



Non-standard Norms and Robust Estimates for Saddle Point Problems

Walter Zulehner

Institute of Computational Mathematics, Johannes Kepler University
Altenberger Str. 69, 4040 Linz, Austria

NuMa-Report No. 2010-07

November 2010

Technical Reports before 1998:

1995

- 95-1 Hedwig Brandstetter
Was ist neu in Fortran 90? March 1995
- 95-2 G. Haase, B. Heise, M. Kuhn, U. Langer
Adaptive Domain Decomposition Methods for Finite and Boundary Element Equations. August 1995
- 95-3 Joachim Schöberl
An Automatic Mesh Generator Using Geometric Rules for Two and Three Space Dimensions. August 1995

1996

- 96-1 Ferdinand Kickingger
Automatic Mesh Generation for 3D Objects. February 1996
- 96-2 Mario Goppold, Gundolf Haase, Bodo Heise und Michael Kuhn
Preprocessing in BE/FE Domain Decomposition Methods. February 1996
- 96-3 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. February 1996
- 96-4 Bodo Heise und Michael Jung
Robust Parallel Newton-Multilevel Methods. February 1996
- 96-5 Ferdinand Kickingger
Algebraic Multigrid for Discrete Elliptic Second Order Problems. February 1996
- 96-6 Bodo Heise
A Mixed Variational Formulation for 3D Magnetostatics and its Finite Element Discretisation. May 1996
- 96-7 Michael Kuhn
Benchmarking for Boundary Element Methods. June 1996

1997

- 97-1 Bodo Heise, Michael Kuhn and Ulrich Langer
A Mixed Variational Formulation for 3D Magnetostatics in the Space $H(\text{rot}) \cap H(\text{div})$ February 1997
- 97-2 Joachim Schöberl
Robust Multigrid Preconditioning for Parameter Dependent Problems I: The Stokes-type Case. June 1997
- 97-3 Ferdinand Kickingger, Sergei V. Nepomnyaschikh, Ralf Pfau, Joachim Schöberl
Numerical Estimates of Inequalities in $H^{\frac{1}{2}}$. August 1997
- 97-4 Joachim Schöberl
Programmbeschreibung NAOMI 2D und Algebraic Multigrid. September 1997

From 1998 to 2008 technical reports were published by SFB013. Please see

<http://www.sfb013.uni-linz.ac.at/index.php?id=reports>

From 2004 on reports were also published by RICAM. Please see

<http://www.ricam.oeaw.ac.at/publications/list/>

For a complete list of NuMa reports see

<http://www.numa.uni-linz.ac.at/Publications/List/>

NON-STANDARD NORMS AND ROBUST ESTIMATES FOR SADDLE POINT PROBLEMS

WALTER ZULEHNER*

Abstract. In this paper we discuss how to find norms for parameter-dependent saddle point problems which lead to robust (i.e.: parameter-independent) estimates of the solution in terms of the data. In a first step a characterization of such norms is given for a general class of symmetric saddle point problems. Then, for special cases, explicit formulas for these norms are derived. Finally, we will apply these results to distributed optimal control problems for elliptic equations and for the Stokes equations. The norms which lead to robust estimates turn out to differ from the standard norms typically used for these problems. This will lead to block diagonal preconditioners for the corresponding discretized problems with mesh-independent and robust convergence rates if used in preconditioned Krylov subspace methods.

Key words. saddle point problems, PDE-constrained optimization, optimal control, robust estimates, block diagonal preconditioners

AMS subject classifications. 65F08, 65N22, 65K10, 49K40

1. Introduction. In this paper we consider mixed variational problems of the following form: Find $u \in V$ and $p \in Q$ such that

$$\begin{aligned} a(u, v) + b(v, p) &= f(v) \quad \text{for all } v \in V, \\ b(u, q) - c(p, q) &= g(q) \quad \text{for all } q \in Q, \end{aligned} \tag{1.1}$$

where V and Q are Hilbert spaces, a , b , and c are bounded bilinear forms on $V \times V$, $V \times Q$, and $Q \times Q$, respectively, and f , g are bounded linear functionals. Additionally, we will assume throughout the paper that a and c are symmetric, i.e.:

$$a(w, v) = a(v, w) \quad \text{for all } v, w \in V, \quad c(r, q) = c(q, r) \quad \text{for all } q, r \in Q, \tag{1.2}$$

and a and c are non-negative, i.e.:

$$a(v, v) \geq 0 \quad \text{for all } v \in V, \quad c(q, q) \geq 0 \quad \text{for all } q \in Q, \tag{1.3}$$

by which (1.1) becomes a symmetric and indefinite problem.

Examples of such problems are the Stokes problem in fluid mechanics, mixed formulations of elliptic boundary value problems and the optimality system of optimal control problems, where (in all these examples) V and Q are infinite-dimensional function spaces, as well as discretized versions of these problems, where V and Q are finite-dimensional finite element spaces. In particular, we consider problems which involve some critical model parameter, say α , like the time step size in the time-discretized Stokes problem, the diffusion coefficient in a diffusion-reaction equation or a regularization parameter occurring in optimal control problems. In the discretized version an additional parameter is always involved: the mesh size, say h , of the underlying subdivision of the computational domain.

A fundamental issue is the question whether (1.1) is well-posed: Does there exist a unique solution (u, p) in $X = V \times Q$ for any data (f, g) from X^* , the dual space of X , and does the solution depend continuously on the data, or equivalently, is the

*Institute of Computational Mathematics, Johannes Kepler University, 4040 Linz, Austria (zulehner@numa.uni-linz.ac.at). The work was supported in part by the Austrian Science Foundation (FWF) under the grant W1214/DK12.

norm of the solution (u, p) in X bounded from above by the norm of the data (f, g) in X^* ? Of course, the answer depends on the choice of the norm or better the inner product in X . We will concentrate on inner products in X of the particular form

$$((v, q), (w, r))_X = (v, w)_V + (q, r)_Q, \quad (1.4)$$

where $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_Q$ are inner products of the Hilbert spaces V and Q , respectively.

Since we have assumed that the bilinear forms a , b , and c are bounded, the norm of the solution (u, p) can always be estimated from below by the norm of the data (f, g) . The focus of this paper is the more specific question whether these estimates from above and from below are independent of the model parameter α and, for the discretized version, of the discretization parameter h . We will address this issue by characterizing those inner products in V and Q which lead to robust estimates. Additionally, for some classes of saddle point problems, we will derive explicit representations of such inner products.

Knowledge of robust estimates does not only contribute to the question of well-posedness but also to discretization error estimates and the construction of efficient solvers for the discretized problem. In the discretized case, having robust estimates for an inner product of the form (1.4) translates to having a block diagonal preconditioner for the linear operator describing the left-hand side of (1.1) with robust estimates for the condition number. This would immediately imply that Krylov subspace methods like the minimal residual method, see [17], converge with convergence rates independent of α and h .

Of course, for a wide range of problems robust estimates have been developed already, see the survey article [14] and the many references contained there. Most of the norms involved in these estimates were found on a case-by-case basis. Following the spirit of the survey article [14] the present paper is to be understood as a further contribution to a more systematic search for the "right" norms.

Remark 1. Observe that Condition (1.3) characterizes exactly the case that the functional $\mathcal{L}(v, q)$, given by

$$\mathcal{L}(v, q) = \frac{1}{2} a(v, v) + b(v, q) - \frac{1}{2} c(q, q) - f(v) - g(q),$$

is a convex function of $v \in V$ and a concave function of $q \in Q$. Such a functional \mathcal{L} is called a saddle function, see [20]. If, additionally, Condition (1.2) holds, it is easy to see that (u, p) solves (1.1) if and only if (u, p) is a saddle point of \mathcal{L} , i.e.:

$$\mathcal{L}(u, q) \leq \mathcal{L}(u, p) \leq \mathcal{L}(v, p) \quad \text{for all } v \in V, q \in Q.$$

A first preliminary version of the contents of this paper was presented in a talk given by the author at the BIRS Workshop Advances and Perspectives on Numerical Methods for Saddle Point Problems, April 2009, Banff, Canada. The work of the author on this topic has been strongly influenced by a series of three lectures given by R. Winther at the London Society Durham Symposium on Computational Linear Algebra for Partial Differential Equations, July 2008, see [14] for the corresponding survey article.

The paper is organized as follows: Section 2 contains the abstract framework. The main abstract result is presented in Theorem 2.6 describing necessary and sufficient conditions on the involved inner products for obtaining robust estimates. In Section 3

several special cases are discussed for which inner products leading to robust estimates are explicitly known. Section 4 deals with the application to optimal control problems. The paper ends with a few concluding remarks in Section 5.

2. The abstract theory. Throughout the paper we will use the following notational convention:

Notation 1. Let H be a Hilbert space with inner product $(\cdot, \cdot)_H$ and associated norm $\|\cdot\|_H$, given by

$$\|x\|_H = \sqrt{(x, x)_H}.$$

The dual space of H is denoted by H^* with norm $\|\cdot\|_{H^*}$, given by

$$\|\ell\|_{H^*} = \sup_{0 \neq x \in H} \frac{\ell(x)}{\|x\|_H}.$$

The duality pairing $\langle \cdot, \cdot \rangle_H$ on $H^* \times H$ is given by

$$\langle \ell, x \rangle_H = \ell(x) \quad \text{for all } \ell \in H^*, x \in H.$$

We will usually drop the subscript H and write instead $\langle \cdot, \cdot \rangle$ for a duality pairing.

Let $\mathcal{I}_H: H \rightarrow H^*$ be given by

$$\langle \mathcal{I}_H x, y \rangle = (x, y)_H.$$

It is well-known that \mathcal{I}_H is an isometric isomorphism between H and its dual space H^* . The inverse $\mathcal{R}_H = \mathcal{I}_H^{-1}$ is called the Riesz-isomorphism, by which functionals in H^* can be identified with elements in H and we have:

$$\langle \ell, x \rangle = (\mathcal{R}_H \ell, x)_H.$$

The set of all linear and bounded operators from H_1 to H_2 , where H_1 and H_2 are normed spaces, is denoted by $L(H_1, H_2)$.

The mixed variational problem (1.1) in V and Q can also be written as a variational problem on the product space $X = V \times Q$: Find $x = (u, p) \in X$ such that

$$\mathcal{B}(x, y) = \mathcal{F}(y) \quad \text{for all } y \in X \tag{2.1}$$

with

$$\mathcal{B}(z, y) = a(w, v) + b(v, r) + b(w, q) - c(r, q), \quad \mathcal{F}(y) = f(v) + g(q)$$

for $y = (v, q)$, $z = (w, r)$.

Since we have assumed that the bilinear forms a , b , and c are bounded, there is a constant, say \bar{c}_x , such that

$$\sup_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{\mathcal{B}(z, y)}{\|z\|_X \|y\|_X} \leq \bar{c}_x < \infty. \tag{2.2}$$

A classical result due to Babuška, see [2], [3], reads in our symmetric situation: Problem (2.1) is well-posed if and only if there is a constant, say \underline{c}_x , such that

$$\inf_{0 \neq z \in X} \sup_{0 \neq y \in X} \frac{\mathcal{B}(z, y)}{\|z\|_X \|y\|_X} \geq \underline{c}_x > 0 \tag{2.3}$$

and we have the following estimates from above and from below:

$$\frac{1}{\bar{c}_x} \|\mathcal{F}\|_{X^*} \leq \|x\|_X \leq \frac{1}{\underline{c}_x} \|\mathcal{F}\|_{X^*}.$$

Condition (2.3) is usually referred to as the inf-sup condition or the Babuška-Brezzi condition.

To each of the three bilinear forms a , b , and c we associate a corresponding linear operator $A \in L(V, V^*)$, $B \in L(V, Q^*)$, and $C \in L(Q, Q^*)$, respectively, given by

$$\langle Aw, v \rangle = a(w, v), \quad \langle Bw, q \rangle = b(w, q), \quad \langle Cr, q \rangle = c(r, q).$$

Additionally, $B^* \in L(Q, V^*)$ denotes the adjoint of $B \in L(V, Q^*)$, given by

$$\langle B^*r, v \rangle = \langle Bv, r \rangle.$$

The problem (1.1) now reads in operator notation:

$$\begin{aligned} Au + B^*p &= f, \\ Bu - Cp &= g. \end{aligned} \tag{2.4}$$

In a similar way, we associate a linear operator $\mathcal{A} \in L(X, X^*)$ to the bilinear form \mathcal{B} , given by

$$\langle \mathcal{A}x, y \rangle = \mathcal{B}(x, y).$$

Then the problem (2.1), which is equivalent to (1.1), reads

$$\mathcal{A}x = \mathcal{F}. \tag{2.5}$$

In operator notation the important conditions (2.2) and (2.3) can be written in the following form:

$$\underline{c}_x \|z\|_X \leq \|\mathcal{A}z\|_{X^*} \leq \bar{c}_x \|z\|_X \quad \text{for all } z \in X. \tag{2.6}$$

The aim in this paper is to find inner products in V and Q which lead to such estimates with coefficients \underline{c}_x and \bar{c}_x independent of the model parameter α and, for the discretized version, also independent of the mesh size h . An immediate consequence of (2.6) is an estimation of the condition number $\kappa(\mathcal{A})$:

$$\kappa(\mathcal{A}) = \|\mathcal{A}\|_{L(X, X^*)} \|\mathcal{A}^{-1}\|_{L(X^*, X)} \leq \frac{\bar{c}_x}{\underline{c}_x}.$$

So, robust estimates of the form (2.6) imply a robust estimate for the condition number, an important property in connection with convergence rates of iterative methods for solving (2.5).

We start the analysis of (1.1) by a very simple and helpful observation:

LEMMA 2.1. *Let $\ell_1 \in V^*$ and $\ell_2 \in Q^*$. Then*

$$\|\ell\|_{X^*}^2 = \|\ell_1\|_{V^*}^2 + \|\ell_2\|_{Q^*}^2$$

for $\ell \in X^*$, given by $\ell(v, q) = \ell_1(v) + \ell_2(q)$.

Proof. By using the Cauchy inequality we obtain

$$\begin{aligned} \|\ell\|_{X^*}^2 &= \sup_{0 \neq (v,q) \in X} \frac{(\langle \ell_1, v \rangle + \langle \ell_2, q \rangle)^2}{\|(v,q)\|_X^2} \\ &\leq \sup_{0 \neq (v,q) \in X} \frac{(\|\ell_1\|_{V^*} \|v\|_V + \|\ell_2\|_{Q^*} \|q\|_Q)^2}{\|v\|_V^2 + \|q\|_Q^2} \leq \|\ell_1\|_{V^*}^2 + \|\ell_2\|_{Q^*}^2. \end{aligned}$$

Equality follows for the choice $v = \mathcal{R}_V \ell_1$ and $q = \mathcal{R}_Q \ell_2$. \square

As a consequence of an estimate of the form (2.6) in X we obtain two simple estimates, one in V and one in Q :

THEOREM 2.2. *If (2.6) holds for constants $\underline{c}_x, \bar{c}_x > 0$, then*

$$\underline{c}_x^2 \|w\|_V^2 \leq \|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2 \leq \bar{c}_x^2 \|w\|_V^2 \quad \text{for all } w \in V \quad (2.7)$$

and

$$\underline{c}_x^2 \|r\|_Q^2 \leq \|Cr\|_{Q^*}^2 + \|B^*r\|_{V^*}^2 \leq \bar{c}_x^2 \|r\|_Q^2 \quad \text{for all } r \in Q. \quad (2.8)$$

Proof. For $z = (w, r)$ we have:

$$\begin{aligned} \|\mathcal{A}z\|_{X^*} &= \sup_{0 \neq (v,q) \in X} \frac{\mathcal{B}((w,r), (v,q))}{\|(v,q)\|_X} \\ &= \sup_{0 \neq (v,q) \in X} \frac{a(w,v) + b(v,r) + b(w,q) - c(r,q)}{\|(v,q)\|_X} = \sup_{0 \neq (v,q) \in X} \frac{\ell_1(v) + \ell_2(q)}{\|(v,q)\|_X} \end{aligned}$$

with

$$\begin{aligned} \ell_1(v) &= a(w,v) + b(v,r) = \langle Aw + B^*r, v \rangle, \\ \ell_2(q) &= b(w,q) - c(r,q) = \langle Bw - Cr, q \rangle. \end{aligned}$$

Therefore, by Lemma 2.1, we obtain

$$\|\mathcal{A}z\|_{X^*} = (\|Aw + B^*r\|_{V^*}^2 + \|Bw - Cr\|_{Q^*}^2)^{1/2}.$$

Then the estimates (2.7) and (2.8) immediately follow from (2.6) for $r = 0$ and for $w = 0$, respectively. \square

So, (2.7) and (2.8) are necessary conditions for (2.6). Next we will show that (2.7) and (2.8), not necessarily with the same constants, are also sufficient:

THEOREM 2.3. *If there are constants $\underline{c}_v, \bar{c}_v, \underline{c}_q, \bar{c}_q > 0$ such that*

$$\underline{c}_v^2 \|w\|_V^2 \leq \|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2 \leq \bar{c}_v^2 \|w\|_V^2 \quad \text{for all } w \in V$$

and

$$\underline{c}_q^2 \|r\|_Q^2 \leq \|Cr\|_{Q^*}^2 + \|B^*r\|_{V^*}^2 \leq \bar{c}_q^2 \|r\|_Q^2 \quad \text{for all } r \in Q,$$

then there are constants $\underline{c}_x, \bar{c}_x > 0$ such that

$$\underline{c}_x \|z\|_X \leq \|\mathcal{A}z\|_{X^*} \leq \bar{c}_x \|z\|_X \quad \text{for all } z \in X,$$

where \underline{c}_x and \bar{c}_x depend only on $\underline{c}_v, \underline{c}_q, \bar{c}_v$, and \bar{c}_q .

Proof. For $z = (w, r)$ we have:

$$\begin{aligned} \|\mathcal{A}z\|_{X^*} &= (\|Aw + B^*r\|_{V^*}^2 + \|Bw - Cr\|_{Q^*}^2)^{1/2} \\ &\leq (2\|Aw\|_{V^*}^2 + 2\|B^*r\|_{V^*}^2 + 2\|Bw\|_{Q^*}^2 + 2\|Cr\|_{Q^*}^2)^{1/2} \\ &\leq (2\bar{c}_v^2\|w\|_V^2 + 2\bar{c}_q^2\|r\|_Q^2)^{1/2} \leq \sqrt{2} \max(\bar{c}_v, \bar{c}_q) \|z\|_X, \end{aligned}$$

which proves the upper bound in (2.6) with $\bar{c}_x = \sqrt{2} \max(\bar{c}_v, \bar{c}_q)$.

For showing a lower bound, we start with the following estimate based on the triangle inequality in X^* :

$$\begin{aligned} \|\mathcal{A}z\|_{X^*} &= (\|Aw + B^*r\|_{V^*}^2 + \|Bw - Cr\|_{Q^*}^2)^{1/2} \\ &\geq (\|B^*r\|_{V^*}^2 + \|Bw\|_{Q^*}^2)^{1/2} - (\|Aw\|_{V^*}^2 + \|Cr\|_{Q^*}^2)^{1/2} \\ &= (\eta - \xi) \|z\|_X \end{aligned}$$

for $z = (w, r)$ with

$$\xi = \frac{(\|Aw\|_{V^*}^2 + \|Cr\|_{Q^*}^2)^{1/2}}{\|(w, r)\|_X}, \quad \eta = \frac{(\|B^*r\|_{V^*}^2 + \|Bw\|_{Q^*}^2)^{1/2}}{\|(w, r)\|_X}.$$

A second lower bound follows from:

$$\|\mathcal{A}z\|_{X^*} = \sup_{0 \neq (v, q) \in X} \frac{\mathcal{B}((w, r), (v, q))}{\|(v, q)\|_X} \geq \frac{\mathcal{B}((w, r), (w, -r))}{\|(w, -r)\|_X} = \frac{a(w, w) + c(r, r)}{\|z\|_X}.$$

Since

$$a(w, v)^2 \leq a(w, w) a(v, v) \leq a(w, w) \|Av\|_{V^*} \|v\|_V \leq \bar{c}_v a(w, w) \|v\|_V^2,$$

we have

$$\|Aw\|_{V^*}^2 = \sup_{0 \neq v \in V} \frac{a(w, v)^2}{\|v\|_V^2} \leq \bar{c}_v a(w, w).$$

Analogously, we obtain

$$\|Cr\|_{Q^*}^2 \leq \bar{c}_q c(r, r).$$

Hence

$$\begin{aligned} a(w, w) + c(r, r) &\geq \frac{1}{\bar{c}_v} \|Aw\|_{V^*}^2 + \frac{1}{\bar{c}_q} \|Cr\|_{Q^*}^2 \geq \frac{1}{\max(\bar{c}_v, \bar{c}_q)} (\|Aw\|_{V^*}^2 + \|Cr\|_{Q^*}^2) \\ &= \frac{\sqrt{2}}{\bar{c}_x} \xi^2 \|(w, r)\|_X^2. \end{aligned}$$

With this estimate we obtain for the second lower bound:

$$\|\mathcal{A}z\|_{X^*} \geq \frac{\sqrt{2}}{\bar{c}_x} \xi^2 \|z\|_X.$$

Taking the maximum of the two lower bounds we immediately obtain

$$\|\mathcal{A}z\|_{X^*} \geq \varphi(\xi, \eta) \|z\|_X \quad \text{with} \quad \varphi(\xi, \eta) = \max \left[\eta - \xi, \frac{\sqrt{2}}{\bar{c}_x} \xi^2 \right].$$

Observe that

$$\xi^2 + \eta^2 \geq \underline{c}_v^2 + \underline{c}_q^2 > 0.$$

Elementary calculations show that

$$\min \{ \varphi(\xi, \eta) : \xi^2 + \eta^2 \geq \underline{c}_v^2 + \underline{c}_q^2 \} \geq \frac{3 - \sqrt{5}}{4} \frac{\underline{c}_v^2 + \underline{c}_q^2}{\max(\bar{c}_v, \bar{c}_q)} = \underline{c}_x,$$

which concludes the proof. \square

In the following two lemmas it will be shown that the conditions (2.7) and (2.8) of the last theorem can be replaced by two other conditions which will turn out to be more easy to work with.

LEMMA 2.4. *If there are constants $\underline{\gamma}_v, \bar{\gamma}_v > 0$ such that*

$$\underline{\gamma}_v \|w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \bar{\gamma}_v \|w\|_V^2 \quad \text{for all } w \in V, \quad (2.9)$$

then (2.7) is satisfied with constants $\underline{c}_v, \bar{c}_v > 0$ that depend only on $\underline{\gamma}_v, \bar{\gamma}_v$.

And, vice versa, if there are constants $\underline{c}_v, \bar{c}_v > 0$ such that (2.7) is satisfied, then (2.9) is satisfied with constants $\underline{\gamma}_v, \bar{\gamma}_v > 0$ that depend only on $\underline{c}_v, \bar{c}_v$.

Proof. Assume that (2.9) is satisfied. Then we have

$$a(w, v)^2 \leq a(w, w)a(v, v) \leq \bar{\gamma}_v a(w, w)\|v\|_V^2,$$

which implies

$$\|Aw\|_{V^*}^2 \leq \bar{\gamma}_v a(w, w).$$

Therefore,

$$\begin{aligned} \|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2 &\leq \bar{\gamma}_v a(w, w) + \|Bw\|_{Q^*}^2 \\ &\leq \max(\bar{\gamma}_v, 1) (a(w, w) + \|Bw\|_{Q^*}^2) \leq \max(\bar{\gamma}_v, 1) \bar{\gamma}_v \|w\|_V^2. \end{aligned}$$

This shows the upper bound in (2.7) for $\bar{c}_v^2 = \max(\bar{\gamma}_v, 1) \bar{\gamma}_v$.

For the lower bound observe that, for all $\varepsilon > 0$:

$$a(w, w) \leq \|Aw\|_{V^*} \|w\|_V \leq \frac{1}{2\varepsilon} \|Aw\|_{V^*}^2 + \frac{\varepsilon}{2} \|w\|_V^2, \quad (2.10)$$

which implies

$$\underline{\gamma}_v \|w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \frac{1}{2\varepsilon} \|Aw\|_{V^*}^2 + \frac{\varepsilon}{2} \|w\|_V^2 + \|Bw\|_{Q^*}^2,$$

and, therefore,

$$\left(\underline{\gamma}_v - \frac{\varepsilon}{2}\right) \|w\|_V^2 \leq \frac{1}{2\varepsilon} \|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2 \leq \max\left(\frac{1}{2\varepsilon}, 1\right) (\|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2).$$

For $\varepsilon = \underline{\gamma}_v$ we obtain the lower bound in (2.7) with $\underline{c}_v = \min(\underline{\gamma}_v, 1/2)\underline{\gamma}_v$.

Now assume that (2.7) is satisfied. Then we have, see the proof of the last theorem:

$$\|Aw\|_{V^*}^2 \leq \bar{c}_v a(w, w)$$

and, therefore,

$$\begin{aligned} a(w, w) + \|Bw\|_{Q^*}^2 &\geq \bar{c}_v^{-1} \|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2 \\ &\geq \min(1, \bar{c}_v^{-1}) (\|Aw\|_{V^*}^2 + \|Bw\|_{Q^*}^2) \geq \min(1, \bar{c}_v^{-1}) \underline{c}_v^2 \|w\|_V^2 \end{aligned}$$

showing the lower bound in (2.9) for $\underline{\gamma}_v = \min(1, \bar{c}_v^{-1}) \underline{c}_v^2$.

For the upper bound we use (2.10) for $\varepsilon = 1/2$ and obtain:

$$a(w, w) + \|Bw\|_{Q^*}^2 \leq \|Aw\|_{V^*}^2 + \frac{1}{4} \|w\|_V^2 + \|Bw\|_{Q^*}^2 \leq \left(\bar{c}_v^2 + \frac{1}{4} \right) \|w\|_V^2.$$

So, the upper bound in (2.9) is satisfied for $\bar{\gamma}_v = \bar{c}_v^2 + 1/4$. \square

Completely analogously, we have

LEMMA 2.5. *If there are constants $\underline{\gamma}_q, \bar{\gamma}_q > 0$ such that*

$$\underline{\gamma}_q \|r\|_Q^2 \leq c(r, r) + \|B^*r\|_{V^*}^2 \leq \bar{\gamma}_q \|r\|_Q^2 \quad \text{for all } r \in Q, \quad (2.11)$$

then (2.8) is satisfied with constants $\underline{c}_q, \bar{c}_q > 0$ that depend only on $\underline{\gamma}_q, \bar{\gamma}_q$.

And, vice versa, if there are constants $\underline{c}_q, \bar{c}_q > 0$ such that (2.8) is satisfied, then (2.11) is satisfied with constants $\underline{\gamma}_q, \bar{\gamma}_q > 0$ that depend only on $\underline{c}_q, \bar{c}_q$.

By summarizing the results of the last two theorems and lemmas we finally obtain

THEOREM 2.6. *If there are constants $\underline{\gamma}_v, \bar{\gamma}_v, \underline{\gamma}_q, \bar{\gamma}_q > 0$ such that*

$$\underline{\gamma}_v \|w\|_V^2 \leq a(w, w) + \|Bw\|_{Q^*}^2 \leq \bar{\gamma}_v \|w\|_V^2 \quad \text{for all } w \in V \quad (2.12)$$

and

$$\underline{\gamma}_q \|r\|_Q^2 \leq c(r, r) + \|B^*r\|_{V^*}^2 \leq \bar{\gamma}_q \|r\|_Q^2 \quad \text{for all } r \in Q, \quad (2.13)$$

then

$$\underline{c}_x \|z\|_X \leq \|Az\|_X \leq \bar{c}_x \|z\|_X \quad \text{for all } z \in X \quad (2.14)$$

is satisfied with constants $\underline{c}_x, \bar{c}_x > 0$ that depend only on $\underline{\gamma}_v, \bar{\gamma}_v, \underline{\gamma}_q, \bar{\gamma}_q$. And, vice versa, if the estimates (2.14) are satisfied with constants $\underline{c}_x, \bar{c}_x > 0$, then the estimates (2.12) and (2.13) are satisfied with constants $\underline{\gamma}_v, \bar{\gamma}_v, \underline{\gamma}_q, \bar{\gamma}_q > 0$ that depend only on $\underline{c}_x, \bar{c}_x$.

Remark 2. *In the case $C = 0$ (i.e. $c(v, v) \equiv 0$) the lower estimate in condition (2.11) has the special form*

$$\underline{\gamma}_q \|r\|_Q^2 \leq \|B^*r\|_{V^*}^2 \quad \text{for all } r \in Q. \quad (2.15)$$

From the lower estimate in (2.9) it immediately follows that

$$\underline{\gamma}_v \|w\|_V^2 \leq a(w, w) \quad \text{for all } w \in \ker B = \{v \in V : Bv = 0\}. \quad (2.16)$$

On the other hand, from (2.15) and (2.16) the lower estimate in (2.9) easily follows, using the fact that (2.15) implies

$$\underline{\gamma}_q \|w\|_V^2 \leq \|Bw\|_{Q^*}^2 \quad \text{for all } w \in (\ker B)^\perp,$$

where $(\ker B)^\perp$ denotes the orthogonal complement of $\ker B$. So we have recovered a classical result by Brezzi [7], [8]: Let a and b bounded bilinear forms and $c \equiv 0$. Then the problem (2.1) is well-posed if and only if a is coercive on $\ker B$, see (2.16), and the inf-sup condition for b is satisfied, see (2.15).

Next we want to rewrite (2.12) and (2.13) in a more convenient form. Using the definition of \mathcal{I}_Q it follows that

$$\begin{aligned} \|Bw\|_{Q^*}^2 &= \sup_{0 \neq q \in Q} \frac{\langle Bw, q \rangle^2}{\|q\|_Q^2} = \sup_{0 \neq q \in Q} \frac{(\mathcal{I}_Q^{-1}Bw, q)_Q^2}{\|q\|_Q^2} \\ &= \|\mathcal{I}_Q^{-1}Bw\|_Q^2 = (\mathcal{I}_Q^{-1}Bw, \mathcal{I}_Q^{-1}Bw)_Q = \langle Bw, \mathcal{I}_Q^{-1}Bw \rangle = \langle B^* \mathcal{I}_Q^{-1}Bw, w \rangle \end{aligned}$$

and, similarly

$$\|B^*r\|_{V^*}^2 = \langle B\mathcal{I}_V^{-1}B^*r, r \rangle.$$

Then the conditions (2.12) and (2.13) read:

$$\underline{\gamma}_v \langle \mathcal{I}_V w, w \rangle \leq \langle (A + B^* \mathcal{I}_Q^{-1} B) w, w \rangle \leq \bar{\gamma}_v \langle \mathcal{I}_V w, w \rangle \quad \text{for all } w \in V$$

and

$$\underline{\gamma}_q \langle \mathcal{I}_Q r, r \rangle \leq \langle (C + B\mathcal{I}_V^{-1}B^*)r, r \rangle \leq \bar{\gamma}_q \langle \mathcal{I}_Q r, r \rangle \quad \text{for all } r \in Q$$

or, in short,

$$\mathcal{I}_V \sim A + B^* \mathcal{I}_Q^{-1} B \quad \text{and} \quad \mathcal{I}_Q \sim C + B\mathcal{I}_V^{-1} B^*, \quad (2.17)$$

by using the following notation:

Notation 2. Let $M, N \in L(H, H^*)$ be two linear self-adjoint operators. Then

1. $M \leq N$, if and only if

$$\langle Mx, x \rangle \leq \langle Nx, x \rangle \quad \text{for all } x \in H.$$

2. $M \lesssim N$, if and only if there is a constant $c \geq 0$ such that $M \leq cN$.

3. $M \sim N$, if and only if $M \lesssim N$ and $N \lesssim M$. In this case we call M and N spectrally equivalent.

If the operators M and N depend on some parameters (like α and h , see the introduction), then we additionally assume that the involved constants are independent of those parameters.

It is clear that (2.17) is equivalent to

$$\mathcal{I}_V \sim A + B^*(C + B\mathcal{I}_V^{-1}B^*)^{-1}B \quad \text{and} \quad \mathcal{I}_Q \sim C + B\mathcal{I}_V^{-1}B^* \quad (2.18)$$

and also to

$$\mathcal{I}_Q \sim C + B(A + B^*\mathcal{I}_Q^{-1}B)^{-1}B^* \quad \text{and} \quad \mathcal{I}_V \sim A + B^*\mathcal{I}_Q^{-1}B. \quad (2.19)$$

The equivalent pairs of conditions (2.17), (2.18), and (2.19) are conditions for \mathcal{I}_V and \mathcal{I}_Q , or in other words, for the inner products in V and Q . In (2.18) as well as in (2.19) the first condition involves only one unknown operator (\mathcal{I}_V in (2.18) and \mathcal{I}_Q in (2.19)). The second condition (in (2.18) as well as in (2.19)) serves as a sort of definition of the second operator in terms of the first one.

3. Special cases. In this section we will discuss the conditions (2.17) for several important cases. We will focus on the finite-dimensional case, on the one hand since we are mainly interested in the construction of preconditioners for discretized problems, on the other hand to avoid some technical difficulties in the infinite-dimensional case. Nevertheless, all results for the finite-dimensional case can be carried over to the infinite-dimensional case under appropriate conditions.

If V and Q are finite-dimensional, then all operators can be represented as matrices acting on vectors of real numbers representing the elements in V and Q with respect to some chosen bases. In this matrix-vector notation problem (2.5) becomes a linear system of the following form

$$\mathcal{A} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{with} \quad \mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}, \quad (3.1)$$

where B^T denotes the transposed matrix. For the matrix \mathcal{I}_X representing the inner product in X we have the special form

$$\mathcal{I}_X = \begin{bmatrix} \mathcal{I}_V & 0 \\ 0 & \mathcal{I}_Q \end{bmatrix}$$

because of (1.4). Condition (2.6) is satisfied for all constants $\underline{c}_x, \bar{c}_x > 0$ with

$$\underline{c}_x \leq |\lambda_{\min}| \quad \text{and} \quad |\lambda_{\max}| \leq \bar{c}_x,$$

where λ_{\max} and λ_{\min} are eigenvalues of the generalized eigenvalue problem

$$\mathcal{A}x = \lambda \mathcal{I}_X x$$

of maximal and minimal modulus, respectively. For the condition number we then obtain

$$\kappa(\mathcal{A}) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \leq \frac{\bar{c}_x}{\underline{c}_x}.$$

In the survey article [4] a wide range of preconditioners for linear systems of the form (3.1) are discussed, among other topics. Following the notation of [4] our focus is the case of block diagonal preconditioners $\mathcal{P} = \mathcal{I}_X$ and our particular interest is the issue of robustness.

We first consider two simple and well-known cases:

3.1. A and $S = C + BA^{-1}B^T$ are non-singular. The matrix S is called the (negative) Schur complement. In this case

$$\mathcal{I}_V = A \quad \text{and} \quad \mathcal{I}_Q = S = C + BA^{-1}B^T \quad (3.2)$$

satisfy (2.17): Since

$$0 \leq B^T(C + B\mathcal{I}_V^{-1}B^T)^{-1}B \leq \mathcal{I}_V \quad \text{for all } \mathcal{I}_V,$$

we have for $\mathcal{I}_V = A$:

$$\mathcal{I}_V = A \leq A + B^T\mathcal{I}_Q^{-1}B = A + B^T(C + B\mathcal{I}_V^{-1}B^T)^{-1}B \leq A + \mathcal{I}_V = 2A = 2\mathcal{I}_V.$$

So, (2.12) holds for $\underline{\gamma}_v = 1$ and $\bar{\gamma}_v = 2$, while (2.13) trivially holds for $\underline{\gamma}_q = \bar{\gamma}_q = 1$. Then robust estimates directly follow from Theorem 2.6.

Remark 3. For $C = 0$ we have direct access to the constants in (2.6). The generalized eigenvalue problem

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \lambda \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}$$

has exactly 3 eigenvalues, namely 1 and $(1 \pm \sqrt{5})/2$ (due to, e.g., Yu. Kuznetsov, 1990). Therefore, (2.6) is satisfied for

$$\underline{c}_x = \frac{\sqrt{5}-1}{2} \quad \text{and} \quad \bar{c}_x = \frac{\sqrt{5}+1}{2}.$$

3.2. C and $R = A + B^T C^{-1} B$ are non-singular. Analogous to the first case one can show that

$$\mathcal{I}_V = R = A + B^T C^{-1} B \quad \text{and} \quad \mathcal{I}_Q = C \quad (3.3)$$

satisfy (2.17).

3.3. A and C are non-singular. Then, of course, both solutions, (3.2) and (3.3), are available. It is easy to conclude from (2.18) and (2.19) that for all possible choices of \mathcal{I}_V and \mathcal{I}_Q satisfying (2.17) we have

$$A \lesssim \mathcal{I}_V \lesssim A + B^T C^{-1} B \quad \text{and} \quad C \lesssim \mathcal{I}_Q \lesssim C + B A^{-1} B^T.$$

So, the solutions (3.2) and (3.3) determine the lower and upper bounds for the range of all possible solutions.

We will now construct additional solutions to (2.17). With

$$\hat{\mathcal{I}}_V = A^{-1/2} \mathcal{I}_V A^{-1/2}, \quad \hat{\mathcal{I}}_Q = C^{-1/2} \mathcal{I}_Q C^{-1/2}, \quad \text{and} \quad \hat{B} = C^{-1/2} B A^{-1/2}$$

condition (2.17) takes the following form:

$$\hat{\mathcal{I}}_V \sim I + \hat{B}^T \hat{\mathcal{I}}_Q^{-1} \hat{B} \quad \text{and} \quad \hat{\mathcal{I}}_Q \sim I + \hat{B} \hat{\mathcal{I}}_V^{-1} \hat{B}^T. \quad (3.4)$$

We first discuss the special case where the equivalence relation \sim is replaced by equality:

$$\hat{\mathcal{I}}_V = I + \hat{B}^T \hat{\mathcal{I}}_Q^{-1} \hat{B} \quad \text{and} \quad \hat{\mathcal{I}}_Q = I + \hat{B} \hat{\mathcal{I}}_V^{-1} \hat{B}^T.$$

Eliminating $\hat{\mathcal{I}}_Q$ from the first equation with the help of the second equation yields:

$$\hat{\mathcal{I}}_V = I + \hat{B}^T (I + \hat{B} \hat{\mathcal{I}}_V^{-1} \hat{B}^T)^{-1} \hat{B}. \quad (3.5)$$

By using the Sherman-Morrison-Woodbury formula, see [9], we have

$$(I + \hat{B} \hat{\mathcal{I}}_V^{-1} \hat{B}^T)^{-1} = I - \hat{B} (\hat{\mathcal{I}}_V + \hat{B}^T \hat{B})^{-1} \hat{B}^T.$$

This allows to rewrite the condition (3.5):

$$\hat{\mathcal{I}}_V = I + \hat{\mathcal{I}}_V - \hat{\mathcal{I}}_V (\hat{\mathcal{I}}_V + \hat{B}^T \hat{B})^{-1} \hat{\mathcal{I}}_V,$$

which simplifies to the quadratic matrix equation

$$\left(\hat{\mathcal{I}}_V\right)^2 - \hat{\mathcal{I}}_V - \hat{B}^T \hat{B} = 0.$$

It is easy to see that

$$\hat{\mathcal{I}}_V = f(\hat{B}^T \hat{B}) \quad \text{with} \quad f(x) = \frac{1}{2} + \sqrt{\frac{1}{4} + x}$$

solves this equation, see [10] for matrix functions. For \mathcal{I}_Q we obtain similarly:

$$\hat{\mathcal{I}}_Q = f(\hat{B} \hat{B}^T).$$

We now return from the equality conditions to the original equivalence conditions (3.4). Obviously, all matrices $\hat{\mathcal{I}}_V$ and $\hat{\mathcal{I}}_Q$ which are spectrally equivalent to $f(\hat{B}^T \hat{B})$ and $f(\hat{B} \hat{B}^T)$, respectively, satisfy the equivalence conditions (3.4). Since

$$\sqrt{1+x} \leq f(x) \leq \frac{2}{\sqrt{3}} \sqrt{1+x} \quad \text{for all } x \geq 0,$$

we have

$$f(\hat{B}^T \hat{B}) \sim (I + \hat{B}^T \hat{B})^{1/2} \quad \text{and} \quad f(\hat{B} \hat{B}^T) \sim (I + \hat{B} \hat{B}^T)^{1/2}.$$

Therefore,

$$\hat{\mathcal{I}}_V = (I + \hat{B}^T \hat{B})^{1/2} \quad \text{and} \quad \hat{\mathcal{I}}_Q = (I + \hat{B} \hat{B}^T)^{1/2}$$

satisfy (3.4). If expressed in terms on the original matrices, we obtain

$$\mathcal{I}_V = A^{1/2} \left(A^{-1/2} S A^{-1/2} \right)^{1/2} A^{1/2} \quad \text{and} \quad \mathcal{I}_Q = C^{1/2} \left(C^{-1/2} R C^{-1/2} \right)^{1/2} C^{1/2},$$

or, in short:

$$\mathcal{I}_V = [A, S]_{1/2} \quad \text{and} \quad \mathcal{I}_Q = [C, R]_{1/2},$$

using the following notation:

Notation 3. Let M and N be symmetric and positive definite n -by- n matrices. Then, for all $\theta \in [0, 1]$, the symmetric and positive definite matrix $[M, N]_\theta$ is given by

$$[M, N]_\theta = M^{1/2} \left(M^{-1/2} N M^{-1/2} \right)^\theta M^{1/2}.$$

Remark 4. Each of the symmetric and positive definite matrices M and N represents an inner product and, therefore, a Hilbert space structure on \mathbb{R}^n . For each $\theta \in (0, 1)$ an intermediate Hilbert space structure on \mathbb{R}^n can be defined using the so-called real method, see [5], whose inner product is represented by the symmetric and positive definite matrix $[M, N]_\theta$ up to a scaling factor.

More generally, we obtain solutions to (3.4) of the form

$$\hat{\mathcal{I}}_V = (I + \hat{B}^T \hat{B})^\theta \quad \text{and} \quad \hat{\mathcal{I}}_Q = (I + \hat{B} \hat{B}^T)^{1-\theta}$$

for each $\theta \in [0, 1]$. This is a direct consequence of the inequalities:

$$(1+x)^\theta \leq 1 + \frac{x}{(1+x)^{1-\theta}} \leq 2(1+x)^\theta \quad \text{for all } x \geq 0.$$

Translating this result to the untransformed quantities leads to the following solutions to (2.17):

$$\mathcal{I}_V = [A, S]_\theta \quad \text{and} \quad \mathcal{I}_Q = [C, R]_{1-\theta} = [R, C]_\theta. \quad (3.6)$$

Remark 5. This family of solutions can also be derived from the solutions (3.2) and (3.3) by using the interpolation theorem, see [5].

Since

$$\frac{1}{2^{1-\theta}}(1+x^\theta) \leq (1+x)^\theta \leq 1+x^\theta \quad \text{for all } x \geq 0,$$

it easily follows that

$$(I + \hat{B}^T \hat{B})^\theta \sim I + (\hat{B}^T \hat{B})^\theta \quad \text{and} \quad (I + \hat{B} \hat{B}^T)^{1-\theta} \sim I + (\hat{B} \hat{B}^T)^{1-\theta},$$

which implies that

$$\mathcal{I}_V = A + [A, B^T C^{-1} B]_\theta \quad \text{and} \quad \mathcal{I}_Q = C + [C, B A^{-1} B^T]_{1-\theta}. \quad (3.7)$$

also satisfy (2.17). This form of a solution will turn out to be particularly useful for the optimal control problems in Section 4.

3.4. $C = 0$. In this case we will formulate a general strategy for constructing inner products if

$$A = A_1 + A_2$$

with symmetric and positive semi-definite matrices A_i under the assumption that we already know robust estimates for

$$\mathcal{A}_i = \begin{bmatrix} A_i & B^T \\ B & 0 \end{bmatrix} \quad \text{for } i = 1, 2.$$

That is, we assume that \mathcal{I}_{V_i} and \mathcal{I}_{Q_i} are available such that

$$\mathcal{I}_{V_i} \sim A_i + B^T \mathcal{I}_{Q_i}^{-1} B \quad \text{and} \quad \mathcal{I}_{Q_i} \sim B \mathcal{I}_{V_i}^{-1} B^T \quad \text{for } i = 1, 2.$$

Then, by adding the first equivalence conditions, we immediately

$$\mathcal{I}_V \sim A + B^T \mathcal{I}_Q^{-1} B$$

for

$$\mathcal{I}_V = \mathcal{I}_{V_1} + \mathcal{I}_{V_2} \quad \text{and} \quad \mathcal{I}_Q = \left(\mathcal{I}_{Q_1}^{-1} + \mathcal{I}_{Q_2}^{-1} \right)^{-1}. \quad (3.8)$$

So, the first condition in (2.17) holds for this choice of \mathcal{I}_V and \mathcal{I}_Q . The second condition in (2.17) can be split into two parts:

$$B \mathcal{I}_V^{-1} B^T \lesssim \mathcal{I}_Q \quad \text{and} \quad \mathcal{I}_Q \lesssim B \mathcal{I}_V^{-1} B^T.$$

The first part follows easily: We have

$$B\mathcal{I}_{V_i}^{-1}B^T \lesssim \mathcal{I}_{Q_i} \quad \text{for } i = 1, 2,$$

which is equivalent to

$$B^T\mathcal{I}_{Q_i}^{-1}B \lesssim \mathcal{I}_{V_i} \quad \text{for } i = 1, 2.$$

By adding these conditions we immediately obtain

$$B^T\mathcal{I}_Q^{-1}B \lesssim \mathcal{I}_V,$$

which is equivalent to

$$B\mathcal{I}_V^{-1}B^T \lesssim \mathcal{I}_Q.$$

For the second part we need the additional assumption that there is a left inverse T of B^T with

$$T^T\mathcal{I}_{Q_i}T \lesssim I_{V_i}^{-1} \quad \text{for } i = 1, 2. \quad (3.9)$$

These conditions are equivalent to

$$T\mathcal{I}_{V_i}T^T \lesssim I_{Q_i}^{-1} \quad \text{for } i = 1, 2.$$

By adding we obtain

$$T\mathcal{I}_VT^T \lesssim I_Q^{-1},$$

or, equivalently,

$$T^T\mathcal{I}_QT \lesssim I_V^{-1}.$$

By multiplying with B from the left and with B^T from the right we immediately obtain

$$\mathcal{I}_Q \lesssim B\mathcal{I}_V^{-1}B^T.$$

The strategy presented here resembles exactly the approach presented in [14] for the case of the generalized Stokes problem, for which the existence of T satisfying (3.9) is known.

Summarizing the results in this section, we have identified several pairs $(\mathcal{I}_V, \mathcal{I}_Q)$ of matrices, see (3.2), (3.3), (3.6), (3.7), and (3.8), representing inner products, which satisfy (2.17) under appropriate assumptions. Which of these pairs are more appropriate for preconditioning than others depend on the particular application. Two applications from optimal control will be discussed next.

4. Applications. Let Ω be an open and bounded domain in \mathbb{R}^d for $d \in \{1, 2, 3\}$ with Lipschitz-continuous boundary Γ and let $L^2(\Omega)$, $H^1(\Omega)$, and $H_0^1(\Omega)$ be the usual Lebesgue space and Sobolev spaces of functions on Ω . The inner product and the norm in $L^2(\Omega)$ are denoted by $(\cdot, \cdot)_{L^2}$ and $\|\cdot\|_{L^2}$, respectively.

4.1. Distributed optimal control of elliptic equations. We consider the following optimal control problem: Find the state $y \in H_0^1(\Omega)$ and the control $u \in L^2(\Omega)$ that minimizes the cost functional

$$J(y, u) = \frac{1}{2} \|y - y_d\|_{L^2}^2 + \frac{\alpha}{2} \|u\|_{L^2}^2$$

subject to the state equation

$$\begin{aligned} -\Delta y &= u & \text{in } \Omega, \\ y &= 0 & \text{on } \Gamma, \end{aligned}$$

or, more precisely, subject to the state equation in its weak form, given by

$$(\text{grad } y, \text{grad } z)_{L^2} = (u, z)_{L^2} \quad \text{for all } z \in H_0^1(\Omega).$$

Here $y_d \in L^2(\Omega)$ is the given (desired) state and $\alpha > 0$ is a regularization parameter.

Remark 6. For ease of notation we will use the symbols $(\cdot, \cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ not only for the case of scalar functions but also for the case of vector-valued functions and later on also for the case of matrix-valued functions.

The Lagrangian functional associated to this optimization problem is given by:

$$\mathcal{L}(y, u, p) = J(y, u) + (\text{grad } y, \text{grad } p)_{L^2} - (u, p)_{L^2},$$

leading to following optimality system

$$\begin{aligned} (y, z)_{L^2} + (\text{grad } z, \text{grad } p)_{L^2} &= (y_d, z)_{L^2} & \text{for all } z \in H_0^1(\Omega), \\ \alpha (u, v)_{L^2} - (v, p)_{L^2} &= 0 & \text{for all } v \in L^2(\Omega), \\ (\text{grad } y, \text{grad } q)_{L^2} - (u, q)_{L^2} &= 0 & \text{for all } q \in H_0^1(\Omega), \end{aligned}$$

which characterizes the solution $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ of the optimal control problem with Lagrangian multiplier (or co-state) $p \in H_0^1(\Omega)$, see, e.g., [11], [23].

From the second equation we learn that $u = \alpha^{-1} p$, which allows to eliminate the control resulting in the reduced optimality system

$$\begin{aligned} (y, z)_{L^2} + (\text{grad } z, \text{grad } p)_{L^2} &= (y_d, z)_{L^2} & \text{for all } z \in H_0^1(\Omega), \\ (\text{grad } y, \text{grad } q)_{L^2} - \alpha^{-1} (p, q)_{L^2} &= 0 & \text{for all } q \in H_0^1(\Omega). \end{aligned}$$

As an example of a discretization method we discuss the finite element method on a simplicial subdivision of Ω with continuous and piecewise linear functions for both the state and the co-state. This leads to the linear system

$$\begin{bmatrix} M & K \\ K & -\alpha^{-1} M \end{bmatrix} \begin{bmatrix} \underline{y} \\ \underline{p} \end{bmatrix} = \begin{bmatrix} \underline{f} \\ \underline{0} \end{bmatrix}$$

for the unknown vectors $\underline{y}, \underline{p}$ of coefficients of the approximate solutions relative to the nodal basis. Here M denotes the mass matrix representing the L^2 inner product and K denotes the stiffness matrix representing the elliptic operator of the state equation on the finite element space.

This linear system fits into the general framework of Sections 2 and 3 with

$$A = M, \quad B = K, \quad C = \alpha^{-1} M.$$

One particular pair of matrices \mathcal{I}_V and \mathcal{I}_Q satisfying (2.17) is given by (3.7) with $\theta = 1/2$:

$$\mathcal{I}_V = A + [A, B^T C^{-1} B]_{1/2} = M + \alpha^{1/2} [M, K M^{-1} K]_{1/2}$$

and

$$\mathcal{I}_Q = C + [C, B A^{-1} B^T]_{1/2} = \alpha^{-1} M + \alpha^{-1/2} [M, K M^{-1} K]_{1/2}.$$

Now

$$\begin{aligned} [M, K M^{-1} K]_{1/2} &= M^{1/2} \left(\underbrace{M^{-1/2} K M^{-1} K M^{-1/2}} \right)^{1/2} M^{1/2} = K. \\ &= (M^{-1/2} K M^{-1/2})^2 \end{aligned}$$

Hence we obtain

$$\mathcal{I}_V = M + \alpha^{1/2} K \quad \text{and} \quad \mathcal{I}_Q = \alpha^{-1} M + \alpha^{-1/2} K. \quad (4.1)$$

From the analysis in Sections 2 and 3 it follows that

$$\mathcal{P} = \begin{bmatrix} M + \alpha^{1/2} K & 0 \\ 0 & \alpha^{-1} M + \alpha^{-1/2} K \end{bmatrix}$$

is a robust block diagonal preconditioner for

$$\mathcal{A} = \begin{bmatrix} M & K \\ K & -\alpha^{-1} M \end{bmatrix}. \quad (4.2)$$

The application of the preconditioner \mathcal{P} requires an efficient evaluation of $\mathcal{P}^{-1}r$ for some given vector r . Up to a scaling factor both diagonal blocks of \mathcal{P} are of the form $\gamma M + K$. This matrix is the stiffness matrix of the second-order elliptic differential operator of the state equation perturbed by a zero-order term. Multigrid or multilevel preconditioners which work robustly in γ are well-known, see, e.g., [16], [6]. So, in practice, the block matrices of the theoretical preconditioner \mathcal{P} are replaced by such efficient preconditioners.

The same analysis can be carried out not only on the discrete level but also on the continuous level leading to the following inner products in $V = H_0^1(\Omega)$ for the state

$$(y, z)_V = (y, z)_{L^2} + \alpha^{1/2} (\text{grad } y, \text{grad } z)_{L^2}$$

and in $Q = H_0^1(\Omega)$ for the co-state

$$(p, q)_Q = \alpha^{-1} (p, q)_{L^2} + \alpha^{-1/2} (\text{grad } p, \text{grad } q)_{L^2}.$$

Well-posedness with robust estimates then follows for the associated norms

$$\|z\|_V = \left(\|z\|_{L^2}^2 + \alpha^{1/2} \|\text{grad } z\|_{L^2}^2 \right)^{1/2}$$

and

$$\|q\|_Q = \left(\alpha^{-1} \|q\|_{L^2}^2 + \alpha^{-1/2} \|\text{grad } q\|_{L^2}^2 \right)^{1/2},$$

which differ from the standard H^1 -norms.

The particular form of the elliptic operator for y in the state equation does not play an essential role as long as the associated bilinear form of the weak formulation is symmetric, bounded and coercive. See [22] for a related elliptic optimal control problem, where robust block-preconditioners have been constructed using the non-standard inner products and norms from above, and [21], where robust all-at-once multigrid methods were analyzed based on these non-standard norms. The form of the control and the cost functional, however, was essential for constructing the inner products. The results cannot be easily extended to other cases.

Remark 7. If instead of (3.7) one uses the solution (3.2), then one obtains

$$\mathcal{I}_V = M \quad \text{and} \quad \mathcal{I}_Q = \alpha^{-1} M + KM^{-1}K,$$

also leading to a robust block diagonal preconditioner. While \mathcal{I}_V is here much easier than before, the matrix \mathcal{I}_Q requires more work. It can be interpreted as matrix representation of a fourth-order differential operator. For preconditioners developed along this lines see, e.g., [18]. These preconditioners were shown to be robust with respect to h (but not with respect to the regularization parameter).

Remark 8. More generally, for any 2-by-2 block matrix \mathcal{A} of the form (4.2), the matrix \mathcal{P} of the form (4.1) is a robust preconditioner as long as M and K are symmetric and positive semi-definite with $\ker M \cap \ker K = \{0\}$. This follows from a simple analysis of the generalized eigenvalue problem $\mathcal{A}x = \lambda \mathcal{P}x$ for which it can be easily shown that

$$|\lambda_{\min}| \geq \frac{1}{\sqrt{2}} \quad \text{and} \quad |\lambda_{\max}| \leq 1,$$

resulting in a condition number estimate $\kappa(\mathcal{A}) \leq \sqrt{2}$.

4.2. Distributed optimal control of the Stokes equations. The second example of an optimal control problem is the so-called velocity tracking problem for Stokes flow with distributed control: Find the velocity $u \in H_0^1(\Omega)^d$, the pressure $p \in L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\}$ and the control $f \in L^2(\Omega)^d$ that minimizes the cost functional

$$J(u, f) = \frac{1}{2} \|u - u_d\|_{L^2}^2 + \frac{\alpha}{2} \|f\|_{L^2}^2$$

subject to the state equations

$$\begin{aligned} -\Delta u + \text{grad } p &= f & \text{in } \Omega, \\ \text{div } u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned}$$

or, more precisely again, subject to the state equations in its weak form:

$$\begin{aligned} (\text{grad } u, \text{grad } v)_{L^2} - (\text{div } v, p)_{L^2} &= (f, v)_{L^2} & \text{for all } v \in H_0^1(\Omega)^d, \\ -(\text{div } u, q)_{L^2} &= 0 & \text{for all } q \in L_0^2(\Omega). \end{aligned}$$

Here $u_d \in L^2(\Omega)^d$ is the desired velocity and $\alpha > 0$ is a regularization parameter.

The Lagrangian functional associated to this optimization problem is given by:

$$\begin{aligned} \mathcal{L}(u, p, f, \hat{u}, \hat{p}) \\ = J(u, f) + (\text{grad } u, \text{grad } \hat{u})_{L^2} - (\text{div } \hat{u}, p)_{L^2} - (\text{div } u, \hat{p})_{L^2} - (f, \hat{u})_{L^2} \end{aligned}$$

leading to following optimality system

$$\begin{aligned}
(u, v)_{L^2} & + (\nabla v, \nabla \hat{u})_{L^2} - (\operatorname{div} v, \hat{p})_{L^2} = (u_d, v)_{L^2}, \\
& - (\operatorname{div} \hat{u}, q)_{L^2} = 0, \\
\alpha (f, \phi)_{L^2} - (\phi, \hat{u})_{L^2} & = 0, \\
(\nabla u, \nabla \hat{v})_{L^2} - (\operatorname{div} \hat{v}, p)_{L^2} - (f, \hat{v})_{L^2} & = 0, \\
- (\operatorname{div} u, \hat{q})_{L^2} & = 0
\end{aligned}$$

for all test functions $v, \hat{v} \in H_0^1(\Omega)^d$, $q, \hat{q} \in L_0^2(\Omega)$, and $\phi \in L^2(\Omega)^d$. This system characterizes the solution $(y, p, f) \in H_0^1(\Omega)^d \times L_0^2(\Omega) \times L^2(\Omega)^d$ of the optimal control problem with Lagrangian multipliers $(\hat{u}, \hat{p}) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$.

As in the elliptic case the control f can be eliminated (using the third equation) resulting in the reduced optimality system:

$$\begin{aligned}
(u, v)_{L^2} & + (\nabla v, \nabla \hat{u})_{L^2} - (\operatorname{div} v, \hat{p})_{L^2} = (u_d, v)_{L^2}, \\
& - (\operatorname{div} \hat{u}, q)_{L^2} = 0, \\
(\nabla u, \nabla \hat{v})_{L^2} - (\operatorname{div} \hat{v}, p)_{L^2} - \alpha^{-1} (\hat{u}, \hat{v})_{L^2} & = 0, \\
- (\operatorname{div} u, \hat{q})_{L^2} & = 0
\end{aligned}$$

for all test functions $v, \hat{v} \in H_0^1(\Omega)^d$ and $q, \hat{q} \in L_0^2(\Omega)$.

As an example of a discretization method we discuss the Taylor-Hood element on a simplicial subdivision of Ω consisting of continuous and piecewise quadratic functions for u and \hat{u} and continuous and piecewise linear functions for p and \hat{p} . This leads to the linear system

$$\begin{bmatrix} M & & K & -D^T \\ & 0 & D & \\ K & -D^T & -\alpha^{-1} M & \\ -D & & & 0 \end{bmatrix} \begin{bmatrix} u \\ p \\ \hat{u} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} M u_d \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here M denotes the mass matrix representing the standard inner product in $L^2(\Omega)^d$ and K denotes the stiffness matrix representing the vector Laplace operator on the finite element space. Additionally, D denotes the matrix representation of the divergence operator on the involved finite element spaces. Observe that D is of full rank, since the Taylor-Hood element satisfies a discrete inf-sup condition (under mild conditions on the underlying mesh).

This linear system fits into the general framework of Section 2 with

$$A = \begin{bmatrix} M & \\ & 0 \end{bmatrix}, \quad B = \begin{bmatrix} K & -D^T \\ -D & \end{bmatrix}, \quad C = \begin{bmatrix} \alpha^{-1} M & \\ & 0 \end{bmatrix} = \alpha^{-1} A,$$

but it does not fit into any of the special cases discussed in Section 3. So we have to go back to the original conditions (2.17). Because of

$$C = \alpha^{-1} A \quad \text{and} \quad B^T = B,$$

it is natural to make the ansatz

$$\mathcal{I}_Q = \alpha^{-1} \mathcal{I}_V. \tag{4.3}$$

Then both conditions in (2.17) reduce to one condition for \mathcal{I}_V , namely:

$$A + \alpha B\mathcal{I}_V^{-1}B \sim \mathcal{I}_V. \quad (4.4)$$

We will now try to find such a matrix \mathcal{I}_V which is of a block diagonal form:

$$\mathcal{I}_V = \begin{bmatrix} \mathcal{I}_u & \\ & \mathcal{I}_p \end{bmatrix}.$$

Then condition (4.4) reads

$$\begin{bmatrix} M + \alpha K\mathcal{I}_u^{-1}K + \alpha D^T\mathcal{I}_p^{-1}D & -\alpha K\mathcal{I}_u^{-1}D^T \\ -\alpha D\mathcal{I}_u^{-1}K & \alpha D\mathcal{I}_u^{-1}D^T \end{bmatrix} \sim \begin{bmatrix} \mathcal{I}_u & \\ & \mathcal{I}_p \end{bmatrix} \quad (4.5)$$

after expanding the matrix expression on the left-hand side. For discussing this equivalence relation the following lemma is very helpful:

LEMMA 4.1. *Let \mathcal{M} be a symmetric and positive definite 2-by-2 block matrix and \mathcal{D} a 2-by-2 block diagonal matrix with symmetric and positive definite diagonal blocks. Then*

$$\mathcal{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \sim \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} = \mathcal{D}$$

if and only if

$$M_{11} \sim D_{11}, \quad \text{and} \quad M_{22} \sim D_{22}, \quad \text{and} \quad M_{11} \lesssim M_{11} - M_{12}M_{22}^{-1}M_{21} \quad (4.6)$$

A proof of this result can be found in [1] in combination with [24].

If Lemma 4.1 is applied to (4.5), the first two conditions in (4.6) read

$$M + \alpha K\mathcal{I}_u^{-1}K + \alpha D^T\mathcal{I}_p^{-1}D^T \sim \mathcal{I}_u \quad \text{and} \quad \alpha D\mathcal{I}_u^{-1}D^T \sim \mathcal{I}_p. \quad (4.7)$$

The second condition in (4.7) is trivially satisfied for the choice

$$\mathcal{I}_p = \alpha D\mathcal{I}_u^{-1}D^T. \quad (4.8)$$

Then the first condition in (4.7) simplifies to

$$M + \alpha K\mathcal{I}_u^{-1}K \sim \mathcal{I}_u, \quad (4.9)$$

since

$$0 \leq \alpha D^T\mathcal{I}_p^{-1}D = D^T (D\mathcal{I}_u^{-1}D^T)^{-1} D \leq \mathcal{I}_u.$$

It is easy to check that

$$\mathcal{I}_u = M + \alpha^{1/2} K \quad (4.10)$$

satisfies Condition (4.9). It remains to verify the third condition in (4.6), which reads here:

$$\mathcal{I}_u \lesssim R_u \quad (4.11)$$

with

$$R_u = M + \alpha K\mathcal{I}_u^{-1}K + \alpha D^T\mathcal{I}_p^{-1}D - \alpha K\mathcal{I}_u^{-1}D^T (D\mathcal{I}_u^{-1}D^T)^{-1} D\mathcal{I}_u^{-1}K.$$

THEOREM 4.2. *Condition (4.11) is satisfied for the matrices \mathcal{I}_u and \mathcal{I}_p , given by (4.10) and (4.8), respectively.*

Proof. We have

$$R_u = M + \alpha D^T \mathcal{I}_p^{-1} D + \alpha \left[K \mathcal{I}_u^{-1} K - K \mathcal{I}_u^{-1} D^T (D \mathcal{I}_u^{-1} D^T)^{-1} D \mathcal{I}_u^{-1} K \right].$$

Since

$$D^T (D \mathcal{I}_u^{-1} D^T)^{-1} D \leq \mathcal{I}_u, \quad (4.12)$$

it follows that

$$K \mathcal{I}_u^{-1} K - K \mathcal{I}_u^{-1} D^T (D \mathcal{I}_u^{-1} D^T)^{-1} D \mathcal{I}_u^{-1} K \geq K \mathcal{I}_u^{-1} K - K \mathcal{I}_u^{-1} \mathcal{I}_u \mathcal{I}_u^{-1} K = 0.$$

Therefore, we obtain as a first estimate

$$R_u \geq M + \alpha D^T \mathcal{I}_p^{-1} D. \quad (4.13)$$

For deriving a second estimate we start with the following observation:

$$\begin{aligned} K \mathcal{I}_u^{-1} D^T (D \mathcal{I}_u^{-1} D^T)^{-1} D \mathcal{I}_u^{-1} K &= \alpha K \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} K \\ &= (\alpha^{1/2} K) \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} (\alpha^{1/2} K) = (\mathcal{I}_u - M) \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} (\mathcal{I}_u - M) \\ &= D^T \mathcal{I}_p^{-1} D - M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D - D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M + M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M. \end{aligned}$$

Therefore,

$$\begin{aligned} R_u &= M + \alpha K \mathcal{I}_u^{-1} K - \alpha M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M \\ &\quad + \alpha (M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D + D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M) \end{aligned}$$

From (4.12) it follows that

$$\alpha \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} = \mathcal{I}_u^{-1} D^T (D \mathcal{I}_u^{-1} D^T)^{-1} D \mathcal{I}_u \leq \mathcal{I}_u^{-1} \mathcal{I}_u \mathcal{I}_u^{-1} = \mathcal{I}_u^{-1}.$$

Hence

$$\begin{aligned} M + \alpha K \mathcal{I}_u^{-1} K - \alpha M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M \\ \geq M + \alpha K \mathcal{I}_u^{-1} K - M \mathcal{I}_u^{-1} M = \sqrt{\alpha} K. \end{aligned} \quad (4.14)$$

Furthermore, we have for all vectors v :

$$\begin{aligned} \alpha v^T (M \mathcal{I}_u^{-1} D^T \mathcal{I}_p^{-1} D + D^T \mathcal{I}_p^{-1} D \mathcal{I}_u^{-1} M) v \\ &= 2v^T M \mathcal{I}_u^{-1} (\alpha D^T \mathcal{I}_p^{-1} D) v \\ &= 2v^T M \mathcal{I}_u^{-1/2} \mathcal{I}_u^{-1/2} (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} v \\ &\geq -2 \left\| \mathcal{I}_u^{-1/2} M v \right\| \left\| \mathcal{I}_u^{-1/2} (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} v \right\| \left\| (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} v \right\| \\ &\geq -2 \left\| \mathcal{I}_u^{-1/2} M v \right\| \left\| (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} v \right\| \geq - \left\| \mathcal{I}_u^{-1/2} M v \right\|^2 - \left\| (\alpha D^T \mathcal{I}_p^{-1} D)^{1/2} v \right\|^2 \\ &= -v^T M \mathcal{I}_u^{-1} M v - v^T (\alpha D^T \mathcal{I}_p^{-1} D) v \geq -v^T M v - v^T (\alpha D^T \mathcal{I}_p^{-1} D) v. \end{aligned}$$

This shows that

$$\alpha (M\mathcal{I}_u^{-1}D^T\mathcal{I}_p^{-1}D + D^T\mathcal{I}_p^{-1}D\mathcal{I}_u^{-1}M) \geq -(M + \alpha D^T\mathcal{I}_p^{-1}D). \quad (4.15)$$

From (4.14) and (4.15) we obtain the second estimate for R_u :

$$R_u \geq \sqrt{\alpha} K - (M + \alpha D^T\mathcal{I}_p^{-1}D).$$

Finally, by combining this estimate with (4.13) it follows that

$$\begin{aligned} 3R_u &= 2R_u + R_u \geq 2(M + \alpha D^T\mathcal{I}_p^{-1}D) + \sqrt{\alpha} K - (M + \alpha D^T\mathcal{I}_p^{-1}D) \\ &= M + \alpha D^T\mathcal{I}_p^{-1}D + \sqrt{\alpha} K \geq M + \sqrt{\alpha} K = \mathcal{I}_u, \end{aligned}$$

which completes the proof. \square

Therefore, summarizing (4.3), (4.10), and (4.8), we have shown that

$$\mathcal{P} = \begin{bmatrix} \mathcal{I}_u & & & \\ & \alpha D\mathcal{I}_u^{-1}D^T & & \\ & & \alpha^{-1}\mathcal{I}_u & \\ & & & D\mathcal{I}_u^{-1}D^T \end{bmatrix} \quad \text{with } \mathcal{I}_u = M + \alpha^{1/2} K \quad (4.16)$$

is a robust block-diagonal preconditioner of

$$\mathcal{A} = \begin{bmatrix} M & & K & -D^T \\ & 0 & D & \\ K & -D^T & -\alpha^{-1}M & \\ -D & & & 0 \end{bmatrix}.$$

Up to a scaling factor the diagonal blocks of \mathcal{P} are of the form $\gamma M + K$ or $D(\gamma M + K)^{-1}D^T$. As already mentioned for the elliptic optimal control problem multigrid or multilevel preconditioners for $\gamma M + K$ are available which work robustly in γ . This is also the case for the matrix $D(\gamma M + K)^{-1}D^T$, which is the Schur complement of a discretized generalized Stokes problem, see [15], [12], [13]. So, in practice, the block matrices of the theoretical preconditioner \mathcal{P} are replaced by such efficient preconditioners.

Remark 9. As for the elliptic problem the same analysis can also be done on the continuous level leading the corresponding non-standards norms in $H_0^1(\Omega) \times L_0^2(\Omega)$ for u and p as well as for the Lagrangian multipliers \hat{u} and \hat{p} .

Remark 10. If the objective functional of the optimal control problem for the Stokes equations contains an additional L^2 -term for the pressure, then the (1,1) block of the system matrix becomes non-singular, and, as an alternative, the block preconditioner based on (3.2) could be used as a theoretical preconditioner. As in the elliptic case the Schur complement S can be interpreted as discretization matrix of a system of partial differential equations including fourth-order differential operators. For preconditioners developed along this lines see, e.g., [19]. These preconditioners were shown to be robust with respect to h (but not with respect to the regularization parameter).

5. Concluding Remarks. The equivalence relations in (2.17) are necessary and sufficient for obtaining robust estimates, or, in the context of iterative methods, for obtaining robust block diagonal preconditioners. So, they can be used to check whether a particular preconditioner is robust or not. How to resolve these conditions, i.e., how to find \mathcal{I}_V and \mathcal{I}_Q satisfying (2.17), is a much harder problem. In Section

3 we have demonstrated how to resolve (2.17) in special cases. These cases cover the distributed optimal control problem for elliptic equations, but not the distributed optimal control problem for the Stokes equations. Nevertheless, also for the Stokes control problem robust preconditioners could be constructed by a specialized analysis of (2.17). Numerical experiments for elliptic optimal control problem can be found in [22] and [21]. In a forthcoming paper we will report on numerical results for the Stokes optimal control problem.

REFERENCES

- [1] O. AXELSSON AND I. GUSTAFSSON, *Preconditioning and two-level multigrid methods of arbitrary degree of approximation.*, Math. Comput., 40 (1983), pp. 219–242.
- [2] I. BABUŠKA, *Error-bounds for finite element method.*, Numer. Math., 16 (1971), pp. 322–333.
- [3] I. BABUŠKA AND A. K. AZIZ, *Survey lectures on the mathematical foundations of the finite element method*, in The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations, A. K. Aziz, ed., New York-London: Academic Press, 1972, pp. 3–343.
- [4] M. BENZI, G. H. GOLUB, AND J. LIESEN, *Numerical Solution of Saddle Point Problems.*, Acta Numerica, 14 (2005), pp. 1–137.
- [5] J. BERGH AND J. LÖFSTRÖM, *Interpolation spaces. An introduction.*, Grundlehren der mathematischen Wissenschaften. 223. Berlin-Heidelberg-New York: Springer-Verlag, 1976.
- [6] J. H. BRAMBLE, J. E. PASCIAK, AND P. S. VASSILEVSKI, *Computational scales of Sobolev norms with application to preconditioning.*, Math. Comput., 69 (2000), pp. 463–480.
- [7] F. BREZZI, *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, R.A.I.R.O. Anal.Numer., 2 (1974), pp. 129–151.
- [8] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, 1991.
- [9] W. W. HAGER, *Updating the inverse of a matrix*, SIAM Rev., 31 (1989), pp. 221–239.
- [10] N. J. HIGHAM, *Functions of matrices. Theory and computation.*, Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2008.
- [11] J. L. LIONS, *Optimal Control of Systems Governed by Partial Differential Equations*, Berlin-Heidelberg-New York: Springer-Verlag., 1971.
- [12] K.-A. MARDAL AND R. WINTHER, *Uniform preconditioners for the time dependent Stokes problem.*, Numer. Math., 98 (2004), pp. 305–327.
- [13] ———, *Uniform preconditioners for the time dependent Stokes problem.*, Numer. Math., 103 (2006), pp. 171–172.
- [14] ———, *Preconditioning discretizations of systems of partial differential equations*, Numer. Linear Algebra Appl., (2010). To appear.
- [15] M. A. OLSHANSKII, J. PETERS, AND A. REUSKEN, *Uniform preconditioners for a parameter dependent saddle point problem with application to generalized Stokes interface equations.*, Numer. Math., 105 (2006), pp. 159–191.
- [16] M. A. OLSHANSKII AND A. REUSKEN, *On the convergence of a multigrid method for linear reaction-diffusion problems.*, Computing, 65 (2000), pp. 193–202.
- [17] C. C. PAIGE AND M. A. SAUNDERS, *Solution of sparse indefinite systems of linear equations.*, SIAM J. Numer. Anal., 12 (1975), pp. 617–629.
- [18] T. REES, H. S. DOLLAR, AND A. J. WATHEN, *Optimal solvers for pde-constrained optimization*, SIAM J. Sci. Comput., 32 (2010), pp. 271 – 298.
- [19] T. REES AND A. J. WATHEN, *Preconditioning iterative methods for the optimal control of the stokes equation*, Technical Report NA-10/04, University of Oxford, June 2010.
- [20] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Landmarks in Mathematics. Princeton, NJ: Princeton University Press, 1997.
- [21] J. SCHÖBERL, R. SIMON, AND W. ZULEHNER, *A robust multigrid method for elliptic optimal control problems*, NuMa-Report 2010-01, Institute of Computational Mathematics, Johannes Kepler University Linz, Austria, 2010.
- [22] J. SCHÖBERL AND W. ZULEHNER, *Symmetric Indefinite Preconditioners for Saddle Point Problems with Applications to PDE-Constrained Optimization Problems*, SIAM J. Matrix Anal. Appl., 29 (2007), pp. 752 – 773.
- [23] F. TRÖLTZSCH, *Optimal control of partial differential equations. Theory, methods and applications.*, Graduate Studies in Mathematics 112. Providence, RI: American Mathematical Society (AMS), 2010.
- [24] P. S. VASSILEVSKI, *Hybrid V-cycle algebraic multilevel preconditioners.*, Math. Comput., 58

(1992), pp. 489–512.

Latest Reports in this series

2009

[..]

- 2009-07 Ulrich Langer, Huidong Yang and Walter Zulehner
A Grid-enabled Solver for the Fluid-Structure Interaction (FSI) Problem August 2009
- 2009-08 Stefan Takacs and Walter Zulehner
Multigrid Methods for Elliptic Optimal Control Problems with Neumann Boundary Control October 2009
- 2009-09 Dylan M. Copeland and Ulrich Langer
Domain Decomposition Solvers for Nonlinear Multiharmonic Finite Element Equations November 2009
- 2009-10 Huidong Yang and Walter Zulehner
A Newton Based Fluid-structure Interaction (FSI) Solver with Algebraic Multigrid Methods (AMG) on Hybrid Meshes November 2009
- 2009-11 Peter Gruber, Dorothee Knees, Sergiy Nesenenko and Marita Thomas
Analytical and Numerical Aspects of Time-dependent Models with Internal Variables November 2009
- 2009-12 Clemens Pechstein and Robert Scheichl
Weighted Poincaré Inequalities and Applications in Domain Decomposition November 2009
- 2009-13 Dylan Copeland, Michael Kolmbauer and Ulrich Langer
Domain Decomposition Solvers for Frequency-Domain Finite Element Equations December 2009
- 2009-14 Clemens Pechstein
Shape-Explicit Constants for Some Boundary Integral Operators December 2009
- 2009-15 Peter G. Gruber, Johanna Kienesberger, Ulrich Langer, Joachim Schöberl and Jan Valdman
Fast Solvers and A Posteriori Error Estimates in Elastoplasticity December 2009

2010

- 2010-01 Joachim Schöberl, René Simon and Walter Zulehner
A Robust Multigrid Method for Elliptic Optimal Control Problems January 2010
- 2010-02 Peter G. Gruber
Adaptive Strategies for High Order FEM in Elastoplasticity March 2010
- 2010-03 Sven Beuchler, Clemens Pechstein and Daniel Wachsmuth
Boundary Concentrated Finite Elements for Optimal Boundary Control Problems of Elliptic PDEs June 2010
- 2010-04 Clemens Hofreither, Ulrich Langer and Clemens Pechstein
Analysis of a Non-standard Finite Element Method Based on Boundary Integral Operators June 2010
- 2010-05 Helmut Gfrerer
First-Order Characterizations of Metric Subregularity and Calmness of Constraint Set Mappings July 2010
- 2010-06 Helmut Gfrerer
Second Order Conditions for Metric Subregularity of Smooth Constraint Systems September 2010
- 2010-07 Walter Zulehner
Non-standard Norms and Robust Estimates for Saddle Point Problems November 2010

From 1998 to 2008 reports were published by SFB013. Please see

<http://www.sfb013.uni-linz.ac.at/index.php?id=reports>

From 2004 on reports were also published by RICAM. Please see

<http://www.ricam.oeaw.ac.at/publications/list/>

For a complete list of NuMa reports see

<http://www.numa.uni-linz.ac.at/Publications/List/>