Analysis and Simulation of Signorini's Problem in Linear Elasticity

Diplomarbeit

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To the memory of my friend Gerald Scheinmayr

Abstract

The aim of this diploma thesis is to derive an efficient solution strategy for Signorini's problem. After introducing into the fields of continuum mechanics and linear elasticity, general nonlinear contact conditions are derived and linearized. The resulting problem is brought into a variational form and analysed using abstract results for variational inequalities. In order to solve Signorini's problem numerically, a finite element formulation of the problem is derived and analysed with respect to existence of a unique solution and convergence.

Special attention is paid to the efficient numerical solution of the discrete problem. Step by step an efficient solution method is derived from a Gauß-Seidel relaxation by enlarging the space splitting using the multilevel nodal basis. In order to preserve the optimal order of the implementation, suitable approximations of the obstacle have to be introduced. The asymptotic convergence properties are improved even further using truncated base functions.

Zusammenfassung

Das Ziel dieser Diplomarbeit ist die Entwicklung einer effizienten Lösungsstrategie für das Signorini-Problem. Nach einer allgemeinen Einleitung in die Kontinuumsmechanik und die lineare Elastizitätstheorie, werden allgemeine, nichtlineare Kontaktbedingungen hergeleitet und linearisiert. Das entstehende Problem wird in eine Variationsformulierung übergeführt und unter Verwendung abstrakter Resultate für Variationsungleichungen analysiert. Zur numerischen Lösung des Problems wird die Finite-Element-Methode verwendet. Eine Analyse des diskreten Problems hinsichtlich Existenz einer eindeutigen Lösung und Konvergenz wird präsentiert.

Besonderes Augenmerk wird der effizienten Auflösung des diskreten Problems geschenkt. Ausgehend von einer Gauß-Seidel Relaxation wird schrittweise ein effizientes Lösungsverfahren hergeleitet, indem die Teilraumzerlegung unter Verwendung der Multilevel-Knotenbasis erweitert wird. Um die optimale Ordnung des Verfahrens zu erhalten, muß das Hindernis in geeigneter Weise approximiert werden. Die Konvergenzgeschwindigkeit kann durch Verwendung abgeschnittener Basisfunktionen weiter erhöht werden.

Preface

The impulse which led to the writing of this diploma thesis has emerged from a trainee-program at the research department of the Hilti-Corporation at Schaan, Principality of Liechtenstein. This was made possible by the endeavour of Prof. Dr. Heinz Engl and Dr. Wilhelm Grever, to whom I want to express my thanks.

Hilti is one of world's market leaders in the fields of fastening, demolition, installation and construction chemicals. There under the supervision of Dr. Christian Dietrich I got a clear insight view not only of the applicability of mathematics in industry, but also of the practicalities of contact problems in real life problems. Special thanks go to Horst-Detlef Gassmann, an about 55 year old research engineer, with whom I shared one office. He gave me a very clear insight into the dynamics of fixenation and also passed much of his experience to me.

Back at university I quickly found out, that I had to downsize the real life situation of dynamic multibody contact problems to a much simpler problem, in order to focus my interest on its efficient solution and to work out an efficient solver.

In this context I want to express my thanks to Prof. Dr. Ulrich Langer for giving me the chance of writing this diploma thesis and to his co-workers at the Department of Numerical Mathematics and Optimization at the Johannes Kepler University Linz, especially Dr. Alfred Tamme and Dipl. Ing. Joachim Schöberl for their patience to answer all the arising questions and for their hints during the development.

Last but not least I want to thank my friend Dipl. Ing. Ulrich Bodenhofer for his LaTeX-support, especially his help during the typesetting of the different algorithms.

August 1997, Wolfram Mühlhuber

Contents

Chapter 1

Introduction

In many situations in solid body mechanics the contact between two or more bodies plays an important role. Hardly any production process in industry can be thought of without cutting, milling, lathing, drilling, grinding or similar processes. Typical for each of these production steps is the contact between a tool and the work piece. In a situation like this several aspects have to be taken into account:

- The tool should not be strained to much, as otherwise the life time of the tool is shortened strongly or the tool itself breaks.
- On the other hand the process should need as little time as possible in order to keep the production costs low.
- In many situations additional requirements have to be met, e. g. the work piece should not be heated locally or the surface of the work piece has to stay within certain tolerances.

In other production processes the tool itself does not play such an important role. The manufacturing of cars for example can not be imagined without deep drawing of sheet metal nowadays. Therefore the piston and the cast can be considered as nearly rigid due to their massive construction. The deformation of the blank sheet is of much greater interest. The metal sheets are pressed into the form of mudguards, engine bonnets, etc. During the deformation process the metal sheet becomes thin and may loose its stabiliy at points of very strong deformation.

The list of examples above could be lengthened by adding many other problems, in which the contact between several bodies is of major importance.

In the past many of these situations were handled by *trail and error* during the development process. In recent years *simulation tools* entered the development departments in order to reduce the costs for building prototypes and to speed up the development process.

Contact problems were treated rather often in history. First mainly engineers worked on this topic. That is why much stuff on the analysis and simulation of contact problems can be found in engineering literature. Kikuchi and Oden [19] present many references on articles having this background.

Later on contact problems entered also the mathematical literature. DUVAUT AND LIONS [6] and later KINDERLEHRER AND STAMPACCHIA [20] laid cornerstones of the analysis of contact problems using variational inequalities.

The break through in the simulation of problems in structural mechanics and especially contact problems was the development of the finite element method during the 1970's. Many well-known persons worked on this topic, e.g. ZIENKIEWICS [34], KIKUCHI [18] or CIARLET [4].

In the 1980's efficient solution methods based on multigrid and multilevel ideas were developed. They were successfully applied to various problems with unilateral boundary conditions for example by MANDEL [28], BRANDT AND CRYER [2], HACKBUSCH AND MITTELMANN [14], HOPPE [16], HOPPE AND KORNHUBER [17] or KORNHUBER [24]. Recently SCHÖBERL [31] developed a very efficient solver for contact problems based on Domain-Decomposition techniques with multilevel preconditioning.

In this work the requirements and demands of industry or production induced contact problems are downsized to the rather simple situation of an elastic body undergoing small deformations coming in contact with a rigid foundation. For this problem an efficient solver based on a multilevel stategy is developed. Nevertheless its applicability can be generalized to some more complex contact problems, e. g. the deep drawing of sheet metal described above.

The work is organized as follows:

- In Chapter 2 the mathematical model for an elastic body undergoing small deformations is presented. Furthermore several contact conditions are derived for the contact between an elastic body and a rigid foundation.

- Chapter 3 deals with the derivation and analysis of a variational formulation of the problem modelled in Chapter 2, including results for existence and uniqueness of a solution for the variational formulation.

- In Chapter 4 a finite element approximation of the variational problem of Chapter 3 is derived and analysed, including existence and uniqueness of a disrete solution. Furthermore a result is presented giving conditions for the convergence of the discrete solution to the solution of the continuous problem.

- Chapter 5 deals with the derivation of an efficient solver for the discrete problem stated in Chapter 4. The solver itself is based on a multilevel stategy and can be seen as a generalisation of the classical V-cycle to contact problems.

- Some numerical results are presented in Chapter 6.

- Last but not least Chapter 7 presents various concluding remarks.

Chapter 2

Derivation of Signorini's Problem in Elasticity

2.1 Introduction

In many practical situations in solid mechanics it is important to model the situation of two or more bodies coming into contact with each other. The aim of this chapter is to derive the contact conditions for a classical and rather easy problem in this context, namely the contact between an elastic body undergoing small deformations and a rigid, frictionless foundation. In the following this problem is referenced as *Signorini's problem*, which is also the name of this problem most often used in literature.

An introduction into the mechanics of continua can be found in Ciarlet [5] or in Engl [8]. A detailed view of linear elasticity is presented by Necas and Hlavacek [29]. Problems with elastic-plastic or plastic material laws are investigated by Korneev and Langer [21] or Necas and Hlavacek [29]. The problem of an elastic body coming in contact with a rigid foundation was first modelled by Signorini [32] in 1933. The derivation of the contact conditions presented here is heavily based on the book of Kikuchi and Oden [19]. A generalization of this approach to two body contact problems can be found in Hlavacek, Haslinger, Necas and Lovisek [15].

In Section 2.2 an introduction of the foundations of the mechanics of continua is presented. Section 2.3 deals with the derivation of general contact conditions for Signorini's problem. These conditions are the basis for the derivation of incremental contact conditions in Section 2.4 and linearized contact conditions in Section 2.5.

2.2 Foundation of the Mechanics of Continua

Let a domain $\Omega \subseteq \mathbb{R}^3$ represent the *reference configuration* of a material body. It is usually convenient to think of the reference configuration Ω as the *rest* or *unstressed* configuration of the body, but we will not restrict ourselves to this case. The material points in Ω are denoted by X. Then, the motion of the body is characterized by the mapping

$$x = P(X,t), \qquad X \in \Omega, t \geq 0, \tag{2.1}$$

with the *deformation*

$$P : \overline{\Omega} \times [0,T] \to \mathbb{R}^3. \tag{2.2}$$

This kind of representation is often called *material representation*. We would like to restrict the deformations to be one-to-one mappings so that the material does not overlap or interpenetrate when it deforms. However, such a constraint is hard to treat so it will be ignored. Nevertheless P is assumed to be injective in Ω, sufficiently smooth and orientation preserving. In most applications the displacement of a particle X at time t given by

$$u(X,t) = x - X = P(X,t) - X \tag{2.3}$$

is of much greater interest than the deformation itself.

As P is assumed to be smooth, the constraint that P preserves orientation, can be expressed by the pointwise inequality

$$J(X,t) = \det\Big(\frac{\partial P_i}{\partial X_j}\Big)(X,t) > 0, \qquad X \in \overline{\Omega},\ t \geq 0. \tag{2.4}$$

In the following the deformation gradient $\big(\frac{\partial P_i}{\partial X_j}\big)_{i,j=1,\dots,3}$ will often be designated by F. (2.1) can be inverted to give

$$X = p(x,t) = P^{-1}(x,t). \tag{2.5}$$

This is often called the *spatial representation* of the deformed body.

The external forces are responsed by stress- and acceleration terms. This leads to the following equations describing the motion of a body in spatial coordinates:

$$\begin{aligned} \operatorname{div}_x \hat{t}(x,t) + f(x,t) &= \rho(x,t)\,\frac{\mathrm{D}\,v}{\mathrm{D}\,t}(x,t), && x \in P(\Omega,t), t > 0,\\ \hat{t}(x,t)\,n_x &= g(x,t), && x \in P(\Gamma_N,t),\\ u(x,t) &= u_0(x,t), && x \in P(\Gamma_D,t), \end{aligned} \tag{2.6}$$

with Cauchy's stress tensor

$$\hat{t}(x,t) = \hat{t}^T(x,t), \tag{2.7}$$

the velocity

$$v(x,t) = \frac{\partial P(X,t)}{\partial t}, \tag{2.8}$$

the material derivative

$$\frac{\mathrm{D}\,y}{\mathrm{D}\,t}(x,t) = \frac{\partial y}{\partial t}(x,t) + \langle \operatorname{grad}_x y, v\rangle, \tag{2.9}$$

the density $\rho(x,t)$ and the volume force density $f(x,t)$. $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^3$. Γ_N denotes that part of the boundary with given surface tractions g, Γ_D that part with prescribed displacements u_0. n_x is the outer unit normal to $P(\Omega,t)$ at point $x(t)$. The equations (2.6) can be rewritten in material coordinates resulting in

$$\begin{aligned} \operatorname{div}_X T(X,t) + \hat{F}(X,t) &= \rho_0(X)\,\frac{\partial^2 P}{\partial t^2}(X,t), && X \in \Omega, t \geq 0,\\ T(X,t)\,n_X &= G(X,t), && X \in \Gamma_N,\\ U(X,t) &= U_0(X,t), && X \in \Gamma_D, \end{aligned} \tag{2.10}$$

with the first Piola-Kirchhoff stress tensor

$$T(X,t) = J(X,t)\,\hat{t}(x,t)\,F(X,t)^{-T}, \tag{2.11}$$

the density ρ_0 of the reference configuration of Ω, the volume force density

$$\hat{F}(X,t) = J(X,t)\,f(x,t) \tag{2.12}$$

and the surface tractions

$$G(X,t) = g(x,t)\,\|J(X,t)\,F(x,t)^{-T} n_x\|. \tag{2.13}$$

n_X is the outer unit normal to Ω at point X. As $\hat{t}$ is symmetric, T fulfills the equation

$$T(X,t)\,F(X,t)^T = F(X,t)\,T(X,t)^T. \tag{2.14}$$

In many applications the acceleration term can be neglected. This leads to the equilibrium equations in spatial coordinates

$$\begin{aligned} \operatorname{div}_x \hat{t}(x,t) + f(x,t) &= 0, && x \in P(\Omega,t), t > 0,\\ \hat{t}(x,t)\,n_x &= g(x,t), && x \in P(\Gamma_N,t),\\ u(x,t) &= u_0(x,t), && x \in P(\Gamma_D,t), \end{aligned} \tag{2.15}$$

and to the equilibrium equations in material coordinates

$$\begin{aligned} \operatorname{div}_X T(X,t) + \hat{F}(X,t) &= 0, && X \in \Omega, t \geq 0,\\ T(X,t)\,n_X &= G(X,t), && X \in \Gamma_N,\\ U(X,t) &= U_0(X,t), && X \in \Gamma_D. \end{aligned} \tag{2.16}$$

The equations (2.6), (2.10), (2.15), (2.16) possess 12 variables each. Formally there are 12 equations (9 symmetry equalities and 3 motion equations) but only 6 of these are linearly independent. The other 6 equations are given by a material law linking the stresses and the strains in Ω, respectively in $P(\Omega,t)$.

In the following we will restrict ourselves to elastic materials only. In this case the material law is given by

$$(F^{-1}T)_{ij} = a_{ijkl}E_{kl} \tag{2.17}$$

with the elasticity tensor a_{ijkl} and the Green-St.Venant strain tensor

$$E = \frac{1}{2}(F^T F - I). \tag{2.18}$$

For isotropic materials the elasticity tensor is given by

$$a_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.19}$$

λ and μ denote Lame's parameters and are both greater 0. They are connected with Young's modulus Y and Poisson's ration ν by

$$\begin{aligned} Y &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \\ \nu &= \frac{\lambda}{2(\lambda + \mu)}. \end{aligned} \tag{2.20}$$

Materials fulfilling the material law (2.17) are called *St. Venant-Kirchhoff* materials. (2.17) is linear in E, but not in the displacements U.

Assuming only very small deformations, the equations (2.16) with the material law (2.17) can be linearized. Ciarlet [5] showed that the operator

$$\mathcal{A} : u \to \begin{pmatrix} -\operatorname{div}\Big(\big(I + \operatorname{grad} u\big)\big(F^{-1}T\big)(E)\Big) \\ \big(I + \operatorname{grad} u\big)\big(F^{-1}T\big)(E) n|_{\Gamma_N} \end{pmatrix} \tag{2.21}$$

is Frechet-differentiable at $u = 0$ and

$$\mathcal{A}'(0)u = \begin{pmatrix} -\operatorname{div} \sigma(u) \\ \sigma(u) n|_{\Gamma_N} \end{pmatrix}. \tag{2.22}$$

σ denotes the linearized stress tensor

$$\sigma_{ij} = a_{ijkl} \epsilon_{kl} \tag{2.23}$$

with the linearized Green-St.Venant strain tensor

$$\epsilon(u) = \frac{1}{2}\big(\operatorname{grad} u + (\operatorname{grad} u)^T\big). \tag{2.24}$$

The material law (2.23) is named *Hooke's law*. Collecting all the results the following equations can be deduced:

$$\begin{aligned} \operatorname{div} \sigma(X,t) + \hat{F}(X,t) &= \rho_0(X) \frac{\partial^2 p}{\partial t^2}(X,T), \qquad && X \in \Omega, t > 0, \\ \sigma(X,t)\, n_X &= G(X), && X \in \Gamma_N, \\ u(X) &= 0, && X \in \Gamma_D, \end{aligned} \tag{2.25}$$

with σ given by (2.23). It can be seen easily, that (2.25) is a system of partial differential equations of second order. In the special case of an isotropic body, in which (2.19) is valid, the left hand side of (2.25) has a relatively simple form as the following holds:

$$\operatorname{div} \sigma = (\lambda + \mu) \operatorname{grad} \operatorname{div} u + \mu \Delta u. \tag{2.26}$$

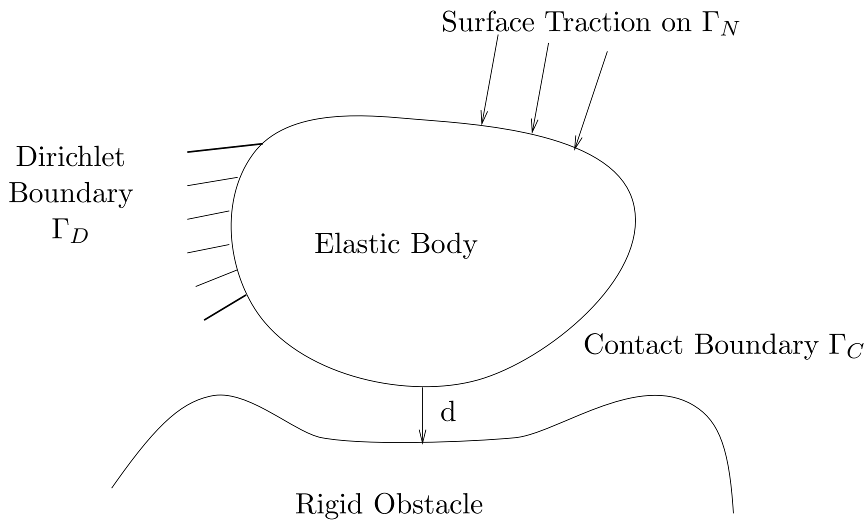

Figure 2.1: Situation before Contact

In situations, in which the acceleration terms can be neglected, (2.25) simplifies to

$$\begin{aligned} \operatorname{div} \sigma(X,t) + \hat{F}(X,t) &= 0, && X \in \Omega, t > 0, \\ \sigma(X,t)\, n_X &= G(X), && X \in \Gamma_N, \\ u(X) &= 0, && X \in \Gamma_D. \end{aligned} \tag{2.27}$$

2.3 Derivation of the General Contact Conditions

From now on we will consider the special geometrical situation illustrated in Figure 2.1. A rigid and fixed body $\mathcal{F}$, called the foundation is located below the body Ω. That is why the displacements of particles in Ω are restricted by $\mathcal{F}$. It is assumed, that $\mathcal{F}$ is an unbounded, semi-infinite half space in $\mathbb{R}^3$. We are interested in the deformation of the body Ω due to the contact between the body and the foundation. In the following a static or quasi static problem is looked at and therefore the dependence on t is neglected. Nevertheless most results hold also for the dynamic problem.

Suppose that the surface of the foundation is given by

$$s(x) = 0 \tag{2.28}$$

and that the material surface Γ_C of Ω contains that portion of the total surface Γ of the body which comes in contact with $\mathcal{F}$. Furthermore assume that s is sufficiently smooth and

$$\begin{aligned} s(x) &> 0, && x \in \mathbb{R}^3 \setminus \mathcal{F}, \\ s(x) &< 0, && x \in \mathcal{F}. \end{aligned} \tag{2.29}$$

Then it is clear that for every $X \in \Gamma_C$

$$s(X + u(X)) \geq 0, \tag{2.30}$$

where $u(X)$ denotes the displacement of X. We refer to inequality (2.30) as the *kinematical contact condition* for a body constrained by a fixed rigid foundation. This condition must also be compatible with the stresses on the contact surface. For the stress tensor $\sigma(x)$ at a particle x on the material surface the normal and tangential components of σ are defined by

$$\left.\begin{aligned} \sigma_n(x) &= \langle \sigma(x)\, n, n\rangle, \\ \sigma_T(x) &= \sigma(x)\, n - \sigma_n(x)\, n, \end{aligned}\right\} \qquad x \in P(\Gamma_C), \tag{2.31}$$

where n denotes the outer unit normal at x. If no contact occures at x, $\sigma(x)\, n = 0$. For those points where contact occures, $\sigma(x)\, n$ must be a compressive normal stress, if frictionless contact is assumed. Otherwise special friction laws connecting the normal and the tangential components of σ and the displacements have to be considered. This leads to the following equations:

$$\left.\begin{aligned} &\sigma_n(x) = 0 && \text{if } s(x) > 0, \\ &\sigma_n(x) \le 0 && \text{if } s(x) = 0, \\ &\sigma_T(x) = 0. && \end{aligned}\right\} \qquad x \in P(\Gamma_C). \tag{2.32}$$

Collecting these results, we arrive at the *general contact condition*

$$\left.\begin{aligned} s(x) &\ge 0 \\ \sigma(x) n &= -\lambda\, n, \qquad \lambda \ge 0, \\ \sigma_n(x) s(x) &= 0 \end{aligned}\right\} \qquad x \in P(\Gamma_C). \tag{2.33}$$

2.4 Incremental Contact Conditions

In order to derive an incremental form of the contact condition (2.33), let us suppose that the body is displaced relative to its current configuration by a small displacement Δu. The resulting configuration also has to fulfill the contact conditions (2.33), i.e.

$$s(x + \Delta u) \ge 0. \tag{2.34}$$

As s is sufficiently smooth and Δu is small, the left hand side of (2.34) can be linearized, leading to

$$s(x + \Delta u) = s(x) + \langle \operatorname{grad} s(x), \Delta u\rangle + o(\|\Delta u\|). \tag{2.35}$$

Introducing the inner normal

$$N(x) = -\frac{\operatorname{grad} s(x)}{\|\operatorname{grad} s(x)\|} \tag{2.36}$$

to isosurfaces of s (surfaces, where $s(x) = \text{const}$ holds) and neglecting the term $o(\|\Delta u\|)$, (2.34) results in

$$\langle \Delta u, N(x)\rangle \le \frac{s(x)}{\|\operatorname{grad} s(x)\|}. \tag{2.37}$$

Of course, to complete the descripion of the contact constraint one must add the corresponding contact conditions (2.32) on stress components to (2.37). The left hand side of (2.37) can be interpreted as the displacement normal to the surface of $\mathcal{F}$, the right hand side as the gap between the body and $\mathcal{F}$. We refer to (2.37) as an *incremental contact condition.* It is useful in describing the motion of a body in terms of increments Δu of a configuration characterized by the displacement field $u(x)$. This is especially important considering contact problems with finite displacements on the one hand or contact problems involving friction on the other hand. In both cases the motion of a body is handled as a sequence of displacement increments. If $u^n(X)$ is the displacement accumulated after n such increments, Δu^n must satisfy the contact condition

$$\langle \Delta u^n, N(X + u^n)\rangle \leq \frac{s(X + u^n)}{\| \operatorname{grad} s(X + u^n)\|}. \tag{2.38}$$

The only drawback of (2.37) is that it involves the component of u normal to an isosurface of s and not to the material surface itself.

2.5 Linearized Contact Conditions

In order to withdraw the necessity of evaluating u normal to an isosurface of s one uses (2.35) as a starting point. Again introducing the inward normal $N(x)$ to isosurfaces of s as in (2.36) and the outward normal $n(x)$ to the contact surface of the body Ω, (2.34) can be rewritten in the form

$$\frac{s(x)}{\| \operatorname{grad} s(x)\|} - \langle N(x), \Delta u\rangle + \langle n(x), \Delta u(x)\rangle - \langle n(x), \Delta u(x)\rangle + o(\|\Delta u\|) \geq 0, \tag{2.39}$$

which can be simplified to

$$\langle n(x), \Delta u(x)\rangle + \langle N(x) - n(x), \Delta u(x)\rangle + o(\|\Delta u\|) \leq \frac{s(x)}{\| \operatorname{grad} s(x)\|}. \tag{2.40}$$

If the body Ω and the foundation $\mathcal{F}$ are sufficiently close to each other, the surfaces of Ω and $\mathcal{F}$ are essentially parallel, as both surfaces are assumed to be sufficiently smooth. That is why the difference $N(x) - n(x)$ is small and tends to 0 as the restriction becomes active and Δu tends to 0. Neglecting the term $o(\|\Delta u\|)$ and taking the above fact into account, (2.40) results in

$$\langle n(x), \Delta u(x)\rangle \leq \frac{s(x)}{\| \operatorname{grad} s(x)\|}. \tag{2.41}$$

The left hand side of (2.41) is the displacement normal to the material surface of the body Ω, the right hand side represents the gap between Ω and the foundation $\mathcal{F}$. Similar to (2.37), (2.41) represents only a nonpenetration condition and the corresponding contact conditions (2.32) on the stress components have to be added in order to complete the description of the contact constraint. Then, the complete description of the contact conditions looks as follows:

$$\left.\begin{aligned} \langle u(x), n(x)\rangle &\leq d(x) \\ \sigma(x)n &= -\lambda\, n, \qquad \lambda \geq 0 \\ \sigma_n(x)\big(\langle u(x), n(x)\rangle - g(x)\big) &= 0 \end{aligned}\right\} \qquad x \in P(\Gamma_C). \tag{2.42}$$

with the gap function

$$d(x) = \frac{s(x)}{\|\operatorname{grad} s(x)\|}. \tag{2.43}$$

A detailed derivation with the proofs for the assertions made above can be found in Eck [7].

Chapter 3

Mathematical Analysis of Signorini's Problem

3.1 Introduction

This chapter deals with the mathematical analysis of Signorini's problem. First of all a variational formulation of this contact problem will be derived. This will result in a variational inequality due to the nonpenetration condition (2.30) respectively (2.41). The actual surface, on which the body comes in contact with the foundation, is not known in advance and is therefore part of the solution. Furthermore the existence of a unique solution will be shown.

Many results on analysing variational problems can be found in Duvaut and Lions [6]. Dealing with variational inequalities especially the monograph by Kinderlehrer and Stampacchia [20] must be noted. Kikuchi and Oden [19] present many results for different kinds of contact problems.

Section 3.2 presents the derivation of a variational formulation of Signorini's problem, resulting in a variational inequality due to the nonpenetration condition. In Section 3.3 some abstract results for variational inequalities are presented. Using these theorems the existence and uniqueness of a solution of the variational inequality describing Signorini's problem is deduced in Section 3.4.

3.2 Derivation of a Variational Formulation

In the following a variational formulation of the equilibrium conditions of a linear elastic body coming in contact with a frictionless rigid foundation is derived. From the equations in Chapter 2 it is clear that the displacement field

u of the body satisfies the following system of equalities and inequalities:

$$\begin{aligned}
\operatorname{div}\sigma + f &= 0 && \text{in } \Omega \\
u &= u_0 && \text{on } \Gamma_D \\
\sigma\, n &= g && \text{on } \Gamma_N \\
\sigma_T &= 0 && \text{on } \Gamma_C \\
\sigma_n &\le 0 && \text{on } \Gamma_C \\
u_n &\le d && \text{on } \Gamma_C \\
(u_n - d)\sigma_n &= 0 && \text{on } \Gamma_C \\
\sigma_{ij} &= a_{ijkl}\epsilon_{kl} && \text{in } \Omega \\
\epsilon &= \frac{1}{2}\big(\operatorname{grad} u + (\operatorname{grad} u)^T\big) && \text{in } \Omega
\end{aligned} \tag{3.1}$$

σ_n and σ_T are defined in (2.31), u_n stands short for $\langle u, n\rangle$. From now on it is assumed that $\Gamma_D \cap \Gamma_N = \Gamma_D \cap \Gamma_C = \Gamma_N \cap \Gamma_C = \emptyset$, $\Gamma_D \cup \Gamma_N \cup \Gamma_C = \partial\Omega$. Γ_D is assumed to be closed, Γ_N to be open.

In the following a variational formulation for the problem stated in (3.1) will be derived.

Definition 3.1. *Let*

$$K = \left\{ v \in V \;\middle|\; v_n \le d \text{ on } \Gamma_C,\ v = u_0 \text{ on } \Gamma_D \right\}, \tag{3.2}$$

where V denotes a vector space, which will be specified later, e.g. $H^1(\Omega)$. K is called the set of admissible displacements*.*

For each function $u \in K$ let the set $I(u)$ denote the actual contact surface

$$I(u) = \left\{ x \in \Gamma_C \;\middle|\; u_n(x) = d(x) \right\}, \tag{3.3}$$

and the set $\Lambda(u)$

$$\Lambda(u) = \left\{ x \in \Gamma_C \;\middle|\; u_n(x) < d(x) \right\}. \tag{3.4}$$

Furthermore often just I and Λ is written.

For the derivation of a variational formulation, we proceed formally and assume $v \in K$ and Ω to be smooth. Let u denote the solution of Signorini's problem (3.1), and v an arbitrary element of K. Then by partial integration the following holds:

$$\int_\Omega \langle \operatorname{div}\sigma(u), v - u\rangle \,\mathrm{d}x = \int_\Gamma \langle \sigma(u)n, v - u\rangle \,\mathrm{d}s - \int_\Omega \sigma(u) : \operatorname{grad}(v - u)\,\mathrm{d}x, \tag{3.5}$$

with

$$A : B = \sum_{i,j=1}^{3} a_{ij} b_{ij}, \qquad A = (a_{ij})_{i,j=1,\dots,3}, \quad B = (b_{ij})_{i,j=1,\dots,3}. \tag{3.6}$$

Due to the symmetry of σ

$$\int_{\Omega} \sigma(u) : \operatorname{grad}(v-u)\,\mathrm{d}x = \int_{\Omega} \sigma(u) : \epsilon(v-u)\,\mathrm{d}x \tag{3.7}$$

holds. The surface integral in (3.5) can be simplified in the following way:

$$\begin{aligned}
\int_{\Gamma} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s &= \int_{\Gamma_D \cup \Gamma_N \cup \Gamma_C} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \\
&= \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s + \int_{\Gamma_C} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \\
&= \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s + \int_{I(u)} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \\
&\quad + \int_{\Lambda(u)} \langle \underbrace{\sigma(u)\,n}_{=0}, v-u\rangle\,\mathrm{d}s \\
&= \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s + \int_{I(u)} \langle \underbrace{\sigma_T}_{=0} + \sigma_n\,n, v-u\rangle\,\mathrm{d}s \\
&= \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s + \int_{I(u)} \sigma_n \langle n, v-u\rangle\,\mathrm{d}s \\
&= \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s + \int_{I(u)} \underbrace{\sigma_n(v_n-d)}_{\geq 0}\,\mathrm{d}s \\
&\geq \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s.
\end{aligned} \tag{3.8}$$

Thus any classical solution of (3.1) satisfies the variational inequality

$$\int_{\Omega} \sigma(u) : \epsilon(v-u)\,\mathrm{d}x \geq \int_{\Omega} \langle f, v-u\rangle\,\mathrm{d}x + \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s \qquad v \in K. \tag{3.9}$$

Now the variational inequality (3.9) is assumed to hold for a function $u \in K$. It will be shown that u is also a classical solution of (3.1), provided that u is sufficiently smooth.

For any open set A let

$$\mathcal{D}(A) = \left\{ f \in C^{\infty}(A) \;\middle|\; f \text{ has compact support in } A \right\}. \tag{3.10}$$

Let $\Phi \in \mathcal{D}(\Omega)$ be arbitrary, $v = u \pm \Phi$. Then with (3.9) the following holds:

$$\begin{aligned}
0 &\le \int\limits_{\Omega} \sigma(u) : \epsilon(v-u)\,\mathrm{d}x - \int\limits_{\Omega} \langle f, v-u\rangle\,\mathrm{d}x - \int\limits_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s \\
&= \int\limits_{\Omega} \sigma(u) : \epsilon(v-u)\,\mathrm{d}x - \int\limits_{\Omega} \langle f, v-u\rangle\,\mathrm{d}x \qquad (\Phi = 0 \text{ on } \partial\Omega) \\
&= \int\limits_{\partial\Omega} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s - \int\limits_{\Omega} \langle \operatorname{div}\sigma + f, v-u\rangle\,\mathrm{d}x \qquad \big(\text{by (3.6), (3.7)}\big) \\
&= \pm \int\limits_{\Omega} \langle \operatorname{div}\sigma + f, \Phi\rangle\,\mathrm{d}x
\end{aligned} \tag{3.11}$$

which implies

$$\operatorname{div}\sigma + f = 0 \qquad \text{in } \Omega. \tag{3.12}$$

Using (3.12), (3.9) can be simplified, as

$$\begin{aligned}
\int\limits_{\Omega} \sigma(u) : \epsilon(v-u)\,\mathrm{d}s - \int\limits_{\Omega} \langle f, v-u\rangle\,\mathrm{d}x &= \int\limits_{\partial\Omega} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \\
&= \int\limits_{\Gamma_N \cup \Gamma_C} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s,
\end{aligned} \tag{3.13}$$

resulting in the variational inequality

$$\int\limits_{\Gamma_N \cup \Gamma_C} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \ge \int\limits_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s. \tag{3.14}$$

Suppose that $\Phi \in \mathcal{D}(\Omega \cup \Gamma_N)$, $v = u \pm \Phi$. Then (3.14) gives

$$0 \le \pm \int\limits_{\Gamma_N} \langle \sigma(u)\,n - g, \Phi\rangle\,\mathrm{d}s, \tag{3.15}$$

which implies

$$\sigma\,n = g \qquad \text{on } \Gamma_N. \tag{3.16}$$

(3.14) can now again be simplified leading to

$$\int\limits_{\Gamma_C} \langle \sigma(u)\,n, v-u\rangle\,\mathrm{d}s \ge 0. \tag{3.17}$$

By using similar arguments as above it can be shown that

$$\sigma\,n = 0 \qquad \text{on } \Lambda(u), \tag{3.18}$$

which simplifies (3.17) to

$$\int\limits_{I(u)} \langle \sigma(u)\, n, v - u \rangle \, \mathrm{d}s \geq 0. \tag{3.19}$$

Let $\Phi \in \mathcal{D}\big(\Omega \cup I(u)\big)$ be arbitrary with $\Phi_n = \langle \Phi, n \rangle \leq 0$ on $I(u)$. Then $v = u + \Phi \in K$ and (3.19) gives

$$\begin{aligned} 0 \leq \int\limits_{I(u)} \langle \sigma\, n, v - u \rangle \, \mathrm{d}s &= \int\limits_{I(u)} \langle \sigma\, n, \Phi \rangle \, \mathrm{d}s = \int\limits_{I(u)} \langle \sigma_T + \sigma_n\, n, \Phi_T + \Phi_n n \rangle \, \mathrm{d}s \\ = \int\limits_{I(u)} \langle \sigma_T, \Phi_T \rangle \, \mathrm{d}s &+ \int\limits_{I(u)} \sigma_n \Phi_n \, \mathrm{d}s \end{aligned} \tag{3.20}$$

with $\Phi_T = \Phi - \Phi_n n$. Choosing Φ such that $\Phi_n = 0$ implies $\sigma_T = 0$ on $I(u)$. Furthermore as $\Phi_n \leq 0$, (3.20) implies that $\sigma_n \leq 0$ on $I(u)$. Summing up, it can be said, that u is a classical solution of (3.1).

Definition 3.2. *Let*

$$K = \left\{ v \in H^1(\Omega) \;\middle|\; v = u_0 \text{ on } \Gamma_D,\; v_n \leq d \text{ on } \Gamma_C \right\}, \tag{3.21}$$

where $H^\alpha(\Omega)$ denotes the Sobolev space of order α and the inequality $v_n \leq d$ on Γ_C should hold almost everywhere. If the variational inequality (3.9) *holds for $u \in K$, $f \in H^{-1}(\Omega)$, $g \in H^{-\frac{1}{2}}(\Gamma_N)$, then u is called a* weak solution of Signorini's problem.

Remark 3.3.

- *In the variational inequality* (3.9) *the contact surface Γ_C does not appear explicitly at the price of solving an inequality instead of the equality* (3.1)*.*
- *The variational inequality* (3.9) *can also be derived from a minimization principle. The potential energy of a body with displacement field u is given by*

$$E_{pot}(u) = \int\limits_{\Omega} \sigma(u) : \epsilon(u) \, \mathrm{d}x - \int\limits_{\Omega} \langle f, u \rangle \, \mathrm{d}x - \int\limits_{\Gamma_N} \langle g, u \rangle \, \mathrm{d}s. \tag{3.22}$$

Introducing the foundation $\mathcal{F}$, the solution u must have minimal potential energy of all admissible displacement fields, i.e.

$$E_{pot}(u) \leq E_{pot}(v) \qquad v \in K. \tag{3.23}$$

That is why for any $v \in K$ the function

$$h\colon [0,1] \to \mathbb{R}, \quad \theta \mapsto E_{pot}\big(u + \theta(v - u)\big) \tag{3.24}$$

has its minimum at $\theta = 0$, resulting in

$$
\begin{aligned}
h'(0) &= \lim_{\substack{\theta \to 0 \\ \theta > 0}} \frac{E_{pot}\big(u + \theta(v-u)\big) - E_{pot}(u)}{\theta} = E'_{pot}(u)(v-u) \\
&= \int_\Omega \sigma(u) : \epsilon(v-u)\,\mathrm{d}x - \int_\Omega \langle f, v-u\rangle\,\mathrm{d}x - \int_{\Gamma_N} \langle g, v-u\rangle\,\mathrm{d}s \qquad (3.25) \\
&\geq 0.
\end{aligned}
$$

In the following the existence of a unique solution of (3.9) in the set of admissible displacements K is shown. Therefore some basic and abstract theorems for variational inequalities are needed, which are given in Section 3.3.

3.3 Some abstract Results for Variational Inequalities

This section deals with two basic results for variational inequalities. The latter is a generalization of the Theorem of Lax and Milgram giving an instrument for proofing existence and uniqueness of solutions of variational inequalities.

Theorem 3.4 (Projection onto convex closed sets in Hilbert spaces). *Let K be a nonempty, closed, convex subset of a Hilbert space V. Then for each $a \in V$ there exists a unique element $b := Pa \in K$, named the projection of a onto K such that the following equivalent assertions are valid:*

$$\|b - a\| \leq \|c - a\| \qquad c \in K, \tag{3.26}$$

respectively

$$\langle b - a, c - a\rangle \geq 0 \qquad c \in K. \tag{3.27}$$

Furthermore for all $a_1, a_2 \in K$ the following holds:

$$\|Pa_1 - Pa_2\| \leq \|a_1 - a_2\|. \tag{3.28}$$

Proof. Binder [1], Kinderlehrer and Stampacchia [20] □

Using Theorem 3.4 the Theorem of Lions and Stampacchia can be proofed. In the Theorem of Lions and Stampacchia, there is no demand for symmetry of the appearing operator . This is especially important if the underlying differential operator is nonsymmetric. Furthermore nonlinear problems can be dealt with. An important demand will be the ellipticity of the underlying operator. The Theorem of Lax and Milgram results as a corollary of the following theorem:

Theorem 3.5 (Lions, Stampacchia). *Let V be a Hilbert space, $K \subseteq V$ a closed, convex, nonempty subset of the space V and $A\colon V \to V'$ a continuous and coercive, not necessarily linear operator, i.e. constants $M, \alpha > 0$ exist with*

$$
\begin{aligned}
\|Au - Av\| &\leq M\,\|u - v\|, \qquad u, v \in K, \\
\langle Au - Av, u - v\rangle &\geq \alpha\,\|u - v\|^2, \qquad u, v \in K,
\end{aligned} \tag{3.29}
$$

where V' denotes the dual of the space V.

Then for each $L \in V'$ there exists a unique solution $u \in K$ of the variational inequality

$$\langle Au - L, v - u \rangle \geq 0. \tag{3.30}$$

Furthermore, the nonlinear solution operator is Lipschitz continuous with constant $1/\alpha$, i.e.

$$\|u_1 - u_2\| \leq \frac{1}{\alpha} \|L_1 - L_2\|. \tag{3.31}$$

Proof. BINDER [1], KINDERLEHRER AND STAMPACCHIA [20] □

Corollary 3.6.

- *(Stampacchia). Let V denote a real Hilbert space, $K \subseteq V$ a closed, convex, nonempty subset of the space V and $a(\cdot,\cdot)$ a continuous, coercive bilinear form. Then a unique solution $u \in K$ exists for every $L \in V'$ fulfilling the variational inequality*

$$a(u, v - u) \geq \langle L, v - u \rangle, \qquad v \in K. \tag{3.32}$$

- *(Theorem of Lax and Milgram). Let L, $a(\cdot,\cdot)$ be as above. If $K = V$, then the variational equality*

$$a(u, v) = \langle L, v \rangle, \qquad v \in V, \tag{3.33}$$

is uniquely solvable.

3.4 Existence of a unique Solution of the Variational Formulation of Signorini's Problem

In this section the existence of a unique solution of the variational inequality (3.9) describing Signorini's problem is shown by using the theorems of Section 3.3. In order to apply Theorem 3.5 or Corollary 3.6 the coercivity of the bilinear form

$$a(u, v) = \int_\Omega \sigma(u) : \epsilon(v) \, \mathrm{d}x \tag{3.34}$$

has to be shown. From now on it is assumed that the elasticity tensor a_{ijkl} has properties of symmetry

$$a_{ijkl} = a_{jilk} = a_{klij} \tag{3.35}$$

and of ellipticity

$$a_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq \alpha\epsilon_{ij}\epsilon_{ij}, \qquad \alpha > 0, \epsilon \in \mathbb{R}^{3\times 3}, \epsilon = (\epsilon_{ij})_{i,j=1,\ldots,3}. \tag{3.36}$$

Then the following theorem holds:

Theorem 3.7. *Let Ω be a bounded domain with a boundary constisting of a finite number of smooth parts. Furthermore let $\Gamma_D \subseteq \Gamma$ and Γ_D has positive measure. Let*

$$V_0 = \left\{ v \in H^1(\Omega) \;\middle|\; v = 0 \text{ on } \Gamma_D \right\}. \tag{3.37}$$

Then there exists an $\alpha_0 > 0$ such that

$$a(v,v) \geq \alpha_0 \left\| v \right\|^2_{H^1(\Omega)}, \qquad v \in V_0. \tag{3.38}$$

Proof. A proof can be found in Duvaut and Lions [6] and uses strongly Korn's inequality. □

Combining the results of Corollary 3.6 and Theorem 3.7 the following consequences can be deduced:

Theorem 3.8. *Let Ω be a bounded domain with a boundary constisting of a finite number of smooth parts, $\Gamma_D \subseteq \Gamma$ with positive measure. Let*

$$K = \left\{ v \in H^1(\Omega) \;\middle|\; v = u_0 \text{ on } \Gamma_D,\; v_n \leq d \text{ on } \Gamma_C \right\}, \tag{3.39}$$

$f \in H^{-1}(\Omega)$, $g \in H^{-\frac{1}{2}}(\Gamma_N)$.

Then the variational inequality (3.9) *has a unique solution in K.*

Chapter 4

Finite Element Analysis of Signorini's Problem

4.1 Introduction

In this chapter a finite element discretisation of the variational inequality (3.9) is presented and analysed. Therefore the solution space $H^1(\Omega)$ is replaced by the discrete space of piecewise linear finite elements. Furthermore a discrete, polygonal approximation of the admissable set K is introduced. In Section 4.3 existence and uniqueness of a discrete solution is shown. Furthermore a result giving conditions for the convergence of the discrete solution to the solution of the continuous problem (3.9) is presented in Section 4.4.

On finite element methods many monographs were published during the last twenty years. Only some are mentioned here, e.g. Zienkiewics [34], Goering, Roos and Tobiska [12], Kikuchi [18], Ciarlet [4] or Krizek and Neittaanmäki [25]. In the context of variational inequalities the book of Glowinski, Lions and Trémolières [11] or Brezzi, Hager and Raviart [3] must be noted. Many references on various aspects of the finite element method especially in the context with multilevel methods can be found in Langer [26].

4.2 Derivation of the Discrete Problem

In the following a finite element space is constructed by discretizing the domain Ω and using linear finite elements. Later on a discrete variational inequality approximating (3.9) is presented.

Let $(\mathcal{T}_h)_{h\in H}$ denote a family of triangulations of Ω into triangles t. Each triangulation $\mathcal{T}_h$ is assumed to be regular in the sense, that the intersection of two triangles $t, \bar{t} \in \mathcal{T}_h$, $t \neq \bar{t}$ consists of a common edge, a common vertex or is empty. A situation like in Figure 4.1 is forbidden for example. The index h is a parameter of the triangulation denoting the mesh width. Furthermore the smallest inner angle of each triangle in $\mathcal{T}_h$ is uniformly bounded away from 0, i. e. the used triangles must not become sharper and sharper as h tends to 0. $\mathcal{N}_h$ denotes the set of nodes in $\mathcal{T}_h$, $\mathcal{E}_h$ the set of edges. Ω_h denotes the union of all triangles in $\mathcal{T}_h$.

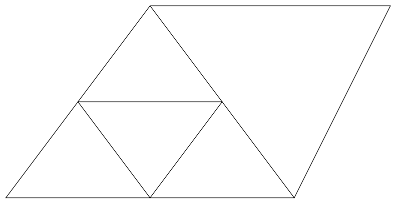

Figure 4.1: Triangulation with a hanging node

For each triangulation $\mathcal{T}_h$ the finite element space $\mathcal{S}_h \subseteq H^1(\Omega_h)$ consists of the continuous functions in $H^1(\Omega_h)$, which are linear on each triangle $t \in \mathcal{T}_h$ for each component. $\mathcal{S}_h$ is spanned by the nodal basis

$$\Lambda_h = \left\{ \lambda_1^{(i),h}, \dots, \lambda_d^{(i),h} \mid p_i \in \mathcal{N}_h \right\}. \tag{4.1}$$

In many cases $\lambda^{(i),h}$ stands short for the vector $\left(\lambda_1^{(i),h}, \dots, \lambda_d^{(i),h}\right)$, where d denotes the dimension of Ω_h. The base functions $\lambda_j^{(i),h} \in \mathcal{S}_h$ are characterized by the fact that for each $p \in \mathcal{N}_h$

$$\lambda_j^{(i),h}(p) = \begin{cases} 1 & p = p_i \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

In order to derive a discrete model also the admissable set K has to be approximated. One possibility would be

$$K_h = K \cap \mathcal{S}_h. \tag{4.3}$$

For numerical computations, this approximation is not useful. In order to check the admissability of a function $v_h \in \mathcal{S}_h$, v_h has to be evaluated at points not in $\mathcal{N}_h$. In order to avoid this drawback K is approximated by

$$\begin{aligned} K_h = \big\{ v \in \mathcal{S}_h \bigm| v(p_i) = u_0(p_i) \text{ for } p_i \in \mathcal{N}_h \cap \Gamma_D, \\ \langle v, n\rangle(p_i) \leq d(p_i) \text{ for } p_i \in \mathcal{N}_h \cap \Gamma_C \big\}. \end{aligned} \tag{4.4}$$

A test on the admissability of $v_h \in \mathcal{S}_h$ only involves evaluations of v_h at nodes in $\mathcal{N}_h$ at the price of K_h being no more an inner approximation of K. That is why the analysis of the convergence of the discrete solution to the solution of the continuous problem becomes more difficult.

By using the terms defined above, the following finite element approximation of the variational inequality (3.9) can be defined:

$$a_h(u_h, v_h - u_h) \geq \langle L_h, v_h - u_h \rangle, \qquad v_h \in K_h, \tag{4.5}$$

where u_h denotes the solution of (4.5) with

$$a_h(u_h, v_h) = \int_{\Omega_h} \sigma(u_h) \colon \epsilon(v_h) \, \mathrm{d}x, \tag{4.6}$$

and

$$\langle L_h, v_h \rangle = \int\limits_{\Omega_h} \langle f, v_h \rangle \, \mathrm{d}x + \int\limits_{\Gamma_{Nh}} \langle g, v_h \rangle \, \mathrm{d}s. \tag{4.7}$$

The existence and uniqueness of a solution will be dealt with in the next section.

4.3 Existence and Uniqueness of a Discrete Solution

In this section the existence and uniqueness of a discrete solution of the variational inequality (4.5) is shown by using the results of Section 3.3 and Section 3.4. As $\mathcal{S}_h \subseteq H^1(\Omega_h)$ and Ω_h has piecewise linear boundary, the ellipticity of the bilinearform (4.6) can be shown by using Theorem 3.7 and is inherited to $\mathcal{S}_h$. Similar to Theorem 3.8 the following theorem can be deduced from Corollary 3.6:

Theorem 4.1. *Let Ω be a bounded domain with regular boundary, $\Gamma_D \subseteq \Gamma$ with positive measure. Let $\mathcal{T}_h$ denote a regular triangulation of Ω by using linear triangle elements, Ω_h be the union of all triangles in $\mathcal{T}_h$. Furthermore let K_h be as in* (4.4) *with a finite element space $\mathcal{S}_h$ like in Section 4.2. Furthermore let $f \in H^{-1}(\Omega)$, $g \in H^{-\frac{1}{2}}(\Gamma_N)$.*

Then the variational inequality (4.5) *with $a_h(\cdot,\cdot)$ defined in* (4.6) *and $\langle L_h, \cdot \rangle$ defined in* (4.7) *has a unique solution in K_h.*

4.4 Convergence of the Discrete Solution

In order to complete the analysis of the discrete problem, a result showing the convergence of the discrete solution u_h to the solution of the continuous problem (3.9) is presented.

Let us suppose that $\Omega_h \subseteq \Omega$ and let $\tilde{v}_h$ denote an extension to the domain Ω of the function v_h defined on $\overline{\Omega}_h$ constructed so that the value of $\tilde{v}_h$ in $\overline{\Omega} \setminus \overline{\Omega}_h$ is defined by constant extension in the directions normal to the boundary Γ_h. Furthermore let $\tilde{K}_h$ be the set of all such extensions of functions in K_h. By definition it is clear that the following identities hold for all $\tilde{u}_h, \tilde{v}_h \in \tilde{K}_h$:

$$\begin{aligned} a_h(u_h, v_h) &= a_h(\tilde{u}_h, \tilde{v}_h) \\ \langle L_h, v_h \rangle &= \langle L_h, \tilde{v}_h \rangle \end{aligned} \tag{4.8}$$

Using this extension the following theorem can be shown:

Theorem 4.2. *Suppose that Ω has sufficiently smooth boundary and the solution $u \in K$ of the variational inequality* (3.9) *is regular in the sense that $u \in H^2(\Omega)$. Suppose that the function d, which characterizes the distance of the body from the rigid foundation, belongs to $H^{\frac{3}{2}}(\Gamma_C)$. Let $\tilde{u}_h \in \tilde{K}_h$ be the extension of the approximate solution $u_h \in K_h$ of* (4.5) *onto the whole domain Ω. Further suppose that the bilinear form $a(\cdot,\cdot)$ defined by* (3.34) *is coercive and continuous on V. In addition, suppose that $f \in L^\infty(\Omega)$ and $g \in L^\infty(\Gamma_N)$.*

Then, the sequence of extended finite element approximations $(\tilde{u}_h)_{h \in H}$ converges in V to the solution $u \in K$ of the variational inequality (3.9) *as h tends to zero. Indeed, a constant C independent of h exists, such that*

$$\|\tilde{u}_h - u\|_{H^1(\Omega_h)} \leq Ch \tag{4.9}$$

where h is the mesh parameter for regular refinements of the mesh.

Proof. KIKUCHI AND ODEN [19] □

Remark 4.3. *The assumption $u \in H^2(\Omega)$ made in Theorem 4.2 is a restriction and not automatically fulfilled by solutions of the variational inequality* (3.9). *Especially at points with changing boundary conditions the regularity is less than $H^2(\Omega)$ in general.*

Chapter 5

Numerical Solution of Signorini's Problem

5.1 Introduction

In Chapter 4 a convergent finite element approximation of Signorini's problem was derived. In the following an iterative method for solving the resulting discrete problem is derived. The starting point for the following is the fact, that the variational inequality

$$a_h(u_h, v_h - u_h) \geq \langle L_h, v_h - u_h \rangle, \qquad v_h \in K_h,$$

is a necessary and sufficient condition for a minimum of

$$J(v_h) = \frac{1}{2}\, a_h(v_h, v_h) - \langle L_h, v_h \rangle, \qquad v_h \in K_h.$$

The solution method for the resulting algebraic system is a relaxation method of Gauß-Seidel type. The unsatisfactory convergence rates of the classical Gauß-Seidel type iterations are caused by the bad representation of the low-frequency contribution of the error. In order to improve that, suitable coarse grid functions are added to the space splitting.

Many references on multigrid and multilevel methods for elliptic problems can be found in Hackbusch [13] or in Langer [27]. They have also been applied to various kinds of variational inequalities for example by Mandel [28], Brandt and Cryer [2], Hackbusch and Mittelmann [14], Hoppe [16], Hoppe and Kornhuber [17], Kornhuber [24] or Schöberl [31]. An introduction into the field of truncated base functions can be found in Kornhuber [24]. Kornhuber [24] also provides a long list of references on the field of multigrid methods for variational inequalities.

Section 5.2 presents an introduction on Gauß-Seidel relaxation methods including a proof of their convergence. In order to improve the convergence properties additional coarse grid functions are added to the space splitting. These extended relaxation methods are presented in Section 5.3 including a proof of their convergence. One possibility for choosing the coarse grid functions is the multilevel nodal basis. The resulting algorithm is presented in Section 5.4.

As this algorithm can not be implemented with optimal order, also the obstacle has to be approximated. This has to be done in a suitable manner in order to preserve the global convergence of the method. One possibility for such an approximation is presented in Section 5.5. In order to improve the convergence rates of the algorithms presented in Section 5.5 an approach using truncated base functions is presented in Section 5.6.

5.2 Gauß-Seidel Relaxation Methods

The starting point for the following considerations is the fact, that solving

$$a_h(u_h, v_h - u_h) \geq \langle L_h, v_h - u_h \rangle, \qquad v_h \in K_h, \tag{5.1}$$

is equivalent to minimizing

$$J(v_h) = \frac{1}{2}\, a_h(v_h, v_h) - \langle L_h, v_h \rangle \tag{5.2}$$

in K_h. The basic idea of relaxation methods is to decompose the global minimization of $J(v_h)$ into a sequence of local minimizations. The Gauß-Seidel relaxation method results from the successive minimization of J in the subspaces spanned by $\left\{\lambda_1^{(i)}, \ldots, \lambda_d^{(i)}\right\}$. $\lambda_j^{(i)}$ represents the nodal base function at point $p_i \in \mathcal{T}_h$ in space direction j, d is the dimension of Ω_h.

The whole procedure can be found in Algorithm 5.1, where $u_j^{(i)}$, $i = 1, \ldots, n$ denote intermediate iterates and $\alpha^{(i)}\lambda^{(i)}$ stands short for $\sum_{k=1}^{d} \alpha_k^{(i)} \lambda_k^{(i)}$.

One iteration step of Algorithm 5.1 will be abbreviated by

$$u_{j+1} = \mathcal{M}(u_j). \tag{5.3}$$

As the functional J is strictly convex, the minimization step

$$J(u_j^{(i-1)} + \beta\lambda^{(i)}) \longrightarrow \min_{u_j^{(i-1)} + \beta\lambda^{(i)} \in K_h} \tag{5.4}$$

has a unique solution and therefore the algorithm is welldefined. Furthermore the energy is monotonically decreasing by construction. In the following theorem the global convergence of the Gauß-Seidel relaxation is shown:

Theorem 5.1. *For any initial guess $u_0 \in K_h$ the sequence of iterates $(u_j)_{j\in\mathbb{N}}$ generated by the Algorithm 5.1 converges to the solution u_h of the variational inequality* (5.1).

Proof. The sequence of iterates $(u_j)_{j\in\mathbb{N}}$ is bounded because of

$$J(u_0) \geq J(u_j) = \frac{1}{2}\, a_h(u_j, u_j) - \langle L_h, u_j \rangle \geq \frac{\alpha}{2}\, \|u_j\|^2 - \|L_h\| \|u_j\| \geq \text{const}, \tag{5.5}$$

with $\alpha > 0$ denoting the coercivity constant of $a_h(\cdot,\cdot)$, i.e.

$$a_h(u, u) \geq \alpha\, \|u\|^2, \qquad u \in V_h. \tag{5.6}$$

Let $u_0 \in K_h$ *denote an arbitrary initial guess.*
$j := 0$

(∗ Loop until convergence ∗)
WHILE *not converged* **DO**
BEGIN
  $u_j^{(0)} := u_j$

  (∗ Minimize the energy J in every subspace ∗)
  FOR $i := 1$ **TO** n **DO**
  BEGIN
    (∗ Minimize the energy $J(u_j^{(i-1)} + \cdot)$ in the subspace ∗)
    (∗ spanned by $\left\{\lambda_1^{(i)}, \ldots, \lambda_d^{(i)}\right\}$ ∗)

$$J(u_j^{(i-1)} + \alpha^{(i)}\lambda^{(i)}) \leq J(u_j^{(i-1)} + \beta\lambda^{(i)}), \quad u_j^{(i-1)} + \beta\lambda^{(i)} \in K_h$$

    $u_j^{(i)} := u_j^{(i-1)} + \alpha^{(i)}\lambda^{(i)}$
  END

  $u_{j+1} := u_j^{(n)}$
  $j := j + 1$
END

Algorithm 5.1: Gauß-Seidel relaxation

Let $(u_{j_k})_{k\in\mathbb{N}}$ be an arbitrarily chosen converging subsequence with

$$\tilde{u} = \lim_{k\to\infty} u_{j_k}, \qquad \tilde{u} \in K_h. \tag{5.7}$$

$(u_{j_k})_{k\in\mathbb{N}}$ exists as $(u_j)_{j\in\mathbb{N}}$ is bounded and V_h is of finite dimension. As

$$J\big(u_{j_{k+1}}\big) \le J\big(u_{j_k+1}\big) = J\big(\mathcal{M}(u_{j_k})\big) \le J\big(u_{j_k}\big) \tag{5.8}$$

the continuity of $J(\cdot)$ and $\mathcal{M}(\cdot)$ implies

$$J\big(\mathcal{M}(\tilde{u})\big) = J(\tilde{u}). \tag{5.9}$$

From (5.9) it is easy to follow that

$$J(\tilde{u}^{(i)}) = J(\tilde{u}), \qquad i = 1, \ldots, n. \tag{5.10}$$

$\tilde{u}^{(i)}$ denotes an intermediate iterate of Algorithm 5.1 with starting point $\tilde{u}$. As the solution of

$$J\big(\tilde{u}^{(i-1)} + \alpha\lambda^{(i)}\big) \to \min_{\substack{\alpha\in\mathbb{R} \\ \tilde{u}^{(i-1)}+\alpha\lambda^{(i)}\in K_h}} \tag{5.11}$$

is unique and

$$J(\tilde{u}^{(i)}) = J(\tilde{u}^{(i-1)}) = J(\tilde{u}) \tag{5.12}$$

the equality

$$\tilde{u}^{(i)} = \tilde{u}^{(i-1)}, \qquad i = 1, \ldots, n \tag{5.13}$$

holds, resulting in

$$\mathcal{M}(\tilde{u}) = \tilde{u}. \tag{5.14}$$

It remains to show that $\tilde{u}$ is a solution of the variational inequality (5.1). Let $y \in K_h$ be arbitrary. y and $\tilde{u}$ can be written as

$$y = \sum_{i=1}^{n} \beta^{(i)}\lambda^{(i)}, \qquad \tilde{u} = \sum_{i=1}^{n} \tilde{\beta}^{(i)}\lambda^{(i)}. \tag{5.15}$$

As $\tilde{u}$ is a fixpoint of $\mathcal{M}$, the variational inequality

$$a_h\big(\tilde{u} + \alpha\lambda^{(j)}, (\beta-\alpha)\lambda^{(j)}\big) \ge \langle L_h, (\beta-\alpha)\lambda^{(j)}\rangle, \qquad \tilde{u}+\alpha\lambda^{(j)}, \tilde{u}+\beta\lambda^{(j)} \in K_h \tag{5.16}$$

has the solution $\alpha = 0$ for $j = 1, \ldots, n$ leading to

$$a_h\big(\tilde{u}, \beta\lambda^{(j)}\big) \ge \langle L_h, \beta\lambda^{(j)}\rangle, \qquad \tilde{u} + \beta\lambda^{(j)} \in K_h. \tag{5.17}$$

Due to the special form of the restrictions in K_h it is easy to see that for any $j \in \{1, \ldots, n\}$

$$\tilde{u} + \big(\beta^{(j)} - \tilde{\beta}^{(j)}\big)\lambda^{(j)} \in K_h. \tag{5.18}$$

Combining (5.17) and (5.18) and summing up (5.17) with respect to j it follows that

$$\sum_{j=1}^{n} a_h\big(\tilde{u}, (\beta^{(j)} - \tilde{\beta}^{(j)})\lambda^{(j)}\big) \geq \sum_{j=1}^{n} \langle L_h, (\beta^{(j)} - \tilde{\beta}^{(j)})\lambda^{(j)}\rangle \tag{5.19}$$

resulting in

$$a_h(\tilde{u}, y - \tilde{u}) \geq \langle L_h, y - \tilde{u}\rangle. \tag{5.20}$$

As $(u_{j_k})_{k\in\mathbb{N}}$ was an arbitrarily chosen convergent subsequence of $(u_j)_{j\in\mathbb{N}}$, the whole sequence has to converge to $\tilde{u}$. As the solution of the variational inequality is unique, $(u_j)_{j\in\mathbb{N}}$ converges to the solution of the variational inequality, which completes the proof.

□

The last part of the proof makes strong usage of the nodal decoupling of the unknowns, respectively the inequalities describing K_h. In fact, this decoupling is necessary for the global convergence of relaxation methods of Gauß-Seidel type, as simple counter examples show (see Glowinski [10]).

5.3 Extended Relaxation Methods

Similar to the unrestricted case, the Gauß-Seidel procedure of Section 5.2 has unsatisfactory convergence properties for decreasing mesh width, although it is globally convergent. This originates in the fact, that it uses only high frequency basis functions for the space splitting. Therefore the low-frequency contribution to the error is represented badly. In order to improve the convergence speed, the set of search directions is extended by additional coarse grid functions $\mu^{(i)}$ with large support, in contrast to the fine grid functions used up to now.

Extended relaxation methods of Gauß-Seidel type result by the successive minimization of J first in the subspaces spanned by $\left\{\lambda_1^{(i)}, \dots, \lambda_d^{(i)}\right\}$, $i = 1, \dots, n$ and afterwards in the subspaces spanned by $\left\{\mu_1^{(i)}, \dots, \mu_d^{(i)}\right\}$, $i = 1, \dots, m$. Neither the case $\mu_j^{(i)} = \mu_j^{(k)}$, $j = 1, \dots, d$, $i \neq k$ nor the case $\mu_j^{(i)} = \lambda_j^{(k)}$, $j = 1, \dots, d$ is excluded in this setup. The whole minimization procedure can be found in Algorithm 5.2.

$u_j^{(i)}$ denotes an intermediate iterate. The first inner iteration of Algorithm 5.2 denotes the fine grid correction and is analogous to Algorithm 5.1, the second inner iteration denotes the coarse grid correction.

Theorem 5.2. *For any initial guess $u_0 \in K_h$ the sequence of iterates $(u_j)_{j\in\mathbb{N}}$ generated by Algorithm 5.2 converges to the solution u_h of the variational inequality* (5.1)*.*

Proof. The proof can be done in a similar way to the proof of Theorem 5.1.

Let $u_0 \in K_h$ *denote an arbitrary initial guess.*
$j := 0$

(∗ Loop until convergence ∗)
WHILE *not converged* **DO**
BEGIN
  $u_j^{(0)} := u_j$

  (∗ Fine grid relaxation ∗)
  FOR $i := 1$ **TO** n **DO**
  BEGIN
    (∗ Minimize the energy $J(u_j^{(i-1)} + \cdot)$ in the subspace ∗)
    (∗ spanned by $\left\{\lambda_1^{(i)}, \ldots, \lambda_d^{(i)}\right\}$ ∗)

$$J(u_j^{(i-1)} + \alpha^{(i)}\lambda^{(i)}) \leq J(u_j^{(i-1)} + \beta\lambda^{(i)}), \quad u_j^{(i-1)} + \beta\lambda^{(i)} \in K_h$$

$$u_j^{(i)} := u_j^{(i-1)} + \alpha^{(i)}\lambda^{(i)}$$

  END

  (∗ Coarse grid relaxation ∗)
  FOR $i := n+1$ **TO** $n+m$ **DO**
  BEGIN
    (∗ Minimize the energy $J(u_j^{(i-1)} + \cdot)$ in the subspace ∗)
    (∗ spanned by $\left\{\mu_1^{(i-n)}, \ldots, \mu_d^{(i-n)}\right\}$ ∗)

$$J(u_j^{(i-1)} + \alpha^{(i)}\mu^{(i-n)}) \leq J(u_j^{(i-1)} + \beta\mu^{(i-n)}), \quad u_j^{(i-1)} + \beta\mu^{(i-n)} \in K_h$$

$$u_j^{(i)} := u_j^{(i-1)} + \alpha^{(i)}\mu^{(i-n)}$$

  END

  $u_{j+1} := u_j^{(n+m)}$
  $j := j+1$
END

Algorithm 5.2: Extended Gauß-Seidel relaxation

The first part showing $(u_j)_{j\in\mathbb{N}}$ being bounded can be done in analogy to Theorem 5.1. Let $(u_{j_k})_{k\in\mathbb{N}}$ denote an arbitrarily chosen subsequence converging to $\tilde{u}$. As the energy of the iterate is decreasing in each inner iteration step

$$J(u_{j_{k+1}}) \leq J(u_{j_k+1}) \leq J\big(\mathcal{M}(u_{j_k})\big) \leq J(u_{j_k}), \tag{5.21}$$

inducing by use of the continuity of $\mathcal{M}$ and J

$$J\big(\mathcal{M}(\tilde{u})\big) = J(\tilde{u}). \tag{5.22}$$

In analogy to the proof of Theorem 5.1, it can be shown that $\tilde{u}$ is a fixpoint of $\mathcal{M}$ and solves the variational inequality (5.1). □

The proof of the above theorem strongly relies on two facts:

- Each fine grid base function is used in the relaxation.
- Each coarse grid correction should not increase the energy.

The great advantage of the above theorem is its flexibility on choosing the coarse grid functions as there are hardly any restrictions for their selection.

5.4 Multilevel Gauß-Seidel Relaxation

Motivated by the convergence speed up of unrestricted multilevel methods for elliptic selfadjoint problems compared to unrestricted Gauß-Seidel procedures, a convergent relaxation method was derived in the previous section. The question, how to choose the additional coarse grid functions, was left unanswered. It is easily seen that the classical multigrid method with a Gauß-Seidel smoother can be regarded as an extended relaxation method induced by the multilevel nodal basis Λ_S, specified in detail later. The resulting algorithm can be implemented in a similar form to the classical V-cycle.

Let $\mathcal{T}_0$ denote the initial triangulation of Ω_h. The triangulation $\mathcal{T}_0$ is refined serveral times, providing a sequence of triangulations $\mathcal{T}_1, \ldots, \mathcal{T}_j$. It is assumed that the initial triangulation $\mathcal{T}_0$ is regular, especially that no hanging nodes exist. The refining should be done in such a way, that it can be guaranteed that

- the refined triangulation still has no hanging nodes, and
- that the inner angles are bounded away from 0, even for j tending to infinity.

Both assumptions can be fulfilled for example by the following refinement stategy for 2D problems, which is often referenced as *red-green refinement* (see e. g. Kornhuber [24] or Verfürth [33]).

A triangle $t \in \mathcal{T}_k$ is refined either by subdividing it into four congruent subtriangles or by connecting one of its vertices with the midpoint of the opposite side. The first case is called *regular (red) refinement* and the resulting triangles are regular as they are similar to t. The second case is called *irregular (green)*

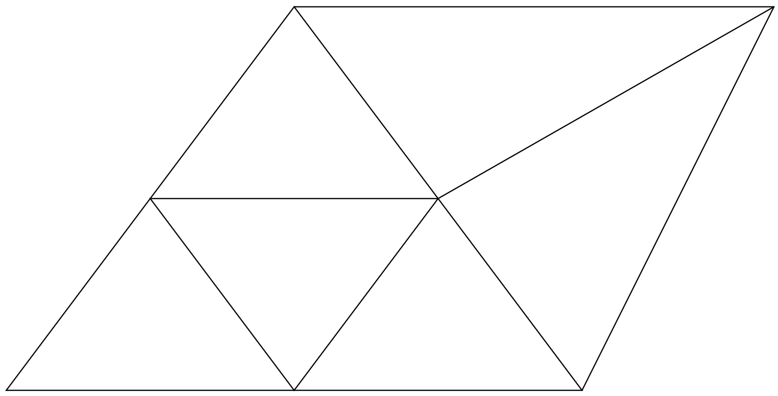

Figure 5.1: Regular refinement and irregular closure

refinement and results in two irregular triangles. In order to fulfill the two assumptions made above, irregular triangles must not be further refined.

The construction of the refined triangulations should be based on some adaptive strategy. For the refinement method described above this looks as follows: Triangles marked for refinement by the error estimator are refined regularly. Irregular refinements are only used to eliminate hanging nodes, generated by the previous regular refinement. Irregular refined triangles must not be refined any further.

The sequence $\mathcal{T}_0, \ldots, \mathcal{T}_j$ of triangulations gives rise to a sequence of finite element spaces $\mathcal{S}_0, \ldots, \mathcal{S}_j$. Let

$$\Lambda_k = \left\{ \lambda^{(i),k} \mid p_i \in \mathcal{N}_k \right\} \tag{5.23}$$

denote the set of nodal basis functions, $k = 0, \ldots, j$. The multilevel nodal basis Λ_S is then defined by

$$\Lambda_S = \left\{ \lambda^{(1),j}, \ldots, \lambda^{(n_j),j}, \ldots, \lambda^{(1),0}, \ldots, \lambda^{(n_0),0} \right\}, \tag{5.24}$$

where n_k denotes the number of nodes in $\mathcal{N}_k$. In the case of unrestricted, elliptic, selfadjoint problems the classical V-cycle with Gauß-Seidel presmoothing and no postsmoothing and the canonical restrictions and prolongations can be seen as extended relaxation method induced by the constant set of search directions Λ_S.

The iteration procedure is given in Algorithm 5.3. As postsmoothing iterations proofed useful in the unrestricted case, they are also added in this context in order to get a symmetric iteration operator.

While the fine grid corrections in the direction of $\lambda^{(1),j}, \ldots, \lambda^{(n_j),j}$ can be calculated explicitly, the exact calculation of the coarse grid correction is rather expensive, as every intermediate iterate has to be evaluated on nodes of the finest grid in order to test its admissability. That is why additional prolongations are needed for every local coarse grid correction. As a consequence such a procedure can no longer be implemented with $O(n_j)$ operations for one global iteration step.

Let $u_0 \in K_h$ *denote an arbitrary initial guess.*
$l := 0$

(∗ Loop until convergence ∗)
WHILE *not converged* **DO**
BEGIN
  $u_l^{(0)} := u_l$
  $\nu := 0$

  (∗ Presmoothing on each level ∗)
  FOR $k := j$ **TO** 0 **STEP** -1 **DO**
  BEGIN
    FOR $i := 1$ **TO** n_k **DO**
    BEGIN
      (∗ Minimize the energy $J(u_l^{(\nu)} + \cdot)$ in the subspace ∗)
      (∗ spanned by $\left\{\lambda_1^{(i),k}, \dots, \lambda_d^{(i),k}\right\}$ ∗)

      $J(u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}) \leq J(u_l^{(\nu)} + \beta\lambda^{(i),k}), \quad u_l^{(\nu)} + \beta\lambda^{(i),k} \in K_h$

      $u_l^{(\nu+1)} := u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}$
      $\nu := \nu + 1$
    END
  END

  (∗ Postsmoothing on each level ∗)
  FOR $k := 0$ **TO** j **DO**
  BEGIN
    FOR $i := n_k$ **TO** 1 **STEP** -1 **DO**
    BEGIN
      (∗ Minimize the energy $J(u_l^{(\nu)} + \cdot)$ in the subspace ∗)
      (∗ spanned by $\left\{\lambda_1^{(i),k}, \dots, \lambda_d^{(i),k}\right\}$ ∗)

      $J(u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}) \leq J(u_l^{(\nu)} + \beta\lambda^{(i),k}), \quad u_l^{(\nu)} + \beta\lambda^{(i),k} \in K_h$

      $u_l^{(\nu+1)} := u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}$
      $\nu := \nu + 1$
    END
  END

  $u_{l+1} := u_l^{\nu}$
END

Algorithm 5.3: Multilevel Gauß-Seidel relaxation

5.5 Restriction of the Obstacle

In the previous section it was shown that a straight forward transfer of the multigrid method to the variational inequality (5.1) results in a method which is no more of optimal order, as the number of operations per global iteration step is no longer bounded by $O(n_j)$. One way to overcome this problem is to calculate the coarse grid correction only approximately. Such an approximation has to be done in a suitable manner in order to preserve the global convergence of the algorithm.

5.5.1 General Considerations

On the coarser grids $\mathcal{T}_k$, $k = 0, \ldots, j-1$ linear functions can be represented by their values on the coarse grid nodes. This property gives rise to the canonical restrictions of the stiffness matrix and the residual occuring in the implementation of linear subspace corrections as the V-cycle.

To take advantage of the simple representation of linear functions on coarse grid spaces, the coarse grid corrections are restricted to the neighborhood of u_j in the following sense: The initial fine grid correction defines the set of active restrictions $\mathcal{A}^j$. This set is not changed during the coarse grid correction process, which means, that

$$\langle u_{j+1}(p_i) - u_j(p_i), n(p_i) \rangle = 0, \qquad i \in \mathcal{A}^j. \tag{5.25}$$

The necessity of the active index set will become clear in the context of truncated base functions used later.

One way to get the effort of one global iteration step under control is to approximate the obstacle in an appropriate fashion. In Algorithm 5.3 one originally had to minimize the energy J on K_h, i.e.

$$J(u_l^{(\nu-1)} + \beta\lambda^{(i),k}) \to \min_{\substack{\beta\in\mathbb{R} \\ u_l^{(\nu-1)}+\beta\lambda^{(i),k}\in K_h}} . \tag{5.26}$$

The reason for the large effort per iteration step of Algorithm 5.3 is the fact that $\lambda^{(i),k} \in \Lambda_k$ has to be evaluated on fine grid points in order to test the admissability of the intermediate iterate, although $\lambda^{(i),k}$ can be represented by its values on the nodes of the k-th level $\mathcal{N}_k$.

In order to get this problem under control, instead of (5.26) the problem

$$J(u_l^{(\nu-1)} + \beta\lambda^{(i),k}) \to \min_{\substack{\beta\in\mathbb{R} \\ u_l^{(\nu-1)}+\beta\lambda^{(i),k}\in K_h^k}} \tag{5.27}$$

is solved, where K_h^k denotes an approximation of K_h containing only restrictions on nodes of the k-th level, but not on nodes of finer levels. In most cases K_h^k is a subset of K_h.

K_h contains inequality restrictions of the form

$$\langle v(p_i), n(p_i) \rangle \le d(p_i), \qquad i \in I, \tag{5.28}$$

K_h^k should contain only restrictions on nodes of the k-th level $\mathcal{N}_k$, that is

$$\langle v(p_i), n(p_i)\rangle \leq \tilde{d}(p_i), \qquad i \in I \cap \mathcal{N}_k, \tag{5.29}$$

where $\tilde{d}$ has to be chosen suitably.

5.5.2 Multilevel Gauß-Seidel Relaxation for Flat Obstacles

In the following it is assumed that all $n(p_i)$, $i \in I$ are equal resulting in

$$n(p_i) = n_0, \qquad i \in I, \tag{5.30}$$

in order to derive a suitable restriction of the obstacle. As the algorithm will be implemented similar to a V-cycle it is enough to derive a restriction of the obstacle on level $k+1$ to that on the k-th level.

Suppose that on level $k+1$ the inequality restrictions have the form

$$\langle v(p_i), n_0\rangle \leq d(p_i), \qquad i \in I \cap \mathcal{N}_{k+1}. \tag{5.31}$$

From now on μ^k denotes an arbitrary coarse grid function of level k, that is

$$\mu^k \in \left\{\lambda^{(1),k}, \dots, \lambda^{(n_k),k}\right\}. \tag{5.32}$$

Then the restrictions on the correction $\alpha\mu^k$ look as follows:

$$\langle u_j^{(\nu-1)}(p_i) + \alpha\mu^k(p_i), n_0\rangle \leq d(p_i), \qquad i \in I \cap \mathcal{N}_{k+1}. \tag{5.33}$$

It is easily seen, that for choosing α several restrictions need to be considered. In detail

$$\langle \alpha\mu^k(p_i), n_0\rangle \leq d(p_i) - \langle u_j^{(\nu-1)}(p_i), n_0\rangle, \qquad i \in I \cap \mathcal{N}_{k+1} \cap \operatorname{supp}\mu^k. \tag{5.34}$$

As the obstacle on level k should only contain restrictions on nodes of level k, one could use

$$\langle \alpha\mu^k(p_i), n_0\rangle \leq \min\left\{d(p_k) - \langle u_j^{(\nu-1)}(p_k), n_0\rangle \;\middle|\; p_k \in \operatorname{supp}(\mu^k) \cap I \cap \mathcal{N}_{k+1}\right\}, \quad i \in I \cap \mathcal{N}_k. \tag{5.35}$$

As $\mu^k \in \Lambda_k$, there is only one $p_i \in \mathcal{N}_k$ with $\mu^k(p_i) \neq 0$. This connection will be expressed in the term $\mu_{p_i}^k$, which denotes for given p_i the coarse grid function $\mu^k \in \Lambda_k$ with $\mu^k(p_i) \neq 0$.

(5.35) delivers only one restriction on the choice of α and can be rewritten in the form

$$\langle \alpha\mu^k(p), n_0\rangle \leq \tilde{d}(p), \qquad p \in I \cap \mathcal{N}_k, \tag{5.36}$$

with

$$\tilde{d} = r_{\mathrm{k+1}}^{\mathrm{k}}\left(d - \langle u_j^{(\nu-1)}, n_0\rangle\right) \tag{5.37}$$

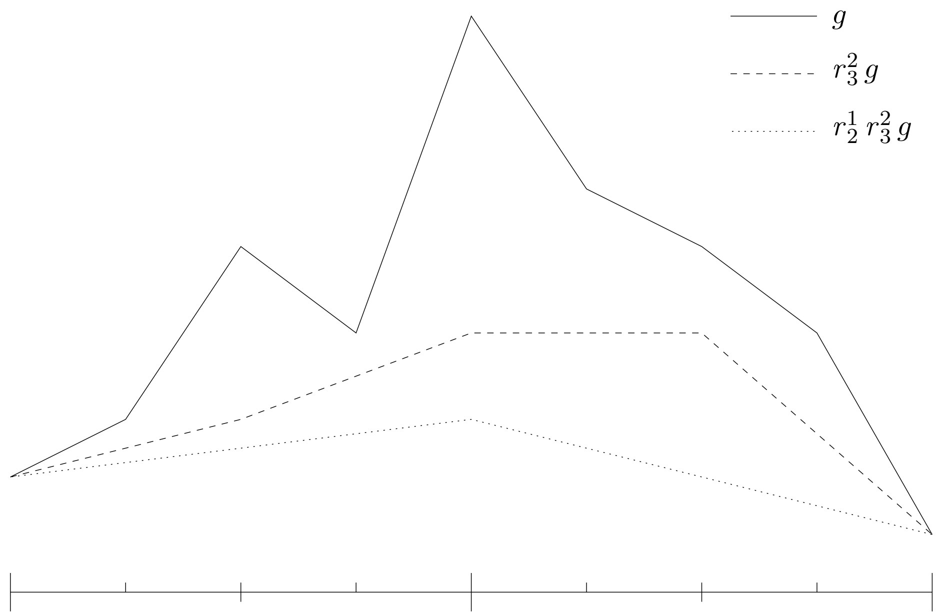

Figure 5.2: Restriction operator $r_{\mathrm{k+1}}^{\mathrm{k}}$

and

$$r_{\mathrm{k+1}}^{\mathrm{k}}\, h(p) = \min \left\{ h(q) \;\middle|\; q \in I \cap \mathcal{N}_{k+1} \cap \operatorname{supp}(\mu_p^k) \right\}, \qquad p \in I \cap \mathcal{N}_k. \quad (5.38)$$

The effect of this restriction can be seen in Figure 5.2.

Although the definition of $r_{\mathrm{k+1}}^{\mathrm{k}}$ looks quite natural, it can be seen, that $r_{\mathrm{k+1}}^{\mathrm{k}}$ delivers rather pessimistic bounds for the corrections. This can be improved by taking the fact into account, that linear finite elements are used.

An improved restriction can be found in KORNHUBER [24]. Let $\mathcal{E}_k' \in \mathcal{E}_k$ denote the set of bisected edges and $\mathcal{E}_k$ the set of all the edges of level k, $p_e \in \mathcal{N}_{k+1}$ is the midpoint of $e \in \mathcal{E}_k'$. Selecting a certain order in $\mathcal{E}_k'$, this set can be written as $\mathcal{E}_k' = (e_1, \ldots, e_s)$. Then the restriction operator $\mathcal{R}_{\mathrm{k+1}}^{\mathrm{k}}$ is defined according to

$$\mathcal{R}_{\mathrm{k+1}}^{\mathrm{k}}\, v = \mathcal{R}_{e_s} \circ \cdots \circ \mathcal{R}_{e_1}\, v. \quad (5.39)$$

For each $e \in \mathcal{E}_k'$ the operator $\mathcal{R}_e$ has the form

$$\mathcal{R}_e\, v = v + v^1 \mu_{p_1}^{k+1} + v^2 \mu_{p_2}^{k+1}, \quad (5.40)$$

with $p_1, p_2 \in \mathcal{N}_k$ denoting the vertices of $e = (p_1, p_2) \in \mathcal{E}_k'$. The scalars $v^1, v^2 \in \mathbb{R}$ are chosen such that

$$\mathcal{R}_e\, v(p) \leq v(p), \qquad p = p_1, p_e, p_2. \quad (5.41)$$

This results in the following two cases:

- If $v(p_e) \geq \frac{1}{2}\big(v(p_1) + v(p_2)\big)$, then

$$(\mathcal{R}_e\, v)(p_1) = v(p_1) \qquad \text{and} \qquad (\mathcal{R}_e\, v)(p_2) = v(p_2). \quad (5.42)$$

$v(p_1) \le v(p_e)$	$v(p_e) \le v(p_2)$	$(\mathcal{R}_e\, v)(p_1) = v(p_1)$ $(\mathcal{R}_e\, v)(p_2) = 2\, v(p_e) - v(p_1)$
	$v(p_e) \ge v(p_2)$	$(\mathcal{R}_e\, v)(p_1) = v(p_1)$ $(\mathcal{R}_e\, v)(p_2) = v(p_2)$
$v(p_1) \ge v(p_e)$	$v(p_e) \le v(p_2)$	$(\mathcal{R}_e\, v)(p_1) = v(p_e)$ $(\mathcal{R}_e\, v)(p_2) = v(p_e)$
	$v(p_e) \ge v(p_2)$	$(\mathcal{R}_e\, v)(p_1) = 2\, v(p_e) - v(p_2)$ $(\mathcal{R}_e\, v)(p_2) = v(p_2)$

Table 5.1: Calculation of $\mathcal{R}^{\mathrm{k}}_{\mathrm{k+1}}$ for $v(p_e) \le \frac{1}{2}\big(v(p_1) + v(p_2)\big)$

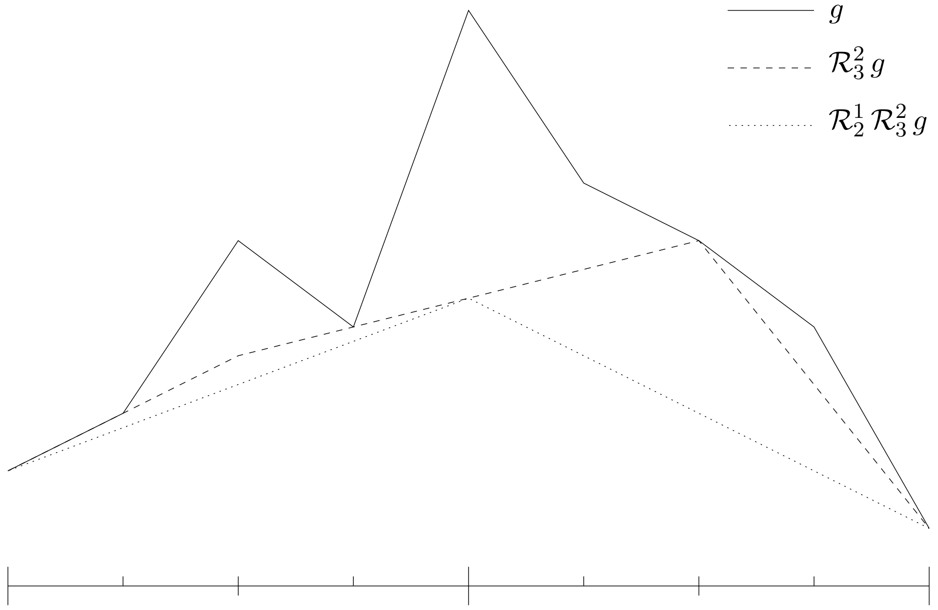

Figure 5.3: Restriction operator $\mathcal{R}^{\mathrm{k}}_{\mathrm{k+1}}$

- Otherwise the restrictions are given by Table 5.1.

Decomposing $r^{\mathrm{k}}_{\mathrm{k+1}}$ in local restriction operators r_e, $e \in \mathcal{E}'_k$ in the same way as done above, it can be shown that

$$r^{\mathrm{k}}_{\mathrm{k+1}}\, v \le \mathcal{R}^{\mathrm{k}}_{\mathrm{k+1}}\, v \tag{5.43}$$

holds for all nonnegative v. In analogy to Figure 5.2 the effect of this restriction can be seen in Figure 5.3. For a better comparison to $r^{\mathrm{k}}_{\mathrm{k+1}}$ the fine grid function is the same as in Figure 5.2.

Due to (5.43) one can expect less damping of the coarse grid corrections, using $\mathcal{R}^{\mathrm{k}}_{\mathrm{k+1}}$ instead of $r^{\mathrm{k}}_{\mathrm{k+1}}$ for restricting the obstacle, providing faster convergence of the whole algorithm.

The whole multilevel relaxation procedure for flat obstacles can be found in Algorithm 5.4, respectively Algorithm 5.5.

Algorithm 5.4. *In this algoritm a multilevel Gauß-Seidel relaxation method with obstacle restriction for flat obstacles is presented. j denotes the number of levels used.*

Let $u_0 \in K_h$ *denote an arbitrary initial guess.*
$l := 0$

(∗ *Loop until convergence* ∗)
WHILE *not converged* **DO**
BEGIN
$u_l^{(0)} := u_l$
$\nu := 0$

(∗ *Presmoothing on each level* ∗)
FOR $k := j$ **TO** 0 **STEP** -1 **DO**
BEGIN
Calculate K_h^k *by restricting the obstacle*

FOR $i := 1$ **TO** n_k **DO**
BEGIN
(∗ *Minimize the energy* $J(u_l^{(\nu)} + \cdot)$ *in the subspace* ∗)
(∗ *spanned by* $\left\{\lambda_1^{(i),k}, \ldots, \lambda_d^{(i),k}\right\}$ ∗)

$$J(u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}) \leq J(u_l^{(\nu)} + \beta\lambda^{(i),k}), \quad u_l^{(\nu)} + \beta\lambda^{(i),k} \in K_h^k$$

$$u_l^{(\nu+1)} = u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}$$

$\nu := \nu + 1$
END
END

(∗ *Postsmoothing on each level* ∗)
FOR $k := 0$ **TO** j **DO**
BEGIN
FOR $i := n_k$ **TO** 1 **STEP** -1 **DO**
BEGIN
(∗ *Minimize the energy* $J(u_l^{(\nu)} + \cdot)$ *in the subspace* ∗)
(∗ *spanned by* $\left\{\lambda_1^{(i),k}, \ldots, \lambda_d^{(i),k}\right\}$ ∗)

$$J(u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}) \leq J(u_l^{(\nu)} + \beta\lambda^{(i),k}), \quad u_l^{(\nu)} + \beta\lambda^{(i),k} \in K_h^k$$

$$u_l^{(\nu+1)} = u_l^{(\nu)} + \alpha^{(\nu)}\lambda^{(i),k}$$

$\nu := \nu + 1$
END
END

$u_{l+1} := u_l^\nu$
END

Algorithm 5.5. *In this algorithm a recursive version of a multilevel Gauß-Seidel relaxation with obstacle restriction for flat obstacles is presented. Again j denotes the number of levels used.*

Let $u_0 \in K_h$ denote an arbitrary initial guess.
$l := 0$

(∗ *Loop until convergence* ∗)
WHILE *not converged* **DO**
BEGIN
$u_l^{(0)} := u_l$

(∗ *Presmoothing* ∗)
PreSmooth$(j, u_l^{(0)}, u_l^{(1)})$

Calculate the set of active restrictions of $u_l^{(1)}$

(∗ *Calculation of the residual* ∗)
$r_l := \langle L_h, \cdot\rangle - a_h(u_l^{(1)}, \cdot)$

(∗ *Calculation of the restricted obstacle for the correction* ∗)
$d^{j-1} := \mathcal{R}_j^{j-1}\left(d^j - \langle u_l^{(1)}, n\rangle\right)$

(∗ *Canonical restriction* ∗)
$r^{j-1} := r_l|_{\mathcal{S}_{j-1}}; \quad a_h^{j-1} := a_h(\cdot,\cdot)|_{\mathcal{S}_{j-1}\times\mathcal{S}_{j-1}}$

$p^{j-1} := 0$
MGM$(j-1, a_h^{j-1}, r^{j-1}, p^{j-1}, d^{j-1})$

(∗ *Canonical interpolation* ∗)
$p^j := p^{j-1}|_{\mathcal{S}_j}$

$u_l^{(2)} := u_l^{(1)} + p^j$

(∗ *Postsmoothing* ∗)
PostSmooth$(j, u_l^{(2)}, u_l^{(3)})$

$u_{l+1} := u_l^{(3)}$
$l := l + 1$
END

PROCEDURE MGM(k, a^k, L^k, u^k, d^k)
BEGIN

IF $k = 1$ **THEN**
BEGIN
$u^{k,(0)} := u^k$
PreSmooth$(k, u^{k,(0)}, u^{k,(1)})$
PostSmooth$(k, u^{k,(1)}, u^{k,(2)})$
$u^k := u^{k,(2)}$
END
ELSE
BEGIN
$u^{k,(0)} := u^k$
PreSmooth$(k, u^{k,(0)}, u^{k,(1)})$

(∗ *Calculation of the residual* ∗)
$r^k := \langle L^k, \cdot \rangle - a^k(u^{k,(1)}, \cdot)$

(∗ *Calculation of the restricted obstacle for the correction* ∗)
$d^{k-1} := \mathcal{R}_k^{k-1} \left(d^k - \langle u^{k,(1)}, n \rangle \right)$

(∗ *Canonical restriction* ∗)
$r^{k-1} := r^k|_{\mathcal{S}_{k-1}}; \quad a^{k-1} := a^k(\cdot, \cdot)|_{\mathcal{S}_{k-1} \times \mathcal{S}_{k-1}}$

$p^{k-1} := 0$
MGM$(k-1, a^{k-1}, r^{k-1}, p^{k-1}, d^{k-1})$

(∗ *Canonical interpolation* ∗)
$p^k := p^{k-1}|_{\mathcal{S}_k}$

$u^{k,(2)} := u^{k,(1)} + p^k$

(∗ *Postsmoothing* ∗)
PostSmooth$(k, u^{k,(2)}, u^{k,(3)})$
$u^k := u^{k,(3)}$
END

END

PROCEDURE PreSmooth(k, u, v)
BEGIN
$u^0 := u$
FOR $i := 1$ **TO** n **DO**
BEGIN
(∗ *Minimize the energy* $J(u^{(i-1)} + \cdot)$ *in the subspace* ∗)

(∗ *spanned by* $\left\{\lambda_1^{(i),k},\ldots,\lambda_d^{(i),k}\right\}$ ∗)

$J(u^{(i-1)} + \alpha^{(i)}\lambda^{(i)}) \leq J(u^{(i-1)} + \beta\lambda^{(i)}), \quad u^{(i-1)} + \beta\lambda^{(i)} \in K_h^k$

$u^{(i)} := u^{(i-1)} + \alpha^{(i)}\lambda^{(i)}$

END

$v := u^{(n)}$

END

PROCEDURE PostSmooth(k, u, v)

BEGIN

$u^0 := u$

FOR $i := n$ **TO** 1 **STEP** -1 **DO**

BEGIN

(∗ *Minimize the energy* $J(u^{(n-i)} + \cdot)$ *in the subspace* ∗)

(∗ *spanned by* $\left\{\lambda_1^{(i),k},\ldots,\lambda_d^{(i),k}\right\}$ ∗)

$J(u^{(n-i)} + \alpha^{(i)}\lambda^{(i)}) \leq J(u^{(n-i)} + \beta\lambda^{(i)}), \quad u^{(n-i)} + \beta\lambda^{(i)} \in K_h^k$

$u^{(n-i+1)} := u^{(n-i)} + \alpha^{(i)}\lambda^{(i)}$

END

$v := u^{(n)}$

END

5.5.3 Multilevel Gauß-Seidel Relaxation for Arbitrary Obstacles

An arbitrary smooth obstacle is handled in a similar manner to flat obstacles. For restricting the obstacles the same procedures as in Algorithm 5.5 are used, as the normals of two adjacent points do not differ much as both the obstacle and the body have smooth surfaces. The corrections are again calculated on a coarser grid. As K_h^k is no longer a subset of K_h^{k+1} the convergence is no more assured by Theorem 5.2 as the corrected solution need not be a member of K_h^{k+1}. That is why an additional underrelaxation step is needed in order to assure the admissability of the intermediate iterate and as a consequence to guarantee the convergence of the scheme. The recursive version of the complete procedure can be found in Algorithm 5.6. It can be seen, that a standard V-cycle can easily be adopted to the requirements needed in this context by using projected Gauß-Seidel smoothers and by adding the obstacle restriction.

Algorithm 5.6. *In this algorithm a recursive version of a multilevel Gauß-Seidel relaxation with obstacle restriction for arbitrary smooth obstacles is presented.* j *denotes the number of levels used.*

Let $u_0 \in K_h$ *denote an arbitrary initial guess.*
$l := 0$

(∗ *Loop until convergence* ∗)
WHILE *not converged* **DO**
BEGIN
$u_l^{(0)} := u_l$

(∗ *Presmoothing* ∗)
PreSmooth$(j, u_l^{(0)}, u_l^{(1)})$

Calculate the set of active restrictions of $u_l^{(1)}$

(∗ *Calculation of the residual* ∗)
$r_l := \langle L_h, \cdot \rangle - a_h(u_l^{(1)}, \cdot)$

(∗ *Calculation of the restricted obstacle for the correction* ∗)
$d^{j-1} := \mathcal{R}_j^{j-1} \left(d^j - \langle u_l^{(1)}, n \rangle \right)$

(∗ *Canonical restriction* ∗)
$r^{j-1} := r_l|_{\mathcal{S}_{j-1}}; \quad a_h^{j-1} := a_h(\cdot, \cdot)|_{\mathcal{S}_{j-1} \times \mathcal{S}_{j-1}}$

$p^{j-1} := 0$
MGM$(j-1, a_h^{j-1}, r^{j-1}, p^{j-1}, d^{j-1})$

(∗ *Canonical interpolation* ∗)
$p^j := p^{j-1}|_{\mathcal{S}_j}$

(∗ *Underrelaxation* ∗)
$\omega = \max \left\{ \alpha \in [0,1] \;\middle|\; u_l^{(1)} + \alpha p^j \in K_h \right\}$
$u_l^{(2)} := u_l^{(1)} + \omega p^j$

(∗ *Postsmoothing* ∗)
PostSmooth$(j, u_l^{(2)}, u_l^{(3)})$

$u_{l+1} := u_l^{(3)}$
$l := l + 1$
END

PROCEDURE MGM$(k,\ a^k,\ L^k,\ u^k,\ d^k)$
BEGIN

IF $k = 1$ **THEN**
BEGIN
$u^{k,(0)} := u^k$
PreSmooth$(k, u^{k,(0)}, u^{k,(1)})$
PostSmooth$(k, u^{k,(1)}, u^{k,(2)})$
$u^k := u^{k,(2)}$
END
ELSE
BEGIN
$u^{k,(0)} := u^k$
PreSmooth$(k, u^{k,(0)}, u^{k,(1)})$

(∗ *Calculation of the residual* ∗)
$r^k := \langle L^k, \cdot\rangle - a^k(u^{k,(1)}, \cdot)$

(∗ *Calculation of the restricted obstacle for the correction* ∗)
$d^{k-1} := \mathcal{R}_k^{k-1}\left(d^k - \langle u^{k,(1)}, n\rangle\right)$

(∗ *Canonical restriction* ∗)
$r^{k-1} := r^k|_{\mathcal{S}_{k-1}}; \quad a^{k-1} := a^k(\cdot,\cdot)|_{\mathcal{S}_{k-1}\times\mathcal{S}_{k-1}}$

$p^{k-1} := 0$
MGM$(k-1, a^{k-1}, r^{k-1}, p^{k-1}, d^{k-1})$

(∗ *Canonical interpolation* ∗)
$p^k := p^{k-1}|_{\mathcal{S}_k}$

(∗ *Underrelaxation* ∗)
$\omega = \max\left\{\alpha \in [0,1] \;\middle|\; u^{k,(1)} + \alpha p^k \in K_h^k\right\}$
$u^{k,(2)} := u^{k,(1)} + p^k$

(∗ *Postsmoothing* ∗)
PostSmooth$(k, u^{k,(2)}, u^{k,(3)})$
$u^k := u^{k,(3)}$
END

END

The pre- and postsmoothing routines are implemented in analogy to those in Algorithm 5.5.

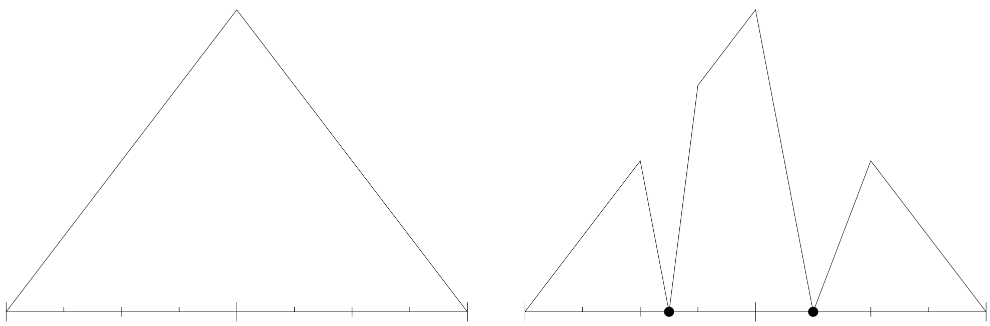

Figure 5.4: Standard and Trucated Nodal Base Functions

5.6 Improvements by Using Truncated Base Functions

For a special class of obstacle problems, the so-called membrane problems, which can be formulated as

$$\frac{1}{2}\, a_h(v_h, v_h) - \langle L_h, v_h \rangle \to \min_{v_h \le d} \tag{5.44}$$

Brandt and Cryer [2] and later on Kornhuber [22, 23] developed a fast converging algorithm using truncations of the nodal base functions. They observed that using standard nodal base functions they got weaker convergence rates due to the fact, that those coarse grid functions, whose support had nonempty intersection with the set of active restrictions, did not contribute to the coarse grid correction. That is why the asymptotic convergence rates were worse compared to the classical linear multigrid methods. Using truncated base functions they could improve the convergence rates for problem (5.44). For problems with more complex inequality restrictions the standard way for solving is to use active set strategies combined with projected gradients and Hessians (see Gill, Murray and Wright [9]). In this section it will be tried to bring both approaches under one umbrella.

First of all the usage of truncated base functions for problem (5.44) is described in order to motivate the following. A detailed derivation and description can be found in Kornhuber [22, 23, 24].

Consider a problem with restrictions like those in (5.44). For any point of the active set $\mathcal{A}$ the coarse grid correction must not change the value of the iterate. This implies that if $x_i \in \mathcal{A}$ then

$$\alpha^k \mu^k = 0 \qquad \text{for } \mu^i(x_i) \neq 0, \tag{5.45}$$

where μ^k denotes an arbitrary coarse grid function of level k. In Figure 5.4 the difference between a coarse grid and a truncated base function on the coarse grid is shown. The two dots denote points in the active set $\mathcal{A}$.

It can be seen that truncated base functions may have a rather strange shape. In particular their support does not need to be connected.

It is clear that the truncation process is not included into the assembling of the stiffness matrix, but into the restrictions and prolongations needed for the stiffness matrix and the residual. In detail this looks as follows:

- Modified restriction of the residual: Set all entries form active nodes to 0 and use linear restriction afterwards.
- Modified restriction of the stiffness matrix: Set all entries form active nodes to 0 and use linear restriction afterwards.
- Modified restriction of the upper bound: Handle all entries from active nodes like entries from unrestricted nodes.
- Modified prolongation of the correction: Set all prolongations to active nodes to 0.

In order to generalize this procedure, this point of view can not be used. But form the view point of projected gradients a generalization is clear. Setting entries belonging to active nodes to 0 is equivalent to eliminating parts not belonging to the null space of the active restrictions used in projected gradient methods.

This strategy can be easily generalized to the kind of boundary conditions used in the context of Signorini's problem. As the restrictions couple the displacements nodewise, not only the smoothing procedure, but also the restrictions and prolongations have to work nodewise. In most cases this requires a restructuring of the existing restriction and prolongation routines, but does not increase the cost of the procedures.

The Gauß-Seidel smoother requires just little changes. The only difference to the classical Gauß-Seidel smoother with projections is the introduction of an active set. Nodes in the active set are only moved in tangential direction, i. e. in the null space of the active restrictions.

Changing the restriction procedures requires more effort. As mentioned above the routine works nodewise. Let r_i denote the nodal residual at node p_i. From the view point of restrictions only those p_i are relevant, that are in the actual grid and not in the coarser one. Then p_i lies in the middle of an edge between p_{i_1} and p_{i_2}. If p_i is active only the part of r_i lying in the null space of the restriction belonging to p_i is restricted. Otherwise the whole part of r_i is restricted.

The restiction of the stiffness matrix works similar to the restriction of the residual. Similar to the nodal representation of the residual r a nodal representation of the stiffness matrix A is needed. As A is a block matrix, i.e.

$$A = \begin{pmatrix} A_{11} & \dots & A_{1d} \\ \vdots & \ddots & \vdots \\ A_{d1} & \dots & A_{dd} \end{pmatrix}, \tag{5.46}$$

where d denotes the number of space dimensions of Ω_h, the nodal part a_{ij} is defined as

$$a_{ij} = \begin{pmatrix} (A_{11})_{ij} & \dots & (A_{1d})_{ij} \\ \vdots & \ddots & \vdots \\ (A_{d1})_{ij} & \dots & (A_{dd})_{ij} \end{pmatrix}. \tag{5.47}$$

If neither i nor j is active, then a_{ij} is restricted by the canonical restricion.

If only i is active, the matrix a_{ij} is first transformed to

$$a_{ij} \to (I - n_i n_i^T) a_{ij}, \tag{5.48}$$

where I denotes the identity matrix, and is then restricted by the canonical restriction. (5.48) can be derived as follows: First of all a_{ij} is transformed into a space spanned by n_i and t_i^k, $k = 1, \dots, d-1$ by a base transform. t_i^k, $k = 1, \dots, d-1$ denote a basis of the null space of the restriction induced by n_i. For the transformed matrix the situation is in analogy to that in (5.44). The entries of the transformed matrix connected with n_i are set to 0 and the resulting matrix is transformed back into the original coordinate system.

The case, in which j is active, is treated in a similar manner leading to

$$a_{ij} \to a_{ij} (I - n_j n_j^T). \tag{5.49}$$

The resulting matrix is restricted in a canonical way.

If both i and j are active, the two cases above are combined leading to

$$a_{ij} \to (I - n_i n_i^T) a_{ij} (I - n_j n_j^T). \tag{5.50}$$

Again the restriction of the resulting matrix is done canonically.

As metioned above the described matrix transforms can be seen as an analogon to projected Hessians. By a suitable renumbering A can be written as

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \tag{5.51}$$

with n denoting the number of nodes and the active set $\mathcal{A} = \{1, \dots, m\}$ and a_{ij} like in (5.47). Then the transform described above is given by

$$A \to \begin{pmatrix} B_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & B_m & \ddots & & \vdots \\ \vdots & & \ddots & I & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & I \end{pmatrix} A \begin{pmatrix} B_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & B_m & \ddots & & \vdots \\ \vdots & & \ddots & I & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & I \end{pmatrix} \tag{5.52}$$

with

$$B_k = I - n_k n_k^T. \tag{5.53}$$

The prolongations are changed in the following way. First the corrections are prolongated in the canonical way to the finer grid. On the finer grid, the interpolated correction is projected onto the active restrictions in order to filter those parts out not belonging to the null space of the active restrictions. Furthermore underrelaxation of the projected correction is used not to violate any of the inequality restrictions.

The implementation of the method described above looks similar to Algorithm 5.6. The only difference is the usage of the modified restrictions and prolongations and of the modified obstacle restriction.

Chapter 6

Numerical Results

In Chapter 5 a method for solving Signorini's problem was constructed. In order to show its efficiency it was implemented at the Johannes Kepler University Linz in the framework of FE++, a C++ version of a multilevel finite element program developed by SCHÖBERL [30]. A method proposed by ZIENKIEWICS AND ZHU [35] was used for estimating the error. Many aspects on error estimation and adaptive mesh refinement can be found in VERFÜRTH [33]. As only the refinement strategy for 2D works well, only 2D examples are presented.

6.1 Example 1: Rectangle Supported by a Step

In the first example the deformation of a rectangle supported by a step of a stair is calculated. The geometrical situation before the deformation is shown in Figure 6.1. The mathematical model for the situation shown in Figure 6.1 looks as follows:

Let $\Omega = [0,1] \times [0,1]$. On the boundary $\Gamma_{x=1}$ the displacements in x-direction are restricted to zero. According to Chapter 2 the deformation can be described by the following equations:

$$\begin{aligned} \operatorname{div}\sigma + f &= 0 && \text{in } \Omega, \\ \sigma\, n &= 0 && \text{on } \Gamma_{x=0} \cup \Gamma_{y=1}, \\ f &= (0,0,0.1)^T && \text{in } \Omega. \end{aligned} \tag{6.1}$$

The stress-strain relationship is described by a linear elastic material law with Young's modulus $Y = 1$ and Poisson's ration $\nu = 0.2$. On the possible contact surface $\Gamma_{y=0}$ Ω is supported by a "stair step". This can be described by

$$\langle u, n \rangle \le g \tag{6.2}$$

with

$$g = \begin{cases} \frac{1}{10} & \text{if } x \in [0, 0.42] \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

The numerical results are given in Table 6.1. For a better comparison the

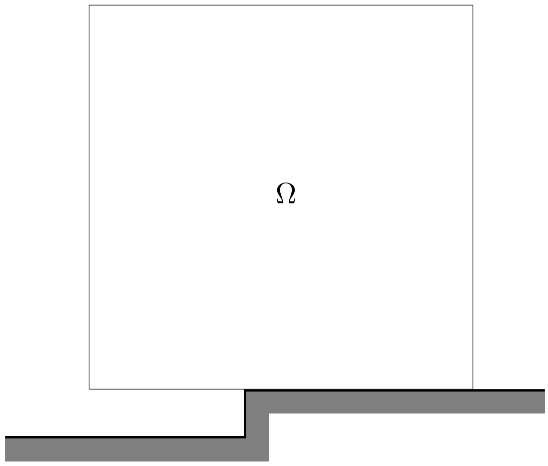

Figure 6.1: Reference configuration

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	4	1	1	1	20
2	9	2	9	12	70
3	25	2	15	16	160
4	47	1	15	15	270
5	78	1	15	25	440
6	108	1	17	26	530
7	156	1	13	27	800
8	219	2	12	25	870
9	367	2	13	24	1530
10	507	1	14	20	1130

Table 6.1: Results for Example 1, Nested Iteration

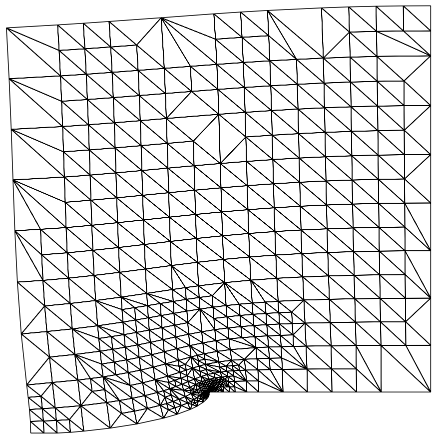

Figure 6.2: Deformed Mesh for Level 10

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	4	1	1	1	20
2	9	1	9	14	80
3	25	2	18	19	200
4	47	2	19	21	360
5	78	2	18	32	560
6	108	2	21	35	830
7	156	2	20	37	1110
8	219	3	19	38	> 2000
9	367	4	18	39	
10	507	4	20	38	

Table 6.2: Results for Example 1, initial guess $u_l = 0$

iteration was stopped, when the iterate fulfilled the Kuhn-Tucker conditions for the problem approximately. Each method uses a nested approach in order to reduce the number of iterations until the correct active set is known. The deformed mesh for level 10 can be found in Figure 6.2. For comparison, results without using nested iteration can be found in Table 6.2.

It can be seen that the method based on truncated base functions has advantages compared to using standard base functions. The difference in the iteration numbers of both methods, especially when not using nested iteration techniques, can be explained as follows:

The change in the boundary condition is not represented by the coarsest grid. That is why the coarse grid correction is bad for the nodes lying in $[0,1] \times \{0\}$. This can be seen best looking at the bounds for the normal displacements induced by the restriction of the obstacle. In Figure 6.3 three levels of the

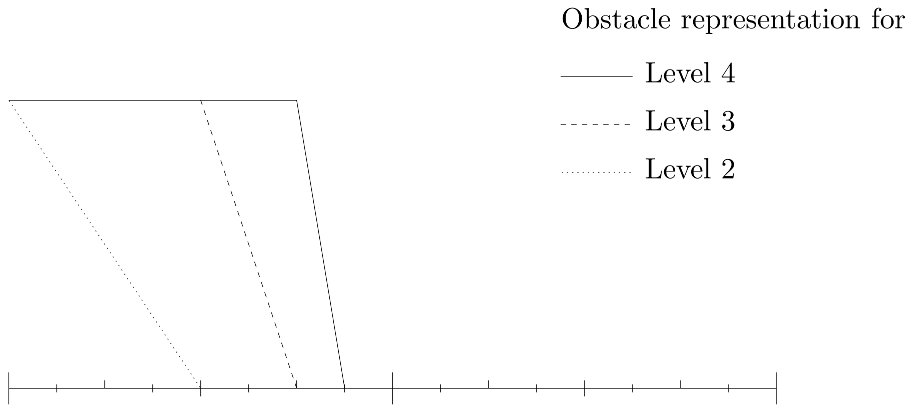

Figure 6.3: Restriction of the obstacle for standard base functions

obstacle restriction are plotted. It can be seen that the obstacle becomes wider from step to step until it supports Ω on the whole boundary $\Gamma_{y=0}$ inducing the weaker convergence rates of the standard multilevel relaxation.

A comparison of the convergence rates of the multilevel Gauß-Seidel relaxation using truncated and standard base functions and standard Gauß-Seidel relaxation can be found in Figure 6.4 and in Figure 6.5.

6.2 Example 2: Smooth Piston

In contrast to the problem stated in Example 1, in this example the contact surface is not known in advance. This is the case almost always found in practical situations. The geometry before the deformation can be seen in Figure 6.6. As possible contact surface the boundary of the semi-circle forming the lower part of the piston is taken. The algorithms of Chapter 5 were applied to the following problem:

$$\begin{aligned} \operatorname{div} \sigma &= 0 \qquad \text{in } \Omega, \\ \sigma\, n &= 0 \qquad \text{on } \Gamma_N, \end{aligned} \tag{6.4}$$

as neither surface tractions nor volume forces were assumed. On Γ_D the displacements in y-direction were prescribed by

$$u_y = -0.4. \tag{6.5}$$

In order to ensure the uniqueness of the solution the x-displacements of the midpoint of Γ_D were restricted to zero.

The stress-strain relationship is described by a linear elastic material law with Young's modulus $Y = 1$ and Poisson's ration $\nu = 0.2$.

Level 7 of the deformed configuration can be found in Figure 6.7. It can be seen, that the refinement mainly concentrates at points, where the contact boundary condition changes. In these points not only the stresses are rather high, as can be seen in Figure 6.8, but also the regularity of the solution is less than in their neighborhood.

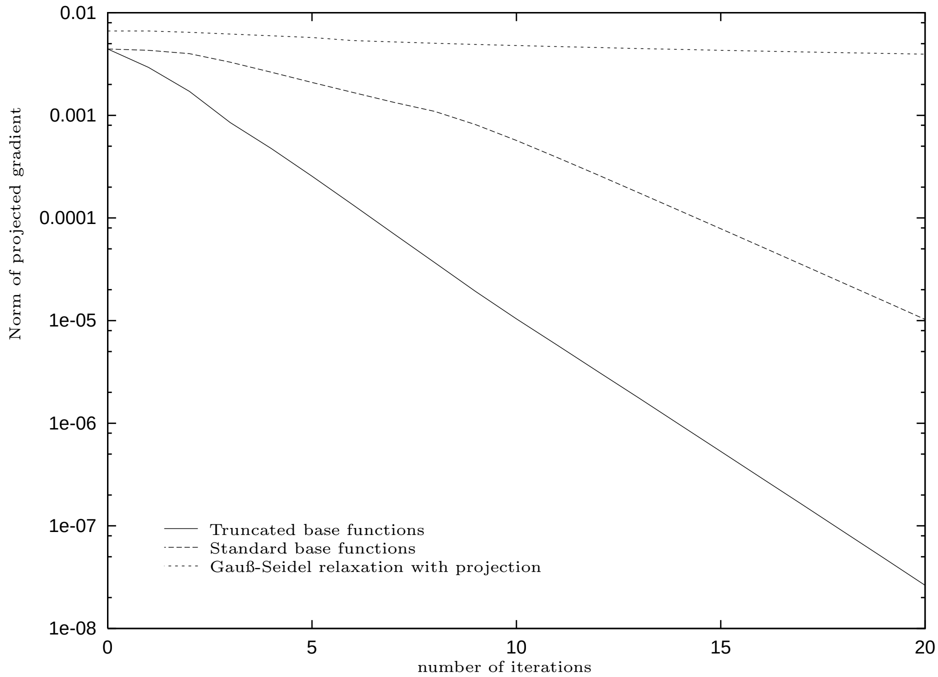

Figure 6.4: Example 1: Comparison of the convergence rates, initial guess $u_l = 0$

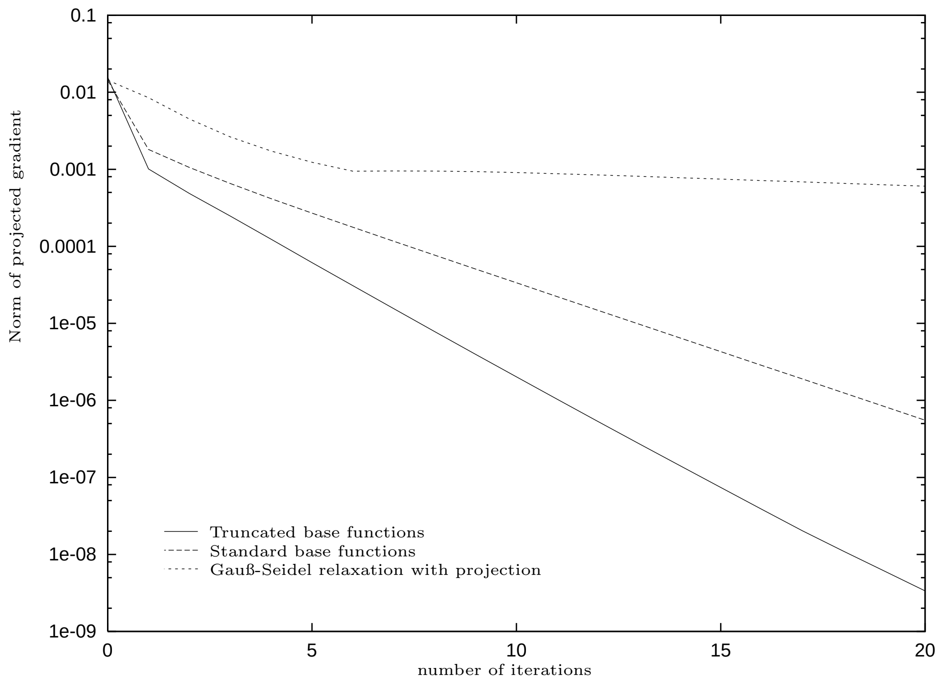

Figure 6.5: Example 1: Comparison of the convergence rates for nested iteration

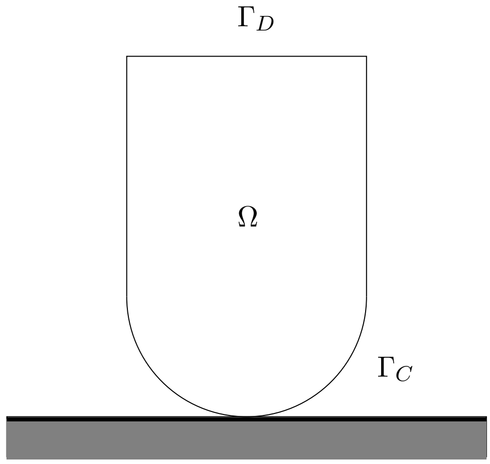

Figure 6.6: Example 2: Reference configuration

The numerical results are given in Table 6.3 using a nested iteration principle and in Table 6.4, starting the relaxation procedures stated in Chapter 5 with the initial guess $u_l = 0$. It can be easily seen, that the use of truncated base functions has severe advantages compared to standard base function. Taking Figure 6.9 and Figure 6.10 into account it can be seen, that the convergence rates of the multilevel Gauß-Seidel relaxation using truncated base functions are much better compared to using standard base functions. The Gauß-Seidel relaxation is only given for completeness, as it has completely unsatisfactory convergence properties.

The large initial residual seen in Figure 6.10 can be explained as follows: The initial solution does not fulfill the Dirichlet boundary conditions on Γ_D. During the calculation these boundary conditions are modelled by a penalty approach, resulting in a very large residual, when these conditions are violated. It can be seen, that the iterate of both multilevel methods fulfills the Dirichlet boundary conditions within one iteration step, whereas the simple Gauß-Seidel relaxation needs much longer. Then both multilevel procedures try to find the right set of contact nodes, which is managed within a few iteration steps. In this phase they behave similar to each other. The last part of the iteration is dominated by the asymptotic convergence behaviour, showing severe advantages of the multilevel relaxation using truncated base functions.

6.3 Example 3: Rigid Punch

In our third example the indentation of a rigid rectangular punch into an elastic underground is calculated. A sketch of the geometry of the punch and the underground can be found in Figure 6.11. Similar to Example 2, no volume forces and no surface tractions are assumed. Then the problem can be modelled

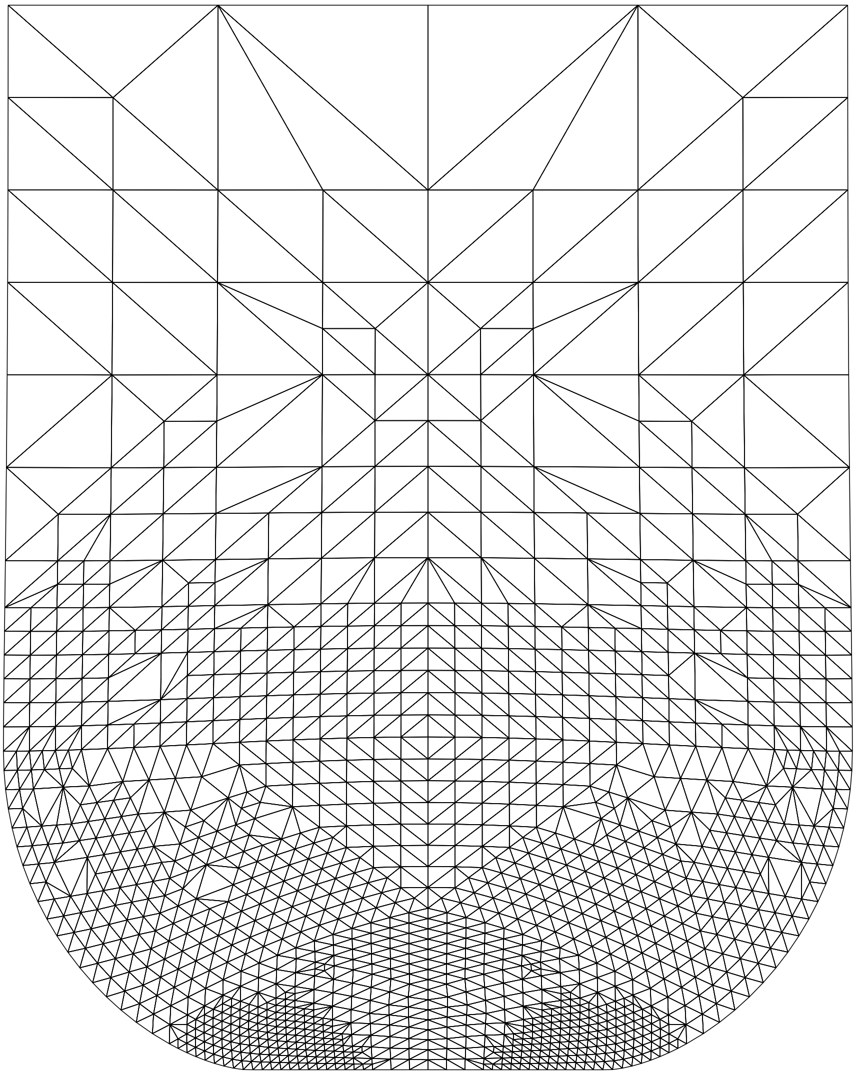

Figure 6.7: Example 2: Deformed mesh of level 7

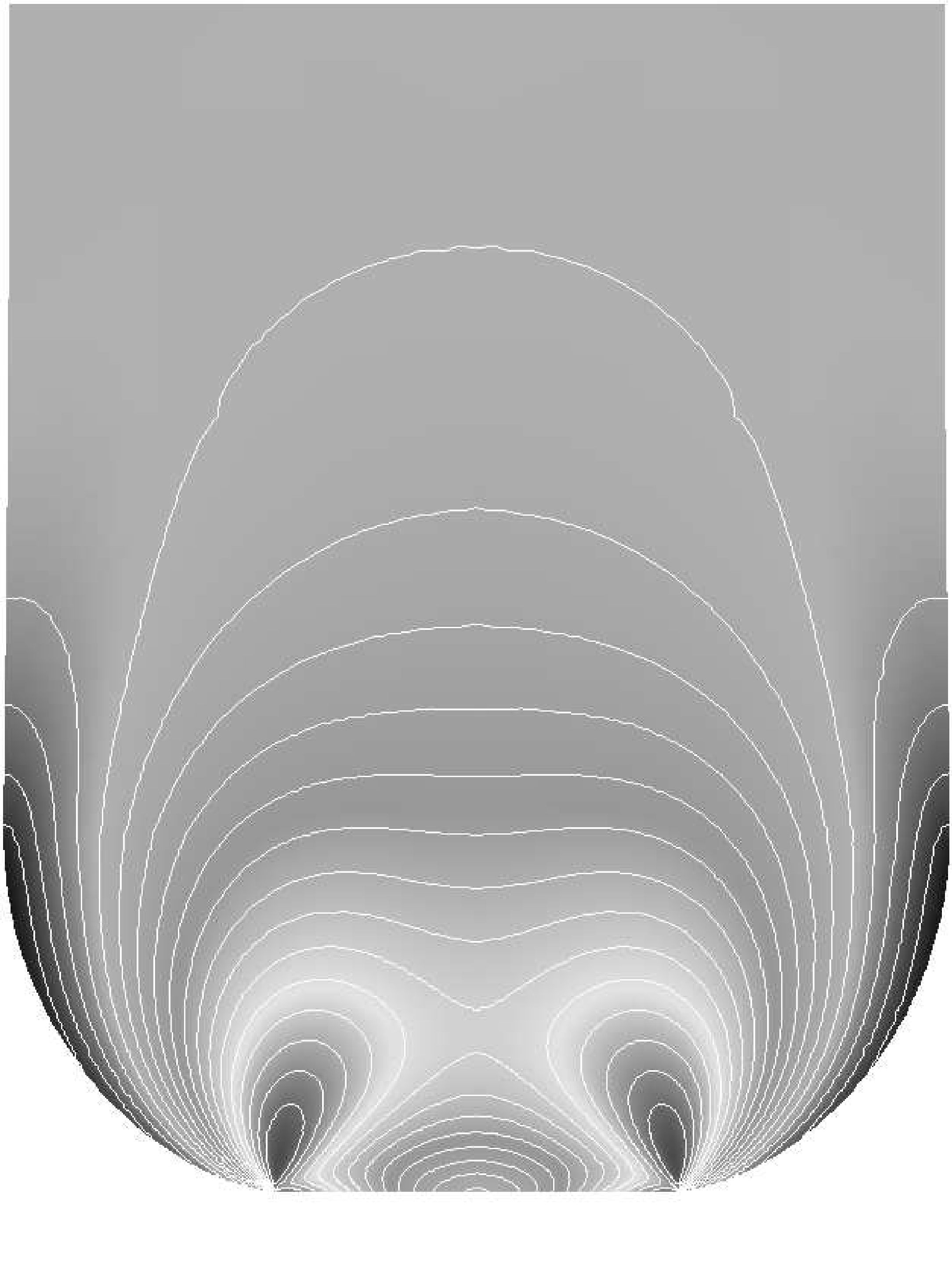

Figure 6.8: Example 2: Van Mieses stress

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	10	1	2	1	50
2	29	1	7	7	150
3	45	1	7	22	240
4	85	1	9	17	300
5	243	2	9	21	360
6	521	1	7	9	430
7	1342	1	7	11	620
8	3533	1	7	15	760
9	8500	3	8	14	660
10	17955	4	6	10	190

Table 6.3: Results for Example 2, Nested Iteration

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	10	1	2	1	50
2	29	1	13	13	280
3	45	2	11	31	400
4	85	3	12	29	630
5	243	6	15	35	1250
6	521	8	17	39	> 2000
7	1342	8	18	42	
8	3533	12	22	42	
9	8500	15	23	41	
10	17955	19	25		

Table 6.4: Results for Example 2, initial guess $u_l = 0$

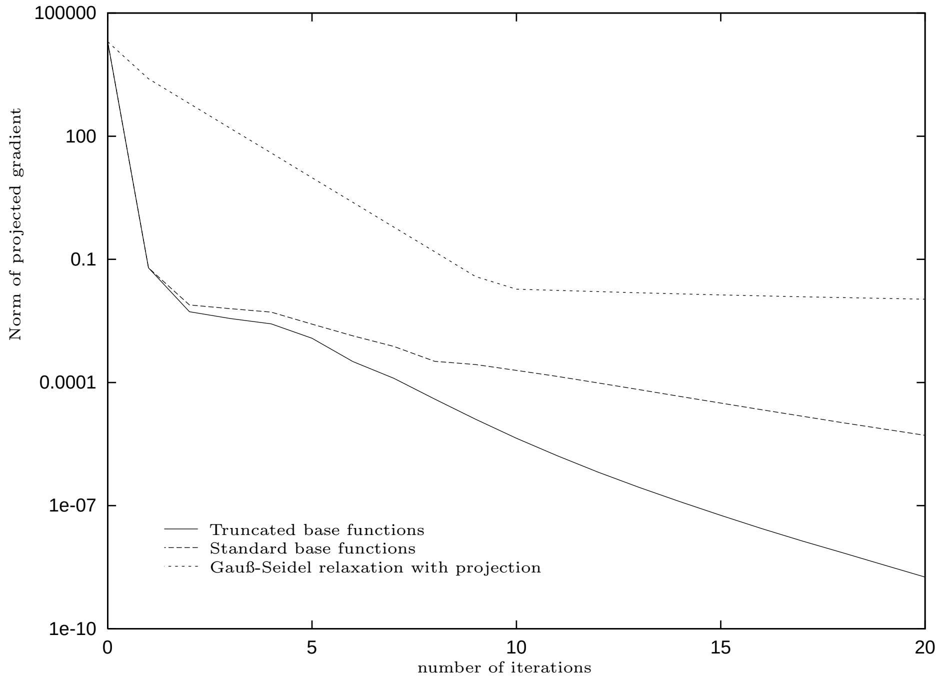

Figure 6.9: Example 2: Comparison of the convergence rates, initial guess $u_l = 0$

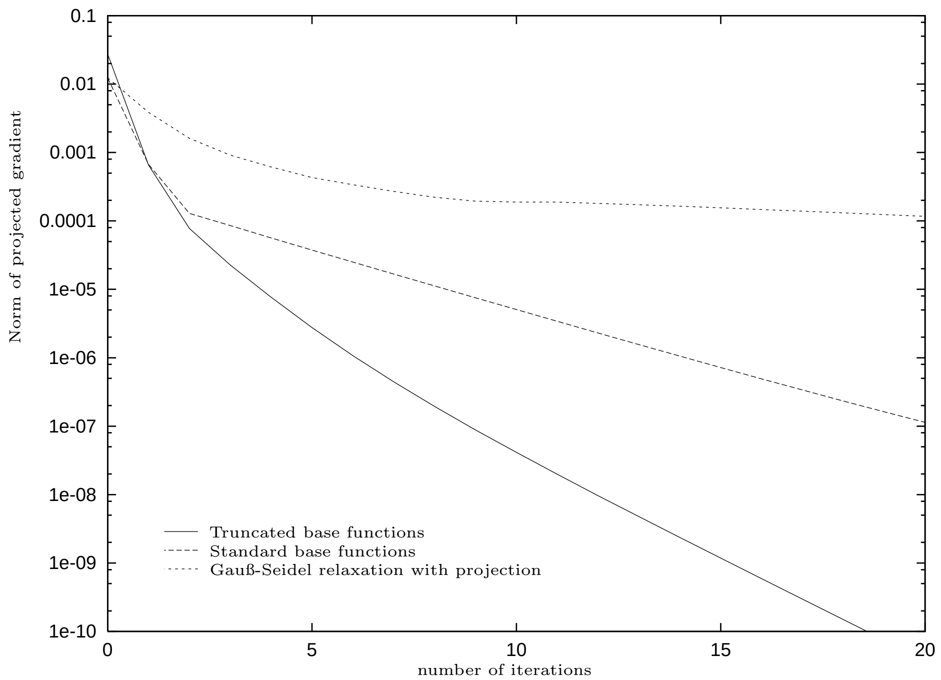

Figure 6.10: Example 2: Comparison of the convergence rates for nested iteration

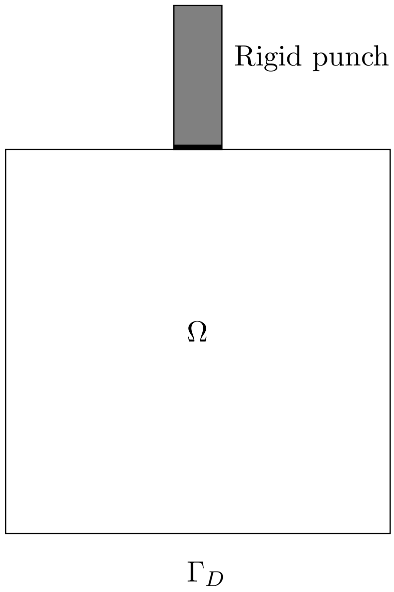

Figure 6.11: Example 3: Reference configuration

by the following equations:

$$\begin{aligned} \operatorname{div} \sigma &= 0 \qquad \text{in } \Omega, \\ \sigma\, n &= 0 \qquad \text{on } \Gamma_N. \end{aligned} \tag{6.6}$$

On Γ_D the displacements in y-direction were prescribed by

$$u_y = 0.4. \tag{6.7}$$

In order to ensure the uniqueness of the solution the x-displacements of the midpoint of Γ_D were restricted to zero.

The stress-strain relationship is described by a linear elastic material law with Young's modulus $Y = 1$ and Poisson's ration $\nu = 0.2$.

In Figure 6.12 the deformed mesh of level 10 is shown. It can be seen that the refinement is concentrated around the corners of the punch induced by the singularity of the solution in these points. Nevertheless the iteration numbers presented in Table 6.5 and Table 6.6 and the convergence rates presented in Figure 6.13 and Figure 6.14 show a similar convergence behaviour of the three iteration methods compared to the examples presented above.

Again it must be noted, that the Dirichlet boundary conditions are modelled by a penalty approach explaining the large residual at the beginning of each iteration procedure shown in Figure 6.14.

6.4 Summary

Summing up the results of the previous examples the multilevel relaxation method using truncated base functions developed in Chapter 5 showed its effi-

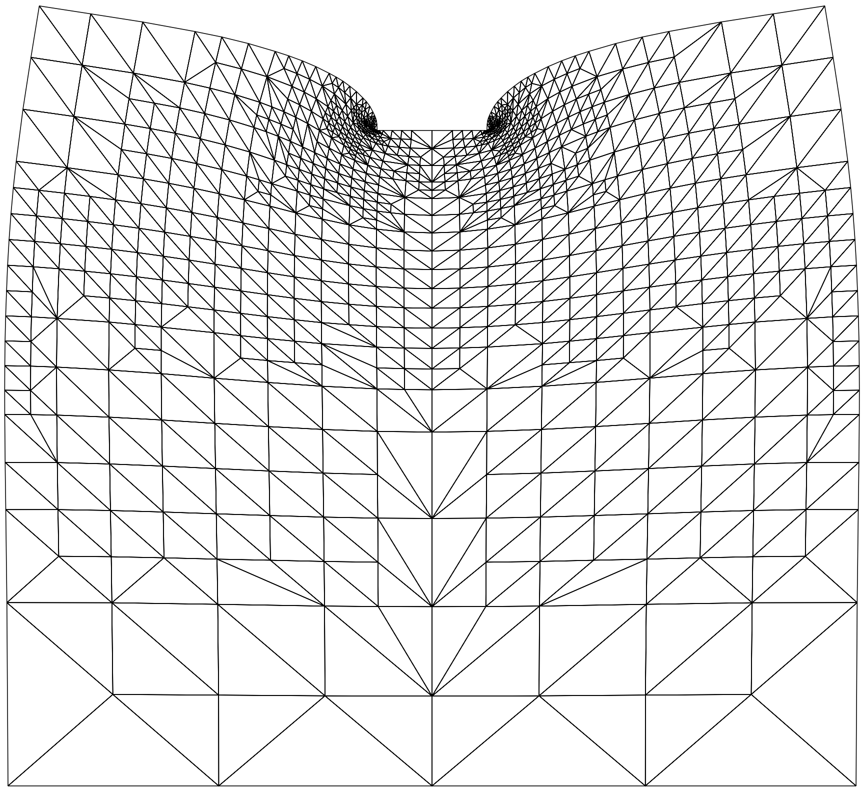

Figure 6.12: Example 3: Deformed mesh of level 10

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	9	1	1	1	20
2	25	1	10	10	480
3	37	1	12	12	910
4	49	1	11	22	1120
5	131	1	12	22	820
6	213	1	12	21	580
7	311	1	13	20	740
8	405	1	12	29	900
9	628	1	12	26	1150
10	926	1	12	22	1610

Table 6.5: Results for Example 3, Nested Iteration

		Truncated Basefunctions		Standard Basefunctions	Gauß-Seidel Iteration
Level	Nodes	Matrix-generations	Iterations	Iterations	Iterations
1	9	1	1	1	20
2	25	1	11	11	550
3	37	2	14	14	1060
4	49	2	13	25	1230
5	131	2	14	26	> 2000
6	213	2	17	28	
7	311	2	18	29	
8	405	2	17	37	
9	628	2	17	37	
10	926	2	18	35	

Table 6.6: Results for Example 3, initial guess $u_l = 0$

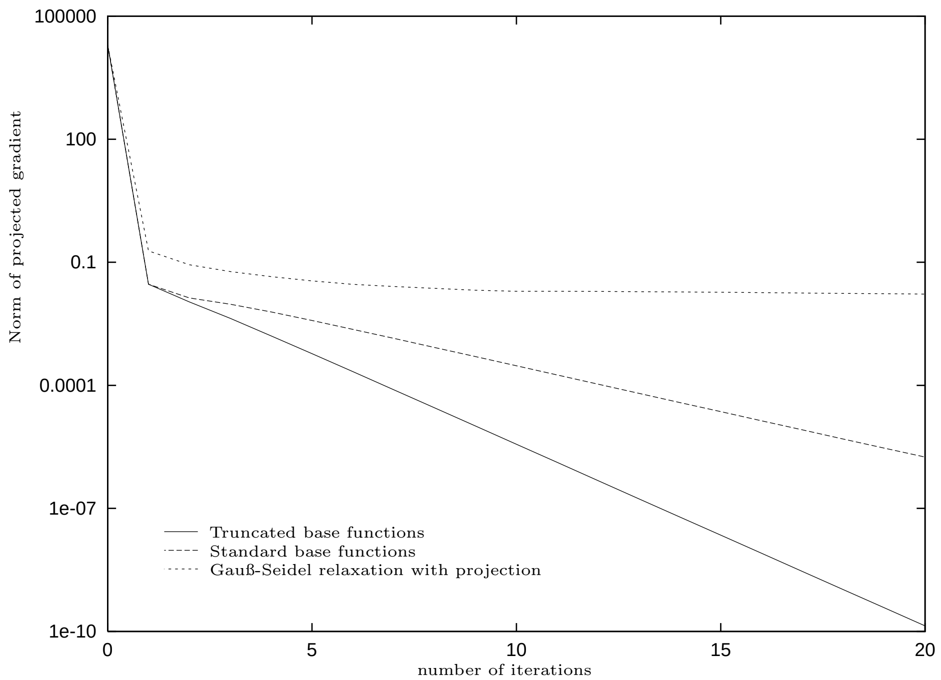

Figure 6.13: Example 3: Comparison of the convergence rates, initial guess $u_l = 0$

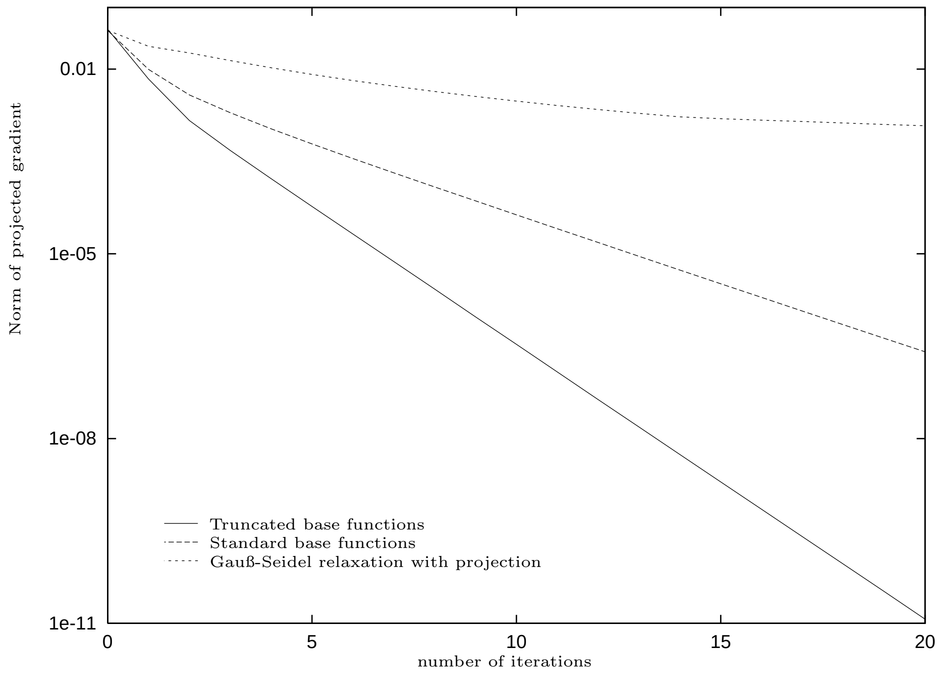

Figure 6.14: Example 3: Comparison of the convergence rates for nested iteration

ciency in various situations. In each of the examples the convergence behaviour of this relaxation method can be split in two parts:

- During the first part the method tries to determine the correct active set. During this phase the difference between the method using truncated base functions and the method using standard ones is small.

- During the second part the correct active set or is at least a very good approximation is known. During this phase the convergence behaviour is dominated by the asymptotic convergence bahaviour of the method. Here the usage of truncated base functions results in severe improvement of the convergence rates compared to the usage of standard base functions.

If nested iteration principles are used, the number of iterations needed to estimate the correct active set is strongly reduced. In this case the asymptotic convergence behaviour dominates the convergence behaviour of the whole method.

Although no proof of convergence rates was found up to now, it can be expected, that the number of iterations needed to solve a given problem grows logarithmically with the number of levels used. Using nested iteration principles the solution of the problem can be obtained within an even slower growing number of iterations or even hardly growing number of iterations. Both assertions are supported by the results presented above.

Chapter 7

Conclusion

In the previous chapters various results on the analysis and simulation of Signorini's problem were presented.

After an introductory section on the foundation of the mechanics of continua including the derivation of the state equations of linear elasticity, general nonlinear contact conditions for an elastic body coming into contact with a rigid foundation were given and simplified, resulting in a system of linear inequalities representing the nonpenetration conditions. Furthermore conditions on the stresses on the contact surface, assuming frictionless contact, were presented.

The resulting system of differential equations and inequalities was transformed into a variational form, resulting in a variational inequality due to the nonpenetration conditions. Using some abstract results, conditions for the existence of a unique solution were given.

In order to solve Signorini's problem numerically, a finite element formulation of the variational problem was derived and analysed. Results on the unique solvability of the discrete problem and on the convergence of the discrete solution were presented.

Special attention was paid to the efficient solution of the discrete problem. Relaxation methods of Gauß-Seidel-type were used as a starting point. In order to improve their unsatisfactory convergence properties, the space splitting was enlarged by using the multilevel nodal basis. As the straight forward transfer of multilevel relaxation methods of Gauß-Seidel-type onto Signorini's problem could no more be implemented with optimal order, a suitable representation of the rigid obstacle was introduced.

In order to improve the convergence properties of the resulting method even further, an approach using truncated base functions was presented.

Although there is no proof of convergence rates up to now, various examples show the efficiency of the presented method, resulting in a slowly increasing number of iterations the finer the used meshes are. If nested iteration principles were used, the growth of the number of iterations could again be reduced.

In the future the algorithm will be extended into two different directions: On the one hand to three-dimensional problems and on the other hand to time dependent problems. Both generalizations can be done in a straight forward manner.

Despite the good convergence properties, the presented method also has some drawbacks: On the one hand not every multilevel code supports the efficient projection of matrices onto coarser grids needed for the implementation of the coarse grid correction using truncated base functions. Furthermore the efficiency depends strongly on the number of matrix generations needed. But the more important drawback is the fact, that no straight forward generalization of the method to the very important class of body-body contact problems is known up to now.

Bibliography

[1] A. Binder, *Freie Randwertprobleme*, Institut für Mathematik, JKU Linz, 1994, Lecture notes.

[2] A. Brandt and C. W. Cryer, *Multigrid algorithms for the solution of linear complementary problems arising from free boundary problems*, SIAM J. Sci. Statist. Comput. **4** (1983), 655 – 684.

[3] F. Brezzi, W. W. Hager, and P. A. Raviart, *Error estimates for the finite element solution of variational inequalities, part 1: Primal theory*, Numer. Math. **28** (1977), 431 – 443.

[4] P. G. Ciarlet, *The finite element method for elliptic problems*, second ed., A Series of Comprehensive Studies in Mathematics, North-Holland Publishing Company, Amsterdam-New York-Oxford-Tokyo, 1987.

[5] ______, *Mathematical elasticity: Three-dimensional elasticity*, Studies in Mathematics and Its Applications, vol. 20, North-Holland Publishing Company, Amsterdam-New York-Oxford-Tokyo, 1994.

[6] G. Duvaut and J. L. Lions, *Inequalities in mechanics and physics*, A Series of Comprehensive Studies in Mathematics, vol. 219, Springer-Verlag, New York-Berlin-Heidelberg, 1976.

[7] C. Eck, *Existenz und Regularität der Lösungen für Kontaktprobleme mit Reibung*, Ph.D. thesis, Universität Stuttgart, Stuttgart, Germany, 1996.

[8] H. W. Engl, *Kontinuumsmechanik*, Institut für Mathematik, JKU Linz, 1993, Lecture notes.

[9] P. E. Gill, W. Murray, and M. H. Wright, *Practical optimization*, Academic Press, London-San Diego-New York, 1981.

[10] R. Glowinski, *Numerical methods for nonlinear variational problems*, Springer-Verlag, New York-Berlin-Heidelberg, 1984.

[11] R. Glowinski, J. L. Lions, and R. Trémolières, *Numerical analysis of variational inequalities*, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1981.

[12] H. Goering, H. G. Roos, and L. Tobiska, *Finite-Element-Methode, Eine Einführung*, Academic Press, Berlin, 1985.

[13] W. Hackbusch, *Multi-grid methods and applications*, Springer-Verlag, Berlin, 1985.

[14] W. Hackbusch and H. D. Mittelmann, *On multi-grid methods for variational inequalities*, Numer. Math. **42** (1983), 65–76.

[15] I. Hlavacek, J. Haslinger, J. Necas, and J. Lovicek, *Solution of variational inequalities in mechanics*, Applied Mathematical Sciences, vol. 66, Springer-Verlag, New York-Berlin-Heidelberg, 1988.

[16] R. Hoppe, *Multigrid algorithms for variational inequalities*, SIAM J. Numer. Anal. **24** (1987), no. 5, 1046–1065.

[17] R. H. W. Hoppe and R. Kornhuber, *Adaptive multilevel methods for obstacle problems*, SIAM J. Numer. Anal. **31** (1994), no. 2, 301–323.

[18] N. Kikuchi, *Finite element methods in mechanics*, Cambridge University Press, Cambridge, 1986.

[19] N. Kikuchi and J. T. Oden, *Contact problems in elasicity: A study of variational inequalities and finite element methods*, SIAM, Philadelphia, 1988.

[20] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[21] V. G. Korneev and U. Langer, *Approximate solution of plastic flow theory problems*, B. G. Teubner, Leipzig, 1984.

[22] R. Kornhuber, *Monotone multigrid methods for elliptic variational inequalities i*, Numer. Math. **69** (1994), 167 – 184.

[23] ———, *Monotone multigrid methods for elliptic variational inequalities ii*, Numer. Math. **72** (1996), 481 – 499.

[24] ———, *Adaptive monotone multigrid methods for nonlinear variational problems*, Advances in Numerical Mathematics, Teubner, Stuttgart, 1997.

[25] M. Krizek and P. Neittaanmäki, *Finite element approximation of variational problems and applications*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 50, Longman Scientific & Technical, Essex, 1990.

[26] U. Langer, *Numerik II*, Institut für Mathematik, JKU Linz, 1994, Lecture notes.

[27] ———, *Multigrid Methoden*, Institut für Mathematik, JKU Linz, 1995, Lecture notes.

[28] J. Mandel, *A multilevel iterative method for symmetric, positive definite linear complementarity problems*, Appl. Math. Optim. **11** (1984), 77–95.

[29] J. Necas and I. Hlavacek, *Mathematical theory of elastic and plastic bodies: An introduction*, Studies in Applied Mechanics, Elsevier, Amsterdam-Oxford-New York, 1983.

[30] J. Schöberl, *Analysis and realization of mixed finite element models in computational mechanics*, Master's thesis, Johannes Kepler Universität Linz, Linz, Austria, 1996.

[31] ______, *Solving the signorini problem on the basis of domain decomposition techniques*, Tech. Report 505, Department of Mathematics, Johannes Kepler Universität Linz, Austria, October 1996.

[32] A. Signorini, *Sopra alcune questioni di elastostatica*, Atti Soz. Ital. Progr. Sci. (1933).

[33] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques*, Wiley-Teubner, 1996.

[34] O. Zienkiewics, *The finite element method in engineering science*, third ed., McGraw-Hill, London, 1977.

[35] O. C. Zienkiewics and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis*, Internat. J. Numer. Methods Engrg. **24** (1987), 337–357.