/2,\Gamma_\pi}\|\mu_1\|_Q + \|v_T^R\|_{-1/2,\Gamma_\pi}\|\mu_2\|_Q \leq 2c_\gamma\|v\|_V\|\mu\|_Q.$$

We will need that b satisfies an inf-sup condition

$$\inf_{\mu\in\Lambda}\sup_{v\in V_0}\frac{b(v,\mu)}{\|v\|_V\|\mu\|_Q} \geq \beta_1 > 0$$

This relation holds true, since for a sufficiently smooth domain Ω, the norm in $H^{-1/2}(\partial\Omega)$ can be defined by

$$\|\mu\|_{H^{-1/2}(\partial\Omega)} = \sup_{v\in H^1(\Omega)} \frac{\int_{\partial\Omega} v\cdot\mu\; ds}{\|v\|_1}$$

As Q is isomorphic to $H^{-1/2}(\Gamma_C)^2$, we get the inf-sup condition by estimating

$$\begin{aligned}
\sup_{v\in V}\frac{b(v,\mu)}{\|v\|_V} &= \sup_{v\in H^1(\Omega)} \frac{\int_{\Gamma_\pi} v_n^R\mu_1\; ds + \int_{\Gamma_\pi} v_T^R\cdot\mu_2\; ds}{\|v\|_1} \\
&= \sup_{v\in H^1(\Omega)} \frac{\int_{\Gamma_C} v\cdot(\mu_1 n+\mu_2)\; ds}{\|v\|_1} \\
&= \|\mu_1 n+\mu_2\|_{H^{-1/2}(\Gamma_C)} \geq c\|\mu\|_Q.
\end{aligned}$$

We will use Theorem 2 to show that a solution (u,λ) exists. Therefore, we consider the Lagrangian

$$\mathcal{L}(v,\mu) = \frac{1}{2}a(v,v) + b(v,\mu) - \langle F,v\rangle - \langle G,\mu\rangle_Q.$$

As the bilinear form a is coercive on the subspace V_0, the mapping $v\mapsto\mathcal{L}(v,\mu)$ is strictly convex.

The mapping $\mu\mapsto\mathcal{L}(v,\mu)$ is linear, thus it is concave. One can show easily, that the Lagrangian is weakly lower semi-continuous in v. As it is linear and continuous in μ, we have that $\mathcal{L}$ is weakly continuous in μ. Also the third condition, namely the coercivity of $\mathcal{L}$ with respect to v, follows directly from the coercivity of the bilinear form a on V_0. To prove the last condition

$$\lim_{\|\mu\|\to\infty,\mu\in\Lambda}\ \inf_{v\in V_0}\mathcal{L}(v,\mu) = -\infty,$$

take $\bar\mu\in\Lambda$ arbitrary but fixed. Then there exists a unique $\bar u\in V_0$ such that

$$a(\bar u,v) + b(v,\bar\mu) = \langle F,v\rangle \qquad \forall v\in V_0.$$

For this $\bar u$ we have

$$\alpha_2\|\bar u\|_V \geq -\frac{b(v,\mu)}{\|v\|_V} - \|F\|_{V^*} \qquad \forall v\in V$$

and further

$$\alpha_2\|\bar u\|_V \geq \beta_1\|\bar\mu\|_Q - \|F\|_{V^*}.$$

Using the above equality and the estimate for $\|\bar u\|_V$, we get

$$\begin{aligned}
\inf_{v\in V_g}\mathcal{L}(v,\bar\mu) &\leq \mathcal{L}(\bar u,\bar\mu) \\
&= -\frac{1}{2}a(\bar u,\bar u) - \langle G,\bar u\rangle_Q \\
&\leq -\frac{\alpha_1}{2}\|\bar u\|_V^2 + \|G\|_{Q^*}\|\bar\mu\|_Q \to -\infty \quad \text{as} \quad \|\bar\mu\|_Q\to\infty.
\end{aligned}$$

We have shown that all conditions in Theorem 2 hold true for our application to the contact problem. Therefore we obtain existence of a saddle point (u, λ) of the Lagrangian $\mathcal{L}$. This saddle point is then a solution to the mixed variational formulation (2.47).

As the bilinear form a is coercive, the Lagrangian is strictly convex in the first variable. Therefore, the first component u of the solution is unique. We will prove uniqueness of the second component λ: Assume that (u, λ^1) and (u, λ^2) are saddle points of $\mathcal{L}$ on $V_0 \times \Lambda$. This implies for $v \in V$

$$\begin{aligned} a(u,v) + b(v,\lambda^1) &= \langle F, v\rangle \\ a(u,v) + b(v,\lambda^2) &= \langle F, v\rangle. \end{aligned}$$

Subtracting the first equation from the second, we get

$$b(v, \lambda^2 - \lambda^1) = 0 \qquad \forall v \in V_0.$$

Using the inf-sup condition for b, we obtain

$$0 = b(v, \lambda^2 - \lambda^1) \geq \beta_1 \|v\|_V \|\lambda^1 - \lambda^2\|_Q \qquad \forall v \in V_0.$$

which implies $\lambda^1 = \lambda^2$.

Chapter 3

Approximation of the Contact Problem

In this chapter, we are interested in finding an approximative solution to the contact problem. Therefore we consider a discrete approximation to the contact problem with given friction. We shall prove existence and uniqueness of a solution to this finite dimensional problem. Afterwards we will show that, for small coefficients of friction, there exists a solution to the discretized problem obeying Coulomb's law of friction. If this coefficient satisfies an even stronger smallness condition, we will even show uniqueness of the solution. Furthermore, we shall study the behavior of the approximative solutions, as the parameter of discretization tends to zero.

As the boundaries of both Ω_a and Ω_b are Lipschitzian, we can assume that the problem is regular, i. e. there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, $g_n \in H^{-1/2+\varepsilon}(\Gamma_\pi)$ there holds $u \in H^{1+\varepsilon}(\Omega)^2$ and

$$\begin{aligned} \|u\|_{1+\varepsilon} &\leq c_1 \|g_n\|_{-1/2+\varepsilon} + c\|F\|_{-1} \\ \|\sigma_n(u)\|_{-1/2+\varepsilon} &\leq c_1 \|g_n\|_{-1/2+\varepsilon} + c\|F\|_{-1} \end{aligned}$$

where $c > 0$ is constant, and $c_1 > 0$ depends on $\mathcal{F}$ such that $c_1 \to 0$ as $\mathcal{F} \to 0$.

3.1 Contact problem with given friction

We shall now find an approximation to the contact problem with given friction. We refer to [7] for further details.

In this approach, we use triangular finite elements for the discretization of V. The parameter of discretization h shall correspond to the length of an edge of such a triangle. We see that the boundaries $\partial\Omega_a, \partial\Omega_b$ of our domains are not polygonal. We define the polygonal domains $\Omega_{ha} \subset \Omega_a, \Omega_{hb} \subset \Omega_b$ such that they are close to Ω_a and Ω_b, respectively:

$$|\Omega_a \backslash \Omega_{ha}| = O(h^2), \ |\Omega_b \backslash \Omega_{hb}| = O(h^2).$$

As we will see later, the error of order h^2 we obtain by using the approximative domains is much smaller than the error we get from the discretization of the problem. Thus, we will further neglect the fact that $\Omega_h \neq \Omega$.

By $(\mathcal{T}_h)$ with $h \to 0^+$, we denote a regular family of triangulations of $\bar{\Omega}$. We define finite dimensional spaces V_h, which contain all piecewise linear functions:

$$V_h := \{v_h \in C(\bar{\Omega})^2 \mid v_{h|T} \in P_1(T) \text{ for all } T \in \mathcal{T}_h\}$$

Analogously, we set

$$\begin{aligned} V_{0h} &= \{v_h \in V_h| \ v_h = 0 \text{ on } \Gamma_D, v_{h1} = 0 \text{ on } \Gamma_S\} \\ V_{gh} &= \{v_h \in V_h| \ v_h = u_g \text{ on } \Gamma_D, v_{h1} = 0 \text{ on } \Gamma_S\}. \end{aligned}$$

For the discretization of Q, we consider a system of partitions $(\mathcal{T}_H), H \to 0^+$ of Γ_π. The parameter H corresponds to the length of the intervals in the partition. Let $b_1, \ldots b_{m_H}$ denote the nodes of this partitions. Then each interval $I_i \in \mathcal{T}_H$ is given by $I_i = (b_i, b_{i+1}), i = 1 \ldots m_H - 1$. We define $H_i := |I_i|$ as the length of such an interval. We assume that the system $(\mathcal{T}_H)$ is regular in the sense that there exists a constant $\bar{\beta} > 0$ such that

$$\min_{i \le m_H} H_i \ge \bar{\beta} H$$

where $H = \max_i H_i$. With $\mathcal{T}_H$ we associate finite dimensional sets of piecewise constant functions on Γ_π:

$$\begin{aligned} Q_{1H} &:= \{\mu_{1H} \in L^2(\Gamma_\pi)| \ \mu_{1H|I} \in P_0(I) \text{ for all } I \in \mathcal{T}_H\} \\ Q_{2H} &:= \{\mu_{2H} \in Q_2| \ \mu_{2H|I} \in P_0(I)^2 \text{ for all } I \in \mathcal{T}_H\} \\ \Lambda_{1H} &:= \{\mu_{1H} \in Q_{1H}| \ \mu_{1H} \ge 0\} \\ \Lambda_{2H} &:= \{\mu_{2H} \in Q_{2H}| \ |\mu_{2H}| \le \mathcal{F} g_n\} \end{aligned}$$

Then $Q_H = Q_{1H} \times Q_{2H}$ is a subset of Q, whereas $\Lambda_H = \Lambda_{1H} \times \Lambda_{2H} \subset \Lambda$. Throughout the following, we consider the case that, as $h \to 0^+$, also $H \to 0^+$. Furthermore, we assume that there exist constants $\tau_1, \tau_2 > 0$ independent of h, H such that

$$\tau_1 \le \frac{h}{H} \le \tau_2.$$

We want to find $u_h \in V_{gh}, \lambda_H \in \Lambda_H$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_H) &= \langle F, v_h \rangle & \forall v_h \in V_{0h} \\ b(u_h, \mu_H - \lambda_H) &\le \langle G, \mu_H - \lambda_H \rangle_Q & \forall \mu_H \in \Lambda_H. \end{aligned} \tag{3.1}$$

Again, this problem is equivalent to a variational inequality of the first kind

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \geq \langle F, v_h - u_h \rangle \qquad \forall v_h \in K_{hH}, \qquad (3.2)$$

where

$$K_{hH} = \{v_h \in V_{gh} | \ \langle v_{hn}^R - g, \mu_{1H} \rangle_{Q_1} \leq 0 \quad \forall \mu_{1H} \in \Lambda_{1H}\}$$

is an external approximation of K. Indeed, $v_h \in K_{hH}$, if the mean value of $v_{hn}^R - g$ over each segment $I \in \mathcal{T}_H$ is smaller or equal to zero.

In order to prove existence and uniqueness of a solution, we suppose that the following condition is satisfied:

$$b(v_h, \mu_H) = 0 \ \forall v_h \in V_{0h} \qquad \Longleftrightarrow \qquad \mu_H = 0$$

This ensures that

$$\|\mu_H\|_h := \sup_{v_h \in V_{0h}} \frac{b(v_h, \mu_H)}{\|v_h\|_V}$$

forms a mesh-dependent norm on Q_H. As on a finite dimensional space all norms are equivalent, there holds an inf-sup condition for b with a constant β_{hH} depending on h, H in general:

$$\inf_{\mu_H \in \Lambda_H} \sup_{v_h \in V_{0h}} \frac{b(v_h, \mu_H)}{\|\mu_H\|_Q \|v_h\|_V} \geq \beta_{hH} \qquad (3.3)$$

To show existence and uniqueness of a solution to the discrete problem, we also need coercivity of a on V_{0h}. We immediately get

$$a(v_h, v_h) \geq \alpha_1 \|v_h\|_V \quad \forall v_h \in V_{0h},$$

as there holds $V_{0h} \subset V_0$. Analogously to the continuous case, we can prove existence and uniqueness of a solution (u_h, λ_H) to the discrete mixed problem (3.1) for any $h, H > 0$ by applying Theorem 2.

To find convergence results, we will need that the inf-sup condition (3.3) is satisfied with $\beta_{hH} = \beta$ independent of h, H:

$$\inf_{\mu_H \in \Lambda_H} \sup_{v_h \in V_{0h}} \frac{b(v_h, \mu_H)}{\|\mu_H\|_Q \|v_h\|_V} \geq \beta > 0. \qquad (3.4)$$

We cannot assume this only from the fact that the continuous problem satisfies an inf-sup condition. Following [4], we show that the inf-sup condition for b is uniformly satisfied under an additional assumption on h, H.

Lemma 3 *Let h/H be sufficiently small. Then there exists a constant $\beta > 0$ such that*

$$\sup_{v_h \in V_{0h}} \frac{b(v_h, \mu_H)}{\|v_h\|_V} \geq \beta \|\mu_H\|_Q$$

holds for any $\mu_H \in \Lambda_H$.

Proof. Let μ in Λ. Due to the inf-sup condition in the continuous setting, the norm in $Q \cong H^{-1/2}(\Gamma_\pi)^2$ can be defined by

$$\|\mu\|_Q = \sup_{v \in V_0} \frac{b(\mu, v)}{\|v\|_V} = \|\bar{v}\|_V,$$

where $\bar{v} \in V$ is the unique solution to

$$(\nabla \bar{v}, \nabla w)_0 + (\bar{v}, w)_0 \quad = \quad b(w, \mu) \quad \forall w \in V_0 \tag{3.5}$$

As both Ω_a and Ω_b have Lipschitzian boundary, this problem is regular in the following sense: There exists $\varepsilon > 0$ such that for $\mu \in H^{-1/2+\varepsilon}(\Gamma_\pi)^2$ the solution $\bar{v}$ lies in $H^{1+\varepsilon}(\Omega)^2$ and is bounded there by μ:

$$\|\bar{v}\|_{1+\varepsilon} \leq c(\varepsilon)\|\mu\|_{-1/2+\varepsilon}.$$

For $\mu_H \in \Lambda_H$ there holds $\mu_H \in H^{-1/2+\varepsilon}(\Gamma_\pi)^2, \varepsilon \in (0,1)$. This implies $\bar{v} \in H^{1+\varepsilon}(\Omega)^2$ and

$$\|\bar{v}\|_{1+\varepsilon} \leq c(\varepsilon)\|\mu_H\|_{-1/2+\varepsilon}.$$

Let $\bar{v}_h \in V_{0h}$ be the unique finite element approximation to (3.5),

$$(\nabla \bar{v}_h, \nabla w_h)_0 + (\bar{v}_h, w_h)_0 = b(w_h, \mu) \quad \forall w_h \in V_{0h}.$$

For $\bar{v}_h$ we have

$$\sup_{v_h \in V_{gh}} \frac{b(\mu_H, v_h)}{\|v_h\|_V} \geq \frac{b(\mu_H, \bar{v}_h)}{\|\bar{v}_h\|_V} = \|\bar{v}_h\|_V.$$

This together with the triangle inequality implies

$$\|\mu_H\|_Q = \|\bar{v}\|_V \leq \sup_{v_h \in V_{gh}} \frac{b(\mu_H, v_h)}{\|v_h\|_V} + \|\bar{v}_h - \bar{v}\|_V.$$

Error estimates for the solution to the boundary value problem $\bar{v}$ and its approximation $\bar{v}_h$ yield

$$\|\bar{v}_h - \bar{v}\|_V \leq c(\varepsilon)h^\varepsilon \|\bar{v}\|_{1+\varepsilon},$$

whereas an inverse inequality for $\mu_H \in \Lambda_H$ gives

$$\|\mu_H\|_{-1/2+\varepsilon} \leq H^{-\varepsilon}\|\mu_H\|_Q.$$

Using these results, we get

$$\begin{aligned} \|\bar{v}_h - \bar{v}\|_V &\leq c(\varepsilon)h^\varepsilon \|\bar{v}\|_{1+\varepsilon} \\ &\leq c(\varepsilon)h^\varepsilon \|\mu_H\|_{-1/2+\varepsilon} \\ &\leq c(\varepsilon)(h/H)^\varepsilon \|\mu_H\|_Q. \end{aligned}$$

Therefore we can deduce the estimate for $\|\mu_H\|_Q$

$$\left(1 - c(\varepsilon)\left(\frac{h}{H}\right)^{\varepsilon}\right)\|\mu_H\|_Q \leq \sup_{v_h \in V_{gh}} \frac{b(\mu_H, v_h)}{\|v_h\|_V}.$$

If the ratio $(h/H)^{\varepsilon}$ is sufficiently small, then $(1 - c(\varepsilon)(h/H)^{\varepsilon}) > 0$, and the inf-sup condition holds in the discrete setting. $\square$

Following [6], we now give an estimate for the error $(u - u_h, \lambda - \lambda_H)$. Let (u, λ) and (u_h, λ_H) be solutions to the continuous and the discrete problem, respectively.

Lemma 4 *Let b satisfy the inf-sup condition (3.4) with a constant independent of h, H. Then there exists a constant $c > 0$ such that for $v_h \in V_{0h}, \mu_H \in \Lambda_H$ there holds*

$$\begin{aligned} \|u - u_h\|_V^2 &\leq c(\|u - v_h\|_V^2 + \|\lambda - \mu_H\|_Q^2 \\ &\quad + b(\mu_H - \lambda, u) - \langle G, \mu_H - \lambda_H\rangle_Q) \quad (3.6) \\ \|\lambda - \lambda_H\|_Q &\leq c(\|u - u_h\| + \|\lambda - \mu_H\|_Q) \quad (3.7) \end{aligned}$$

Proof. For solutions $(u, \lambda), (u_h, \lambda_H)$ there holds

$$\begin{aligned} a(u, w_h) + b(w_h, \lambda) &= \langle F, w_h\rangle \quad \forall w_h \in V_{0h} \\ a(u_h, w_h) + b(w_h, \lambda_H) &= \langle F, w_h\rangle \quad \forall w_h \in V_{0h}, \end{aligned}$$

and by subtraction

$$a(u - u_h, w_h) + b(w_h, \lambda - \lambda_H) = 0 \quad \forall w_h \in V_{0h}.$$

Replacing w_h by $v_h - u_h, v_h \in V_{gh}$, we obtain

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h) + b(u_h - v_h, \lambda - \lambda_H). \end{aligned}$$

There holds the following inequality:

$$\begin{aligned} b(u_h - v_h, \lambda - \lambda_H) &= b(u - v_h, \lambda - \lambda_H) + b(u_h - u, \lambda - \mu_H) \\ &\quad + b(u_h - u, \mu_H - \lambda_H) \\ &\leq b(u - v_h, \lambda - \lambda_H) + b(u_h - u, \lambda - \mu_H) \\ &\quad + b(u, \lambda_H - \mu_H) - \langle G, \lambda_H - \mu_H\rangle_Q \\ &= b(u - v_h, \lambda - \lambda_H) + b(u_h - u, \lambda - \mu_H) \\ &\quad + b(u, \lambda - \mu_H) - \langle G, \lambda - \mu_H\rangle_Q \\ &\quad + b(u, \lambda_H - \lambda) - \langle G, \lambda_H - \lambda\rangle_Q \\ &\leq b(u - v_h, \lambda - \lambda_H) + b(u_h - u, \lambda - \mu_H) \\ &\quad + b(u, \lambda - \mu_H) - \langle G, \lambda - \mu_H\rangle_Q \end{aligned}$$

This implies

$$
\begin{aligned}
a(u-u_h, u-u_h) &\leq a(u-u_h, u-v_h) + b(u-v_h, \lambda-\lambda_H) + b(u_h-u, \lambda-\mu_H) \\
&\quad + b(u, \lambda-\mu_H) - \langle G, \lambda-\mu_H\rangle_Q
\end{aligned}
$$

and further

$$
\begin{aligned}
\alpha_1\|u-u_h\|_V^2 &\leq \alpha_2\|u-u_h\|_V\|u-v_h\|_V + \beta_2\|u-v_h\|_V\|\lambda-\lambda_H\|_Q \\
&\quad + \beta_2\|u-u_h\|_V\|\lambda-\mu_H\|_Q + b(u,\lambda-\mu_H) - \langle G, \lambda-\mu_H\rangle_Q.
\end{aligned}
$$

We observe that for any $w_h \in V_{0h}, \mu_H \in \Lambda_H$

$$
\begin{aligned}
\|\lambda-\lambda_H\|_Q &\leq \|\lambda-\mu_H\|_Q + \|\mu_H-\lambda_H\|_Q \\
&\leq \|\lambda-\mu_H\|_Q + \frac{1}{\beta\|w_h\|_V} b(w_h, \mu_H-\lambda_H) \\
&= \|\lambda-\mu_H\|_Q + \frac{1}{\beta\|w_h\|_V}(b(w_h, \mu_H-\lambda) - a(u-u_h, w_h)) \\
&\leq \|\lambda-\mu_H\|_Q + \frac{1}{\beta}(\beta_2\|\mu_H-\lambda\|_Q + \alpha_2\|u-u_h\|_V) \\
&\leq c(\|\lambda-\mu_H\|_Q + \|u-u_h\|_V).
\end{aligned}
$$

Inserting this into the estimate above, we obtain

$$
\begin{aligned}
\|u-u_h\|^2 &\leq c(\|u-u_h\|_V\|u-v_h\|_V + \|u-u_h\|_V\|\lambda-\mu_H\|_Q \\
&\quad + b(u,\lambda-\mu_H) - \langle G, \lambda-\mu_H\rangle_Q).
\end{aligned}
$$

Therefore, the assertion of the theorem follows from the ε-inequality $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$.
□

Using this Lemma, we can prove the following theorem.

Theorem 5 *Let b satisfy the inf-sup condition (3.4) with a positive constant $\beta > 0$ independent of h, H. If $u \in H^{1+q}(\Omega)^2$ for some $q > 0$ and $\lambda_1 \in L^2(\Gamma_\pi)$, then there exists a constant c independent of h, H such that*

$$
\begin{aligned}
\|u-u_h\|_V &\leq cH^{\bar{q}} \\
\|\lambda-\lambda_H\|_Q &\leq cH^{\bar{q}},
\end{aligned}
$$

where $\bar{q} = \min\{\frac{1}{4}, q\}$.

Proof. We will make use of Lemma 4 and bound all terms in the right hand side of (3.6). Setting $v_h = I_h u$ the Lagrange interpolate of u, we immediately obtain

$$
\|u - I_h u\|_V \leq ch^q\|u\|_{1+q}.
$$

Since $\lambda_1 \in L^2(\Gamma_\pi)$ and $\lambda_2 \in L^\infty(\Gamma_\pi) \subset L^2(\Gamma_\pi)$, we can choose $\mu_H = P_H\lambda$, where P_H is the L^2 projection of Λ onto Λ_H. We may estimate

$$\|\lambda - \mu_H\|_Q \leq cH^{1/2}\|\lambda\|_0.$$

To bound the last term in (3.6), we use the fact that $b : V \times Q \to \mathbb{R}$ is continuous:

$$b(\mu_H - \lambda, u) \leq \beta_2\|\mu_H - \lambda\|_Q\|u\|_V \leq cH^{1/2}\|\lambda\|_0\|u\|_V.$$

Combining all these estimates, we arrive at

$$\|u - u_h\|_V \leq c(h^q + H^{1/2} + H^{1/4}) \leq cH^{\bar{q}},$$

where we used that $h \leq \tau_2 H$. □

Assuming further regularity of the solution, one can even prove the following theorem, as was done in [4].

Theorem 6 *Let b satisfy the inf-sup condition (3.4) with a positive constant $\beta > 0$ independent of h, H. Let $u \in H^2(\Omega)^2$ and $\lambda_1 \in L^2(\Gamma_\pi)$, and let the set of all points, where u changes from $u_n^R < g$ to $u_n^R = g$ and from $u_T^R = 0$ to $u_T^R \neq 0$ be finite. Then there exists a constant c independent of h, H such that*

$$\begin{aligned} \|u - u_h\|_V &\leq cH^{1/2} \\ \|\lambda - \lambda_H\|_Q &\leq cH^{1/2}. \end{aligned}$$

3.2 Contact problem with Coulomb friction

We shall now study the behavior of the finite element approximation to the contact problem obeying Coulomb's law of friction. Therefore, all assumptions made in the previous section shall hold true. In the continuous setting, we introduced the operator $\Phi : \Lambda_1 \to \Lambda_1, \Phi(g_n) = \lambda_1$, where $(u, \lambda) \in V_g \times \Lambda(g_n)$ were the unique solution to

$$\begin{aligned} a(u, v) + b(v, \lambda) &= \langle F, v\rangle & \forall v \in V_0 \\ b(u, \mu - \lambda) &\leq \langle G, \mu - \lambda\rangle_Q & \forall \mu \in \Lambda(g_n). \end{aligned}$$

Then a solution to the contact problem with Coulomb friction was defined as a pair (u, λ) satisfying the system above, where λ_1 was a fixed point of Φ.

In the previous section, we have shown that, for given contact stress $g_n \in \Lambda_{1H}$, there exists a unique solution $(u_h, \lambda_H) \in V_{gh} \times \Lambda_H$ to the discrete problem. This ensures that the mapping

$$\Phi_H : \Lambda_{1H} \to \Lambda_{1H}, g_n \mapsto \lambda_{1H}$$

is well defined. Analogously to the continuous setting we call a pair (u_h, λ_H) a solution to the contact problem with Coulomb friction, if and only if λ_{1H} is a fixed point of the mapping Φ_H.

In order to prove, that Φ_H is continuous, we will first analyze the mapping $\Psi_H : \Lambda_{1H} \to V_g, \Psi(g_n) = u_h$. We can see that Ψ_H is continuous: Let for $g_n, \bar{g}_n \in \Lambda_{1H}$ be $u_h, \bar{u}_h$ the respective solutions to the primal inequality (3.2) with functionals $j, \bar{j}$:

$$\begin{aligned} a(u_h, v_h - u_h) + j(v_h) - j(u_h) &\geq \langle F, v_h \rangle \qquad \forall v_h \in K_{hH} \\ a(u_h, v_h - \bar{u}_h) + \bar{j}(v_h) - \bar{j}(\bar{u}_h) &\geq \langle F, v_h \rangle \qquad \forall v_h \in K_{hH}. \end{aligned}$$

We set $v_h = \bar{u}_h$ in the first inequality, and $v_h = u_h$ in the second. Adding up the two inequalities yields

$$-a(u_h - \bar{u}_h, u_h - \bar{u}_h) + j(\bar{u}_h) - j(u_h) + \bar{j}(u_h) - \bar{j}(\bar{u}_h) \geq 0.$$

Using the coercivity of a, we obtain

$$\begin{aligned} \alpha_1 \|u_h - \bar{u}_h\|_V^2 &\leq \int_{\Gamma_\pi} \mathcal{F} g_n (|\bar{u}_{hn}^R| - |u_{hn}^R|) dX_1 + \int_{\Gamma_\pi} \mathcal{F} \bar{g}_n (|u_{hn}^R| - |\bar{u}_{hn}^R|) dX_1 \\ &= \int_{\Gamma_\pi} \mathcal{F} (g_n - \bar{g}_n)(|\bar{u}_{hn}^R| - |u_{hn}^R|) dX_1 \\ &\leq \|\mathcal{F}\|_{L^\infty} \|g_n - \bar{g}_n\|_{Q_1} \|u_{hn}^R - \bar{u}_{hn}^R\|_{1/2,\Gamma_C} \\ &\leq c_\gamma \|\mathcal{F}\|_{L^\infty} \|g_n - \bar{g}_n\|_{Q_1} \|u_h - \bar{u}_h\|_V, \end{aligned}$$

where $c_\gamma > 0$ is the bound of the trace operator. Therefore, $\Psi : \Lambda_{1H} \to V_{gh}$ is continuous:

$$\|u_h - \bar{u}_h\|_V \leq \frac{\|\mathcal{F}\|_{L^\infty} c_\gamma}{\alpha_1} \|g_n - \bar{g}_n\|_{Q_1}. \tag{3.8}$$

For Φ_H, there holds the following:

Theorem 7 *The mapping $\Phi_H : \Lambda_{1H} \to \Lambda_{1H}, g_n \mapsto \lambda_{1H}$ is continuous.*

Proof. For any $g_n, \bar{g}_n \in \Lambda_{1H}$, let (u_h, λ_H) and $(\bar{u}_h, \bar{\lambda}_H)$ be the respective solutions. Then for any $v_h \in V_{0h}$, there holds

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_H) &= \langle F, v_h \rangle, \\ a(\bar{u}_h, v_h) + b(v_h, \bar{\lambda}_H) &= \langle F, v_h \rangle. \end{aligned}$$

Subtracting the second equation from the first and using the continuity of a, we get

$$b(v_h, \lambda_H - \bar{\lambda}_H) \leq \alpha_2 \|u_h - \bar{u}_h\|_V \|v_h\|_V.$$

Together with the inf-sup condition for b and the continuity of Ψ, we have

$$\begin{aligned}\beta_{hH}\|\lambda_{1H}-\bar{\lambda}_{1H}\|_Q &\leq \sup_{v_h\in V_{0h}}\frac{b(v,\lambda_H-\bar{\lambda}_H)}{\|v_h\|_V}\\ &\leq \alpha_2\|u_h-\bar{u}_h\|_V\\ &\leq \frac{\|\mathcal{F}\|_{L^\infty}c_\gamma\alpha_2}{\alpha_1}\|g_n-\bar{g}_n\|_{Q_1}.\end{aligned}$$

Thus the discrete operator $\Phi_H:\Lambda_{1H}\to\Lambda_{1H}$ is continuous. □

Let $B_\varepsilon(r)$ denote a ball in the $H^\varepsilon(\Gamma_\pi)$-topology with center in the origin and radius r. Following [7], we can show that Φ_H maps $B_{-1/2}(r)\cap\Lambda_{1H}$ onto itself, given that r is large enough. Furthermore, Φ_H is contractive for small coefficients $\mathcal{F}$.

Theorem 8 *Let $\|\mathcal{F}\|_{L^\infty}\leq\frac{\alpha_1\beta}{\alpha_2 c_\gamma}$, then for any radius r such that*

$$r\geq r_0=\frac{(\alpha_1+\alpha_2)\|F\|_{V^*}}{\alpha_1\beta-\alpha_2 c_\gamma\|\mathcal{F}\|_{L^\infty}},$$

there holds

$$\Phi_H(B_{-1/2}(r)\cap\Lambda_{1H})\subseteq B_{-1/2}(r)\cap\Lambda_{1H}.$$

Proof. Let $g_n\in B_{-1/2}(r)\cap\Lambda_{1H}$, and (u_h,λ_H) a solution to the discrete problem:

$$a(u_h,v_h-u_h)+j(v_h)-j(u_h)\geq\langle F,v_h-u_h\rangle\quad\forall v_h\in K_{hH}.$$

Substituting $v_h=0,2u_h\in K_{hH}$ yields

$$a(u_h,u_h)+j(u_h)=\langle F,u_h\rangle.$$

Therefore it holds

$$\alpha_1\|u_h\|_V^2\leq c_\gamma\|\mathcal{F}\|_{L^\infty}\|g_h\|_{Q_1}\|u_h\|_V+\|F\|_{V^*}\|u_h\|_V.$$

From the mixed formulation of the problem we have

$$a(u_h,v_h)+b(v_h,\lambda_H)=\langle F,v_h\rangle\quad\forall v_h\in V_{0h}.$$

Using this estimate, the inf-sup condition for b and continuity of a, we obtain

$$\begin{aligned}\beta\|\lambda_H\|_Q &\leq \sup_{v_h\in V_h}\frac{b(v_h,\lambda_H)}{\|v_h\|_V}\\ &= \sup_{v_h\in V_h}\frac{-a(u_h,v_h)+\langle F,v_h\rangle}{\|v_h\|_V}\\ &\leq \alpha_2\|u_h\|_V+\|F\|_{V^*}\\ &\leq \frac{\alpha_2 c_\gamma\|\mathcal{F}\|_{L^\infty}}{\alpha_1}\|g_n\|_{Q_1}+\left(1+\frac{\alpha_2}{\alpha_1}\right)\|F\|_{V^*}\end{aligned}$$

As the coefficient of friction satisfies $\|\mathcal{F}\|_{L^\infty} \leq \frac{\alpha_1\beta}{\alpha_2 c_\gamma}$, and $\|g_n\|_Q \leq r$, there holds

$$\|\lambda_{1H}\|_{Q_1} \leq \|\lambda_H\|_Q \leq \frac{1}{\alpha_1\beta}(\alpha_2 c_\gamma \|\mathcal{F}\|_{L^\infty} r + (\alpha_1+\alpha_2)\|F\|_{V^*}) < r$$

after a direct calculation. Therefore, we have shown that $\lambda_{1H} \in B_{-1/2}(r)$. $\square$

Theorem 9 *Given the coefficient of friction $\mathcal{F}$ is small enough, the mapping $\Phi_H : \Lambda_{1H} \to \Lambda_{1H}$ is a contraction.*

Proof. Let $g_n, \bar{g}_n \in \Lambda_{1H}$, and $\lambda_H = \Phi_H(g_n), \bar{\lambda}_H = \Phi_H(\bar{g}_n)$. Analogously to the proof of the continuity of Ψ_H, we obtain

$$-a(u_h - \bar{u}_h, u_h - \bar{u}_h) + j(\bar{u}_h) - j(u_h) + \bar{j}(u_h) - \bar{j}(\bar{u}_h) \geq 0.$$

As $g_n, \bar{g}_n \in L^2(\Gamma_\pi)$, we may estimate

$$\begin{aligned}
\alpha_1\|u_h - \bar{u}_h\|_V^2 &\leq \int_{\Gamma_\pi} \mathcal{F}(g_n - \bar{g}_n)(|\bar{u}_{hn}^R| - |u_{hn}^R|)dX_1 \\
&\leq \|\mathcal{F}\|_{L^\infty}\|g_n - \bar{g}_n\|_0 \|u_{hn}^R - \bar{u}_{hn}^R\|_0 \\
&\leq c_\gamma \|\mathcal{F}\|_{L^\infty}\|g_n - \bar{g}_n\|_0 \|u_h - \bar{u}_h\|_V.
\end{aligned}$$

Therefore, we have

$$\|u_h - \bar{u}_h\|_V \leq \frac{c_\gamma\|\mathcal{F}\|_{L^\infty}}{\alpha_1}\|g_n - \bar{g}_n\|_0.$$

Proceeding as before, we get

$$\|\lambda_{1H} - \bar{\lambda}_{1H}\|_Q \leq \frac{\|\mathcal{F}\|_{L^\infty} c_\gamma \alpha_2}{\alpha_1\beta}\|g_n - \bar{g}_n\|_0.$$

Using the inverse inequality

$$\|g_n - \bar{g}_n\|_0 \leq c_i H^{-1/2}\|g_n - \bar{g}_n\|_{Q_1}$$

we obtain

$$\|\lambda_{1H} - \bar{\lambda}_{1H}\|_Q \leq \frac{c_i c_\gamma \alpha_2 \|\mathcal{F}\|_{L^\infty}}{\alpha_1\beta} H^{-1/2}\|g_n - \bar{g}_n\|_{Q_1}.$$

If $\|\mathcal{F}\|_{L^\infty}$ is sufficiently small, we have $H^{-1/2}c_i c_\gamma \alpha_2\|\mathcal{F}\|_{L^\infty}/(\alpha_1\beta) < 1$, and therefore Φ_H is contractive. The bound for $\mathcal{F}$ depends on H, so we do not get contractivity for $H \to 0$. $\square$

From these results and the Schauder fixed point theorem we see that, given the coefficient of friction $\mathcal{F}$ satisfies the assumption in Theorem 8, there exists at least one fixed point of Φ_H. If $\mathcal{F}$ even satisfies $\|\mathcal{F}\|_{L^\infty} < H^{1/2}\alpha_1\beta/(c_i c_\gamma \alpha_2)$, the mapping Φ_H is contractive, and the Banach fixed point theorem gives existence and uniqueness of a solution (u_h, λ_H). This solution can be computed by the method of successive approximations. As the bound for $\|\mathcal{F}\|_{L^\infty}$ depends on H, we cannot assume uniqueness of (u_h, λ_H) for $H \to 0$.

Next we shall study the behavior of a sequence of solutions (u_h, λ_H) to the contact problem with Coulomb friction if h, H tend to zero. In the previous section we have seen that, if the ratio h/H is sufficiently small, a solution to the discrete problem with given friction g_n converges strongly to the solution of the continuous problem, as $h, H \to 0$. For the Coulomb problem, we will guarantee the existence of a strongly convergent subsequence of solutions. To prove this, we need the following lemma:

Lemma 10 *Let b satisfy the inf-sup condition (3.4) with a constant β independent of h, H. Then for $\mathcal{F}$ sufficiently small, there exists some $r_0 > 0$ such that*

$$\Phi_H(B_{-1/2+\varepsilon_0}(r_0) \cap \Lambda_{1H}) \subseteq B_{-1/2+\varepsilon_0}(r_0) \cap \Lambda_{1H}.$$

Moreover, the sequences $(\|u_h\|_{1+\varepsilon_0}), (\|\lambda_H\|_{-1/2+\varepsilon_0}), h, H \to 0$ remain bounded if $(\|g_n\|_{-1/2+\varepsilon_0})$ is bounded.

Proof. The proof was done in [7]. □

Using this lemma, we can prove existence of a fixed point of Φ_H in $B_{-1/2+\varepsilon_0}(r_0) \cap \Lambda_{1H}$ by the Schauder fixed point theorem. We will denote such a fixed point by λ_{1H}. The pair (u_h, λ_H) shall then be a solution to the discrete contact problem with Coulomb friction. There holds the following theorem:

Theorem 11 *Let all assumptions of Lemma 10 be satisfied. Then there exists a subsequence (u_{h_n}, λ_{h_n}) of (u_h, λ_H) such that*

$$\begin{array}{l} u_{h_n} \to u \text{ in } V \\ \lambda_{h_n} \to \lambda \text{ in } Q \end{array} \quad \text{as} \quad h_n, H_n \to \infty,$$

where (u, λ) is a solution to the contact problem with Coulomb friction.

Proof. From Lemma 10, we have that (u_h) and (λ_H) are bounded in $H^{1+\varepsilon_0}(\Omega)$ and $H^{-1/2+\varepsilon_0}(\Gamma_\pi)$, respectively. From the compactness of the embeddings $H^{1+\varepsilon_0}(\Omega)$ into V, and $H^{-1/2+\varepsilon_0}(\Gamma_\pi)$ into Q, we obtain existence of a subsequence (u_{h_n}, λ_{h_n}) such that

$$\begin{aligned} u_{h_n} &\to u && \text{in } V \\ \lambda_{h_n} &\to \lambda && \text{in } Q, \end{aligned}$$

where (u, λ) lies in $V_g \times \Lambda$.

It remains to show that (u, λ) is a solution to the Coulomb problem: The pair (u_{h_n}, λ_{h_n}) satisfies the mixed variational formulation

$$\begin{array}{lcll} a(u_{h_n}, v_{h_n}) + b(v_{h_n}, \lambda_{H_n}) & = & \langle F, v_{h_n} \rangle & \forall v_{h_n} \in V_{0h_n} \\ b(u_{h_n}, \mu_{H_n} - \lambda_{H_n}) & \leq & \langle G, \mu_{H_n} - \lambda_{H_n} \rangle_Q & \forall \mu_{H_n} \in \Lambda_{H_n}(\lambda_{H_n 1}). \end{array}$$

For each $v \in V_0$ there exists a sequence $v_{h_n} \in V_{0h_n}$ such that $v_{h_n} \to v$ in V. The same property holds true for Q and Q_{H_n}. We still need to show that for $\mu \in \Lambda(\lambda_1)$ there exists a sequence $\mu_{H_n} \in \Lambda_{H_n}(\lambda_{H_n 1})$ such that $\mu_{H_n} \to \mu \in Q$. For $\mu_1 \in \Lambda_1$, this holds true as $\Lambda_1 = \bigcup_{H_n} \Lambda_{H_n 1}$. Take $\mu_2 \in \Lambda_2(\lambda_1)$, then we have

$$\langle |\mu_2|, w \rangle_{Q_2} \leq \langle \mathcal{F} \lambda_1, w \rangle_{Q_2} \quad \forall w \geq 0, w \in H^{1/2}(\Gamma_\pi).$$

As $\lambda_{H_n 1} \to \lambda_1$, there exists a sequence $\tilde{\mu}_{H_n 2}$ in Q, $\tilde{\mu}_{H_n 2} \to \mu_2$ such that

$$\langle |\tilde{\mu}_{H_n 2}|, w \rangle \leq \langle \mathcal{F} \lambda_{H_n 1}, w \rangle \quad \forall w \geq 0, w \in H^{1/2}(\Gamma_\pi).$$

For $\tilde{\mu}_{H_n 2} \in \Lambda_2(\lambda_{H_n 1})$, there exists $\mu_{H_n 2} \in \Lambda_{H_n 2}(\lambda_{H_n 1})$ such that

$$\|\tilde{\mu}_{H_n 2} - \mu_{H_n 2}\|_Q \leq \frac{1}{n}.$$

Thus, we found a sequence $\mu_{H_n} = (\mu_{H_n 1}, \mu_{H_n 2}) \in \Lambda_{H_n}(\lambda_{H_n 1})$ satisfying $\mu_{H_n} \to \mu$. Therefore, and due to the continuity of a, b, F and G, we can pass to the limit in the discrete variational inequality of the second kind, and obtain

$$\begin{array}{lcll} a(u, v) + b(v, \lambda) & = & \langle F, v \rangle & \forall v \in V_0 \\ b(u, \mu - \lambda) & \leq & \langle G, \mu - \lambda \rangle_Q & \forall \mu \in \Lambda(\lambda_1). \end{array}$$

This proves that (u, λ) is a solution to the continuous contact problem with Coulomb friction. $\square$

Remark 12 In the proof of Theorem 11 we have found a convergent subsequence (u_{h_n}, λ_{h_n}) of (u_h, λ_H). For its limit (u, λ) we have seen that it solves the continuous problem. Therefore we have finally shown that there exists a weak solution to the contact problem with Coulomb friction in the infinte dimensional setting.

Chapter 4

Algorithmic Realization

In this chapter we present two algorithms to solve the discrete contact problem obeying Coulomb's law of friction. As we have seen before, the pair $(u_h, \lambda_H) \in V_{gh} \times \Lambda_H$ is a solution if and only if λ_{1H} is a fixed point to the mapping Φ_H, which maps some given slip stress on the contact boundary to the contact stress arising in normal direction. For small coefficients of friction $\mathcal{F}$ this mapping is contractive. We can therefore find the fixed point λ_{1H} by the method of successive approximations.

To do so, we need to compute $\Phi_H(g_n)$ for some given $g_n \geq 0, g_n \in Q_{1H}$. Therefore, we have to solve the problem of finding $u_h \in V_{gh}, \lambda_H \in \Lambda_H$ such that

$$\begin{array}{lcll} a(u_h, v_h) + b(v_h, \mu_H) & = & \langle F, v_h \rangle & \forall v_h \in V_{0h} \\ b(u_h, \mu_H - \lambda_H) & \leq & \langle G, \mu_H - \lambda_H \rangle_Q & \forall \mu_H \in \Lambda_H. \end{array} \tag{4.1}$$

Then we can set $\Phi_H(g_n) = \lambda_{1H}$.

To this end, we introduce some more notation: Let $\{\phi_i, i \leq n_h\}$ be a basis for V_h. We choose $\{\phi_i\}$ to be the nodal finite element basis functions on the triangulation $\mathcal{T}_h$. Then $v_h \in V_h$ can be uniquely represented as a vector $\underline{v} = (v_i), v_i \in \mathbb{R}^2$, such that

$$v_h = \sum_{i \leq n_h} v_i \phi_i.$$

Analogously, let $\{\psi_k, k \leq m_H\}$ denote a basis for Q_H. In our case, we choose ψ_k as the characteristic function of the element I_k in $\mathcal{T}_H$. Then $\mu_H \in Q_H$ has a vector representation $\underline{\mu} = (\mu_k), \mu_k \in \mathbb{R}^2$ such that

$$\mu_H = \sum_{k \leq m_H} \mu_k \psi_k.$$

Note that μ_H lies in Λ_H if and only if the respective vector components satisfy

$$\mu_{1k} \geq 0, \quad |\mu_{2k}| \leq \langle \mathcal{F} g_n, \psi_k \rangle_{Q_1} \quad \forall k \leq m_H.$$

The union of all those vectors $\underline{\mu}$ defines a convex set $\underline{\Lambda} \subset \mathbb{R}^{2m_H}$,

$$\underline{\Lambda} = \{\underline{\mu} : \mu_{1k} \geq 0,\ |\mu_{2k}| \leq \langle \mathcal{F}g_n, \psi_k \rangle_{Q_1} \quad \forall k \leq m_H\}.$$

By $\underline{f}, \underline{g}$ and $\underline{l}$, we denote the vectors induced by the linear functionals F and G, and the bound $\mathcal{F}g_n$ on $|\lambda_2|$, respectively:

$$\begin{aligned} \underline{f} &= (f_i), \qquad f_i = \langle F, \phi_i \rangle \\ \underline{g} &= (g_k), \qquad g_k = \langle G, \psi_k \rangle_Q \\ \underline{l} &= (l_k), \qquad l_k = \langle \mathcal{F}g_n, \psi_k \rangle_Q. \end{aligned}$$

From the bilinear forms a and b we can define matrices A, B via

$$\begin{aligned} A &= (A_{ij}), \qquad A_{ij} = a(\phi_j, \phi_i) \\ B &= (B_{ki}), \qquad B_{ki} = b(\phi_i, \psi_k). \end{aligned}$$

As the bilinear form a is symmetric and coercive, A is symmetric and positive definite. For B, the inf-sup condition in the discrete setting ensures that $rank\ B = m_H$.

Then the discrete mixed variational formulation can be rewritten as a system of equations and inequalities for $\underline{u} \in \mathbb{R}^{2n_h}, \underline{\lambda} \in \mathbb{R}^{2m_H}, \underline{\lambda} \in \underline{\Lambda}$:

$$\begin{array}{lcll} A\underline{u} + B^T\underline{\lambda} & = & \underline{f} & \\ (\underline{\mu} - \underline{\lambda})^T Bu & \leq & (\underline{\mu} - \underline{\lambda})^T \underline{g} & \quad \forall \underline{\mu} \in \underline{\Lambda}, \end{array} \tag{4.2}$$

where

$$\underline{\Lambda} = \{\underline{\mu} : \mu_{1k} \geq 0,\ |\mu_{2k}| \leq l_k \quad \forall k \leq m_H\}.$$

From the first equation, we immediately see that

$$u = -A^{-1}(B^T\underline{\lambda} - \underline{f}).$$

Inserting this into the inequality of our system, we get

$$-(\underline{\mu} - \underline{\lambda})^T(BA^{-1}B^T\underline{\lambda}) \leq (\underline{\mu} - \underline{\lambda})^T(\underline{g} - A^{-1}\underline{f}) \qquad \forall \underline{\mu} \in \underline{\Lambda}.$$

This leaves us with a problem for $\underline{\lambda}$ only. Setting the Schur complement $S = BA^{-1}B^T$ and right hand side $\underline{h} = A^{-1}\underline{f} - \underline{g}$, we obtain

$$(\underline{\mu} - \underline{\lambda})^T(S\underline{\lambda} - \underline{h}) \geq 0 \qquad \forall \underline{\mu} \in \underline{\Lambda}. \tag{4.3}$$

The Schur complement S is symmetric, as A is symmetric. We also have that A is positive definite, and $rank\ B = m_H$. Therefore, also S is positive definite, and there exist positive numbers s_1, s_2 such that

$$\begin{array}{lcl} (S\underline{\mu}, \underline{\mu}) & \geq & s_1\|\underline{\mu}\|^2 \\ (S\underline{\mu}, \underline{\lambda}) & \leq & s_2\|\underline{\mu}\|\|\underline{\lambda}\| \end{array}$$

where $\|.\|$ and $(.,.)$ denote norm and scalar product in $\mathbb{R}^{2m_H}$

Due to this, (4.3) is equivalent to the minimization problem

$$\begin{aligned} \mathcal{S}(\underline{\lambda}) &= \min_{\underline{\mu}\in\underline{\Lambda}} \mathcal{S}(\underline{\mu}) \\ \mathcal{S}(\underline{\mu}) &= \frac{1}{2}\underline{\mu}^T S \underline{\mu} - \underline{h}^T \underline{\mu}. \end{aligned}$$

By r, we denote the gradient of the energy $\mathcal{S}$

$$r(\mu) := \nabla \mathcal{S}(\mu) = S\mu - h.$$

Note that, in general, the number m_H of contact elements is much smaller than the number n_h of nodes in the triangulation of Ω. Therefore, the system (4.3) is of much smaller dimension than the original system (4.2). In the following, we shall present an algorithm for the minimization of $\mathcal{S}$ over $\underline{\Lambda}$. Once we have computed $\underline{\lambda}$ by such a method, we can find the deformation $\underline{u}$ via

$$\underline{u} = -A^{-1}(B^T \underline{\lambda} - \underline{f}).$$

To simplify notation, we will omit all underlines and denote the vector induced by some $\lambda_h \in \Lambda_H$ by λ only. Throughout this chapter, the norm $\|.\|$ and scalar product $(.,.)$ are the Euclidean norm and scalar product in $\mathbb{R}^{2m_H}$.

In the following, we will need the projection operator P_Λ , which acts from $\mathbb{R}^{2m_H}$ onto Λ. It is given by

$$\begin{aligned} P_\Lambda &= (P_{1\Lambda}, P_{2\Lambda}) \\ P_{1\Lambda}(\mu)_k &= \max(\mu_k, 0) \\ P_{2\Lambda}(\mu)_k &= \begin{cases} -l_k & \text{if } \mu_k < -l_k \\ l_k & \text{if } \mu_k > l_k \\ \mu_k & \text{else} \end{cases}. \end{aligned}$$

4.1 Uzawa's method

Uzawa's method is a simple, but effective iterative method for solving mixed systems like (4.2). Assume we are given some initial guess $\lambda^{(0)} \in \Lambda$. Then, in each iterative step, we construct $u^{(k)}$, $\lambda^{(k+1)}$ from $\lambda^{(k)}$ by the following procedure, called Uzawa's method.

Algorithm 13 *(Uzawa's method)*

1. *Compute $u^{(k)}$ such that it satisfies the equilibrium equation*

$$Au^{(k)} + B^T \lambda^{(k)} = f.$$

2. *Starting from* $\lambda^{(k)}$, *find* $\lambda_*^{(k+1)}$ *in direction of the gradient of the Lagrangian* $\mathcal{L}$ *with respect to* λ

$$\lambda_*^{(k+1)} = \lambda^{(k)} + \rho(Bu^{(k)} - g),$$

where ρ *is some given, small positive constant.*

3. *Take* $\lambda^{(k+1)}$ *as the projection of* $\lambda_*^{(k+1)}$ *onto* Λ

$$\lambda^{(k+1)} = P_\Lambda(\lambda_*^{(k+1)}).$$

We repeat the above procedure as long as $\max\{\|u^{(k+1)} - u^{(k)}\|, \|\lambda^{(k+1)} - \lambda^{(k)}\|\} > \varepsilon$.

For an appropriate choice of ρ, one can show that the sequence $(u^{(k)}, \lambda^{(k)})$ converges to the solution (u, λ) of the reduced contact problem (4.2). Using the Schur complement S and right hand side h we introduced before, we can eliminate u from the algorithm. This means, we compute $\lambda^{(k+1)}$ directly from $\lambda^{(k)}$, without finding $u^{(k)}$. From the first step of the k^{th} iteration, we obtain

$$u^{(k)} = -A^{-1}(B^T\lambda^{(k)} - f).$$

Inserting this into the formula for $\lambda_*^{(k+1)}$, we get

$$\begin{aligned} \lambda_*^{(k+1)} &= \lambda^{(k)} + \rho(-BA^{-1}B^T\lambda^{(k)} + A^{-1}f - g) \\ &= \lambda^{(k)} - \rho(S\lambda^{(k)} - h). \end{aligned}$$

Therefore, we can reformulate the algorithm. In each iteration, we compute $\lambda^{(k+1)}$ from $\lambda^{(k)}$ in two steps.

1. Given $\lambda^{(k)}$, define ${\lambda^{(k+1)}}_*$ by

$$\lambda_*^{(k+1)} = \lambda^{(k)} - \rho(S\lambda^{(k)} - h),$$

where ρ is some given, small positive constant.

2. Take $\lambda^{(k+1)}$ as the projection of $\lambda_*^{(k+1)}$ onto Λ

$$\lambda^{(k+1)} = P_\Lambda(\lambda_*^{(k+1)}).$$

Stop the iteration procedure once $\|\lambda^{(k+1)} - \lambda^{(k)}\| < \varepsilon$. Then $u^{(k+1)}$ can be computed via

$$u^{(k+1)} = -A^{-1}(B^T\lambda^{(k+1)} - f).$$

From this reformulation of the algorithm, we can motivate the choice of $\lambda_*^{(k+1)}$ from $\lambda^{(k)}$. We observe that

$$\nabla\mathcal{S}(\lambda) = S^T\lambda - h.$$

This implies that we look for $\lambda_*^{(k+1)}$ in direction of steepest descent from $\lambda^{(k)}$.

To show convergence of the algorithm, we will first present an estimate on the gradient projection. This theorem was proven in [2] in a slightly different setting, we do only minor adaptions to our problem.

Theorem 14 *Let $\lambda \in \Lambda$ be the unique solution of (4.3), and $\mu \in \Lambda$. Moreover, let s_1, s_2 denote the smallest and largest eigenvalues of S, respectively. Then for $\rho \in (0, s_2^{-1})$ there holds*

$$\mathcal{S}(P_\Lambda(\mu - \rho r(\mu))) - \mathcal{S}(\lambda) \leq (1 - \rho s_1)(\mathcal{S}(\mu) - \mathcal{S}(\lambda)).$$

Proof. For some fixed $\mu \in \Lambda$, we define a quadratic function by

$$\bar{\mathcal{S}}(y) = \rho\mathcal{S}(y) + \frac{1}{2}(y-\mu)^T(I - \rho S)(y - \mu) \qquad \forall y \in \mathbb{R}^{2m_H}. \tag{4.4}$$

For $y \in \mathbb{R}^{2m_H}$ there holds

$$\begin{aligned} \bar{\mathcal{S}}(y) &\geq \rho\mathcal{S}(y) \\ \nabla\bar{\mathcal{S}}(y) &= \rho\nabla\mathcal{S}(y) + (I - \rho S)(y - \mu) = y - \mu + \rho r(\mu) \\ \nabla^2\bar{\mathcal{S}}(y) &= I, \end{aligned}$$

whereas for the special choice $y = \mu$ we obtain

$$\begin{aligned} \bar{\mathcal{S}}(\mu) &= \rho\mathcal{S}(\mu) \\ \nabla\bar{\mathcal{S}}(\mu) &= \rho\nabla\mathcal{S}(\mu) = \rho r(\mu). \end{aligned}$$

Furthermore we define the reduced gradient $\tilde{r}$ such that

$$\mu - \rho\tilde{r}(\mu) = P_\Lambda(\mu - \rho r(\mu)).$$

This is the part of the gradient r we actually use in one step of the Uzawa method. It is given explicitely by

$$\begin{aligned} \tilde{r}_{1i} &= \min(\rho^{-1}\mu_{1i}, r_{1i}) \\ \tilde{r}_{2i} &= \min(\rho^{-1}(\mu_{2i} + l_i), \max(\rho^{-1}(\mu_{2i} - l_i), r_{2i})). \end{aligned}$$

For any $y \in \Lambda$ there exists some d such that $y = \mu - \rho\tilde{r}(\mu) + d$. For $i \leq m_H$ there holds $d_{1i} \geq \rho\tilde{r}_{1i} - \mu_{1i}, d_{2i} \in [\rho\tilde{r}_{2i} - \mu_{2i} - l_i, \rho\tilde{r}_{2i} - \mu_{2i} + l_i]$. One can easily check that there always holds

$$d^T(r - \tilde{r}) \geq 0.$$

Therefore, we can estimate

$$\begin{aligned} \bar{\mathcal{S}}(y) &= \bar{\mathcal{S}}(P_\Lambda(\mu - \rho r(\mu)) + d) \\ &= \bar{\mathcal{S}}(P_\Lambda(\mu - \rho r(\mu))) + \nabla\bar{\mathcal{S}}(\mu - \rho\tilde{r}(\mu))^T d + \frac{1}{2}\|d\|^2 \\ &= \bar{\mathcal{S}}(P_\Lambda(\mu - \rho r(\mu))) + \rho(r(\mu) - \tilde{r}(\mu))^T d + \frac{1}{2}\|d\|^2 \\ &\geq \bar{\mathcal{S}}(P_\Lambda(\mu - \rho r(\mu))) \end{aligned}$$

This implies that $P_\Lambda(\mu-\rho r(\mu))$ is a global minimizer for $\bar{\mathcal{S}}$. Next, we set $d=\lambda-\mu$. As Λ is convex, there holds $\mu+td\in\Lambda$ for $t\in[0,1]$. Using our results from above, we can obtain the assertion of the theorem.

$$\begin{aligned}
\mathcal{S}(P_\Lambda(\mu-\rho r(\mu)))-\mathcal{S}(\lambda) &\leq \rho^{-1}\bar{\mathcal{S}}(P_\Lambda(\mu-\rho r(\mu)))-\mathcal{S}(\lambda)\\
&= \min_{y\in\Lambda}(\rho^{-1}\bar{\mathcal{S}}(y)-\mathcal{S}(\lambda))\\
&\leq \min_{t\in[0,1]}(\rho^{-1}\bar{\mathcal{S}}(\mu+td)-\mathcal{S}(\mu+d))\\
&= \min_{t\in[0,1]}(\rho^{-1}\bar{\mathcal{S}}(\mu)+td^Tr(\mu)+\frac{t^2}{2\rho}\|d\|^2\\
&\quad -\mathcal{S}(\mu)-d^Tr(\mu)-\frac{1}{2}d^TSd)\\
&\leq \rho s_2 d^Tr(\mu)+\frac{(\rho s_2)^2}{2\rho}\|d\|^2-d^Tr(\mu)-\frac{1}{2}d^TSd\\
&\leq \rho s_2 d^Tr(\mu)+\frac{\rho s_2}{2}d^TSd-d^Tr(\mu)-\frac{1}{2}d^TSd\\
&= (\rho s_2-1)(d^Tr(\mu)-\frac{1}{2}d^TSd)\\
&= (1-\rho s_2)(\mathcal{S}(\mu)-\mathcal{S}(\lambda))
\end{aligned}$$

□

From this theorem, we easily get convergence for the iterates $(u^{(k)},\lambda^{(k)})$ to the solution (u,λ) of (4.2).

Theorem 15 *Let A,B be defined as above, and (u,λ) be a solution to the system (4.2). Let $(u^{(k)},\lambda^{(k)})$ be defined by Uzawa's method. Then there exists a positive constant $\bar{\rho}>0$ such that for each choice $\rho\in(0,\bar{\rho})$ there holds*

$$u^{(k)}\to u,\quad \lambda^{(k)}\to\lambda.$$

Proof. Let again denote $S=BA^{-1}B^T$ the Schur complement, and s_1,s_2 its smallest and largest eigenvalue. As we have seen before, the iterate $\lambda^{(k+1)}$ can be found as

$$\lambda^{(k+1)}=P(\lambda^{(k)}+\rho r(\lambda^{(k)}))$$

with $\rho<s_2^{-1}=:\bar{\rho}$. As λ is a solution only if

$$(S\lambda-h,\mu-\lambda)\geq 0\qquad\forall\mu\in\Lambda,$$

we have for the special setting $\mu=\lambda^{(k+1)}\in\Lambda$

$$(S\lambda-h,\lambda^{(k+1)}-\lambda)\geq 0.$$

Therefore we get

$$\begin{aligned} s_1\|\lambda^{(k+1)} - \lambda\|^2 &\leq (S(\lambda^{(k+1)} - \lambda), \lambda^{(k+1)} - \lambda) \\ &= 2(\mathcal{S}(\lambda^{(k+1)}) - \mathcal{S}(\lambda) - (S\lambda - h, \lambda^{(k+1)} - \lambda)) \\ &\leq 2(\mathcal{S}(\lambda^{(k+1)}) - \mathcal{S}(\lambda)) \\ &= 2(\mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r(\lambda^{(k)}))) - \mathcal{S}(\lambda)). \end{aligned}$$

Using Theorem 14, we obtain

$$\begin{aligned} s_1\|\lambda^{(k+1)} - \lambda\|^2 &\leq 2(1 - \rho s_2)(\mathcal{S}(\lambda^{(k)}) - \mathcal{S}(\lambda)) \\ &\leq 2(1 - \rho s_2)^{k+1}(\mathcal{S}(\lambda^{(0)}) - \mathcal{S}(\lambda)) \end{aligned}$$

This implies convergence of $\lambda^{(k)} \to \lambda$. As $u^{(k)}$ depends continuously on $\lambda^{(k)}$, we can conclude $u^{(k)} \to u$. □

A more detailed discussion of Uzawa's method can be found in [3].

4.2 Proportioning with reduced gradient projections

We will now study another method for solving the discretized contact problem with given friction. The algorithm we consider was proposed in [2] in a slightly different setting, we have to do small adaptions to fit it to our problem. The method is active-set based. In each iterative step, we will try to minimize the energy $\mathcal{S}$ on the face of the admissible set Λ given by the active constraints. To do so, we use the conjugate gradient method. Once we produce an iterate $\lambda^{(k)}$ which does not satisfy $\lambda^{(k)} \in \Lambda$, we expand the active set by reduced gradient projection with fixed step size. Following [2], we will see that the algorithm terminates in finitely many steps, and give a rate of convergence.

4.2.1 Algorithm

Throughout the following, let $\mathcal{A}_1$ denote the sets of all active indices of $\lambda_1 \in \Lambda_1$,

$$\mathcal{A}_1(\lambda) = \{k \leq m_H : \lambda_{1k} = 0\}.$$

We call $\mathcal{A}_1$ the active set of λ_1. Its complement

$$\mathcal{F}_1(\lambda) = \{k \leq m_H : \lambda_{1k} > 0\}$$

is called the free set of λ_1.

For λ_2, we can split the active set $\mathcal{A}_2$ into two subsets

$$\begin{aligned}\mathcal{A}_2 &= \mathcal{A}_2^+ \cup \mathcal{A}_2^- \\ \mathcal{A}_2^+ &= \{k \leq m_H : \lambda_{2k} = l_k\} \\ \mathcal{A}_2^- &= \{k \leq m_H : -\lambda_{2k} = l_k\}.\end{aligned}$$

Again, $\mathcal{F}_2$ denotes the free set of λ_2

$$\mathcal{F}_2(\lambda) = \{k \leq m_H : |\lambda_{1k}| < l_k\}.$$

As we have seen before, the gradient r of the energy $\mathcal{S}$ is given by

$$r(\lambda) = \nabla\mathcal{S}(\lambda) = S\lambda - h.$$

For $\lambda \in \Lambda$, we introduce the free gradient φ which is defined by

$$\varphi_{ik}(\lambda) = \begin{cases} r_{ik}(\lambda) & k \in \mathcal{F}_i(\lambda) \\ 0 & k \in \mathcal{A}_i(\lambda) \end{cases}.$$

The chopped gradient β is given by

$$\begin{aligned}\beta_{1k}(\lambda) &= \begin{cases} 0 & k \in \mathcal{F}_1(\lambda) \\ r_{1k}^-(\lambda) & k \in \mathcal{A}_1(\lambda) \end{cases} \\ \beta_{2k}(\lambda) &= \begin{cases} 0 & k \in \mathcal{F}_2(\lambda) \\ r_{2k}^-(\lambda) & k \in \mathcal{A}_2^- \\ r_{2k}^+(\lambda) & k \in \mathcal{A}_2^+ \end{cases}.\end{aligned}$$

Note that free and chopped gradient are orthogonal, $\beta^T\varphi = 0$. Then λ is a solution to the constrained minimization problem if and only if the projected gradient $\varphi(\lambda) + \beta(\lambda) = 0$.

In each iterative step of the algorithm, we first decide if we leave the face on which we are minimizing $\mathcal{S}$, or if we try a conjugate gradient step on this face. Therefore, we compare the chopped gradient β and the free gradient φ. If β dominates φ, we do a step in direction of $-\beta$ to leave the current face. Thereby some active constraints are released. If, on the other hand, φ dominates, we compute the next iterate by the conjugate gradient method using φ. Again, there are two possibilities: The iterate may be admissible, therefore we keep the current active set and start the next iteration. If the cg-iterate does not lie in Λ, we do a projected gradient step in direction of $-\varphi$. Then some indices are added to the active sets $\mathcal{A}_i$.

Starting from some given initial value $\lambda^{(0)}$, we compute the iterates $\lambda^{(k)}$ according to the following procedure. Assume that we have already found $\lambda^{(k)}$. Then, to compute $\lambda^{(k+1)}$, we do the following: First we decide whether to leave the current face or try a cg step. Therefore we consider the reduced free and chopped gradients $\tilde{\varphi}$ and $\tilde{\beta}$. For some fixed step size ρ, they are defined such that

$$\begin{aligned}\mu - \rho\tilde{\varphi}(\mu) &= P_\Lambda(\mu - \rho\varphi(\mu))\\ \mu - \rho\tilde{\beta}(\mu) &= P_\Lambda(\mu - \rho\beta(\mu)).\end{aligned}$$

They can be defined explicitely as

$$\begin{aligned}\tilde{\varphi}_{1k}(\mu) &= \min(\mu_{1k}/\rho, \varphi_{1k})\\ \tilde{\varphi}_{2k}(\mu) &= \min((\mu_{2k} - l_k)/\rho, \max((\mu_{2k} + l_k)/\rho, \varphi_{2k}))\\ \tilde{\beta}_{1k}(\mu) &= \min(\mu_{1k}/\rho, \beta_{1k})\\ \tilde{\beta}_{2k}(\mu) &= \min((\mu_{2k} - l_k)/\rho, \max((\mu_{2k} + l_k)/\rho, \beta_{2k}))\end{aligned}$$

If there holds

$$(\beta(\lambda^{(k)}), \tilde{\beta}(\lambda^{(k)})) \leq \Gamma^2(\varphi(\lambda^{(k)}), \tilde{\varphi}(\lambda^{(k)}))$$

we call $\lambda^{(k)}$ strictly proportional. This means that the free component φ of the gradient r shall be reduced. Therefore, we try a conjugate gradient step:

$$\lambda^{(k+1)} = \lambda^{(k)} - \rho_{cg} p^{(k)}$$

where $p^{(k)}$ is the conjugate gradient direction

$$p^{(k)} = \varphi(\lambda^{(k)}) - \gamma p^{(k-1)}, \qquad \gamma = \frac{\varphi(\lambda^{(k-1)})^T S p^{(k-1)}}{{p^{(k-1)}}^T S p^{(k-1)}}$$

which is constructed recurrently. The step size ρ_{cg} is chosen such that the energy $\mathcal{S}$ is minimized on the line $\lambda^{(k)} + \rho p^{(k)}$. It can be computed by

$$\rho_{cg} = \frac{{p^{(k)}}^T r(\lambda^{(k)})}{{p^{(k)}}^T S p^{(k)}}.$$

Each direction $p^{(k)}$ generated by the conjugate gradient method is S-orthogonal to its predecessor $p^{(k-1)}$,

$${p^{(k)}}^T S p^{(k-1)} = 0.$$

If the constructed iterate $\lambda^{(k+1)}$ lies in Λ, we may use it directly. Assuming that from some iterate $\lambda^{(s)}$ on, we only used conjugate gradient steps up to iteration k, and all generated $\lambda^{(i)}, s \leq i \leq k$ were in Λ, there even holds

$${p^{(i)}}^T S p^{(j)} = 0 \qquad \forall s \leq i, j \leq k, i \neq j.$$

Therefore, $\lambda^{(k)}$ minimizes $\mathcal{S}$ over the span of $p^{(s)}, \ldots, p^{(k)}$

$$\mathcal{S}(\lambda^{(k)}) = \min\{\mathcal{S}(\lambda^{(s)} + y) : y \in \operatorname{span}\{p^{(s)}, \ldots p^{(k-1)}\}\}.$$

This way we find a minimum on the face

$$\{\lambda : \lambda_{1i} = 0 \text{ for } i \in \mathcal{A}_1, |\lambda_{2i}| = l_i \text{ for } i \in \mathcal{A}_2\}.$$

As we only use the conjugate gradient method as long as the constructed iterate $\lambda^{(k)}$ lies in Λ, the active sets are not changed from iteration s up to k. Once we have found the active sets $\mathcal{A}_i(\lambda), i = 1, 2$ for the solution λ, the minimization of $\mathcal{S}$ on this face is carried out efficiently by the cg method.

On the other hand, it may happen that an iterate is not admissible, we have $\lambda^{(k)} \notin \Lambda$. This indicates that we need to add indices to the active sets. We do an expansion step, which is given by

$$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} + \rho\varphi(\lambda^{(k)})) = \lambda^{(k)} + \rho\tilde{\varphi}(\lambda^{(k-1)}).$$

The step size $\rho \in (0, s_2^{-1})$ is fixed. Doing this step, we may expand the active sets. We reset the direction $p^{(k+1)}$, in which we search for the next iterate, to $p^{(k+1)} = \varphi(\lambda^{(k+1)})$.

If $\lambda^{(k)}$ is not proportional, i. e.

$$(\beta(\lambda^{(k)}), \tilde{\beta}(\lambda^{(k)})) > \Gamma^2\varphi(\lambda^{(k)})^T\tilde{\varphi}(\lambda^{(k)}),$$

we do a proportioning step. This is used to remove indices from the active sets, and thereby reduce the chopped gradient $\beta(\lambda^{(k)})$. We set

$$\lambda^{(k+1)} = \lambda^{(k)} - \rho\tilde{\beta}(\lambda^{(k)}).$$

For the next iteration, we use the search direction

$$p^{(k+1)} = \varphi(\lambda^{(k+1)}).$$

We stop the iteration as soon as the projected gradient $\varphi(\lambda^{(k)}) + \beta(\lambda^{(k)})$ is small enough, and use $\lambda^{(k)}$ as an approximative solution for λ. Summarizing the motivations above, we can propose the following algorithm.

Algorithm 16 *Let $\lambda^{(0)} \in \Lambda, \rho \in (0, s_2^{-1}), \Gamma > 0$ and $\varepsilon > 0$ be given. For $k \geq 0$ and $\lambda^{(k)} \in \Lambda$ known, choose $\lambda^{(k+1)}$ such that*

- *if $\lambda^{(k)}$ is strictly proportional, $(\beta(\lambda^{(k)}), \tilde{\beta}(\lambda^{(k)})) \leq \Gamma^2\varphi(\lambda^{(k)})^T\tilde{\varphi}(\lambda^{(k)})$, try to generate $\lambda^{(k+1)}$ by a conjugate gradient step:*

  $$\lambda^{(k+1)} = \lambda^{(k)} - \rho_{cg}p^{(k)},\ \rho_{cg} = \frac{{p^{(k)}}^T r(\lambda^{(k)})}{{p^{(k)}}^T S p^{(k)}}$$

  where

  $$p^{(k)} = \varphi(\lambda^{(k)}) - \gamma p^{(k-1)}, \quad \gamma = \frac{\varphi(\lambda^{(k-1)})^T S p^{(k-1)}}{{p^{(k-1)}}^T S p^{(k-1)}}$$

  If $\lambda^{(k+1)} \in \Lambda$, accept it, else compute $\lambda^{(k+1)}$ by the expansion step:

  $$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} - \rho\varphi(\lambda^{(k)})).$$

- *if $\lambda^{(k)}$ is not proportional, $(\beta(\lambda^{(k)}), \tilde{\beta}(\lambda^{(k)})) > \Gamma^2 \varphi(\lambda^{(k)})^T \tilde{\varphi}(\lambda^{(k)})$, define $\lambda^{(k+1)}$ by the proportioning step:*

$$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} - \rho\beta(\lambda^{(k)})).$$

Stop the iteration if $\|\varphi(\lambda^{(k)}) + \beta(\lambda^{(k)})\| < \varepsilon$.

4.2.2 Convergence

Our next aim is to prove that the sequence $\lambda^{(k)}$ produced by Algorithm 16 converges to the solution λ of the variational inequality for the Schur complement (4.3). Following [2], we will even give a rate of convergence for $\lambda^{(k)} \to \lambda$. Therefore, we will analyze each of the three possible steps, namely expansion, conjugate gradient and proportioning step, separately.

Theorem 17 *Let $\lambda \in \Lambda$ denote the unique solution of (4.3), and s_1, s_2 be the smallest and largest eigenvalues of S, respectively. Let $\Gamma > 0$ be a given constant, and $\rho \in (0, s_2^{-1})$ be fixed. For the sequence $\lambda^{(k)}$ produced by Algorithm 16, there holds*

$$\mathcal{S}(\lambda^{(k+1)}) - \mathcal{S}(\lambda) \leq \eta(\mathcal{S}(\lambda^{(k)}) - \mathcal{S}(\lambda))$$

where

$$\eta = 1 - \frac{\rho s_2}{2 + 2\hat{\Gamma}}, \qquad \hat{\Gamma} = \max(\Gamma, \Gamma^{-1}).$$

The error in the S-energy norm is bounded by

$$\|\lambda^{(k)} - \lambda\| \leq 2\eta^k(\mathcal{S}(\lambda^{(0)}) - \mathcal{S}(\lambda)).$$

Proof. We will find estimates for $\mathcal{S}(\lambda^{(k+1)}) - \mathcal{S}(\lambda)$ separately for all three possible methods of generation of $\lambda^{(k+1)}$. Let us first state that, for any vectors μ and d, and the residual $r = r(\mu) = S\mu - h$, there holds

$$\begin{aligned} \mathcal{S}(\mu + d) &= (S(\mu + d), \mu + d) - (h, \mu + d) \\ &= \mathcal{S}(\mu) + (r, d) + \frac{1}{2}(Sd, d) && (4.5) \\ &\geq \mathcal{S}(\mu) + (r, d). && (4.6) \end{aligned}$$

For the gradient projection, this implies

$$\begin{aligned} \mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r(\lambda^{(k)}))) &= \mathcal{S}(\lambda^{(k)} - \rho(\tilde{\varphi}(\lambda^{(k)}) + \beta(\lambda^{(k)}))) \\ &\geq \mathcal{S}(\lambda^{(k)}) - \rho(r(\lambda^{(k)}), \tilde{\varphi}(\lambda^{(k)}) + \tilde{\beta}(\lambda^{(k)})) && (4.7) \\ &= \mathcal{S}(\lambda^{(k)}) - \rho[(\tilde{\varphi}(\lambda^{(k)}), \varphi(\lambda^{(k)})) + (\beta(\lambda^{(k)}), \tilde{\beta}(\lambda^{(k)}))], \end{aligned}$$

where we used the orthogonality of φ and β. Now we treat the three possible cases separately. To simplify notation, we will omit $\lambda^{(k)}$ as an argument of a gradient, and set $r = r(\lambda^{(k)}), \varphi = \varphi(\lambda^{(k)}), \beta = \beta(\lambda^{(k)})$.

1. *Expansion step.* Let us first assume that $\lambda^{(k+1)}$ is generated from $\lambda^{(k)}$ by the expansion step
$$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} - \rho\varphi).$$
Using equation (4.5) for $d = \tilde{\varphi}$, we obtain
$$\begin{aligned}
\mathcal{S}(\lambda^{(k+1)}) &= \mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho\varphi)) \\
&= \mathcal{S}(\lambda^{(k)}) - \rho(\varphi, \tilde{\varphi}) + \frac{\rho^2}{2}(S\tilde{\varphi}, \tilde{\varphi}) \\
&\leq \mathcal{S}(\lambda^{(k)}) - \rho(\varphi, \tilde{\varphi}) + \frac{\rho^2 s_2}{2}\|\tilde{\varphi}\|^2 \\
&\leq \mathcal{S}(\lambda^{(k)}) - \rho(\varphi, \tilde{\varphi}) + \frac{\rho}{2}(\varphi, \tilde{\varphi}) = \mathcal{S}(\lambda^{(k)}) - \frac{\rho}{2}(\varphi, \tilde{\varphi}).
\end{aligned}$$
We only take an expansion step if $\lambda^{(k)}$ is strictly proportional. Therefore we have
$$(\beta, \tilde{\beta}) \leq \Gamma^2(\varphi, \tilde{\varphi}).$$
Inserting this into the estimate on gradient projection (4.7), we get
$$\mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r(\lambda^{(k)})) \geq \mathcal{S}(\lambda^{(k)}) - \rho(1 + \Gamma^2)(\tilde{\varphi}(\lambda^{(k)}), \varphi(\lambda^{(k)})).$$
Thus we may estimate
$$\begin{aligned}
\mathcal{S}(\lambda^{(k+1)}) &\leq \mathcal{S}(\lambda^{(k)}) - \frac{\rho}{2}(\varphi, \tilde{\varphi}) \\
&= \frac{1}{2 + 2\Gamma^2}(\mathcal{S}(\lambda^{(k)}) - \rho(1 + \Gamma^2)(\varphi, \tilde{\varphi}) + (1 + 2\Gamma^2)\mathcal{S}(\lambda^{(k)})) \\
&\leq \frac{1}{2 + 2\Gamma^2}(\mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r(\lambda^{(k)})) + (1 + 2\Gamma^2)\mathcal{S}(\lambda^{(k)}))
\end{aligned}$$
Finally we use the result given in Theorem 14, and get
$$\mathcal{S}(\lambda^{(k+1)}) \leq \frac{2\Gamma^2 + 2 - \rho s_2}{2 + 2\Gamma^2}\mathcal{S}(\lambda^{(k)}) + \frac{\rho s_2}{2 + 2\Gamma^2}\mathcal{S}(\lambda).$$

2. *Conjugate gradient step.* Let us now consider the case that $\lambda^{(k+1)}$ is generated by the conjugate gradient step
$$\lambda^{(k+1)} = \lambda^{(k)} - \rho_{cg} p$$
where ρ_{cg} and $p = p^{(k)}$ are chosen according to Algorithm 16. As we have stated before, there holds
$$\mathcal{S}(\lambda^{(k+1)}) = \min\{\mathcal{S}(\lambda^{(s)} + y) : y \in \text{span } \{p^{(s)}, \ldots, p^{(k)}\}\}$$

if $\lambda^{(s+1)}, \ldots, \lambda^{(k+1)}$ are generated by conjugate gradient steps. Therefore, we may estimate

$$\begin{aligned}
\mathcal{S}(\lambda^{(k+1)}) &\leq \mathcal{S}(\lambda^{(k)} - \rho(\varphi)\varphi) \\
&= \mathcal{S}(\lambda^{(k)}) - \rho(\varphi)\|\varphi\|^2 + \frac{\rho(\varphi)^2}{2}(S\varphi, \varphi)
\end{aligned}$$

where

$$\rho(\varphi) = \frac{(r, \varphi)}{(S\varphi, \varphi)} = \frac{\|\varphi\|^2}{(S\varphi, \varphi)}.$$

This implies

$$\begin{aligned}
\mathcal{S}(\lambda^{(k+1)}) &\leq \mathcal{S}(\lambda^{(k)}) - \frac{\|\varphi\|^4}{2(S\varphi, \varphi)} \\
&\leq \mathcal{S}(\lambda^{(k)}) - \frac{\|\varphi\|^4}{2s_2\|\varphi\|^2} \\
&\leq \mathcal{S}(\lambda^{(k)}) - \frac{\rho\|\varphi\|^2}{2} \\
&\leq \mathcal{S}(\lambda^{(k)}) - \frac{\rho}{2}(\varphi, \tilde{\varphi})
\end{aligned}$$

as we fixed $\rho \in (0, s_2^{-1})$. This is the same estimate we obtained for the expansion step. Again, the conjugate gradient step is only carried out if $\lambda^{(k)}$ is strictly proportional. Therefore, reasoning the same way as for the expansion step, we finally get

$$\mathcal{S}(\lambda^{(k+1)}) \leq \frac{2\Gamma^2 + 2 - \rho s_2}{2 + 2\Gamma^2}\mathcal{S}(\lambda^{(k)}) + \frac{\rho s_2}{2 + 2\Gamma^2}\mathcal{S}(\lambda).$$

3. *Proportioning step.* The last case we consider is the one that $\lambda^{(k+1)}$ is generated by the proportioning step

$$\lambda^{(k+1)} = P_\Lambda(\lambda^{(k)} - \rho\tilde{\beta}).$$

Using equation (4.5), we obtain

$$\begin{aligned}
\mathcal{S}(\lambda^{(k+1)}) &= \mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho\tilde{\beta}) = \mathcal{S}(\lambda^{(k)} - \rho\tilde{\beta}) \\
&= \mathcal{S}(\lambda^{(k)}) - \rho(\beta, \tilde{\beta}) + \frac{\rho^2}{2}(S\tilde{\beta}, \tilde{\beta}) \\
&\leq \mathcal{S}(\lambda^{(k)}) - \rho(\beta, \tilde{\beta}) + \frac{\rho^2}{2}s_2\|\tilde{\beta}\|^2 \\
&\leq \mathcal{S}(\lambda^{(k)}) - \frac{\rho}{2}(\beta, \tilde{\beta}) \\
&= \frac{1}{2 + 2\Gamma^{-2}}\big(\mathcal{S}(\lambda^{(k)}) - \rho(1 + \Gamma^{-2})(\beta, \tilde{\beta}) + (1 + 2\Gamma^{-2})\mathcal{S}(\lambda^{(k)})\big).
\end{aligned}$$

We take the proportioning step if and only if $\lambda^{(k)}$ is not proportional,

$$(\beta, \tilde{\beta}) > \Gamma^2(\varphi, \tilde{\varphi}).$$

Therefore, there holds

$$\begin{aligned} \mathcal{S}(\lambda^{(k)}) - \rho(1+\Gamma^{-2})(\beta, \tilde{\beta}) &< \mathcal{S}(\lambda^{(k)}) - \rho\big((\varphi, \tilde{\varphi}) + (\beta, \tilde{\beta})\big) \\ &= \mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r)). \end{aligned}$$

Inserting this into the estimate for $\mathcal{S}(\lambda^{(k+1)})$ yields

$$\mathcal{S}(\lambda^{(k+1)}) < \frac{1}{2+2\Gamma^{-2}}\big(\mathcal{S}(P_\Lambda(\lambda^{(k)} - \rho r)) + (1+2\Gamma^{-2})\mathcal{S}(\lambda^{(k)})\big).$$

Now we may apply Theorem 14 and get

$$\mathcal{S}(\lambda^{(k+1)}) \le \frac{2\Gamma^{-2} + 2 - \rho s_2}{2+2\Gamma^{-2}}\mathcal{S}(\lambda^{(k)}) + \frac{\rho s_2}{2+2\Gamma^{-2}}\mathcal{S}(\lambda).$$

We have now obtained similar bounds for $\mathcal{S}(\lambda^{(k+1)})$ for all three possible cases. Collecting these results, we find

$$\begin{aligned} \mathcal{S}(\lambda^{(k+1)}) &\le \left(1 - \frac{\rho s_2}{2+2\hat{\Gamma}^2}\right)\mathcal{S}(\lambda^{(k)}) + \frac{\rho s_2}{2+2\hat{\Gamma}^2}\mathcal{S}(\lambda) \\ &= \eta\mathcal{S}(\lambda^{(k)}) - (\eta - 1)\mathcal{S}(\lambda) \end{aligned}$$

and therefore the assertion of the theorem holds. □

The sequence $\lambda^{(k)}$ generated by Algorithm 16 does not only converge to the solution λ, it even terminates within finitely many steps. A detailed proof was done in [2], we will only present the main ideas.

First we assume that the solution λ satisfies the strict complementarity condition,

$$\begin{aligned} \lambda_{1i} = 0 &\quad\Rightarrow\quad r_{1i} > 0 \\ |\lambda_{2i}| = l_i &\quad\Rightarrow\quad r_{2i} \neq 0. \end{aligned}$$

One can show, that there exists some $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ the free and active sets of λ are subsets to the free and active sets of $\lambda^{(k)}$, respectively. As the condition of strict complementarity implies

$$\mathcal{F}_1(\lambda) \cup \mathcal{A}_1(\lambda) = \mathcal{F}_2(\lambda) \cup \mathcal{A}_2(\lambda) = \{1, \dots, m_H\}$$

we see that the free and active sets of λ and $\lambda^{(k)}$ are equal. Then all subsequent steps are conjugate gradient steps. Therefore, the method switches to the cg method, which terminates within finitely many steps. This implies, that once we have found some $\lambda^{(k)}$ with the same free and active sets as the solution λ, we compute λ efficiently.

The finite termination property also holds for λ not satisfying the condition of strict complementarity, given the constant Γ is large enough. More precisely, there has to hold

$$\Gamma > \kappa(S)^{1/2},$$

where $\kappa(S)$ denotes the condition number of the matrix S, $\kappa(S) = \|S\|\|S^{-1}\| = s_2/s_1$.

4.3 Realization

In this section, we derive an algorithm to solve the time-dependent two-body contact problem with Coulomb friction. For time discretization we use the implicit scheme we introduced in Chapter 1. Starting from time $t = 0$, we consider time steps of uniform length Δt. In each step we compute the increase $\bar{u} = u(t + \Delta t) - u(t)$. In general, the relation between deformation u and stress tensor σ is nonlinear. As Δt is small, we may assume that $\bar{u}$ satisfies the equations of linearized elasticity. At time t, the bilinear forms a and b we consider are given as the linearizations of the respective nonlinear operators in $u(t)$. Thereby we obtained a mixed variational formulation of the contact problem

$$\begin{array}{lcll} a(\bar{u}, v) + b(v, \lambda) & = & \langle F, v \rangle & \forall v \in V_0 \\ b(\bar{u}, \mu - \lambda) & \leq & \langle G, \mu - \lambda \rangle_Q & \forall \mu \in \Lambda(\lambda_1). \end{array}$$

This is the problem we have to solve in each time step. Using the finite element discretization above, we get the following system for the vectors $\underline{u}$ for $\bar{u}$, $\underline{\lambda}$ for λ

$$\begin{array}{lcll} A\underline{u} + B^T \underline{\lambda} & = & \underline{f} & \\ (\underline{\mu} - \underline{\lambda})^T B \underline{u} & \leq & (\underline{\mu} - \underline{\lambda})^T \underline{g} & \forall \underline{\mu} \in \underline{\Lambda}(\underline{\lambda}_1). \end{array}$$

Note that we have to recompute the matrices A and B in each time step. Also the right hand sides $\underline{f}, \underline{g}$ change in time.

Although the equations and inequalities are linear, we cannot find the solution $\underline{u}$ directly. The convex set $\underline{\Lambda}$ still depends on the solution $\underline{\lambda}_1$. Therefore we have to do a fixed point iteration. Starting from some given contact pressure g_n, we find a solution $(\underline{u}, \underline{\lambda})$ such that

$$\begin{array}{lcll} A\underline{u} + B^T \underline{\lambda} & = & \underline{f} & \\ (\underline{\mu} - \underline{\lambda})^T B \underline{u} & \leq & (\underline{\mu} - \underline{\lambda})^T \underline{g} & \forall \underline{\mu} \in \underline{\Lambda}(g_n). \end{array}$$

For the next iteration, we set $g_n = \underline{\lambda}_1$, and compute the next iterate $(\underline{u}, \underline{\lambda})$.

This leaves us with the problem of finding a solution to the mixed problem above. Therefore, we can use one of the algorithms presented in Chapter 3.

Collecting all results from above, we obtain a method for the solution of the time-dependent contact problem with Coulomb friction. We describe this method in a language one can understand easily.

Algorithm 18 *(Contact problem with Coulomb friction)*

```
// Initialisation
u0 = 0;
lambda = 0;
time = 0;
while ( time < T )
  {
     // Compute vectors, matrices
     A = A(u0);
     B = B(u0);
     f = f(u0);
     g = g(u0);

     h = B*Inverse(A)*f - g;

     // Fixpoint iteration
     do
        {
           gn = lambda;
           l  = F * gn;
           // Solve mixed problem
           lambda = SolutionMethod(B*Inverse(A)*Transpose(B),
                                   h, l, lambda);
        }
     while ( L2Norm(lambda - gn) > eps*L2Norm(lambda) );

     // Compute deformation
     u = Inverse(A) * (f - Transpose(B)*lambda);
     u0 = u0 + u;

     time = time + deltat;
  }
```

To compute $\underline{\lambda}$, one can use either Uzawa's method as described in Algorithm 13, or the proportioning method Algorithm 16 we analyzed in Section 4.2. In a language closer to actual implementation, these algorithms read as follows

Algorithm 19 *(Uzawa's method)*

```
UzawaMethod(S, h, l, lambda)
  {
     // Initialisation
     r = S*lambda - h;
     e = (S*lambda - h)*(lambda_1, abs(lambda_2) - l)
     while( e > eps )
      {
         r = S*lambda - h;
         lambda = P(lambda - tau*r);
         e = (lambda_1, abs(lambda_2) - l) * (S*lambda - h);
      }
     return lambda;
  }
```

Algorithm 20 *(Modified proportioning with reduced gradient projections)*

```
MPRGP(S, h, l, lambda)
  {
     // Initialisation
     r = S*lambda - h;
     phi = phi(lambda); beta = beta(lambda);
     p = phi;
     e = (S*lambda - h)*(lambda_1, abs(lambda_2) - l)
     while( e > eps )
      {
         if ( beta*beta_tilde < Gamma^2 * phi*phi_tilde )
           {
              alpha_cg = (r*p)/(p*S*p);
              if ( lambda - alpha_cg*p in Lambda)
                {
                   // Conjugate gradient step
                   lambda = lambda - alpha_cg * p;
                   r = r - alpha_cg*S*p;
                   gamma = (phi*S*p)/(p*S*p);
                   p = phi(lambda) - gamma*p;
                }
              else
                {
                   // Expansion step
                   lambda = P(lambda - alpha*p);
                   r = S*lambda - h;
```

```
                p = phi(lambda);
              }
          }
        else
          {
            // Proportioning step
            lambda = P(lambda - alpha*beta);
             r = S*lambda - h;
             p = phi(lambda);
           }
         e = (S*lambda - h)*(lambda_1, abs(lambda_2) - 1)
      }
   return lambda;
}
```

Chapter 5

Numerical Results

We solve the contact problem applying both algorithms presented in the previous chapter. We assume that both bodies are characterized by the same Young's modulus $E = 2 \cdot 10^5$ and Poisson's ratio $\nu = 0.3$. The coefficient of friction shall be given as $\mathcal{F} = 0.5$.

For the finite element model we approximate the roll and strip of steel by a triangular mesh. We used NETGEN [8] for mesh generation and refinement. We consider piecewise linear, continuous nodal finite element basis functions for the displacement. We use a mesh consisting of 114688 elements in our calculations. The finite element solution $\underline{u}$ has $2 \cdot 58113$ degrees of freedom.

We discretize the contact boundary by a partition of Γ_π into intervals of uniform length. We choose the size H of such an interval such that the ratio h/H is sufficiently small. As we have seen in Chapter 3, this ensures that the discrete inf-sup condition holds, and therefore the system (4.2) is solvable. Numerical experiments show that the system is solvable if the ratio h/H is chosen to be approximately 0.7. We use piecewise constant finite element basis functions for the discretization of Q. We get 51 degrees of freedom for both the normal and tangential stress on the contact boundary.

We have implemented both algorithms using NGSolve. The Schur Complement $S = BA^{-1}B^T$ was computed explicitely. This could be done because of its smallness: For the finest mesh we considered, S is a 102×102 matrix. To find $A^{-1}f$ for some $f \in \mathbb{R}^{2n_h}$, we did a sparse Cholesky factorization of A in each time step. For the finest discretization, the computation of S took about 20 seconds, about 5 of which were used for the factorization of A.

For the fixed point iteration, we used the stopping criterion $\|\Phi_H(\underline{\lambda}_1) - \underline{\lambda}_1\|_{l^2} \leq 10^{-10}\|\underline{\lambda}_1\|_{l^2}$ in both cases. We required a maximum of seven iterations to find a fixed point of Φ_H. The number of iterations necessary increased, as the discretization became finer, if we started from $\lambda^{(0)} = 0$. In Figure 5.1, we give a visualization of the convergence of the fixed point iteration

Figure 5.2 shows the deformed configuration of roll and strip for $t = 35$ and

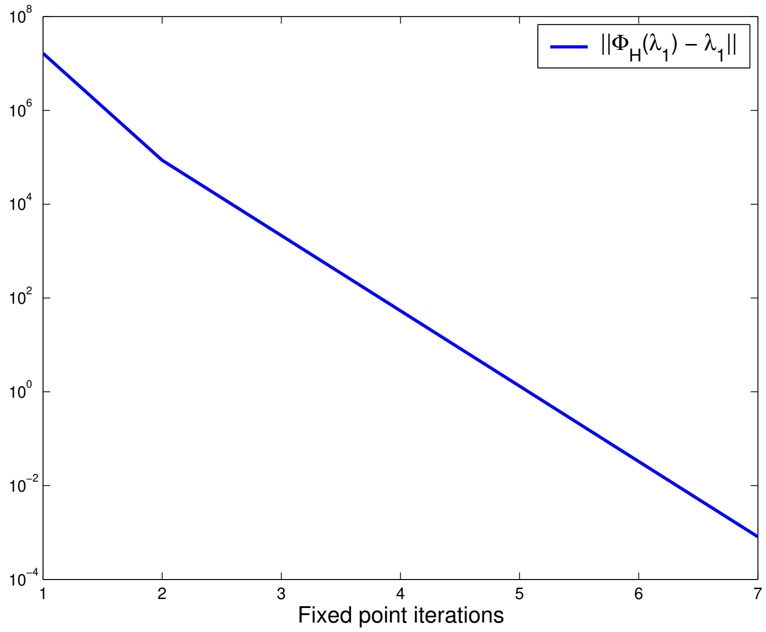

Figure 5.1: Convergence of the fixed point iteration

$t = 45$. The time steps were of uniform length $\Delta t = 0.33$. For the plots, we used a coarser mesh. Comparing the two deformed meshes, one cannot find any difference. This implies that the time dependent deformation u reaches its steady state after some initial oscillations.

For both the normal and tangential stress holds the same as for the deformation: They converge to a steady state. Figure 5.4 shows the normal stress along the contact boundary Γ_c for times $t = 0, 5, 15, 25, 35, 45$. There one can make out that the stress changes in the beginning. After some time, the oscillations we observed in the beginning subside. In Figure 5.3, we put the change of λ in time, namely $\|\underline{\lambda}(t + \Delta t) - \underline{\lambda}(t)\|/\Delta t$. In Figure 5.5 we see the normal and tangential stress along the contact boundary at time $t = 45$. To visualize the Coulomb condition, we also plotted $\mathcal{F}|\lambda_2|$. One can see that it is bounded by λ_1. In Figure 5.6 one finds normal stress and gap function. One can see, that either the bodies touch, i. e. the gap function equals zero, or there is no stress in normal direction, $\lambda_1 = 0$. We also give a plot of the Coulomb condition on the tangential stress: Figure 5.7 shows λ_1, $\mathcal{F}|\lambda_2|$ and the relative deformation in tangential direction. There one can observe, that slipping between the two bodies only occurs if λ_1 is and active constraint for $\mathcal{F}|\lambda_2|$. As long as there holds $\lambda_1 > \mathcal{F}|\lambda_2|$, the relative deformation u_T is zero, and the bodies stick to each other.

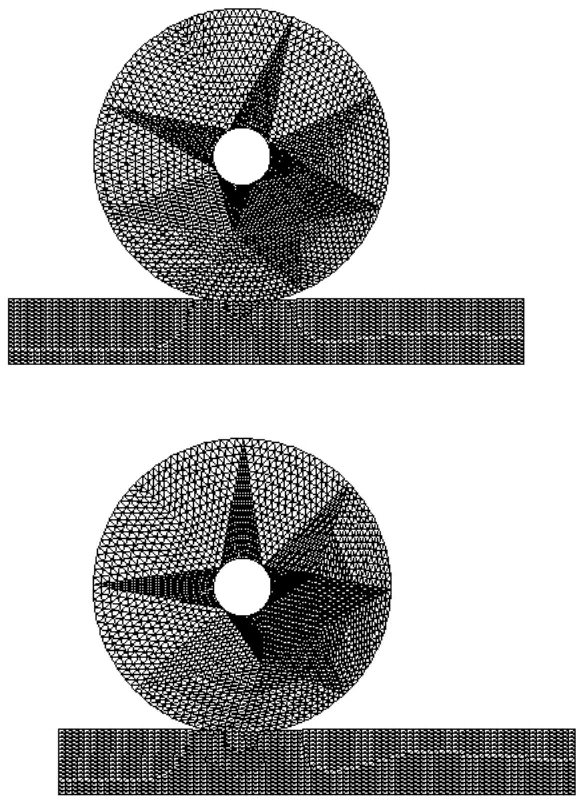

Figure 5.2: Deformed configuration for $t = 35, 45$

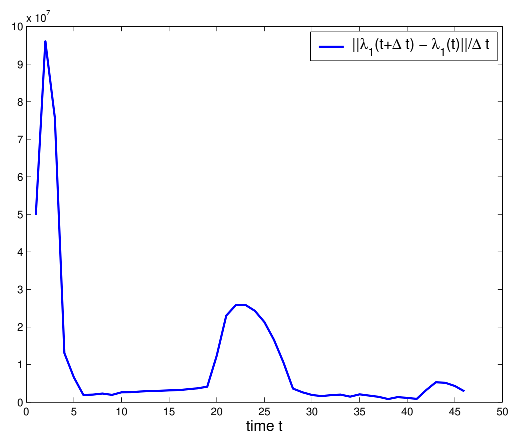

Figure 5.3: Change of λ_1 in time

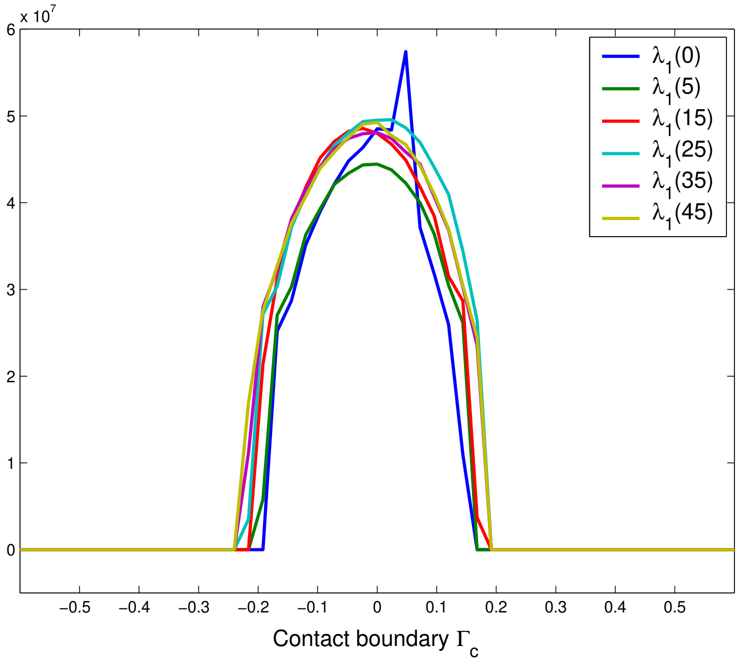

Figure 5.4: λ_1 at times 0, 5, 15, 25, 35, 45

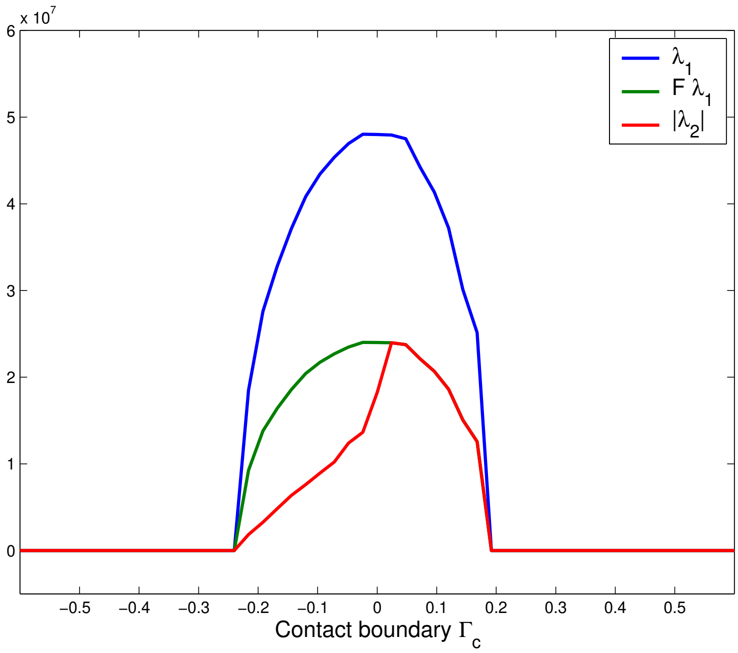

Figure 5.5: Normal and tangential stress along the contact boundary

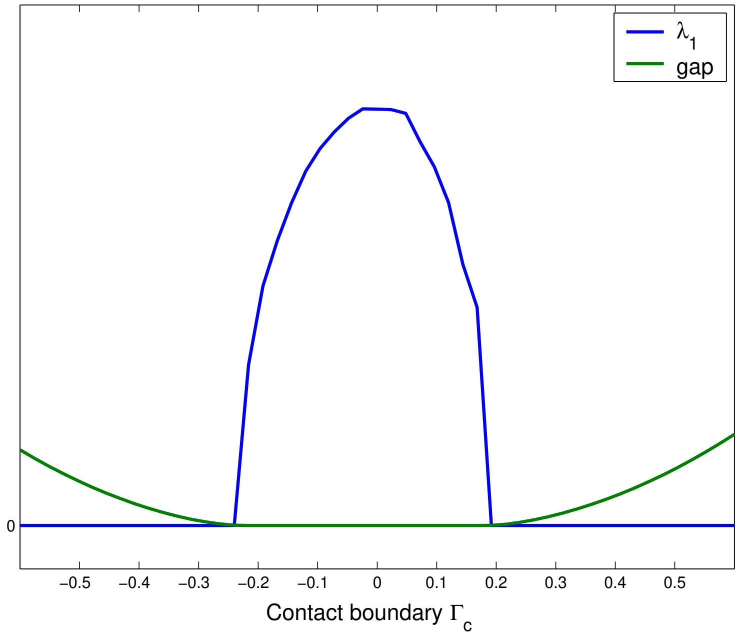

Figure 5.6: Visualization of the non-penetration condition

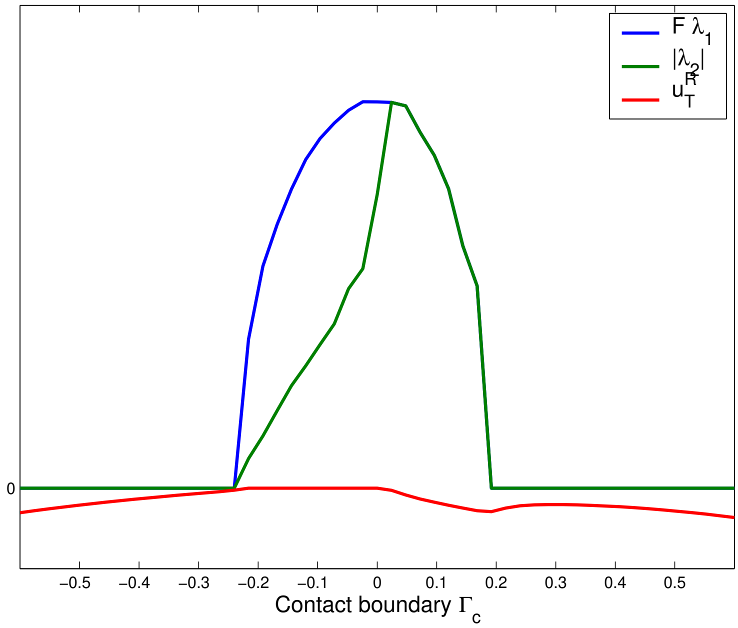

Figure 5.7: Visualization of Coulomb's law

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
Points	993	3777	14721	58113
m_H	6	12	26	51
H	0.2	0.92	0.46	0.23
α	$5 \cdot 10^6$	$5 \cdot 10^7$	$5 \cdot 10^8$	$5 \cdot 10^9$
Uzawa steps	1264	4230	8280	13082
cg steps	7	17	43	92

Table 5.1: Contact problem with given friction

	Mesh 1	Mesh 2	Mesh 3	Mesh 4
fixed point iterations	3	3	7	7
Uzawa steps total	2515	7446	23307	33358
Uzawa steps mean	838.3	2482	3329.6	4765
cg steps total	16	34	174	337
cg steps mean	5.3	11.3	24.9	48.1

Table 5.2: Contact problem with Coulomb friction

To study the dependency of the convergence of the proposed methods on the mesh size h, we use four different refinement levels. The respective meshes consist of 993, 3777, 14721 and 58113 nodes. For the dual variables $\underline{\lambda}_1, \underline{\lambda}_2$, we get 6, 12, 26 and 51 degrees of freedom. As the condition number of $S = BA^{-1}B^T$ depends on h, H, the number of iterations required for one computation of u, λ by either the Uzawa or the proportioning method will increase. In Table 5.1 we put the number of iterations required for one solve of the contact problem with given friction, which we have to do in each fixed point step. As initial value we chose $\lambda^{(0)} = 0$, assuming we have no additional information on λ.

The time the computation of λ by either Uzawa's method of the proportioning algorithm is comparatively small. As the matrix S is only of size 102×102 in the worst case, all matrix-vector operations required in the respective methods can be done fast. Due to this, both algorithms converge within less than one second, if the parameter α is chosen appropriately.

Table 5.2 lists the number of iterations needed to solve the contact problem with Coulomb friction in the first time step. The number of vertices, mesh size H and step size α are the same as in Table 5.1. Again, we used $\lambda^{(0)} = 0$ as an initial guess. We also give the arithmetic mean of the number of steps necessary in one fixed point iteration. These mean values are smaller than the numbers of steps we obtained for the first iteration in Table 5.1. This can be explained by the fact that we have better starting values for the subsequent iterations. As time

t increases, the number of steps necessary for one solve of the contact problem with given friction decreases even further. Then $\lambda(t)$ becomes a better initial guess for computing $\lambda(t + \Delta t)$, when the process converges to its steady state.

Conclusion

Inspired by the process of temper rolling in steel industry, we wanted to describe the deformation arising in two bodies which are pressed against each other. We have found a model for an elastic two-body contact problem with Coulomb friction. This model takes into account the effects of friction which occurs between the contact surfaces. We provided a primal and a mixed variational formulation of the contact problem, both of which included variational inequalities. We could not obtain existence and uniqueness of a solution to the general problem with arbitrary friction. By introducing a fixed point problem for the contact pressure, we could prove existence of a solution, given that the coefficient of friction is small enough.

In order to actually compute the deformation and contact stresses, we discretized the problem. In the discrete setting, we could show that there exists a unique fixed point for the normal stress along the contact boundary. This fixed point determined the arising deformation in both bodies. However, we needed the coefficient of friction to be bounded by a quantity which tended to zero as the discretization became finer.

We proposed two algorithms to solve the variational inequalities induced by the discrete problem. We proved convergence of both algorithms. For the first one, namely Uzawa's method, we obtained linear convergence. So we did for the second algorithm, a proportioning method. But while Uzawa's method uses directions of steepest descent, we have seen that the proportioning method switches to the conjugate gradient method. It does so as soon as it has found the segments along the contact boundary, on which the constraints on the contact stress given by Coulomb's law are active. Therefore, it converges much faster than the first method.

Finally, we realized the problem numerically using NGSolve. We implemented both methods mentioned above to solve the system of equations and inequalities. We found a solution to the contact problem with Coulomb friction for our model geometry. We observed that the system reaches a steady state after some initial oscillations. Then the occurring deformations and pressures do not change any more. Concerning the different algorithms for the solution of the problem, we found that the proportioning method converged much faster than Uzawa's method, as was to be expected. We saw that the convergence depended on the parameter of discretization, as the condition number of the system matrices gets worse for finer meshes. The algorithms could therefore be improved by preconditioning, which we did not consider in this work.

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Eidesstattliche Erklärung

Ich, Astrid Sinwel, erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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