# Exercise sheet 3

1. (Existence for quasilinear PDE for $p \geq d$) In the proof to the theorem of Brezis (see Theorem 5.5) we assumed for simplicity $p < d$. Here, follow the proof and show the existence in the case of $p \geq d$. We have to use case distinctions and different embedding results.

2. (Uniqueness using Brezis' theorem) Under which conditions on the functional $g$ does the quasilinear elliptic PDE $-\text{div}(|\nabla u|^{p-2}\nabla u) + g(u) = f$ (with Dirichlet boundary) have a unique solution?

3. (De Rham)

   (a) Prove that
   $$Y := \{F \in H^{-1}(\Omega; \mathbb{R}^3) : \exists p \in L^2(\Omega) \text{ such that } F = \nabla p \text{ in } H^{-1}(\Omega; \mathbb{R}^3)\}$$
   is closed in $H^{-1}(\Omega; \mathbb{R}^3)$. One could also write $Y = \text{im}(\nabla)$.

   (b) Let $F \in H^{-1}(\Omega; \mathbb{R}^3)$ and $X = \{\phi \in H_0^1(\Omega; \mathbb{R}^3) : \text{div}\phi = 0\}$. Further it holds
   $$\langle F, \varphi \rangle_{H_0^1} = 0 \quad \forall \varphi \in X.$$
   Prove that there exists a function $p \in L^2(\Omega)$ with $\int_\Omega p \, \mathrm{d}x$ such that $F = \nabla p$ in the sense
   $$\langle F, \varphi \rangle_{H_0^1} = \int_\Omega p \text{div}\varphi \, \mathrm{d}x \quad \forall \varphi \in H_0^1(\Omega; \mathbb{R}^3).$$

   (c) Prove that the function $p$ is unique up to a constant.

   (d) Apply the de Rham theorem to the stationary Navier-Stokes equation to associate a pressure to the velocity.

   (e) Prove that it is not possible to associate a pressure to the velocity if the source function is in $X'$.