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Abstract Robust Coarse Spaces for Systems of PDEs via Generalized Eigenproblems in the Overlaps

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ABSTRACT ROBUST COARSE SPACES FOR SYSTEMS OF PDES VIA GENERALIZED EIGENPROBLEMS IN THE OVERLAPS

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Abstract. Coarse spaces are instrumental in obtaining scalability for domain decomposition methods for partial differential equations (PDEs). However, it is known that most popular choices of coarse spaces perform rather weakly in the presence of heterogeneities in the PDE coefficients, especially for systems of PDEs. Here, we introduce in a variational setting a new coarse space that is robust even when there are such heterogeneities. We achieve this by solving local generalized eigenvalue problems in the overlaps of subdomains that isolate the terms responsible for slow convergence. We prove a general theoretical result that rigorously establishes the robustness of the new coarse space and give some numerical examples on two and three dimensional heterogeneous PDEs and systems of PDEs that confirm this property.

Keywords: coarse spaces, overlapping Schwarz method, two-level methods, generalized eigenvectors, problems with large coefficient variation

Mathematics Subject Classification (2000): 65F10, 65N22, 65N30, 65N55

1. Introduction

The effort to reach scalability in domain decomposition methods has led to the design of so called two-level methods. Each of these methods is characterized by two ingredients: a coarse space and a formulation of how this coarse space is incorporated into the domain decomposition method. We will work in the already extensively studied framework of the overlapping additive Schwarz preconditioner [30, 28], and focus on the definition of a suitable coarse space with the aim to achieve robustness with regard to heterogeneities in any of the coefficients in the PDEs. This type of problem arises in many applications, such as subsurface flows or linear elasticity. One way to avoid long stagnation in Schwarz methods is to build the subdomains in such a way that the variations in the coefficients are small or nonexistent inside each subdomain. In this configuration, classical coarse spaces that are based on these subdomain partitions are known to be robust, see e.g. [10, 9, 20, 5, 6]. Recently, some authors also extended these results for scalar elliptic problems to certain classes of coefficients that are not resolved by the subdomain partition and to operator dependent coarse spaces, see e.g. [16, 25, 22, 14, 26] as well as [31] and the references therein.

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However, ideally we would prefer methods that are robust for any partition into subdomains and regardless of the coefficient distribution. A class of such stable coarse spaces has been presented among the literature on Schwarz methods for scalar elliptic PDEs in [15, 13, 27, 12], as well as in earlier work on algebraic multigrid (AMG) methods in [2, 3]. The key ingredient in these coarse spaces is the solution of a generalized eigenvalue problem on each subdomain (or on each coarse element in spectral AMGe). While these spaces are indeed robust for any arbitrary coefficient distribution, they do not discriminate between coefficient variations that influence only the solution in the interior of the subdomains and those that are actually responsible for the lack of robustness of standard coarse spaces. A consequence of this is that the resulting coarse space is often unnecessarily large.

A related but slightly different coarse space for scalar elliptic PDEs, based on generalized eigenvalue problems for the Dirichlet-to-Neumann operator on the boundary of each subdomain, has recently been introduced and analyzed in [8, 7]. However, although the coarse space size is largely reduced in this approach, this gain comes at the expense of being not uniformly robust for arbitrary coefficient variations. The lack of robustness for some coefficient distributions is evident in the theoretical convergence analysis in [7], where a weighted Poincaré inequality is needed to prove a stable weak approximation property for the coarse space in the part of each subdomain that is overlapped by other subdomains. This type of inequality is very powerful, but it does not apply in the very general setting of arbitrary coefficient variations that we wish to maintain (cf. [23]).

The terms in the overlapped regions arise naturally in the convergence calculations so they provide valuable information. In this article, in order to be more general and to avoid the need for weighted Poincaré type inequalities, while still discriminating between coefficient variations that are crucial for robustness and those which are not, we define the generalized eigenproblems directly in the overlapped regions. Apart from the obvious attraction of providing a fully robust coarse space that is in general significantly smaller than that proposed in [15], another significant advantage of the proposed approach is that its fully abstract setting allows us to consider a very wide range of symmetric positive definite systems of equations. The method was first and briefly introduced in [29] by the same authors and we give here a detailed proof of the previously stated convergence result along with some extensive numerical results.

One disadvantage of the earlier methods proposed in [14, 8] is that they need weighted mass matrices either in each subdomain or on its boundary. The mass matrices appear on the right hand side of the generalized eigenproblems in those approaches. In practice, this requires coefficient and mesh information. As in the more recent article [12], we do not need any mass matrices in our approach here. The right hand side of the generalized eigenproblems is constructed from element stiffness matrices and a family of partition of unity functions/operators. Thus, we only have to assume access to some topological information to build a suitable partition of unity and to the element stiffness matrices (as in AMGe methods, cf. [3]), which is reasonable in standard FE packages such as FreeFEM++ [17]. The preconditioners can then be implemented fully algebraically, providing a viable option for simulations on physical problems without the need to rewrite entire codes.

Our approach and our analysis are clearly related to those in [12], but fundamentally different. The analysis in [12] is not given in the fully discrete setting. The analysis assumes that the product of an eigenfunction with a partition of unity function is again in the (abstract) finite element space. This assumption is of course violated for standard spaces, and to get back into the space, an additional interpolation operator is needed. In the practical implementation the authors of [12] use such an interpolation operator, but a theoretical proof of the stability of this interpolation operator is not given. In fact, such a stable interpolation operator does not exist yet to our knowledge. We circumvent the use of interpolation operators by the use of partition of unity *operators* that work directly

on the degrees of freedom. This way, we maintain a fully abstract setting and our theory applies to the fully discrete method.

In Section 2 we define the problem that we solve and introduce the two-level additive Schwarz framework along with some elements of generalized eigenvalue problem theory. In Section 3 we define the abstract procedure to construct robust coarse spaces based on Generalized Eigenproblems in the Overlap (which we will refer to as the *GenEO coarse space*) and give the main convergence result (Theorem 3.21). Section 4 gives detailed guidelines on how to implement the two-level Schwarz preconditioner with the GenEO coarse space in a finite element code. Finally in Section 5 we test our method for Darcy and linear elasticity and make sure that it converges indeed robustly even for highly varying coefficients in two and three dimensions.

2. Preliminaries and notations

2.1. **Problem Description.** Given a Hilbert space V, a symmetric and coercive bilinear form $a : V \times V \to \mathbb{R}$ and an element f in the dual space V', we consider the abstract variational problem: Find $v \in V$ such that

$$a(v,\, w) = \langle f, w\rangle, \quad \text{for all} \quad w \in V, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing. This variational problem is associated with an elliptic boundary value problem (BVP) on a given polygonal (polyhedral) domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with suitable boundary conditions posed in a suitable space V of functions on Ω.

We consider a discretization of the variational problem (1) with finite elements based on a mesh $\mathcal{T}_h$ of Ω:

$$\overline{\Omega} = \textstyle\bigcup_{\tau\in\mathcal{T}_h} \tau.$$

Let $V_h \subset V$ denote the chosen conforming space of finite element functions. In the case where $a(\cdot,\cdot)$ is a bilinear form derived from a system of PDEs, V_h is a space of vector functions. The discretization of (1) then reads: Find $v_h \in V_h$ such that

$$a(v_h, w_h) = \langle f, w_h\rangle, \quad \text{for all} \quad w_h \in V_h. \tag{2}$$

Let $\{\phi_k\}_{k=1}^n$ be a basis for V_h with $n := \dim(V_h)$, then from (2) we can derive a linear system

$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{f}, \tag{3}$$

where the coefficients of the stiffness matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and the load vector $\boldsymbol{f} \in \mathbb{R}^n$ are given by $A_{k,l} = a(\phi_l, \phi_k)$ and $f_k = \langle f, \phi_k\rangle$, where k, $l = 1, \ldots, n$, and $\boldsymbol{v}$ is the vector of coefficients corresponding to the unknown finite element function v_h in (2).

The basis $\{\phi_k\}_{k=1}^n$ can be quite arbitrary but it should fulfil a unisolvence property, such that the basis functions supported on each element $\tau \in \mathcal{T}_h$ are linearly independent when restricted to τ. This is the case for standard finite element bases.

The only significant assumption we make on the problem is that the stiffness matrix $\boldsymbol{A}$ is assembled from positive semi-definite element stiffness matrices.

Assumption 2.1. *Let $V_h(\tau) = \{v|_\tau : v \in V_h\}$. We assume that there exist positive semi-definite bilinear forms $a_\tau : V_h(\tau) \times V_h(\tau) \to \mathbb{R}$, for all $\tau \in \mathcal{T}_h$, such that*

$$a(v, w) = \sum_{\tau\in\mathcal{T}_h} a_\tau(v|_\tau, w|_\tau), \quad \textit{for all} \quad v, w \in V_h.$$

Remark 2.2. *If the variational problem is obtained from integrating local forms on the domain then this is not a problem at all. For instance in the case of the Darcy equation*

we can write for all $v, w \in H_0^1(\Omega)$*:*

$$a(v,w) = \int_\Omega \kappa \nabla v \cdot \nabla w = \sum_{\tau \in \mathcal{T}_h} \int_\tau \kappa \nabla v \cdot \nabla w = \sum_{\tau \in \mathcal{T}_h} a_\tau(v|_\tau, w|_\tau).$$

2.2. **Additive Schwarz setting.** In order to automatically construct a robust two-level Schwarz preconditioner for (3), we first partition our domain Ω into a set of non-overlapping subdomains $\{\Omega_j'\}_{j=1}^N$ resolved by $\mathcal{T}_h$ using for example a graph partitioner such as METIS [18] or SCOTCH [4]. Each subdomain Ω_j' is then extended to a domain Ω_j by adding one or several layers of mesh elements in the sense of Definition 2.3, thus creating an overlapping decomposition $\{\Omega_j\}_{j=1}^N$ of Ω.

Definition 2.3. *Given a subdomain* $D' \subset \Omega$ *which is resolved by* $\mathcal{T}_h$*, the extension of* D' *by one layer of elements is*

$$D = \mathrm{Int}\Big(\bigcup_{k \in \overline{\mathrm{dof}}(D')} \mathrm{supp}(\phi_k) \Big), \quad \textit{where} \quad \overline{\mathrm{dof}}(D') := \{k : \mathrm{supp}(\phi_k) \cap D' \neq \emptyset\},$$

and $\mathrm{Int}(\cdot)$ *denotes the interior of a domain. Extensions by more than one layer can then be defined recursively.*

The proof of the following lemma is a direct consequence of Definition 2.3.

Lemma 2.4. *For every degree of freedom* k*, with* $1 \le k \le n$*, there is a subdomain* Ω_j*, with* $1 \le j \le N$*, such that* $\mathrm{supp}(\phi_k) \subset \overline{\Omega}_j$ *.*

Now, for each $j = 1, \ldots, N$, let

$$V_h(\Omega_j) := \{v|_{\Omega_j} : v \in V_h\}$$

denote the space of restrictions of functions in V_h to Ω_j. Furthermore, let

$$V_{h,0}(\Omega_j) := \{v|_{\Omega_j} : v \in V_h,\ \mathrm{supp}\,(v) \subset \Omega_j\}$$

denote the space of finite element functions supported in Ω_j. By definition, the extension by zero of a function $v \in V_{h,0}(\Omega_j)$ to Ω lies again in V_h. We denote the corresponding extension operator by

$$R_j^\top : V_{h,0}(\Omega_j) \to V_h\,. \tag{4}$$

Lemma 2.4 guarantees that $V_h = \sum_{j=1}^N R_j^\top V_{h,0}(\Omega_j)$. The adjoint of $R_j^\top$

$$R_j : V_h' \to V_{h,0}(\Omega_j)'\,,$$

called the restriction operator, is defined by $\langle R_j g, v\rangle = \langle g, R_j^\top v\rangle$, for $v \in V_{h,0}(\Omega_j)$, $g \in V_h'$. However, for the sake of simplicity, we will often leave out the action of $R_j^\top$ and view $V_{h,0}(\Omega_j)$ as a subspace of V_h.

The final ingredient is a coarse space $V_H \subset V_h$ which will be defined later. Let $R_H^\top : V_H \to V_h$ denote the natural embedding and R_H its adjoint. Then the two-level additive Schwarz preconditioner (in matrix form) reads

$$\boldsymbol{M}_{AS,2}^{-1} = \boldsymbol{R}_H^T \boldsymbol{A}_H^{-1} \boldsymbol{R}_H + \sum_{j=1}^N \boldsymbol{R}_j^T \boldsymbol{A}_j^{-1} \boldsymbol{R}_j, \quad \boldsymbol{A}_H := \boldsymbol{R}_H \boldsymbol{A} \boldsymbol{R}_H^T \quad \text{and} \quad \boldsymbol{A}_j := \boldsymbol{R}_j \boldsymbol{A} \boldsymbol{R}_j^T, \tag{5}$$

where $\boldsymbol{R}_j$, $\boldsymbol{R}_H$ are the matrix representations of R_j and R_H with respect to the basis $\{\phi_k\}_{k=1}^n$ and the chosen basis of the coarse space V_H. As usual for standard elliptic BVPs, $\boldsymbol{A}_j$ corresponds to the original (global) system matrix restricted to subdomain Ω_j with Dirichlet conditions on the artificial boundary $\partial\Omega_j \setminus \partial\Omega$.

To simplify the notation, if D is the union of elements of $\mathcal{T}_h$ and

$$V_h(D) := \{v|_D : v \in V_h\},$$

we write, for any $v, w \in V_h(D)$,

$$a_D(v,w) := \sum_{\tau \in D} a_\tau(v|_\tau, w|_\tau) \quad \text{and} \quad |v|_{a,D} = \sqrt{a_D(v,v)},$$

where the latter is the energy seminorm. The definition of $a_D(\cdot,\cdot)$ extends naturally to $v, w \in V_h(D')$, for any $D \subset D' \subset \Omega$ which simplifies notations. On each of the local spaces $V_{h,0}(\Omega_j)$ the bilinear form $a_{\Omega_j}(\cdot,\cdot)$ is positive definite since

$$a_{\Omega_j}(v,w) = a(R_j^\top v, R_j^\top w), \quad \text{for all } v, w \in V_{h,0}(\Omega_j),$$

and because $a(\cdot,\cdot)$ is coercive on V. For the same reason, the matrix $\boldsymbol{A}_j$ in (5) is invertible. Hence, $|\cdot|_{a,\Omega_j}$ becomes a norm on $V_{h,0}(\Omega_j)$ and so we write

$$\|v\|_{a,\Omega_j} = \sqrt{a_{\Omega_j}(v,v)}, \quad \text{for all } v \in V_{h,0}(\Omega_j).$$

If $D = \Omega$, we omit the domain from the subscript and write $\|\cdot\|_a$ instead of $\|\cdot\|_{a,\Omega}$.

We use here the abstract framework for additive Schwarz (see [30, Chapter 2]). In the following we summarize the most important ingredients.

Definition 2.5. *We define* $k_0 = \max_{\tau \in \mathcal{T}_h} \Big(\#\{\Omega_j : 1 \le j \le N, \tau \subset \Omega_j\} \Big)$.

This means that each point in Ω belongs to at most k_0 of the subdomains Ω_j.

Lemma 2.6. *With k_0 as in Definition 2.5, the largest eigenvalue of $\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}$ satisfies*

$$\lambda_{max}(\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}) \;\le\; k_0 + 1.$$

Proof. See, e.g., [11, Section 4]. □

Definition 2.7 (Stable decomposition)**.** *Given a coarse space $V_H \subset V_h$, local subspaces $\{V_{h,0}(\Omega_j)\}_{1\le j\le N}$ and a constant C_0, a C_0-stable decomposition of $v \in V_h$ is a family of functions $\{z_j\}_{0\le j\le N}$ that satisfies*

$$v = \sum_{j=0}^{N} z_j\,, \qquad \text{with} \quad z_0 \in V_H\,, \quad z_j \in V_{h,0}(\Omega_j), \text{ for } j \ge 1, \tag{6}$$

and

$$\|z_0\|_a^2 + \sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \le C_0^2\, \|v\|_a^2\,. \tag{7}$$

Theorem 2.8. *If every $v \in V_h$ admits a C_0-stable decomposition (with uniform C_0), then the smallest eigenvalue of $\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}$ satisfies*

$$\lambda_{min}(\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}) \;\ge\; C_0^{-2}.$$

Therefore, the condition number of the two-level Schwarz preconditioner (5) *can be bounded by*

$$\kappa(\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}) \;\le\; C_0^2(k_0+1).$$

Proof. The statement is a direct consequence of [30, Lemma 2.5] and Lemma 2.6. □

In the following, we will construct a C_0-stable decomposition in a specific framework, but prior to that we will provide in an abstract setting, a sufficient and simplified condition of stability.

Lemma 2.9. *Using the notations introduced in Definition 2.7, if there exists a constant C_1 such that*

$$\|z_j\|^2_{a,\Omega_j} \leq C_1 |v|^2_{a,\Omega_j}, \quad \text{for all} \quad j = 1, \ldots, N, \tag{8}$$

then the decomposition (6) *is C_0-stable with $C_0^2 = 2 + C_1 k_0 (2k_0 + 1)$ where k_0 is given in Definition 2.5.*

Proof. From (8) and Definition 2.5 we get successively

$$\sum_{j=1}^{N} \|z_j\|^2_{a,\Omega_j} \leq C_1 \sum_{j=1}^{N} |v|^2_{a,\Omega_j} \leq C_1 k_0 \|v\|^2_a . \tag{9}$$

We also have:

$$\|z_0\|^2_a = \left\| v - \sum_{j=1}^{N} z_j \right\|^2_a \leq 2\|v\|^2_a + 2 \left\| \sum_{j=1}^{N} z_j \right\|^2_a , \tag{10}$$

and from Definition 2.5 and (9) we get

$$\left\| \sum_{j=1}^{N} z_j \right\|^2_a \leq k_0 \sum_{j=1}^{N} \|z_j\|^2_{a,\Omega_j} \leq C_1 k_0^2 \|v\|^2_a . \tag{11}$$

Using (11) in (10) yields

$$\|z_0\|^2_a \leq 2(1 + C_1 k_0^2) \|v\|^2_a . \tag{12}$$

By adding (9) and (12) we get (7) with $C_0^2 = 2 + C_1 k_0 (2k_0 + 1)$. □

When $\|z_0\|^2_a$ can be bounded directly in terms of $\|v\|^2_a$ (independently of the coefficient variation), this lemma is superfluous and leads to a suboptimal quadratic dependence on k_0. In general, however, it is not possible to provide such a uniform bound on $\|z_0\|^2_a$, which is why Lemma 2.9 is in fact absolutely crucial for our analysis.

2.3. **Abstract generalized eigenproblems.** In order to construct the coarse space we will use generalized eigenvalue problems in each subdomain. Since several variations of generalized eigenvalue problems exist in the literature (particularly concerning the interpretation of the 'infinite eigenvalue'), we state the definition that we use.

Definition 2.10 (Generalized eigenvalue problem)**.** *Let $\widetilde{V}$ be a finite-dimensional Hilbert space, let $\widetilde{a} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ and $\widetilde{b} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ be two symmetric bilinear forms. Then the generalized eigenvalues associated with the so called 'pencil' $(\widetilde{a}, \widetilde{b})$ are the following values $\lambda \in \mathbb{R} \cup \{+\infty\}$: either $\lambda \in \mathbb{R}$ and there exists $p \in \widetilde{V} \backslash \{0\}$ such that*

$$\widetilde{a}(p, v) = \lambda \, \widetilde{b}(p, v), \quad \text{for all } v \in \widetilde{V}, \tag{13}$$

or $\lambda = +\infty$ and there exists $p \in \widetilde{V} \backslash \{0\}$ such that

$$\widetilde{b}(p, v) = 0, \quad \text{for all } v \in \widetilde{V}, \quad \text{and} \quad \widetilde{a}(p, v) \neq 0, \quad \text{for a certain} \quad v \in \widetilde{V}.$$

In both cases p is called a generalized eigenvector associated with the eigenvalue λ.

The definition above allows for infinite eigenvalues. This results from the fact that if $(+\infty, p)$ is an eigenpair for the pencil $(\widetilde{a}, \widetilde{b})$ then $(0, p)$ is an eigenpair for the pencil $(\widetilde{b}, \widetilde{a})$ and there is no reason to discriminate between both formulations. In cases where the bilinear form $\widetilde{b}$ is positive definite the problem can be simplified and crucial properties on the eigenvalues and eigenvectors arise. In particular, this leads quite naturally to optimal projectors onto subspaces of the functional space as the next lemma shows in an abstract setting.

Lemma 2.11. *Now let $\tilde{a}$ be positive semi-definite and $\tilde{b}$ positive definite, and let the eigenpairs $\{(p_k, \lambda_k)\}_{k=1}^{\dim(\widetilde{V})}$ of the generalized eigenvalue problem* (13) *be ordered such that*

$$0 \leq \lambda_1 \leq \ldots \leq \lambda_{\dim(\widetilde{V})} \quad \textit{and} \quad \widetilde{b}(p_k, p_l) = \delta_{kl}\,, \quad \textit{for any} \quad 1 \leq k, l \leq \dim(\widetilde{V}).$$

Then, for any integer $1 \leq m < \dim(\widetilde{V})$, the projection

$$\widetilde{\Pi}_m v := \sum_{k=1}^{m} \widetilde{b}(v, p_k) p_k$$

is $\widetilde{a}$-orthogonal, and thus

$$|\widetilde{\Pi}_m v|_{\widetilde{a}} \leq |v|_{\widetilde{a}} \quad \textit{and} \quad |v - \widetilde{\Pi}_m v|_{\widetilde{a}} \leq |v|_{\widetilde{a}}, \quad \textit{for all } v \in \widetilde{V}. \tag{14}$$

Additionally, if m is such that $\lambda_{m+1} > 0$, we have the stability estimate

$$\|v - \widetilde{\Pi}_m v\|_{\widetilde{b}}^2 \;\leq\; \frac{1}{\lambda_{m+1}} |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2, \quad \textit{for all } v \in \widetilde{V}.$$

Proof. Due to the additional assumptions on $\widetilde{a}$ and $\widetilde{b}$, the generalized eigenvalue problem can be simplified to a standard eigenvalue problem, for which the existence of eigenvectors $\{p_k\}_{k=1}^{\dim(\widetilde{V})}$ with associated non-negative real eigenvalues $\{\lambda_k\}_{k=1}^{\dim(\widetilde{V})}$ is guaranteed by standard spectral theory. Moreover, $\{p_k\}_{k=1}^{\dim(\widetilde{V})}$ can be chosen such that it is a basis of $\widetilde{V}$ fulfilling the orthogonality conditions:

$$\widetilde{a}(p_k, p_l) = \widetilde{b}(p_k, p_l) = 0 \quad \forall k \neq l, \quad |p_k|_{\widetilde{b}}^2 = 1 \quad \text{and} \quad |p_k|_{\widetilde{a}}^2 = \lambda_k.$$

Now let $v \in \widetilde{V}$ be fixed. From the $\widetilde{b}$-orthonormality of the basis we get

$$v = \sum_{k=1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k) p_k.$$

For any index set $I \subset \{1, ..., \dim(\widetilde{V})\}$, the $\widetilde{a}$-orthogonality implies

$$\left| \sum_{k \in I} \widetilde{b}(v, p_k) p_k \right|_{\widetilde{a}}^2 = \sum_{k \in I} \widetilde{b}(v, p_k)^2 |p_k|_{\widetilde{a}}^2.$$

Thus

$$|v|_{\widetilde{a}}^2 = |\widetilde{\Pi}_m v|_{\widetilde{a}}^2 + |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2.$$

and (14) follows directly. Finally,

$$\begin{aligned}
\|v - \widetilde{\Pi}_m v\|_{\widetilde{b}}^2 \;&=\; \Big\| \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)\, p_k \Big\|_{\widetilde{b}}^2 \\
&= \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 && \text{(by the } \widetilde{b}\text{-orthonormality of } p_k) \\
&= \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 \, \frac{1}{\lambda_k} \, |p_k|_{\widetilde{a}}^2 && \text{(since } \lambda_k = |p_k|_{\widetilde{a}}^2) \\
&\leq \frac{1}{\lambda_{m+1}} \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 \, |p_k|_{\widetilde{a}}^2 && \text{(since } \lambda_1 \leq \ldots \leq \lambda_{\dim(\widetilde{V})}) \\
&= \frac{1}{\lambda_{m+1}} |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2 && \text{(by the } \widetilde{a}\text{-orthogonality of } p_k).
\end{aligned}$$

□

This lemma will be one of the core arguments to prove the existence of a stable decomposition onto the new GenEO (Generalized Eigenproblems in the Overlap) coarse space and the local subspaces.

3. Algebraic construction of a robust coarse space and its analysis

In this section we introduce the coarse space and give a bound on the condition number of the two-level additive Schwarz method with this coarse space along with a rigorous proof of this result. The proof will consist in proving the existence of a stable splitting for any function in V_h in the sense of Definition 2.7.

3.1. The coarse space.

The GenEO coarse space is constructed as follows. In each subdomain we pose a suitable generalized eigenproblem and select a number of low frequent eigenfunctions. These local functions are converted into global coarse basis functions using a partition of unity operator. As mentioned before, the eigenproblems are restricted to the overlapping zone, which is introduced in the next definition. Following this definition, we will then define the partition of unity operator, which will appear both in the eigenproblems themselves and in the construction of the coarse basis functions.

Definition 3.1 (Overlapping zone)**.** *For each subdomain Ω_j ($1 \leq j \leq N$), the overlapping zone is given by*

$$\Omega_j^\circ = \{x \in \Omega_j : \exists\, j' \neq j \quad \text{such that} \quad x \in \Omega_{j'}\}.$$

We will also require the set of degrees of freedom associated with $V_h(\Omega_j)$, as well as those associated with $V_{h,0}(\Omega_j)$, for $1 \leq j \leq N$.

Definition 3.2. *Given a subdomain D that is a union of elements from $\mathcal{T}_h$, let*

$$\overline{\mathrm{dof}}(D) := \{k = 1, \ldots, n : \mathrm{supp}\,(\phi_k) \cap D \neq \emptyset\}$$

denote the set of degrees of freedom that are "active" in D, including those associated with the boundary. Similarly, we denote by

$$\mathrm{dof}(D) := \{k : 1 \leq k \leq n \text{ and } \mathrm{supp}(\phi_k) \subset \overline{D}\}$$

the set of internal degrees of freedom in D.

Remark 3.3. *Since the basis functions ϕ_k of V_h fulfil a unisolvence property on each element they also fulfil a unisolvence property on each subdomain Ω_j, in other words the functions $\{\phi_k|_{\Omega_j}\}_{k \in \overline{\mathrm{dof}}(\Omega_j)}$ (resp. $\{\phi_k|_{\Omega_j}\}_{k \in \mathrm{dof}(\Omega_j)}$) are linearly independent. A direct consequence is that these functions form a basis of $V_h(\Omega_j)$ (resp. $V_{h,0}(\Omega_j)$).*

Now we can introduce the partition of unity operators. Recall that, for any $v \in V_h$, we write $v = \sum_{k=1}^n v_k\, \phi_k$.

Definition 3.4 (Partition of unity)**.** *For any degree of freedom k, $1 \leq k \leq n$, let μ_k denote the number of subdomains for which k is an internal degree of freedom, i.e.*

$$\mu_k := \#\{j : 1 \leq j \leq N \text{ and } k \in \mathrm{dof}(\Omega_j)\}.$$

Then, for $1 \leq j \leq N$, the local partition of unity operator $\Xi_j : V_h(\Omega_j) \to V_{h,0}(\Omega_j)$ is defined by

$$\Xi_j(v) := \sum_{k \in \mathrm{dof}(\Omega_j)} \frac{1}{\mu_k}\, v_k\, \phi_k|_{\Omega_j}, \quad \text{for all } v \in V_h(\Omega_j).$$

Lemma 3.5. *The operators Ξ_j from Definition 3.4 form a partition of unity in the following sense:*

$$\sum_{j=1}^{N} R_j^\top \Xi_j(v|_{\Omega_j}) \;=\; v, \quad \textit{for all } v \in V_h. \tag{15}$$

Moreover,

$$\Xi_j(v)|_{\Omega_j \setminus \Omega_j^\circ} = v|_{\Omega_j \setminus \Omega_j^\circ}, \quad \textit{for all } v \in V_h(\Omega_j) \textit{ and } 1 \le j \le N. \tag{16}$$

Proof. Property (15) follows directly from the definition. To show (16), let $v \in V_h(\Omega_j)$ and recall that by definition

$$\Xi_j(v)|_{\Omega_j \setminus \Omega_j^\circ} \;=\; \sum_{k \in \mathrm{dof}(\Omega_j)} \frac{1}{\mu_k}\, v_k\, \phi_k|_{\Omega_j \setminus \Omega_j^\circ}\,.$$

Now note that if $\mu_k > 1$, then $\phi_k|_{\Omega_j \setminus \Omega_j^\circ} = 0$. Hence,

$$\Xi_j(v)|_{\Omega_j \setminus \Omega_j^\circ} \;=\; \sum_{k \in \mathrm{dof}(\Omega_j) \text{ s.t. } \mu_k = 1} v_k\, \phi_k|_{\Omega_j \setminus \Omega_j^\circ} \;=\; \sum_{k \in \overline{\mathrm{dof}}(\Omega_j \setminus \Omega_j^\circ)} v_k\, \phi_k|_{\Omega_j \setminus \Omega_j^\circ}\,,$$

but this is also the definition of $v|_{\Omega_j \setminus \Omega_j^\circ}$. □

Next we define the local generalized eigenproblems for the GenEO coarse space.

Definition 3.6 (Generalized Eigenproblems in the Overlaps)**.** *For each $j = 1, \ldots, N$, we define the following generalized eigenvalue problem*

$$a_{\Omega_j}(p, v) = \lambda\, b_j(p,\, v), \quad \textit{for all } \; v \in V_h(\Omega_j). \tag{17}$$

where $b_j(p,\, v) := a_{\Omega_j^\circ}(\Xi_j(p), \Xi_j(v))$, for all $p,\; v \in V_h(\Omega_j)$.

Remark 3.7. *Although the form of the bilinear forms $b_j(\cdot,\cdot)$ seems somewhat artificial, we will see below that it actually arises naturally in the analysis. We will also see that we could have chosen a different partition of unity operator, provided it satisfies the properties in Lemma 3.5. The eigenvalues in Lemma 3.19 are* dimensionless *quantities and do not change when the coordinates of the mesh are rescaled.*

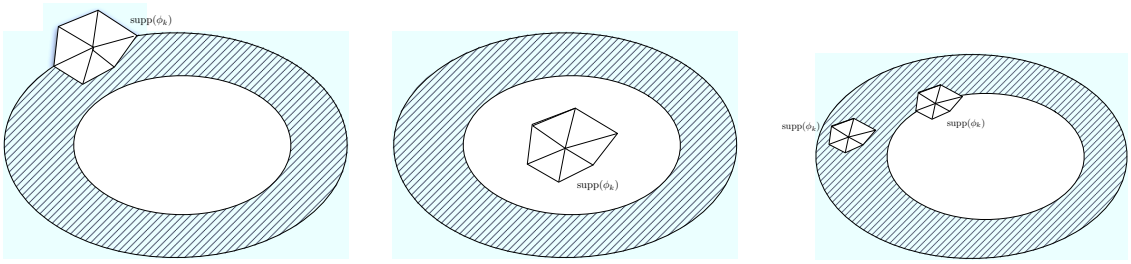
The GenEO coarse space is now constructed (locally) as the span of a suitable subset of eigenfunctions in (17). To obtain a global coarse space we apply the partition of unity operators.

Definition 3.8 (GenEO coarse space)**.** *For each $j = 1, \ldots, N$, let $(p_k^j)_{k=1}^{m_j}$ be the eigenfunctions of the eigenproblem* (17) *in Definition 3.6 corresponding to the m_j smallest eigenvalues. Then,*

$$V_H := \mathrm{span}\{R_j^\top \Xi_j(p_k^j) : k = 1, \ldots, m_j;\; j = 1, \ldots, N\},$$

where Ξ_j are the partition of unity operators from Definition 3.4 and $R_j^\top$ are the extension operators defined in (4)*.*

Consequently, we can also make explicit the final component in Definition 5 of the matrix form $\boldsymbol{M}_{AS,2}^{-1}$ of the additive Schwarz preconditioner, namely the prolongation matrix $\boldsymbol{R}_H^T$. The columns of the rectangular matrix $\boldsymbol{R}_H^T \in \mathbb{R}^{n \times \dim(V_H)}$ are simply the vector representations of the functions $\{R_j^\top \Xi_j(p_k^j) : k = 1, \ldots, m_j;\; j = 1, \ldots, N\}$ with respect to the finite element basis $\{\phi_k\}_{k=1}^n$. Clearly $\dim(V_H) = \sum_{j=1}^N m_j$ and a strategy for selecting m_j will be given below. This completes the definition of $\boldsymbol{M}_{AS,2}^{-1}$.



$k \in \beta_1^j$
$\operatorname{supp}(\phi_k) \not\subset \overline{\Omega}_j$

$k \in \beta_2^j$
$\operatorname{supp}(\phi_k) \subset \overline{\Omega_j \setminus \Omega_j^\circ}$

$k \in \beta_3^j$
$\operatorname{supp}(\phi_k) \subset \overline{\Omega}_j$,
$\operatorname{supp}(\phi_k) \not\subset \Omega_j \setminus \Omega_j^\circ$

FIGURE 1. Three types of finite element basis functions on each subdomain Ω_j. The hashed surface is the overlap Ω_j°.

3.2. **Analysis of the preconditioner.** To confirm the robustness of the above coarse space and to bound the condition number of $\boldsymbol{M}_{AS,2}^{-1}\boldsymbol{A}$ via Theorem 2.8 we will now show that there is a stable splitting for each $v \in V_h$ in the sense of Definition 2.7. First we will give some results on the local subspaces Ω_j, then we use them to show that the eigenproblems from Definition 3.6 are well defined and that the eigenpairs have some particular properties. In order to do this we define a subspace $\widetilde{V}_j$ of each $V_h(\Omega_j)$ on which the restriction of the local generalized eigenproblems satisfy the hypotheses of Lemma 2.11. This leads to local projectors onto subspaces of $V_h(\Omega_j)$ which satisfy stability estimates. These stability estimates will generalize to the whole of $V_h(\Omega_j)$ and enable us to split any $v \in V_h$ in a "C_0-stable" manner.

Definition 3.9. *We partition the set* $\overline{\operatorname{dof}}(\Omega_j)$ *of degrees of freedom in* $V_h(\Omega_j)$ *into three sets (see also Figure 1):*

$$\begin{aligned}
\beta_1^j &:= \overline{\operatorname{dof}}(\Omega_j) \setminus \operatorname{dof}(\Omega_j) && \text{(the DOFs on the boundary of } \Omega_j\text{)},\\
\beta_2^j &:= \operatorname{dof}(\Omega_j \setminus \Omega_j^\circ) && \text{(the interior DOFs in } \Omega_j \setminus \Omega_j^\circ\text{)},\\
\beta_3^j &:= \operatorname{dof}(\Omega_j) \setminus \operatorname{dof}(\Omega_j \setminus \Omega_j^\circ) && \text{(the DOFs in the overlap, incl. the inner boundary)}.
\end{aligned}$$

From these index sets we define subsets of functions of $V_h(\Omega_j)$

$$\mathcal{B}_1^j := \operatorname{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_1^j}, \quad \mathcal{B}_2^j := \operatorname{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_2^j} \quad \text{and} \quad \mathcal{B}_3^j := \operatorname{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_3^j},$$

such that

$$V_h(\Omega_j) = \mathcal{B}_1^j \oplus \mathcal{B}_2^j \oplus \mathcal{B}_3^j.$$

The following simple properties will be used frequently in the following.

Lemma 3.10. *For any* $1 \le j \le N$, *the following properties are true*

(1) $\operatorname{supp}(v) \subset \Omega_j^\circ$, *for all* $v \in \mathcal{B}_1^j$,
(2) $\mathcal{B}_1^j = \operatorname{Ker}(\Xi_j)$,
(3) $\mathcal{B}_2^j = \{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\}$,
(4) a_{Ω_j} *is coercive on* $\mathcal{B}_2^j$.

Proof.

(1) For any basis function ϕ_k with $k \in \beta_1^j$, Lemma 2.4 implies that there is another subdomain $\Omega_{j'}$ with $\operatorname{supp}(\phi_k) \subset \overline{\Omega}_{j'}$, and so $\operatorname{supp}(\phi_k) \cap (\Omega_j \setminus \Omega_j^\circ) = \emptyset$.

(2) Let $v \in V_h(\Omega_j)$. Then

$$v \in \mathrm{Ker}(\Xi_j) \quad \Leftrightarrow \quad v_k = 0, \ \text{ for all } \ k \in \mathrm{dof}(\Omega_j) \quad \Leftrightarrow \quad v = \sum_{k \in \beta_1^j} v_k \phi_k|_{\Omega_j} \in \mathcal{B}_1^j \,.$$

(3) It is clear from the definition of $\mathcal{B}_2^j$ that $\mathcal{B}_2^j \subset \{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\}$. Conversely, if $v|_{\Omega_j^\circ} = 0$, then from the unisolvence property, $v_k = 0$, for all $k \in \overline{\mathrm{dof}}(\Omega_j^\circ) = \beta_1^j \cup \beta_3^j$, and therefore $\{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\} \subset \mathcal{B}_2^j$ also.

(4) The previous property implies that $\mathcal{B}_2^j \subset V_{h,0}(\Omega_j)$ and so

$$a_{\Omega_j}(v, w) = a(R_j^\top v, R_j^\top w) \quad \text{for all } v, w \in \mathcal{B}_2^j.$$

The coercivity of $a_{\Omega_j}(\cdot,\cdot)$ on $\mathcal{B}_2^j$ follows from the coercivity of $a(\cdot,\cdot)$. □

To carry out a robustness analysis we need to make the following two assumptions.

Assumption 3.11. *For any $1 \le j \le N$, a_{Ω_j} is coercive on $\mathcal{B}_1^j$.*

Assumption 3.12. *For any $1 \le j \le N$, $a_{\Omega_j^\circ}$ is coercive on $\mathcal{B}_3^j$.*

Note that by the first property in Lemma 3.10, Assumption 3.11 is equivalent to assuming that, for any $1 \le j \le N$, $a_{\Omega_j^\circ}$ is coercive on $\mathcal{B}_1^j$.

Remark 3.13. *Assumptions 3.11 and 3.12 are not too restrictive. If all the element stiffness matrices are positive definite, then a_{Ω_j} and $a_{\Omega_j^\circ}$ are positive definite on the whole of $V_h(\Omega_j)$. For the Darcy equation or linear elasticity, the element stiffness matrices are not positive definite. However, any function $v \in \mathcal{B}_1^j$ satisfies $v_k = 0$, for $k \notin \beta_1^j$, and any function $v \in \mathcal{B}_3^j$ vanishes on the boundary of Ω_j (by that we mean that $v_k = 0$, for $k \in \beta_j^1$). Therefore, in the Darcy case and in the case of standard H^1-conforming finite elements, Assumptions 3.11 and 3.12 hold if each of the sets β_1^j and β_3^j contains at least one DOF. To make the assumptions hold for linear elasticity, the sets β_1^j and β_3^j need to contain enough DOFs to fix the rigid body modes in Ω_j°, i.e., at least $3(d-1)$ DOFs. Hence, for standard H^1-conforming finite elements, it is sufficient to have d non-collinear points (with associated DOFs for all components of the vector function) that lie on the outer boundary $\partial\Omega_j$, respectively in $\overline{\Omega_j^\circ} \setminus \partial\Omega_j$.*

The final technical hurdle to construct a stable splitting is that we cannot apply the abstract Lemma 2.11 to the specific eigenproblems used in the construction of the GenEO coarse space V_H directly, because the bilinear forms $b_j(\cdot,\cdot) := a_{\Omega_j^\circ}(\Xi_j(\cdot), \Xi_j(\cdot))$ from Definition 17 are not necessarily positive definite on all of $V_h(\Omega_j) \times V_h(\Omega_j)$, for all $1 \le j \le N$. To complete the analysis we thus need to define a suitable subspace $\widetilde{V}_j \subset V_h(\Omega_j)$ such that b_j is positive definite on $\widetilde{V}_j \times \widetilde{V}_j$.

Definition 3.14. *Let the spaces $\widetilde{V}_j$ and $\widetilde{W}_j$ be defined by*

$$\widetilde{V}_j := \{v \in V_h(\Omega_j) : a_{\Omega_j}(v, w) = 0, \ \textit{for all } w \in \widetilde{W}_j\} \quad \textit{where} \quad \widetilde{W}_j := \mathcal{B}_1^j \oplus \mathcal{B}_2^j \,.$$

Lemma 3.15. *Under Assumption 3.11,*

$$V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j.$$

Proof. Since a_{Ω_j} is coercive on $\mathcal{B}_1^j$ (cf. Assumption 3.11) and on $\mathcal{B}_2^j$ (cf. Lemma 3.10 (4)) and since functions in $\mathcal{B}_1^j$ and $\mathcal{B}_2^j$ have disjoint supports, we also have that a_{Ω_j} is coercive on $\widetilde{W}_j$. It follows from the definition of $\widetilde{V}_j$ (via some simple linear algebra) that $\widetilde{V}_j \cap \widetilde{W}_j = \{0\}$ and that $\dim(\widetilde{V}_j) = \dim(V_h(\Omega_j)) - \dim(\widetilde{W}_j)$. □

Remark 3.16. *While this lemma shows that $\widetilde{V}_j$ and $\mathcal{B}_3^j$ contain the same degrees of freedom, it does not imply that $\widetilde{V}_j = \mathcal{B}_3^j$. The functions in $\widetilde{V}_j$ are extended "discrete PDE-harmonically" to the whole of Ω_j, while those in $\mathcal{B}_3^j$ are extended by zero. The harmonic extension into $\Omega_j \setminus \Omega_j^\circ$ is always well defined because of the coercivity of a_{Ω_j} on $\mathcal{B}_2^j$ (cf. Lemma 3.10 (4)). The fact that the harmonic extension onto $\mathcal{B}_1^j$ is well defined is a consequence of Assumption 3.11.*

The role of Assumption 3.12 becomes clear in the next lemma.

Lemma 3.17. *Under Assumptions 3.11 and 3.12, for $j = 1, ..., N$, the bilinear form $b_j(\cdot,\cdot) := a_{\Omega_j^\circ}(\Xi_j(\cdot), \Xi_j(\cdot))$ is positive definite on $\widetilde{V}_j \times \widetilde{V}_j$.*

Proof. Let $v \in \widetilde{V}_j$ such that $\widetilde{b}_j(v, v) = 0$. We need to show that necessarily $v = 0$.

There exists a unique decomposition $v = v_1 + v_2 + v_3$, such that $v_i \in \mathcal{B}_i^j$. The second property in Lemma 3.10 states that $\mathcal{B}_1^j = \mathrm{Ker}(\Xi_j)$, and so

$$\Xi_j(v_1) = 0.$$

From the definition of Ξ_j it is obvious that $\Xi_j|_{\mathcal{B}_2^j} : \mathcal{B}_2^j \to \mathcal{B}_2^j$ is the identity, and so $\Xi_j(v_2) \in \mathcal{B}_2^j$ and in particular from the third property in Lemma 3.10

$$\mathrm{supp}\,(\Xi_j(v_2)) \cap \Omega_j^\circ = \emptyset.$$

From these two remarks and the definition of b_j it follows that

$$b_j(v, v) = a_{\Omega_j^\circ}(\Xi_j(v_3), \Xi_j(v_3)). \tag{18}$$

Moreover, from the definition of Ξ_j it is also obvious that $\Xi_j|_{\mathcal{B}_3^j} : \mathcal{B}_3^j \to \mathcal{B}_3^j$ is a bijection, and so $\Xi_j(v_3) \in \mathcal{B}_3^j$. Now, (18) and Assumption 3.12 imply that $\Xi_j(v_3) = 0$. The fact that $\Xi_j|_{\mathcal{B}_3^j}$ is a bijection in turn implies that $v_3 = 0$, and so $v \in \widetilde{W}_j$. From Lemma 3.15, we know that $\widetilde{V}_j \cap \widetilde{W}_j = \{0\}$, and so $v = 0$ which ends the proof. □

We can now apply Lemma 2.11 to the restriction of the GenEO eigenproblems to $\widetilde{V}_j \times \widetilde{V}_j$ and characterize the entire spectrum (including the infinite eigenvalues).

Lemma 3.18. *For each $j = 1, ..., N$, consider the generalized eigenproblem* (17) *in Definition 3.6.*

(i) There are $\dim(\widetilde{V}_j)$ finite eigenvalues $0 \le \lambda_1^j \le \lambda_2^j \le \ldots \le \lambda^j_{\dim(\widetilde{V}_j)} < \infty$ (counted according to multiplicity) with corresponding eigenvectors denoted by $\{p_k^j\}_{k=1}^{\dim(\widetilde{V}_j)}$ and normalized to form an orthonormal basis of $\widetilde{V}_j$ with respect to $b_j(\cdot,\cdot)$.

(ii) There are $\dim(\widetilde{W}_j)$ infinite eigenvalues $\lambda^j_{\dim(\widetilde{V}_j)+1} = \ldots = \lambda^j_{\dim(V_h(\Omega_j))} = \infty$ with associated eigenvectors denoted by $\{p_k^j\}_{\dim(\widetilde{V}_j)+1}^{\dim(V_h(\Omega_j))}$ forming a basis of $\widetilde{W}_j$.

Proof. Since $V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j$ (cf. Lemma 3.15) and $a_{\Omega_j}(v, w) = b_j(v, w) = 0$, for all $v \in \widetilde{V}_j$ and $w \in \widetilde{W}_j$, the eigenproblem (17) can be decoupled into two eigenproblems: one on $\widetilde{V}_j$ and one on $\widetilde{W}_j$.

Since, according to Lemma 3.17, $b_j(\cdot,\cdot)$ is coercive on $\widetilde{V}_j \times \widetilde{V}_j$, we can apply Lemma 2.11 with $\widetilde{V} \mapsto \widetilde{V}_j$, $\widetilde{a} \mapsto a_{\Omega_j}$, and $\widetilde{b} \mapsto b_j$ to analyse the restriction of (17) to $\widetilde{V}_j$. This completes the proof of (i).

For the restriction of (17) to $\widetilde{W}_j$, we prove that all vectors in $\widetilde{W}_j$ are eigenvectors associated with the eigenvalue $+\infty$ in the sense of Definition 2.10. Let $v \in \widetilde{W}_j$. Then $\Xi_j(v)|_{\Omega_j^\circ} = 0$ and so in particular

$$a_{\Omega_j^\circ}(\Xi_j(v), \Xi_j(w)) = 0 \quad \text{for all } v,\ w \in \widetilde{W}_j\,. \tag{19}$$

Moreover, we have already seen in the proof of Lemma 3.15 that a_{Ω_j} is coercive on $\widetilde{W}_j$, and so

$$a_{\Omega_j}(v, v) \neq 0 \quad \text{for all } v \in \widetilde{W}_j \backslash \{0\}. \tag{20}$$

Due to (19) and (20), any $v \in \widetilde{W}_j$ is indeed an eigenvector to the eigenvalue $+\infty$ in the sense of Definition 2.10. We can use any set of linearly independent vectors in $\widetilde{W}_j$ to form a basis, e.g. $\{p_k^j\}_{k=\dim(\widetilde{V}_j)+1}^{\dim(V_h(\Omega_j))} = \{\phi_k|_{\Omega_j}\}_{k\in\beta_1^j\cup\beta_2^j}$. □

We are now ready to define the crucial projection operators onto the local components of the GenEO coarse space that satisfy suitable stability estimates.

Lemma 3.19 (Local stability estimate)**.** *Let $j \in \{1, ..., N\}$ and let $\{(p_k^j, \lambda_k^j)\}_{k=1}^{\dim(V_h(\Omega_j))}$ be as defined in Lemma 3.18. Suppose that $m_j \in \{1, \ldots, \dim(V_h(\Omega_j)) - 1\}$ such that $0 < \lambda_{m_j+1}^j < \infty$. Then, the local projection operator*

$$\Pi_{m_j}^j v := \sum_{k=1}^{m_j} a_{\Omega_j^\circ}(\Xi_j(v), \Xi_j(p_k^j))\, p_k^j$$

satisfies

$$|\Pi_{m_j}^j v|_{a,\Omega_j} \leq |v|_{a,\Omega_j} \quad \textit{and} \quad |v - \Pi_{m_j}^j v|_{a,\Omega_j} \leq |v|_{a,\Omega_j}\,, \quad \textit{for all } v \in V_h(\Omega_j), \tag{21}$$

as well as the stability estimate

$$\left|\Xi_j(v - \Pi_{m_j}^j v)\right|_{a,\Omega_j^\circ}^2 \leq \frac{1}{\lambda_{m_j+1}^j} \left|v - \Pi_{m_j}^j v\right|_{a,\Omega_j}^2\,, \quad \textit{for all } v \in V_h(\Omega_j). \tag{22}$$

Proof. The condition $\lambda_{m_j+1}^j < \infty$, ensures that $m_j \leq \dim(\widetilde{V}_j)$, so $\Pi_{m_j}^j$ maps to $\widetilde{V}_j$. Therefore, for all $v \in \widetilde{V}_j$, the estimates in (21) and (22) can be deduced from Lemma 2.11 again, with $\widetilde{V} \mapsto \widetilde{V}_j$, $\widetilde{a} \mapsto a_{\Omega_j}$, $\widetilde{b} \mapsto b_j$, and $m \mapsto m_j$.

To prove the result for all $v \in V_h(\Omega_j)$, we use again the fact that $V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j$ and that $a_{\Omega_j}(v, w) = 0$, for all $v \in \widetilde{V}_j$ and $w \in \widetilde{W}_j$. Let $v = v_V + v_W \in V_h(\Omega_j)$ with $v_V \in \widetilde{V}_j$ and $v_W \in \widetilde{W}_j$. Then $\Pi_{m_j}^j v = \Pi_{m_j}^j v_V$ and so (21) follows due to the a_{Ω_j}-orthogonality of $\widetilde{V}_j$ and $\widetilde{W}_j$. Estimate (22) follows similarly from $\Xi_j(v_W)|_{\Omega_j^\circ} = 0$. □

Lemma 3.20 (Stable decomposition)**.** *Let $v \in V_h$ and suppose the definitions and notations of Lemma 3.19 hold. Then, the decomposition*

$$z_0 := \sum_{j=1}^{N} \Xi_j(\Pi_{m_j}^j v|_{\Omega_j}), \qquad z_j := \Xi_j(v|_{\Omega_j} - \Pi_{m_j}^j v|_{\Omega_j}), \quad \textit{for } j = 1, \ldots, N,$$

is C_0-stable with

$$C_0^2 = 2 + k_0(2k_0 + 1) \max_{1\leq j\leq N} \Big(1 + \frac{1}{\lambda_{m_j+1}^j}\Big).$$

Proof. By definition $\|z_j\|^2_{a,\Omega_j} = |\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})|^2_{a,\Omega_j^\circ} + |\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})|^2_{a,\Omega_j\setminus\Omega_j^\circ}$. However, due to property (16) in Lemma 3.5, Ξ_j is the identity for restrictions of functions to $\Omega_j \setminus \Omega_j^\circ$, and so

$$\|z_j\|^2_{a,\Omega_j} = \left|\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})\right|^2_{a,\Omega_j^\circ} + \left|v - \Pi^j_{m_j} v|_{\Omega_j}\right|^2_{a,\Omega_j\setminus\Omega_j^\circ}.$$

Now we can apply Lemma 3.19 to get

$$\|z_j\|^2_{a,\Omega_j} \le \Big(1 + \frac{1}{\lambda^j_{m_j+1}}\Big)\left|v - \Pi^j_{m_j} v|_{\Omega_j}\right|^2_{a,\Omega_j} \le \Big(1 + \frac{1}{\lambda^j_{m_j+1}}\Big)|v|^2_{a,\Omega_j},$$

where in the last step we have used (21). □

With this stable decomposition we can now state our main result on the convergence of the two-level Schwarz preconditioner with the new GenEO coarse space. It follows immediately from Theorem 2.8 and Lemma 3.20.

Theorem 3.21 (Bound on the condition number)**.** *Let Assumptions 2.1, 3.11, and 3.12 hold. Suppose that the coarse space V_H is given by Definition 3.8 and $\boldsymbol{M}^{-1}_{AS,2}$ is as defined in* (5)*. Then we can bound the condition number for the two-level Schwarz method by*

$$\kappa(\boldsymbol{M}^{-1}_{AS,2}\boldsymbol{A}) \le (1 + k_0)\Big[2 + k_0(2k_0 + 1)\max_{1\le j\le N}\Big(1 + \frac{1}{\lambda^j_{m_j+1}}\Big)\Big],$$

where k_0 is given in Definition 2.5.

The only parameters that need to be chosen in our coarse space are the numbers m_j of eigenmodes on each subdomain Ω_j, $1 \le j \le N$, to be included in the coarse space. We suggest the following choice which recovers the condition number estimate for problems with no strong coefficient variation.

Corollary 3.22. *For any j, $1 \le j \le N$, let*

$$m_j := \min\left\{m : \lambda^j_{m+1} > \frac{\delta_j}{H_j}\right\}, \tag{23}$$

where δ_j is a measure of the width of the overlap Ω_j° and $H_j = \operatorname{diam}(\Omega_j)$. Then

$$\kappa(\boldsymbol{M}^{-1}_{AS,2}\boldsymbol{A}) \le (1 + k_0)\Big[2 + k_0(2k_0 + 1)\max_{1\le j\le N}\Big(1 + \frac{H_j}{\delta_j}\Big)\Big].$$

Note that the number of subdomains and the coefficient variations do not appear in this bound on the condition number. This means that we have established rigorously that the algorithm is robust with respect to these two parameters. The size of the coarse space induced by the criterion does however depend on the geometry of the coefficient variation in the overlaps. We will confirm this in Section 5.

4. Implementation

In this section we would like to address implementation issues of the proposed algorithm involving the GenEO coarse space. In the sections above, we have worked with function spaces as they are more convenient in the analysis. However, as we will demonstrate below, our algorithm requires only abstract information of the problem in form of the element stiffness matrices and no further information on the mesh, the finite element spaces, or any coefficients. Indeed, for running the algorithm we need

(i) the list $\overline{\operatorname{dof}}(\tau)$ of degrees of freedom associated with each element $\tau \in \mathcal{T}_h$,
(ii) the element stiffness matrix $\boldsymbol{A}^\tau = (a_\tau(\phi_l, \phi_k))_{k,\, l\in\overline{\operatorname{dof}}(\tau)}$ associated with each element $\tau \in \mathcal{T}_h$.

Unless the overlapping subdomain partition is available a priori, we additionally need

(iii) the number ℓ of layers which determine the amount of overlap.

Before going into details, we note that as for the classical two-level overlapping Schwarz method (see, e.g. [30, Sect. 3]), our algorithm can be parallelized straightforwardly. In particular, the solution of the eigenproblems in the preprocessing step and the subdomain solves during each PCG iteration can be performed fully in parallel.

4.1. **Preprocessing.** We need the overlapping partition $\Omega = \bigcup_{j=1}^{N} \Omega_j$ in form of the list of elements associated with each subdomain Ω_j. To obtain this, we first create the connectivity graph of the elements (using the lists $\overline{\mathrm{dof}}(\tau)$ from (i)) and partition it into disjoint sets of elements which make up the non-overlapping subdomains Ω'_j using for instance METIS [18] or SCOTCH [4]. Then, for each (global) DOF k, we build the list

$$\mathrm{elem}(k) = \{\tau \in \mathcal{T}_h : k \in \overline{\mathrm{dof}}(\tau)\}$$

of elements where DOF k is active. This list realizes $\mathrm{supp}(\phi_k)$ without knowing the basis function ϕ_k itself. In a second step we add ℓ layers to each non-overlapping subdomain Ω'_j according to Definition 2.3, which finally results in a list $\mathcal{T}_h^j$ of elements per (overlapping) subdomain Ω_j. From $\mathcal{T}_h^j$, we construct

$$\overline{\mathrm{dof}}(\Omega_j) = \bigcup_{\tau \in \mathcal{T}_h^j} \overline{\mathrm{dof}}(\tau)$$

(cf. Definition 3.2). Then we can compute the set

$$\mathrm{dof}(\Omega_j) = \{k \in \overline{\mathrm{dof}}(\Omega_j) : \forall \tau \in \mathrm{elem}(k) : \tau \in \mathcal{T}_h^j\}$$

of *internal* degrees of freedom in Ω_j (cf. Definition 3.4). One should keep in mind that $\{\phi_k|_{\Omega_j}\}_{k \in \overline{\mathrm{dof}}(\Omega_j)}$ is a basis for $V_h(\Omega_j)$ and that $\{\phi_k|_{\Omega_j}\}_{k \in \mathrm{dof}(\Omega_j)}$ is a basis for $V_{h,0}(\Omega_j)$.

Finally, by generating the subdomain ownership for each element,

$$\mathrm{owner}(\tau) = \{j = 1, \ldots, N : \tau \in \mathcal{T}_h^j\},$$

it is straightforward to get the list

$$\mathcal{T}_h^{j,\circ} = \left\{\tau \in \mathcal{T}_h^j : \mathrm{owner}(\tau) \setminus \{j\} \neq \emptyset\right\}$$

of elements that make up the overlapping zone Ω_j° for each $j = 1, \ldots, N$.

4.2. **The eigenproblems.** For each subdomain Ω_j, $j = 1, \ldots, N$ we use a local renumbering of the degrees of freedom $\overline{\mathrm{dof}}(\Omega_j)$ of $V_h(\Omega_j)$. By assembling the element stiffness matrices for the selected DOFs over the elements $\tau \in \mathcal{T}_h^j$, we get the subdomain "Neumann" matrix $\widetilde{\boldsymbol{A}}_j$. For the same renumbering of DOFs, we assemble only over the elements $\tau \in \mathcal{T}_h^{j,\circ}$ and obtain the "Neumann" matrix $\widetilde{\boldsymbol{A}}_j^\circ$ associated to the overlapping zone Ω_j°. Note that $\widetilde{\boldsymbol{A}}_j$ and $\widetilde{\boldsymbol{A}}_j^\circ$ have the same format, but $\widetilde{\boldsymbol{A}}_j^\circ$ usually contains a block of zeros.

From Definition 3.4, we see immediately that the action of the operator Ξ_j can be coded by a diagonal matrix $\boldsymbol{X}_j$, where the diagonal entry corresponding to the global DOF k is equal to $1/\mu_k$, if $k \in \mathrm{dof}(\Omega_j)$, and zero otherwise.

With these notations, the eigenproblem given in Definition 3.6 reads: Find the eigenvectors $\boldsymbol{p}_k^j \in \mathbb{R}^{\#\overline{\mathrm{dof}}(\Omega_j)}$ and eigenvalues $\lambda_k^j \in \mathbb{R} \cup \{+\infty\}$ that satisfy

$$\widetilde{\boldsymbol{A}}_j \boldsymbol{p}_k^j = \lambda_k^j \boldsymbol{B}_j \boldsymbol{p}_k^j\,, \tag{24}$$

where

$$\boldsymbol{B}_j = \boldsymbol{X}_j \widetilde{\boldsymbol{A}}_j^\circ \boldsymbol{X}_j.$$

To get the coarse basis functions, we need to solve these eigenproblems (at least we need sufficiently many eigenpairs corresponding to low frequent modes) and to then select m_j of these eigenfunctions for our coarse space. With the criterion suggested in (23), we need

measures δ_j and H_j for the width of the overlapping zone and the subdomain diameter, respectively. If the mesh can be assumed to be quasi-uniform, we may replace the ratio δ_j/H_j by the number of layers of extension we applied in subdomain Ω_j divided by the number of layers Ω_j contains in total (which is available via the connectivity graph).

4.3. **The preconditioner.** Having selected the eigenvectors $\boldsymbol{p}_k^j$, the coarse basis functions are given by the vectors $\widetilde{\boldsymbol{R}}_j^T \boldsymbol{X}_j \boldsymbol{p}_k^j$, where the matrix $\widetilde{\boldsymbol{R}}_j^T$ maps the renumbered DOFs to the global DOFs and fills the rest of the vector with zeros. The columns of the matrix $\boldsymbol{R}_H^T$ are exactly the vectors $\widetilde{\boldsymbol{R}}_j^T \boldsymbol{X}_j \boldsymbol{p}_k^j$, where $j = 1, \ldots, N$, $k = 1, \ldots, m_j$. The coarse matrix $\boldsymbol{A}_H = \boldsymbol{R}_H \boldsymbol{A} \boldsymbol{R}_H^T$ can be efficiently assembled subdomain-wise by using the fact that the coarse basis functions corresponding to two subdomains only interact when the subdomains overlap. Thus, in a parallel regime, we basically only need next-neighbor communication.

For each subdomain, we introduce a renumbering of the DOFs of $V_{h,0}(\Omega_j)$ in $\mathrm{dof}(\Omega_j)$ (i.e. only the interior DOFs). This may differ from the numbering employed in the eigenproblems that were defined on all DOFs. By assembling the element stiffness matrices for these DOFs over the elements $\tau \in \mathcal{T}_h^j$, we get the subdomain "Dirichlet" matrix $\boldsymbol{A}_j$ (alternatively, one can just get the entries of $\boldsymbol{A}_j$ directly from those of the global matrix $\boldsymbol{A}$). The matrix $\boldsymbol{R}_j^T$ simply describes the mapping of the renumbered DOFs to the global DOFs. It differs from $\widetilde{\boldsymbol{R}}_j^T$ in that it only maps from the interior DOFs on Ω_j. This completes the definition of the preconditioner $\boldsymbol{M}_{AS,2}^{-1}$.

Clearly, once the information above is stored and the matrices $\boldsymbol{A}_j$ are factorized, each application of $\boldsymbol{M}_{AS,2}^{-1}$ (within the PCG) can be carried out efficiently.

4.4. **An alternative way of solving the eigenproblems.** The size of the (algebraic) eigenproblem (24) to be solved in each subdomain can be reduced. By rearranging the local DOFs $\overline{\mathrm{dof}}(\Omega_j)$ with respect to the sets β_1^j, β_2^j, and β_3^j (cf. Definition 3.9), the matrices $\widetilde{\boldsymbol{A}}_j$ and $\boldsymbol{B}_j$ take the following block form

$$\widetilde{\boldsymbol{A}}_j = \begin{pmatrix} \widetilde{\boldsymbol{A}}_j^{11} & 0 & \widetilde{\boldsymbol{A}}_j^{13} \\ 0 & \widetilde{\boldsymbol{A}}_j^{22} & \widetilde{\boldsymbol{A}}_j^{23} \\ (\widetilde{\boldsymbol{A}}_j^{13})^T & (\widetilde{\boldsymbol{A}}_j^{23})^T & \widetilde{\boldsymbol{A}}_j^{33} \end{pmatrix}, \qquad \boldsymbol{B}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{B}_j^{33} \end{pmatrix},$$

where $\widetilde{\boldsymbol{A}}_j^{kl} = a_{\Omega_j}(\phi_m, \phi_n)_{n \in \beta_k^j, m \in \beta_l^j}$. The two zero blocks in $\widetilde{\boldsymbol{A}}_j$ are due the fact that the supports of functions in $\mathcal{B}_1^j$ and $\mathcal{B}_2^j$ are always disjoint. Since $\widetilde{\boldsymbol{A}}_j^{11}$ is the matrix version of the bilinear form $a_{\Omega_j^\circ}(\cdot,\cdot) : \mathcal{B}_1^j \times \mathcal{B}_1^j \to \mathbb{R}$, and since Assumption 3.11 states that $a_{\Omega_j^\circ}(\cdot,\cdot)$ is coercive on $\mathcal{B}_1$, it follows that the block $\widetilde{\boldsymbol{A}}_j^{11}$ is positive definite and thus invertible. Similarly, $\boldsymbol{A}_j^{22}$ is positive definite due to Lemma 3.10 (4).

Suppose that $(\boldsymbol{p}_k^j,\ \lambda_k^j)$ is an eigenpair of (24) with $\lambda_k^j < \infty$ and let $\boldsymbol{p}_k^{j,l}$, $l = 1, \ldots, 3$ denote the blocks of $\boldsymbol{p}_k^j$ with respect to β_l^j. Then it follows by block-elimination that

$$\boldsymbol{S}_j \, \boldsymbol{p}_k^{j,3} = \lambda_k^j \boldsymbol{B}_j^{33} \boldsymbol{p}_k^{j,3} \tag{25}$$

with $\boldsymbol{S}_j = \widetilde{\boldsymbol{A}}_j^{33} - \widetilde{\boldsymbol{A}}_j^{13}[\widetilde{\boldsymbol{A}}_j^{11}]^{-1}\widetilde{\boldsymbol{A}}_j^{13} - \widetilde{\boldsymbol{A}}_j^{23}[\widetilde{\boldsymbol{A}}_j^{22}]^{-1}\widetilde{\boldsymbol{A}}_j^{23}$. The two remaining blocks can then be computed from

$$\boldsymbol{p}_k^{j,1} = -[\widetilde{\boldsymbol{A}}_j^{11}]^{-1}\widetilde{\boldsymbol{A}}_j^{13}\, \boldsymbol{p}_k^{j,3},$$
$$\boldsymbol{p}_k^{j,2} = -[\widetilde{\boldsymbol{A}}_j^{22}]^{-1}\widetilde{\boldsymbol{A}}_j^{23}\, \boldsymbol{p}_k^{j,3}$$

(i.e. via discrete harmonic extension). Since we are only interested in the eigenpairs with finite eigenvalues, we can solve eigenproblem (25) instead of (24). Due to the appearance of the Schur complement $\boldsymbol{S}_j$ and because we are interested only in the first few eigenpairs, an iterative eigensolver seems appropriate, e.g., we could use the inverse power method [21] or the LOBPCG method [19], maybe using a suitable regularization of $\widetilde{\boldsymbol{A}}_{jj}^{33}$ or $\boldsymbol{S}_j$ as a preconditioner. This, however, will be the subject of future research. Note finally, that the blocks $\boldsymbol{p}_k^{j,2}$ never need to be calculated in practice as they are annihilated by the matrix $\boldsymbol{X}_j$.

5. Numerical results

We have introduced an algorithm for a wide range of problems. In this section we test its efficiency on the three-dimensional Darcy equation and on the two- and three-dimensional linear elasticity equations with heterogeneous coefficients. We have used FreeFem++ [17] to define the test cases and build all the finite element data. The eigenvalue problems were solved using LAPACK [1]. For the remainder (including the subdomain solves and the coarse solve) we have used Matlab. Throughout this section we compare three methods.

(1) The first one is the one-level additive Schwarz method (referred to as AS), defined by the preconditioner $\boldsymbol{M}_{AS,1}^{-1} = \sum_{j=1}^{N} \boldsymbol{R}_j^T \boldsymbol{A}_j^{-1} \boldsymbol{R}_j$.
(2) The second one (referred to as ZEM for Zero Energy Modes) is the two-level method given by (5) with the coarse space $V_H := \text{span}\{\boldsymbol{R}_j^T \Xi_j(\boldsymbol{q}_k^j)\}_{j,k}$ where the $\boldsymbol{q}_k^j$ span the kernel of the subdomain operator. For the Darcy equation these are the constant functions and for elasticity the rigid body modes. In the floating subdomains that do not touch the Dirichlet boundary, this basically coincides with choosing $m_j = \dim(\ker(a_{\Omega_j}))$ in our GenEO method.
(3) The third method (referred to as GenEO) is the two-level method introduced here where, except for one test where we specify otherwise, for $j = 1, \ldots, N$, the number m_j is chosen according to (23).

For each of these method we use the Preconditioned Conjugate Gradient (PCG) solver. As a stopping criterion we apply $\frac{\|v-\bar{v}\|_\infty}{\|\bar{v}\|_\infty} < 10^{-6}$ where $\bar{v}$ is the solution of (2) obtained *via* a direct solver on the global problem. Of course this criterion is not practical but in this context we have chosen it to ensure a fair comparison.

In the tables below, we provide the number of PCG iterations needed to reach convergence. We have also computed condition number estimates for each of the preconditioned matrices using the Rayleigh-Ritz procedure [24] on the Krylov subspaces within PCG. We do not give any detail on the maximal and minimal eigenvalue. However, we can report that adding/enriching the coarse space leads to larger minimal eigenvalues, whereas the maximal eigenvalue depends only on the geometry. This is in agreement with Lemma 2.6 and Theorem 2.8. Finally, for the ZEM and the GenEO coarse spaces, we display the dimension of the coarse space V_H.

In both three-dimensional cases (Sections 5.1 and 5.2), in order to study scalability, we use the domain $\Omega = [0, L] \times [0, 1] \times [0, 1]$ and a regular tetrahedral mesh of $(10L+1 \times 11 \times 11)$ nodes which we divide into L horizontally side by side subdomains. We will either use a regular partition into L unit cubes (Figure 2 (left)) or an automatic partition into L subdomains using Metis (Figure 2 (right)). The two dimensional case (Section 5.3) will present results for more general partitions.

5.1. The Darcy equation.

With the domain Ω given above, we solve the following problem: Find $v \in H^1(\Omega)$ such that

$$-\boldsymbol{\nabla} \cdot (\kappa(x, y, z) \boldsymbol{\nabla} v) = 0 \quad \text{in} \quad \Omega,$$

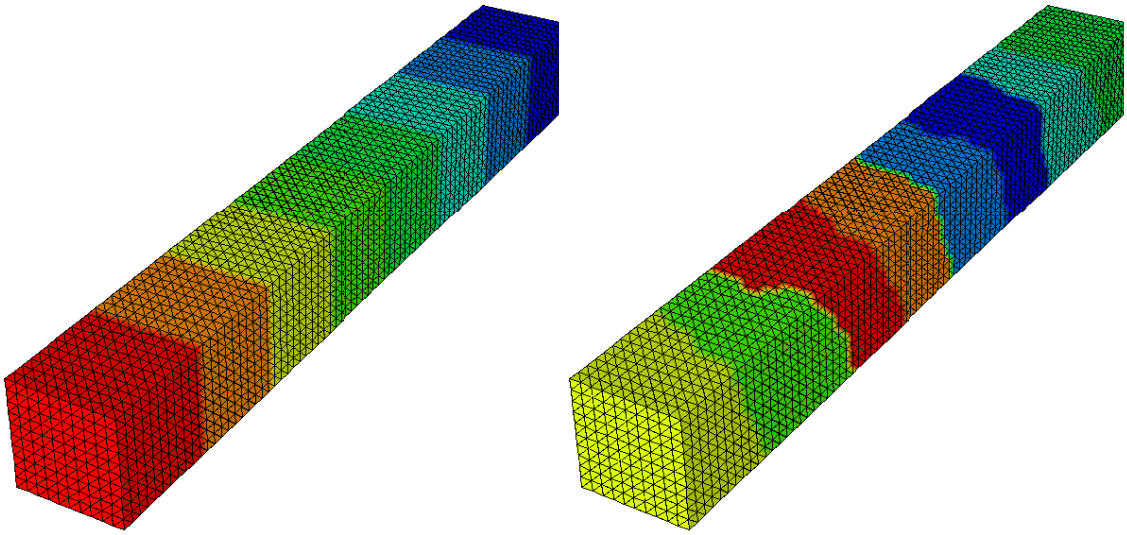


FIGURE 2. Partition of Ω into $L = 8$ subdomains – regular (left) and Metis (right)

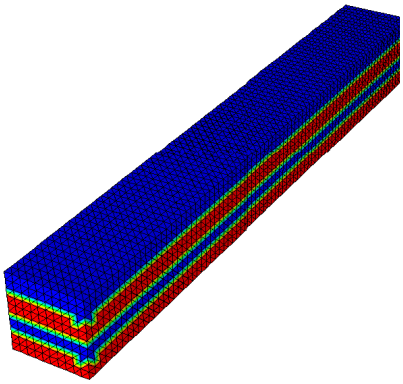


FIGURE 3. Coefficient distribution (four alternating layers)

$v = 0$ on $\partial\Omega_D = \{(x, y, z) \in \partial\Omega : x = 0\}$ and $\kappa\boldsymbol{\nabla} v \cdot \boldsymbol{n} = 0$ on the rest of $\partial\Omega$, where $\boldsymbol{n}$ is the outward unit normal.

As a coefficient distribution we use two different constant values κ_1 and κ_2 of κ and four horizontal layers alternating between κ_1 and κ_2 (as shown in Figure 3). First, we study the robustness of our algorithm with respect to the coefficient variation. We partition Ω into $L = 8$ (non-overlapping) regular subdomains. Each subdomain is then extended by $\ell = 1$ layers in order to create the overlapping partition. Table 1 shows the iteration counts and condition numbers for fixed value $\kappa_1 = 1$ and various κ_2. As expected, for our algorithm the condition number and the number of PCG iterations are robust with respect to the jump κ_2/κ_1. Furthermore, for $\kappa_2 = \kappa_1$, the algorithm automatically selects seven eigenmodes (one per floating subdomain) to build the coarse space, this leads essentially to the same choice as in the ZEM except for the subdomain in which the Dirichlet boundary condition is active, in both cases 11 iterations are needed to reach convergence.

The second test that we conduct is the scalability with regard to the problem size and the number of subdomains. For simplicity, we make the problem parameter L vary. Increasing L elongates the bar-shaped domain and at the same time increases the number of subdomains which equals L. The global number of degrees of freedom is proportional to

	AS		ZEM			GenEO		
κ_2	it	*cond*	it	*cond*	dim	it	*cond*	dim
1	16	*229*	11	*6.3*	8	11	*8.4*	7
10^2	27	*230*	19	*22*	8	13	*8.4*	14
10^4	29	*230*	23	*210*	8	15	*8.4*	14
10^6	26	*230*	22	*230*	8	11	*8.4*	14

TABLE 1. 3D Darcy: number of PCG iterations (it), condition number (*cond*) and coarse space dimension (dim) vs. jump in κ for $\kappa_1 = 1$, $\ell = 1$ added layers, $L = 8$ regular subdomains

		Regular							
		AS		ZEM			GenEO		
sub	glob DOF	it	*cond*	it	*cond*	dim	it	*cond*	dim
4	4840	14	*51*	15	*51*	4	10	*8.4*	6
8	9680	26	*230*	22	*230*	8	11	*8.4*	14
16	19360	51	*980*	36	*970*	16	13	*8.4*	30
32	38720	103	*4000*	61	*3900*	32	13	*8.4*	62
		Metis with criterion given by (26)							
		AS		ZEM			GenEO		
sub	glob DOF	it	*cond*	it	*cond*	dim	it	*cond*	dim
4	4840	21	*67*	18	*63*	4	9	*3.0*	19
8	9680	36	*290*	29	*280*	8	9	*3.0*	40
16	19360	65	*1200*	45	*1200*	16	11	*3.1*	81
32	38720	123	*4900*	79	*4700*	32	11	*3.1*	171

TABLE 2. 3D Darcy: number of PCG iterations (it), condition number (*cond*) and coarse space dimension (dim) vs. problem size for $\kappa_1 = 1$, $\kappa_2 = 10^6$, $\ell = 1$ added layers, L (sub) subdomains

L. Table 2 gives the results for different problem sizes (we display the number of subdomains and the total number of degrees of freedom) and for regular and irregular partitions. For irregular partitions, the choice of m_j becomes more tricky since the diameter of the subdomain is not necessarily relevant and there is no correlation between the diameters of the subdomains for two 'Metis' decompositions into L and L' subdomains as soon as $L \neq L'$. However, for the regular decomposition all floating subdomains are identical. We notice that in these subdomains Ω_j the number of selected eigenvectors is $m_j = 2$ and $\lambda_3 = 0.5$. The quantity that appears in the condition number bound in Theorem 3.21 is $\max_{1\leq j\leq N}\left(\frac{1}{\lambda^j_{m_j+1}}\right)$. In order for the bound on the condition number given by theorem in the irregular case to be at least as strict as in the regular case, for the 'Metis' simulation in Table 2 we define

$$m_j \;:=\; \min\left\{m : \lambda^j_{m+1} > 0.5\right\}, \tag{26}$$

in each subdomain. We would like to point to the low and stable condition numbers in both the regular and irregular subdomain cases.

Finally, Table 3 studies the dependence on the amount of overlap, or equivalently on the number ℓ of layers added to each non-overlapping subdomain. We can see that for this example, increasing the amount of overlap improves convergence without increasing the dimension of the coarse space.

	AS		ZEM			GenEO		
ℓ	it	*cond*	it	*cond*	dim	it	*cond*	dim
1	26	*230*	22	*230*	8	11	*8.4*	14
2	22	*150*	18	*150*	8	9	*5.4*	14
3	16	*110*	15	*110*	8	9	*4.0*	14
4	15	*92*	13	*92*	8	7	*3.3*	14

TABLE 3. 3D Darcy: number of PCG iterations (it), condition number (*cond*) and coarse space dimension (dim) vs. number of added layers ℓ by which each domain is extended, for $L = 8$ regular subdomains, $\kappa_1 = 1$ and $\kappa_2 = 10^6$

		AS		ZEM			GenEO		
L	glob DOF	it	*cond*	it	*cond*	dim	it	*cond*	dim
4	14520	79	$2.4 \cdot 10^3$	54	$2.9 \cdot 10^2$	24	16	*10*	46
8	29040	177	$1.3 \cdot 10^4$	87	$1.0 \cdot 10^3$	48	16	*10*	102
16	58080	378	$1.5 \cdot 10^5$	145	$1.4 \cdot 10^3$	96	16	*10*	214

TABLE 4. 3D Elasticity: number of PCG iterations (it), condition number (*cond*), and coarse space dimension (dim) vs. number of regular subdomains, for $\ell = 1$ added layers, $g = 10$, $(E_1, \nu_1) = (2 \cdot 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \cdot 10^7, 0.45)$.

5.2. **The linear elasticity equations.** For the second family of tests the equations are the following. Find $\boldsymbol{v} = (v_1, v_2, v_3)^{\mathrm{T}} \in H^1(\Omega)^3$ such that

$$-\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{v})) = \boldsymbol{f}, \quad \text{in } \Omega,$$

$\boldsymbol{v} = (0,0,0)^{\mathrm{T}}$ on $\partial\Omega_D = \{(x,y,z) \in \partial\Omega : x = 0\}$ and $\boldsymbol{\sigma}(\boldsymbol{v}) \cdot \boldsymbol{n} = 0$ on the rest of $\partial\Omega$, where the stress tensor $\boldsymbol{\sigma}(\boldsymbol{v})$, the Lamé coefficients λ and μ and the right hand side are given by

$$\begin{cases} \sigma_{ij}(\boldsymbol{v}) = 2\mu\varepsilon_{ij}(\boldsymbol{v}) + \lambda\delta_{ij}\mathrm{div}(\boldsymbol{v}),\ \varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \boldsymbol{f} = (0,0,g)^T, \\ \mu = \frac{E}{2(1+\nu)},\ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \end{cases}$$

Here E and ν denote respectively Young's modulus and Poisson's ratio, and we will let both parameters vary discontinuously over the domain. Again we use four alternating layers as shown in Figure 3 of coefficients between two sets of values (E_1, ν_1) and (E_2, ν_2). Table 4 displays the iteration counts, condition numbers, and coarse space dimensions for partitions into different numbers of regular subdomains (the parameter choices are given below the table). Note that for GenEO, we need only 16 PCG iterations in all cases. As an example, Figure 4 shows the convergence profile for the case where Ω is split into 16 regular subdomains.

5.3. **The two-dimensional linear elasticity equations.** In this subsection we deal with the two-dimensional linear elasticity equations with again a Dirichlet boundary condition at $x = 0$. In this case, the ZEM coarse space consists of three rigid body modes per subdomain. Here we choose $\Omega = (0,1) \times (0,1)$ and use a structured simplicial mesh with 81×81 nodes. The coefficient distribution is sketched on the left hand side of Figure 5: on the two regions indicated by the two different colors, we take the parameters $(E_1, \nu_1) = (2 \cdot 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \cdot 10^7, 0.45)$.

This time, we keep the problem size fixed, but we make the number of subdomains vary. In all cases, we use a Metis partition and extend the non-overlapping subdomains by $\ell = 2$ layers. As shown in Figure 5 (right) for a decomposition into 64 subdomains there are

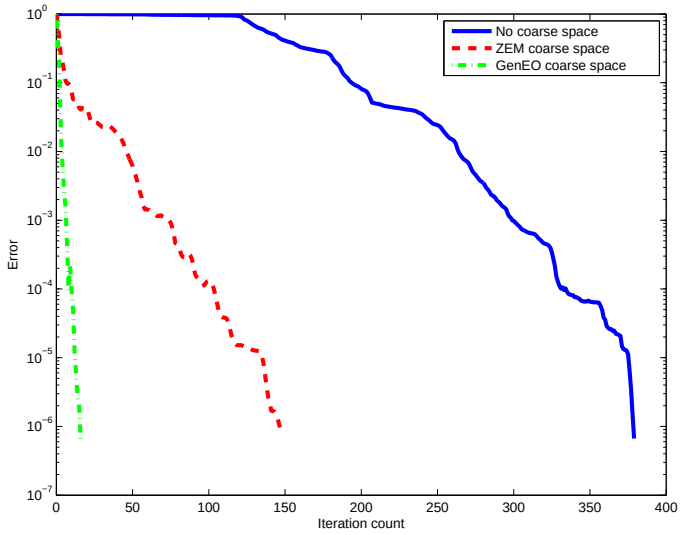


FIGURE 4. (3D Elasticity) Relative error vs. iteration count for $L = 16$ regular subdomains

		AS	ZEM		GenEO	
sub	glob DOF	it	it	dim	it	dim
4	13122	90	94	12	36	36
16	13122	169	179	48	39	112
25	13122	222	157	75	40	166
64	13122	317	196	192	39	343

TABLE 5. 2D Elasticity: number of PCG iterations (it) and coarse space dimension (dim) vs. number of Metis subdomains for fixed problem size

many floating subdomains. Table 5 shows the iteration counts and coarse space dimensions for different Metis partitions (the chosen parameters are given below the table). From the iteration counts we see that the GenEO method is scalable.

It is not surprising that the coarse space dimension grows with the number of subdomains because we construct local coarse basis functions per subdomain. Note however that for the case of 64 subdomains, the coarse space dimension of 343 is still comparable to average dimension 205 of a subdomain problem. To find the optimal partition in terms of CPU time, one must clearly take the cost of the subdomain solves into account, and additionally the cost of the eigensolves in the setup of the method. For this series of tests no estimates for the condition number of the preconditioned matrices are given as in some cases (but for all three types of preconditioners), the Rayleigh-Ritz procedure returned one or a few negative eigenvalues, which is probably due to large floating point error propagation due to the high contrast. For more extensive results for two dimensional elasticity, see [29].

6. CONCLUSION

In this article we have introduced a coarse space for problems given by symmetric positive definite bilinear forms. In order to remain as general as possible, we did so using an abstract formulation. We thoroughly proved a bound for the condition number of the overlapping two-level additive Schwarz preconditioner for this coarse space. This bound does not depend on any of the coefficients in the equations or on the way the domain is split

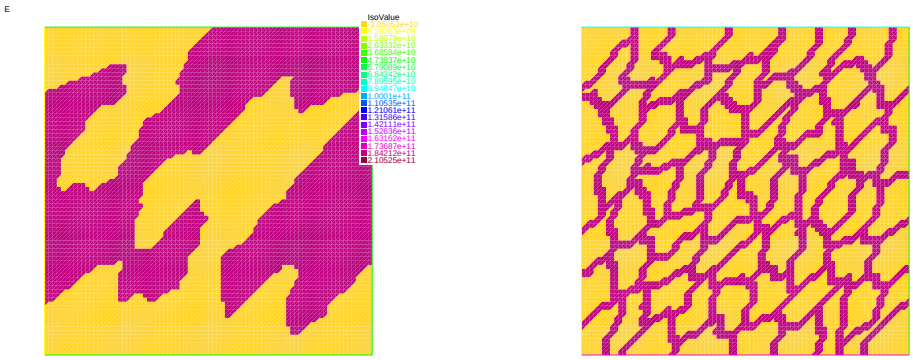


FIGURE 5. 2D Elasticity: coefficient distribution (left) – Metis decomposition into 64 subdomains (right)

into subdomains. Numerical results on two-dimensional and three-dimensional problems are in agreement with the fact that the method is robust with regard to heterogeneities and rather irregular subdomains. We also gave details on how to implement the coarse space construction insisting on the fact that given a finite element code no additional elementary matrices need to be computed. This means that the method is quite easily applicable to simulations of actual physical problems and it is our ambition to do so.

Along the way we have identified promising leads for further improving the efficiency of the method. In the near future there are three main ideas for further investigations. The first one is to take advantage of the fact that the partition of unity can be chosen differently since the proof holds as long as the partition of unity is defined by individual weights per interior degree of freedom in each subdomain. The second idea is to optimize the eigenvalue computations. Although this is a purely parallel task, this is the most costly part in building the coarse space. Finally, the formulation of the GenEO coarse space makes it particularly well suited for a multilevel parallel implementation and this would boost efficiency as it would deal with the 'large' coarse spaces which can occur if we select many low frequency modes.

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