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SECOND ORDER CONDITIONS FOR METRIC SUBREGULARITY OF SMOOTH CONSTRAINT SYSTEMS

HELMUT GFRERER*

Abstract. Metric subregularity (respectively calmness) of multifunctions is a property which is not stable under smooth perturbations, implying that metric subregularity cannot be fully characterized by first order theory. In this paper we derive second order conditions for metric subregularity, both sufficient and necessary, for multifunctions associated with constraint systems as they occur in optimization. We show that the main difference between the necessary and sufficient conditions is the replacement of an inequality by a strict inequality, just as in the case of "no gap" second order optimality conditions in optimization.

Key words. Metric subregularity, calmness, multifunctions, constraint qualification

AMS subject classifications. 90C31, 26E25, 49J53

1. Introduction. Following [3], a multifunction $G : X \rightrightarrows Y$ acting between normed spaces X and Y is called *metrically subregular* at $(\bar{x}, \bar{y}) \in \operatorname{gph} G$, provided there exists a neighborhood U of $\bar{x}$ and a real number $\kappa > 0$ such that

$$\mathrm{d}(x, G^{-1}(\bar{y})) \leq \kappa \mathrm{d}(\bar{y}, G(x)), \quad \forall x \in U. \tag{1.1}$$

The metric subregularity property was introduced by Ioffe [11],[13] using the terminology "regularity at a point".

In this paper we study characterizations of metric subregularity at $(\bar{x}, 0)$ for smooth constraint systems of the form

$$0 \in G(x) := g(x) - C, \tag{1.2}$$

where $g : X \to Y$ is a continuous mapping strictly differentiable at the point $\bar{x}$ under consideration, X and Y denote Banach spaces and $C \subset Y$ is a closed convex set.

A prominent example of such a constraint system is given by the constraints of a possibly infinite dimensional mathematical programming problem with $Y = \hat{Y} \times \mathbb{R}^m$, $C = \{0\} \times \mathbb{R}^m_-$ and $g = (P, h_1, \ldots, h_m)$, i.e.

$$\begin{aligned} P(x) &= 0 \\ h_i(x) &\leq 0, \ i = 1, \ldots, m. \end{aligned} \tag{1.3}$$

Subregularity is a weaker condition than the more familar property of *metric regularity*, where the inequality (1.1) should hold not only for $\bar{y}$ but for all y belonging to some neighborhood of $\bar{y}$, i.e. there is also some neighborhood V of $\bar{y}$ such that

$$\mathrm{d}(x, G^{-1}(y)) \leq \kappa \mathrm{d}(y, G(x)), \quad \forall (x, y) \in U \times V. \tag{1.4}$$

For a survey on the theory of metric regularity and also on the related notions of *pseudo-Lipschitz continuity*, *Aubin property*, *Lipschitz-like property* and *openness with a linear rate* we refer to [13] and to the monographs [15], [18], [20].

It is well known [3] that metric subregularity of G at $(\bar{x}, \bar{y})$ is equivalent to calmness of the inverse multifunction G^{-1} at $(\bar{y}, \bar{x})$. A multifunction $S : Y \rightrightarrows X$ is said to be *calm* at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$, if there exists $\kappa > 0$ along with some neighborhoods U of $\bar{x}$ such that

$$S(y) \cap U \subset S(\bar{y}) + \kappa \|y - \bar{y}\| \mathcal{B}_X, \ \forall y \in Y. \tag{1.5}$$

Usually the definition of calmness requires condition (1.5) to hold for y sufficiently close to $\bar{y}$, but it can be easily verified that this and our definition are equivalent.

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In the special case that the set $S(\bar y)$ reduces locally to a singleton and S is calm at $(\bar y, \bar x)$, i.e. condition (1.5) is replaced by

$$S(y) \cap U \subset \bar x + \kappa \|y - \bar y\| \mathcal{B}_X,\ \forall y \in Y,$$

then S is called *locally upper Lipschitz*.

Of particular importance is the subregularity respectively calmness of constraint systems as this become the key for the existence of nondegenerate multipliers, local error bounds, exact penalty functions or weak sharp local minimizers, see e.g. [2], [11], [13], [16], [17], [21], [22].

An important subclass of multifunctions which are known to be metrically subregular at every point of its graph, is given by polyhedral multifunctions, i.e. multifunctions whose graph is the union of finitely many polyhedral sets. This result is due to Robinson [19]. An important special case of polyhedral multifunctions is given by linear systems, where subregularity is a consequence of Hoffman's error bound [10]. Some extensions to the infinite dimensional case are given in [1, Section 2.5.7].

There is a growing interest in criteria for subregularity and calmness, respectively. We refer to the papers [5], [6],[7], [8], [9], [14], [22]. These papers deal with first order conditions. But it is well-known that the property of metric subregularity is not stable under smooth perurbations and hence we consider second order charcterizations, both necessary and sufficient, for subregularity of (1.2) in this paper. These conditions extend the first order characterizations presented in the recent paper [5]. We will see that the difference between the necessary and the sufficient second order conditions is a change from an inequality to a strict inequality, similar to the 'no-gap' second order optimality conditions of nonlinear optimization as presented in [1].

2. Preliminaries. Throughout this paper we assume that $0 \in g(\bar x) - C$. We consider the case, that Y can be represented as the topological direct sum of two subspaces Y_1, Y_2, i.e. $Y = Y_1 \oplus Y_2$. In what follows we denote by p_i, $i = 1, 2$ the projection from Y onto Y_i, i.e. $p_i \in \mathcal{L}(Y, Y_i)$, $p_i^2 = p_i$ and $y = p_1(y) + p_2(y)$, $\forall y$. Further we denote by G_i, $i = 1, 2$ the multifunction $G_i : X \rightrightarrows Y_i$ given by $G_i(x) = \{p_i(y) \,|\, y \in G(x)\} = p_i(g(x)) - C_i$, where $C_i = p_i(C)$.

Throughout this paper we make the following assumption:

Assumption 1. *The multifunction G_2 is metrically subregular at $(\bar x, 0)$ and $C = C_1 + C_2$.*

Note that the general case also fits into this setting by choosing $Y_1 := Y$, $Y_2 = \{0\}$.

In many applications the constraint system (1.2) can be naturally subdivided into a nonlinear and a linear part. Then in many cases the linear part of G is a proper choice for the multifunction G_2.

Our notation is fairly standard. In a normed space Z, $\mathcal{B}_Z := \{z \in Z \,|\, \|z\| \le 1\}$ denotes the closed unit ball and $\mathcal{S}_Z := \{z \in Z \,|\, \|z\| = 1\}$ denotes the unit sphere. The topological dual space is denoted by Z^*. By $\langle z^*, z\rangle$ we denote the value $z^*(z)$ of the linear functional $z^* \in Z^*$ at $z \in Z$. For a set $D \subset Z$ we denote by $\sigma_D(\cdot)$ its support function, $\sigma_D(z^*) := \sup_{z \in D} \langle z^*, z\rangle$. $\operatorname{lin} D$ denotes the closed linear hull of D. If D is convex, we denote the normal cone respectively the tangent cone at $z \in D$ by $N_D(z)$ and $T_D(z)$, respectively.

If not otherwise stated we endow the product space of normed spaces with the norm given by the sum of the individual norms.

The Fréchet (Gâteaux) derivative of a mapping h differentiable at $\bar x$ is denoted by $Dh(\bar x)$.

Note that for the multifunction $G = g(\cdot) - C$ the so-called contingent derivative at $(x, y) \in \operatorname{gph} G$ is given by the multifunction $CG(x, y) : X \rightrightarrows Y$, $CG(x, y)(u) := Dg(x)u - T_C(g(x) - y)$. In what follows we will denote by $\overline{CG} := CG(\bar x, 0) = Dg(\bar x)(\cdot) - T_C(g(\bar x))$ the contingent derivative at $(\bar x, 0)$. It is well known (cf. [5, Prop.1.3, Remark 1.4]) that metric subregularity of G at $(\bar x, 0)$ implies metric subregularity of $\overline{CG}$ at $(0, 0)$, at least if X is reflexive or $Dg(\bar x)$ has closed range and $\operatorname{lin}(C - g(\bar x))$ is finite dimensional. Further note that $\overline{CG}(\lambda u) = \lambda \overline{CG}(u)$, $\forall u \forall \lambda > 0$ and hence $\overline{CG}$ is metrically subregular at $(0, 0)$ iff

$$\mathrm{d}(u, \overline{CG}^{-1}(0)) \le \kappa \mathrm{d}(0, \overline{CG}(u)), \quad \forall u \in X$$

holds for some $\kappa > 0$.

The contingent derivative $\overline{CG}$ can also be used to characterize upper Lipschitz continuity of G^{-1}. It was shown in [5, Prop.3.4], that G^{-1} is locally upper Lipschitz at $(0,\bar{x})$ if and only if there is no sequence $(u_k)\subset\mathcal{S}_X$ with $\lim_k \mathrm{d}(Dg(\bar{x})u_k, T_C(g(\bar{x}))) = \lim_k \mathrm{d}(0,\overline{CG}(u_k)) = 0$. Furthermore, if the subspace $Dg(\bar{x})X + \mathrm{lin}\,(C-g(\bar{x}))$ is closed in Y and the subspace $Dg(\bar{x})X\cap \mathrm{lin}\,(C-g(\bar{x}))$ is finite dimensional, then G^{-1} is locally upper Lipschitz at $(0,\bar{x})$ if and only if there is no nonzero u with $Dg(\bar{x})u\in T_C(g(\bar{x}))$ or equivalently, $\overline{CG}^{-1}(0)=\{0\}$. Note that besides the cases that X or Y is finite dimensional, the last assumption is fulfilled in particular for the infinite dimensional mathematical programming problem (1.3) provided that $Dg(\bar{x})$ has closed range, since $\mathrm{lin}\,((C-g(\bar{x}))=\{0\}\times\mathbb{R}^m$ is finite dimensional.

Recall that the Banach space Y admits a Fréchet smooth renorm, if there is an equivalent norm on Y that is Fréchet differentiable at any nonzero point. In particular, every reflexive space admits a Fréchet smooth renorm.

THEOREM 2.1. *Assume that Assumption 1 is fulfilled and that Y admits a Fréchet smooth renorm. Further assume that either*

(a) there is no sequence $(u_k)\subset\mathcal{S}_X$ with $\lim_k \mathrm{d}(Dg(\bar{x})u_k, T_C(g(\bar{x}))) = \lim_k \mathrm{d}(0,\overline{CG}(u_k)) = 0$,
or

(b) there is no sequence $(y_k^*)\subset\mathcal{S}_{Y^*}$ with $\lim_k Dg(\bar{x})^*y_k^*=0$, $\lim_k\langle y_k^*, g(\bar{x})\rangle-\sigma_C(y_k^*)=0$, $\liminf_k \|y^*_{k|_{Y_1^*}}\|>0$.

Then G is metrically subregular at $(\bar{x},0)$.

Proof. The assertion follows from [5, Thm. 3.2, Prop. 3.3, 3.4]. □

We now want to discuss the assumptions of the theorem. As mentioned above, assumption (a) is equivalent to the upper Lipschitz continuity of G^{-1} at $(\bar{y},0)$. Concerning assumption (b), it was shown in [5, Prop.3.4] that if the set $(Dg(\bar{x})\mathcal{B}_X-(C-g(\bar{x})))\cap Y_1$ has nonempty interior in Y_1, then the following statements are equivalent:

(i) There is no sequence $(y_k^*)\subset\mathcal{S}_{Y^*}$ with $\lim_k Dg(\bar{x})^*y_k^*=0$, $\lim_k\langle y_k^*, g(\bar{x})\rangle-\sigma_C(y_k^*)=0$, $\liminf_k \|y^*_{k|_{Y_1^*}}\|>0$,

(ii) There is no nonzero $y^*\in Y^*$ satisfying $Dg(\bar{x})^*y^*=0$, $\langle y^*, g(\bar{x})\rangle-\sigma_C(\tilde{y}^*)=0$ and $\|y^*_{|_{Y_1}}\|>0$.

Remark 2.2. In case that $Y_1=Y$ the following statements are equivalent: (cf.[4])

1. There is no sequence $(y_k^*)\subset\mathcal{S}_{Y^*}$ with $\lim_k Dg(\bar{x})^*y_k^*=0$, $\lim_k\langle y_k^*, g(\bar{x})\rangle-\sigma_C(y_k^*)=0$.
2. $g(\cdot)-C$ is metrically regular near $(\bar{x},0)$

Hence if $Y_1=Y$ the assumptions of Theorem 2.1 imply either upper Lipschitz continuity of G^{-1} or metric regularity of G. But these properties imply metric subregularity without the assumption that Y admits a Fréchet smooth renorm and hence Theorem 2.1 remains valid without the existence of a Fréchet smooth renorm. In fact, we conjecture that this is also true in case that $Y_1\neq Y$.

In case that $Y_1=Y$ the sufficient conditions for metric subregularity of Theorem 2.1 are in some sense the best possible sufficient conditions which one can establish using solely first order theory at $\bar{x}$. In fact, it follows from [5, Thm. 2.2,Prop.3.3] that if the assumptions (a) and (b) of Theorem 2.1 are not fulfilled, then there exists a continuously differentiable mapping $h: X\to Y$ with $h(\bar{x})=0$, $Dh(\bar{x})=0$ such that G is not metrically subregular at $(\bar{x},0)$.

3. Second order conditions for smooth constraint systems. In this section we consider second order conditions, both sufficient and necessary for the smooth constraint system (1.2). We denote by

$$\mathcal{C} := \{u\in X \mid Dg(\bar{x})u\in T_C(g(\bar{x}))\}$$

the *cone of critical directions.*

For the remainder of this paper we make the following assumption:

ASSUMPTION 2. *There are positive constants η and R such that*

$$\|g(x)-g(x')-Dg(\bar{x})(x-x')\|\le\eta\max\{\|x-\bar{x}\|,\|x'-\bar{x}\|\}\|x-x'\|,\ \forall x,x'\in\bar{x}+R\mathcal{B}_X.$$

Assumption 2 is for instance fulfilled for $C^{1,1}$-mappings g.

3.1. A general sufficient condition for metric subregularity. The following proposition is the basic tool for verifying metric subregularity.

PROPOSITION 3.1. *Assume that Assumptions 1, 2 are fulfilled and that the contingent derivative* $\overline{CG}$ *is metrically subregular at* $(0,0)$. *If there are positive reals* M, ρ *and a function* $\epsilon : [0,R) \to \mathbb{R}_+$, $\lim_{t\to 0}\epsilon(t) = 0$ *such that*

$$\sup_{y^*\in\mathcal{S}_{Y^*}} \{\langle y^*, g(\bar{x}+tu)\rangle - \sigma_C(y^*) - t^2(M\|Dg(\bar{x})^*y^*\| - \rho\|y^*_{|Y_1}\| - \epsilon(t))\} \leq 0 \tag{3.1}$$

holds for every $(u,t) \in (\mathcal{C}\cap\mathcal{S}_X)\times[0,R)$, *then* G *is metrically subregular at* $(\bar{x},0)$.

Proof. Decreasing R if necessary, since G_2 is subregular at $(\bar{x},0)$ there is a positive constant κ_2 such that for all $x \in \bar{x}+R\mathcal{B}_X$ we have $\mathrm{d}(x, G_2^{-1}(0))) \leq \kappa_2\mathrm{d}(0, G_2(x))$. The constant for subregularity of $\overline{CG}$ is denoted by κ'. From Assumption 2 it follows immediately that g is Lipschitz on $\bar{x}+R\mathcal{B}_X$ with modulus $L := \|Dg(\bar{x})\| + \eta R$. Next we set

$$L_2 := 1+\kappa_2\|p_2\|L, \quad c_1 := (\frac{2M\|p_1\|}{\rho} + \kappa'(1+8\frac{\eta\|p_1\|}{\rho}))L_2 + \kappa_2\|p_2\|$$

and choose the positive constants r and c_2 small enough such that

$$r \leq \frac{R}{3}, \quad c_1c_2 \leq \frac{1}{2}, \quad \kappa'(2L_2c_2+2\eta r) \leq \frac{1}{2}, \quad 6\eta r c_1 + \frac{4\|p_1\|}{\rho}\epsilon(t)L_2 \leq \frac{1}{2}, \forall t\in[0,3r].$$

In order to prove the proposition we show that for all $x_0 \in \bar{x}+r\mathcal{B}_X$ we have $\mathrm{d}(x_0, G^{-1}(0)) \leq \frac{1}{c_2}\mathrm{d}(0, G(x_0))$. Let $x_0 \in \bar{x}+r\mathcal{B}_X$ be arbitrarily fixed. If $d_0 := \mathrm{d}(0, G(x_0))) \geq c_2\|x_0-\bar{x}\|$ then obviously the assertion follows from $\bar{x}\in G^{-1}(0)$. Hence we assume $d_0 < c_2\|x_0-\bar{x}\|$. Starting with $k=0$ we now construct iteratively a sequence $(x_k) \subset \bar{x}+2r\mathcal{B}_X$ such that $d_k := \mathrm{d}(0, G(x_k)) < c_2\|x_k-\bar{x}\|$, $d_k \leq 2^{-k}d_0$, $\|x_k-x_0\| \leq 2(1-2^{-k})c_1d_0 \leq (1-2^{-k})\|x_0-\bar{x}\|$ and $\|x_{k+1}-x_k\| \leq c_12^{-k}d_0$ for all k.

Assume that x_k has already been computed for some $k\geq 0$. Since $\mathrm{d}(0, G_2(x_k)) \leq \|p_2\|d_k$ and G_2 is subregular at $\bar{x}$ we can find some $\tilde{x}_k \in G_2^{-1}(0)$ with $\|\tilde{x}_k-x_k\| \leq \kappa_2\|p_2\|d_k \leq c_1d_k \leq c_12^{-k}d_0$. Hence we obtain $\|\tilde{x}_k-x_0\| \leq \|\tilde{x}_k-x_k\|+\|x_k-x_0\| \leq c_1(2^{-k}+2-2^{-(k-1)})d_0 = 2c_1(1-2^{-(k+1)})d_0 \leq 2c_1c_2(1-2^{-(k+1)})\|x_0-\bar{x}\| \leq (1-2^{-(k+1)})\|x_0-\bar{x}\|$ and consequently $\|\tilde{x}_k-\bar{x}\| < 2\|x_0-\bar{x}\| \leq 2r$. On the other hand we have $\|\tilde{x}_k-\bar{x}\| \geq \|x_k-\bar{x}\| - c_1d_k \geq (1-c_1c_2)\|x_k-\bar{x}\| \geq \frac{1}{2}\|x_k-\bar{x}\|$.

If $\tilde{d}_k := \mathrm{d}(0, G(\tilde{x}_k)) = 0$ then we can simply set $x_{k+1} = \tilde{x}_k$. Hence let $\tilde{d}_k > 0$. By Lipschitz continuity of g we have

$$\begin{aligned}\tilde{d}_k &= \mathrm{d}(g(\tilde{x}_k), C) \leq \mathrm{d}(g(x_k), C) + \|g(\tilde{x}_k)-g(x_k)\| \leq d_k + L\|\tilde{x}_k-x_k\| \\ &\leq (1+\kappa_2\|p_2\|L)d_k = L_2d_k \leq L_2c_2\|x_k-\bar{x}\| \leq 2L_2c_2\|\tilde{x}_k-\bar{x}\|.\end{aligned}$$

For arbitrary $\epsilon > 0$ let $s_k \in G(\tilde{x}_k)$ denote some element with $\tilde{d}_k \leq \|s_k\| \leq \tilde{d}_k+\epsilon$. Since $p_1(s_k) \in G_1(\tilde{x}_k)$ we conclude $\|p_1\|\|s_k\| \geq \|p_1(s_k)\| \geq \mathrm{d}(0, G_1(\tilde{x}_k))$. Since ϵ was arbitrary chosen we obtain $\mathrm{d}(0, G_1(\tilde{x}_k)) \leq \|p_1\|\tilde{d}_k$. From $C = p_1(C)+p_2(C)$ we immediately obtain $G = G_1+G_2$ and therefore for arbitrary $v \in -G_1(\tilde{x}_k)$ we have, since $0\in G_2(\tilde{x}_k)$,

$$-v \in G_1(\tilde{x}_k) + G_2(\tilde{x}_k) = G(\tilde{x}_k).$$

Hence we can find some $v_k \in 2\mathrm{d}(0, G_1(\tilde{x}_k)))\mathcal{B}_{Y_1}$ such that $g(\tilde{x}_k)+v_k \in C$.

Using Assumption 2 with $x = \tilde{x}_k$, $x' = \bar{x}$ we obtain $\|g(\tilde{x}_k) - (g(\bar{x}) + Dg(\bar{x})(\tilde{x}_k-\bar{x}))\| \leq \eta\|\tilde{x}_k-\bar{x}\|^2$ and hence

$$\begin{aligned}\mathrm{d}(Dg(\bar{x})(\tilde{x}_k-\bar{x}), T_C(g(\bar{x}))) &\leq \mathrm{d}(Dg(\bar{x})(\tilde{x}_k-\bar{x}), C-(g(\bar{x}))) \\ &= \mathrm{d}(0, g(\bar{x})+Dg(\bar{x})(\tilde{x}_k-\bar{x})-C) \\ &\leq \tilde{d}_k + \eta\|\tilde{x}_k-\bar{x}\|^2.\end{aligned}$$

Since $\overline{CG}$ is subregular at $(0,0)$, we can find some $x'_k \in \bar{x}+\mathcal{C}$ with

$$\|x'_k-\tilde{x}_k\| \leq \kappa'(\tilde{d}_k+\eta\|\tilde{x}_k-\bar{x}\|^2) \leq \kappa'(2L_2c_2+2\eta r)\|\tilde{x}_k-\bar{x}\| \leq \frac{1}{2}\|\tilde{x}_k-\bar{x}\|.$$

From this inequality we also obtain the bounds $\|\tilde{x}_k - \bar{x}\|/\|x'_k - \bar{x}\| \le 2$ and $\|x'_k - \bar{x}\| \le 3r$. Since $v_k \in Y_1$ we have $\langle y^*, \frac{v_k}{\|v_k\|}\rangle \le \|y^*_{|Y_1}\|$ for $y^* \in Y^*$ and therefore, by using (3.1) with $t = \|x'_k - \bar{x}\|$, $u = (x'_k - \bar{x})/\|x'_k - \bar{x}\|$,

$$\sup_{y^*\in\mathcal{S}_{Y^*}} \{\langle y^*, g(x'_k) - \rho\|x'_k - \bar{x}\|^2 \frac{v_k}{\|v_k\|}\rangle - \sigma_C(y^*) - \|x'_k - \bar{x}\|^2 (M\|Dg(\bar{x})^* y^*\| + \epsilon(\|x'_k - \bar{x}\|)\|y^*\|)\}$$
$$\le \sup_{y^*\in\mathcal{S}_{Y^*}} \{\langle y^*, g(x'_k)\rangle - \sigma_C(y^*) - \|x'_k - \bar{x}\|^2 (M\|Dg(\bar{x})^* y^*\| - \rho\|y^*_{|Y_1}\| + \epsilon(\|x'_k - \bar{x}\|))\} \le 0$$

Using [4, Proposition 2.6] we conclude that

$$0 \in \mathrm{cl}\left(g(x'_k) - \rho\|x'_k - \bar{x}\|^2 \frac{v_k}{\|v_k\|} - C - \|x'_k - \bar{x}\|^2 (M Dg(\bar{x})\mathcal{B}_X + \epsilon(\|x'_k - \bar{x}\|)B_Y)\right)$$

and hence there are some $u_k \in M\|x'_k - \bar{x}\|^2 \mathcal{B}_X$ and $w_k \in 2\|x'_k - \bar{x}\|^2 \epsilon(\|x'_k - \bar{x}\|)\mathcal{B}_Y$ such that

$$g(x'_k) - \rho\|x'_k - \bar{x}\|^2 \frac{v_k}{\|v_k\|} + Dg(\bar{x})u_k + w_k \in C.$$

Note that $\|v_k\| > 0$ due to $\tilde{d}_k > 0$. Now we define $\alpha_k = (1 + \frac{\rho}{\|v_k\|}\|x'_k - \bar{x}\|^2)^{-1}$ and set $x_{k+1} = \tilde{x}_k + \alpha_k(u_k + (x'_k - \tilde{x}_k))$. By using the estimate $\|v_k\| \le 2\mathrm{d}(0, G_1(\tilde{x}_k)) \le 2\|p_1\|\tilde{d}_k$ we obtain

$$\alpha_k\|u_k\| \le \|v_k\| \frac{M\|x'_k - \bar{x}\|^2}{\|v_k\| + \rho\|x'_k - \bar{x}\|^2} \le \frac{2M\|p_1\|}{\rho}\tilde{d}_k,$$

$$\text{(3.2)}\quad \alpha_k\|x'_k - \tilde{x}_k\| \le \frac{\|v_k\|}{\|v_k\| + \rho\|x'_k - \bar{x}\|^2}\kappa'(\tilde{d}_k + \eta\|\tilde{x}_k - \bar{x}\|^2) \le \kappa'(\tilde{d}_k + \|v_k\|\frac{\eta\|\tilde{x}_k - \bar{x}\|^2}{\rho\|x'_k - \bar{x}\|^2})$$
$$\le \kappa'(1 + 8\frac{\eta\|p_1\|}{\rho})\tilde{d}_k \le \kappa'(1 + 8\frac{\eta\|p_1\|}{\rho})L_2 d_k \le c_1 d_k,$$

$$\text{(3.3)}\quad \alpha_k\|w_k\| \le \|v_k\| \frac{2\|x'_k - \bar{x}\|^2}{\|v_k\| + \rho\|x'_k - \bar{x}\|^2}\epsilon(\|x'_k - \bar{x}\|) \le \frac{4\|p_1\|}{\rho}\epsilon(\|x'_k - \bar{x}\|)\tilde{d}_k$$
$$\le \frac{4\|p_1\|}{\rho}\epsilon(\|x'_k - \bar{x}\|)L_2 d_k,$$

$$\text{(3.4)}\quad \|x_{k+1} - \tilde{x}_k\| \le (\frac{2M\|p_1\|}{\rho} + \kappa'(1 + 8\frac{\eta\|p_1\|}{\rho}))\tilde{d}_k \le (\frac{2M\|p_1\|}{\rho} + \kappa'(1 + 8\frac{\eta\|p_1\|}{\rho}))L_2 d_k \le c_1 d_k$$

and consequently

$$\|x_{k+1} - x_k\| \le \|x_{k+1} - \tilde{x}_k\| + \|\tilde{x}_k - x_k\| \le ((\frac{2M\|p_1\|}{\rho} + \kappa'(1 + 8\frac{\eta\|p_1\|}{\rho}))L_2 + \kappa_2\|p_2\|)d_k$$
$$= c_1 d_k \le c_1 2^{-k} d_0$$

follows. We also have that $\|x_{k+1} - x_0\| \le \|x_{k+1} - x_k\| + \|x_k - x_0\| \le c_1(2^{-k} + 2 - 2^{-(k-1)})d_0 = 2c_1(1-2^{-(k+1)})d_0 \le 2c_1c_2(1-2^{-(k+1)})\|x_0-\bar{x}\| \le (1-2^{-(k+1)})\|x_0-\bar{x}\|$, $\|x_{k+1}-\bar{x}\| < 2\|x_0-\bar{x}\| \le 2r$ and $\|x_{k+1} - \bar{x}\| \ge \|x_k - \bar{x}\| - c_1 d_k \ge (1 - c_1c_2)\|x_k - \bar{x}\| \ge \frac{1}{2}\|x_k - \bar{x}\|$. Since $0 \le \alpha_k \le 1$ and C is convex, we have

$$y_k := (1 - \alpha_k)(g(\tilde{x}_k) + v_k) + \alpha_k(g(x'_k) + Dg(\bar{x})u_k - \rho\|x'_k - \bar{x}\|^2 \frac{v_k}{\|v_k\|} + w_k)$$
$$= (1 - \alpha_k)g(\tilde{x}_k) + \alpha_k(g(x'_k) + Dg(\bar{x})u_k + w_k) + (1 - \alpha_k(1 + \frac{\rho}{\|v_k\|}\|x'_k - \bar{x}\|^2)v_k$$
$$= (1 - \alpha_k)g(\tilde{x}_k) + \alpha_k(g(x'_k) + Dg(\bar{x})u_k + w_k) \in C.$$

Using (3.2), (3.3), (3.4) and the relation $\alpha_k Dg(\bar{x})u_k = Dg(\bar{x})(x_{k+1} - \tilde{x}_k) + \alpha_k Dg(\bar{x})(\tilde{x}_k - x'_k)$ we conclude that

$$d_{k+1} = \mathrm{d}(0, G(x_{k+1})) = \mathrm{d}(g(x_{k+1}), C) \le \|g(x_{k+1}) - y_k\|$$

$$
\begin{aligned}
&= \|g(x_{k+1}) - g(\tilde{x}_k) - Dg(\bar{x})(x_{k+1} - \tilde{x}_k) \\
&\qquad + \alpha_k(g(\tilde{x}_k) - g(x'_k) - Dg(\bar{x})(\tilde{x}_k - x'_k)) - \alpha_k w_k\| \\
&\leq \eta \max\{\|x_{k+1} - \bar{x}\|, \|\tilde{x}_k - \bar{x}\|, \|x'_k - \bar{x}\|\}(\|x_{k+1} - \tilde{x}_k\| + \alpha_k\|\tilde{x}_k - x'_k\|) + \alpha_k\|w_k\| \\
&\leq 6\eta r c_1 d_k + \frac{4\|p_1\|}{\rho}\epsilon(\|x'_k - \bar{x}\|)L_2 d_k \leq \frac{1}{2}d_k \leq 2^{-(k+1)}d_0
\end{aligned}
$$

and $d_{k+1} \leq \frac{1}{2}d_k < \frac{1}{2}c_2\|x_k - \bar{x}\| \leq c_2\|x_{k+1} - \bar{x}\|$. Hence, the point x_{k+1} has the required properties and the existence of the sequence (x_k) is established.

Obviously, the sequence (x_k) is a Cauchy sequence and therefore convergent to some element $\tilde{x}$. Then $\mathrm{d}(g(\tilde{x}), C) = \lim_k \mathrm{d}(g(x_k), C) = 0$ showing $\tilde{x} \in G^{-1}(0)$. Hence $\mathrm{d}(x_0, G^{-1}(0)) \leq \|x_0 - \tilde{x}\| = \lim_k \|x_0 - x_k\| \leq \lim_k 2c_1 d_0 \leq \frac{1}{c_2}d(0, G(x_0))$ and our assertion is proved. □

3.2. Second order approximations of convex sets. In order to proceed with a second order analysis we need a tool to describe the possible curvature of a set.

DEFINITION 3.2. *Let S be a closed convex subset of the Banach space Y, $\bar{s} \in S$, and consider a continuous linear mapping $A : X \to Y$ defined on the Banach space X and a direction $z \in A^{-1}(T_S(\bar{s}))$.*

1. *Let ξ be a nonnegative real. A set $\mathcal{I} \subset Y$ is called an* inner second order approximation set *for S at $\bar{s}$ with respect to A, z and ξ if*

$$
\lim_{t \downarrow 0} t^{-2}\mathrm{d}(\bar{s} + tAz + \frac{t^2}{2}w, S + t^2\xi A\mathcal{B}_X) = 0 \tag{3.5}
$$

holds for all $w \in \mathcal{I}$. A multifunction $\mathcal{A}_{S,\bar{s},A,\xi} : A^{-1}(T_S(\bar{s})) \cap \mathcal{S}_X \rightrightarrows Y$ is an inner second order approximation mapping *for S at $\bar{s}$ with respect to A and ξ if for each $z \in A^{-1}(T_S(\bar{s})) \cap \mathcal{S}_X$ the set $\mathcal{A}_{S,\bar{s},A,\eta}(z)$ is an inner second order approximation set with respect to A, z and ξ and the limit (3.5) holds uniformly for all $z \in A^{-1}(T_S(\bar{s})) \cap \mathcal{S}_X$ and all $w \in \mathcal{A}_{S,\bar{s},A,\xi}(z)$.*

2. *Let $\vec{t} := (t_k) \downarrow 0$ denote a sequence of positive numbers converging to 0. The* second order compound tangent set *to S at $(\bar{s}, z)$ with respect to A and $\vec{t}$ is the set*

$$
S''_{A,\vec{t}}(\bar{s}; z) = \{w \in V \mid \exists (z_k) \to z : \mathrm{d}(\bar{s} + t_k A z_k + \frac{t_k^2}{2}w, S) = o(t_k^2)\}
$$

Whereas the concept of second order compound tangent sets plays an important role for deriving second order optimality conditions (see e.g. [4]), our analysis for metric subregularity is mainly based on inner second order approximation mappings.

The following Lemma states a property of inner second order approximation mappings:

LEMMA 3.3. *Let X, Y be Banach spaces, let $S \subset Y$ be closed convex, $\bar{s} \in S$ and let $A : X \to Y$ be a continuous linear operator. Let $\mathcal{A}$ be an inner second order approximation mapping for S at $\bar{s}$ with respect to A and some $\xi \geq 0$. Then* $\mathrm{Range}\,\mathcal{A} \subset \mathrm{cl}\,(T_S(\bar{s}) + AX)$ *and consequently*

$$
\langle y^*, w \rangle \leq 0, \ \forall w \in \mathrm{Range}\,\mathcal{A}\ \forall y^* N_S(\bar{s}) : A^* y^* = 0. \tag{3.6}
$$

Proof. For every $z \in A^{-1}(T_S(\bar{s}))$ and every $w \in \mathcal{A}(z)$ we have

$$
\mathrm{d}(\bar{s} + tAz + \frac{t^2}{2}w, S + t^2\xi A\mathcal{B}_X) = \frac{t^2}{2}\mathrm{d}(w, 2t^{-2}(S - \bar{s}) + A(-2t^{-1}z + 2\xi\mathcal{B}_X)) = o(t^2).
$$

Since for every $t > 0$ we have $2t^{-2}(S - \bar{s}) \subset T_S(\bar{s})$ and $A(-2t^{-1}z + 2\xi\mathcal{B}_X) \subset AX$ we obtain

$$
\mathrm{d}(w, T_S(\bar{s}) + AX) \leq \mathrm{d}(w, 2t^{-2}(S - \bar{s}) + A(-2t^{-1}z + 2\xi\mathcal{B}_X)) = 2t^{-2}o(t^2)
$$

and therefore $\mathrm{d}(w, T_S(\bar{s}) + AX) = 0$, showing $w \in \mathrm{cl}\,(T_S(\bar{s}) + AX)$. □

3.3. "No gap" second order conditions for metric subregularity. In the sequel we use the following assumption which is for instance fulfilled if g is twice Fréchet differentiable at $\bar{x}$.

ASSUMPTION 3. *We assume that the second order directional derivative*

$$g''(\bar{x}; u) := \lim_{t\downarrow 0} 2t^{-2}(g(\bar{x}+tu) - g(\bar{x}) - tDg(\bar{x})u)$$

exists for all $u \in \mathcal{C}$ and convergence is uniform with respect to u in bounded subsets of $\mathcal{C}$

As an immediate consequence of Assumption 2 we have

$$\|g''(\bar{x}; u)\| \leq 2\eta, \ \forall u \in \mathcal{C} \cap \mathcal{S}_X \tag{3.7}$$

THEOREM 3.4 ("No gap" second order conditions). *Assume that Assumptions 1-3 are fulfilled.*

1. *Assume that the contingent derivative $\overline{CG}$ is metrically subregular at $(0,0)$ and that there are reals $\xi \geq 0$, $\rho > 0$ and an inner second order approximation mapping $\mathcal{A} := \mathcal{A}_{C,g(\bar{x}),Dg(\bar{x}),\xi}$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ, such that*

$$\zeta := \sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \inf_{w\in\mathcal{A}(u)} \|w\| < \infty \tag{3.8}$$

and for each sequence $(y_k^*) \subset \mathcal{S}_{Y^*}$ satisfying $\lim_k \langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$ one has

$$\limsup_{k\to\infty}\{\sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \frac{1}{2}\{\langle y_k^*, g''(\bar{x}, u)\rangle - \sigma_{\mathcal{A}(u)}(y_k^*)\} + \rho\|y^*_{k|_{Y_1}}\|\} \leq 0. \tag{3.9}$$

Then G is metrically subregular at $(\bar{x}, \bar{y})$.

2. *Conversely, if G is metrically subregular at $(\bar{x}, 0)$ and*

$$\mathrm{d}(g(\bar{x}) + tDg(\bar{x})u, C) = \mathcal{O}(t^2), \ t > 0 \tag{3.10}$$

holds uniformly for all $u \in \mathcal{C} \cap \mathcal{S}_X$, then there are reals $\xi \geq 0$, $\rho \geq 0$ and an inner second order approximation mapping $\mathcal{A} := \mathcal{A}_{C,g(\bar{x}),Dg(\bar{x}),\xi}$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ, such that (3.8) holds and such that (3.9) holds for each sequence $(y_k^*) \subset \mathcal{S}_{Y^*}$ satisfying $\lim_k\langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$. Moreover, if either X is reflexive or $Dg(\bar{x})$ has closed range and $\mathrm{lin}\,(C - g(\bar{x}))$ is finite dimensional, then $\overline{CG}$ is metrically subregular at $(0,0)$.

Proof. On order to show the sufficient conditions we show that all assumptions of Proposition 3.1 are fulfilled. In fact, there is only (3.1) to show. Assume on the contrary that (3.1) does not hold for sufficiently large M, i.e. for every $M > 0$ the function $\epsilon_M : (0, R) \to \mathbb{R}$ given by

$$\epsilon_M(t) := t^{-2} \sup_{u\in\mathcal{C}\cap\mathcal{S}_X} \sup_{y^*\in\mathcal{S}_{Y^*}} \{\langle y^*, g(\bar{x}+tu)\rangle - \sigma_C(y^*) - t^2(M\|Dg(\bar{x})^*y^*\| - \rho\|{y^*}_{|_{Y_1}}\|)\}$$

fulfills $\limsup_{t\downarrow 0} \epsilon_M(t) > 0$, or equivalently, there are sequences $(t_k) \downarrow 0$, $(u_k) \subset \mathcal{C}\cap\mathcal{S}_X$, $(y_k^*) \subset \mathcal{S}_{Y^*}$ such that

$$\liminf_{k\to\infty} t_k^{-2}(\langle y_k^*, g(\bar{x}+t_k u_k)\rangle - \sigma_C(y_k^*) - t_k^2(k\|Dg(\bar{x})^* y_k^*\| - \rho\|y^*_{k|_{Y_1}}\|)) > 0. \tag{3.11}$$

Since g is Lischitz near $\bar{x}$ this implies $\langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) \geq \mathcal{O}(t_k)$ and together with $\langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) \leq 0$ we obtain $\lim_k\langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = 0$. By the definition of a second order approximation mapping we have

$$\mathrm{d}(g(\bar{x}) + t_k Dg(\bar{x})u_k + \frac{t_k^2}{2}w, C + t_k^2\xi Dg(\bar{x})\mathcal{B}_X) = o(t_k^2)$$

uniformly for all $w \in \mathcal{A}(u_k)$ and therefore

$$\begin{aligned}\langle y_k^*, g(\bar{x}) + t_k Dg(\bar{x})u_k\rangle + \frac{t_k^2}{2}\sigma_{\mathcal{A}(u_k)}(y_k^*) &= \sup_{w\in\mathcal{A}(u_k)} \langle y_k^*, g(\bar{x}) + t_k Dg(\bar{x})u_k + \frac{t_k^2}{2}w\rangle \\ &\leq \sigma_C(y_k^*) + t_k^2\xi\|Dg(\bar{x})^* y_k^*\| + o(t_k^2).\end{aligned}$$

Using this estimate and Assumption 3 we deduce from (3.11)

$$\begin{aligned}
0 < \liminf_k t_k^{-2}\Big(&\langle y_k^*, g(\bar{x}) + t_k Dg(\bar{x})u_k + \frac{t_k^2}{2} g''(\bar{x}; u_k) + o(t_k^2)\rangle \\
&-(\langle y_k^*, g(\bar{x}) + t_k Dg(\bar{x})u_k\rangle + \frac{t_k^2}{2}\sigma_{\mathcal{A}(u_k)}(y_k^*) - t_k^2\xi\|Dg(\bar{x})^* y_k^*\| + o(t_k^2)) \\
&-t_k^2(k\|Dg(\bar{x})^* y_k^*\| - \rho\|y^*_{k|_{Y_1}}\|)\Big) \\
= \liminf_k \frac{1}{2}(&\langle y_k^*, g''(\bar{x}; u_k)\rangle - \sigma_{\mathcal{A}(u_k)}(y_k^*)) - (k-\xi)\|Dg(\bar{x})^* y_k^*\| + \rho\|y^*_{k|_{Y_1}}\|
\end{aligned}$$

Using (3.8) we obtain $\sigma_{\mathcal{A}(u_k)}(y_k^*) \geq -\zeta$. Using (3.7) we conclude that $\lim_k \|Dg(\bar{x})^* y_k^*\| = 0$. Then we can proceed to obtain

$$\begin{aligned}
0 &< \liminf_k \frac{1}{2}(\langle y_k^*, g''(\bar{x}; u_k)\rangle - \sigma_{\mathcal{A}(u_k)}(y_k^*)) + \rho\|y^*_{k|_{Y_1}}\| \\
&\leq \limsup_k \sup_{u \in \mathcal{C}\cap\mathcal{S}_X} \frac{1}{2}\{\langle y_k^*, g''(\bar{x}; u)\rangle - \sigma_{\mathcal{A}(u)}(y_k^*)\} + \rho\|y^*_{k|_{Y_1}}\| \\
&\leq 0,
\end{aligned}$$

a contradiction. Hence (3.1) holds and the sufficient conditions are proved.

Now let us consider the necessary conditions. As already mentioned in the previous section, it was shown in [5, Prop.1.3, Remark 1.4]) that metric subregularity of G at $(\bar{x}, 0)$ implies metric subregularity of $\overline{CG}$ at $(0,0)$, at least if X is reflexive or $Dg(\bar{x})$ has closed range and $\mathrm{lin}\,(C - g(\bar{x}))$ is finite dimensional. Further, it follows from Lemma 3.5 below together with (3.7) that an inner second order approximation mapping $\mathcal{A}$ with the stated properties is given by $\mathcal{A}(u) := \{g''(\bar{x}; u)\}$. □

Lemma 3.5. *Let Assumptions 2 and 3 be fulfilled, assume that G is metrically subregular at $(\bar{x}, 0)$ and consider $u \in \mathcal{C} \cap S_X$. Then for each sequence $\vec{t} = (t_k) \downarrow 0$ we have $g''(\bar{x}; u) \in C''_{Dg(\bar{x}),\vec{t}}(g(\bar{x}), u)$. If*

$$\mathrm{d}(g(\bar{x}) + tDg(\bar{x})u, C) = \mathcal{O}(t^2),\ t > 0 \tag{3.12}$$

then $\{g''(\bar{x}; u)\}$ is an inner second order approximation set at $g(\bar{x})$ with respect to $Dg(\bar{x})$, u and some $\xi > 0$. Moreover, if (3.12) holds uniformly for all $u \in \mathcal{C} \cap S_X$ then the mapping $\mathcal{A}(u) := \{g''(\bar{x}; u)\}$ is an inner second order approximation mapping.

Proof. Since $u \in \mathcal{C}$ we have $\mathrm{d}(g(\bar{x} + t_k u), C) = o(t_k)$ and hence by subregularity there is a sequence $(u_k) \to u$ satisfying $g(\bar{x} + t_k u_k) \in C$. Since

$$\begin{aligned}
&\|g(\bar{x} + t_k u_k) - g(\bar{x}) - t_k Dg(\bar{x})u_k - \frac{t_k^2}{2} g''(\bar{x}; u)\| \\
&\leq \|g(\bar{x} + t_k u_k) - g(\bar{x} + t_k u) - t_k Dg(\bar{x})(u_k - u)\| + \|g(\bar{x} + t_k u) - g(\bar{x}) - t_k Dg(\bar{x})u - \frac{t_k^2}{2} g''(\bar{x}; u)\| \\
&\leq \eta t_k \max\{\|u_k\|, \|u\|\} t_k\|u_k - u\| + o(t_k^2) = o(t_k^2)
\end{aligned}$$

we obtain $\mathrm{d}(g(\bar{x}) + t_k Dg(\bar{x})u_k + \frac{t_k^2}{2} g''(\bar{x}; u), C) = o(t_k^2)$ showing $g''(\bar{x}; u) \in C''_{Dg(\bar{x}),\vec{t}}(g(\bar{x}), u)$. To show the second assertion just note that by subregularity we have $\|u_k - u\| \leq \xi t_k$ for some $\xi > 0$ and therefore

$$\begin{aligned}
&\mathrm{d}(g(\bar{x}) + t_k Dg(\bar{x})u + \frac{t_k^2}{2} g''(\bar{x}; u), C + t_k^2\xi Dg(\bar{x})\mathcal{B}_X) \\
&\leq \mathrm{d}(g(\bar{x}) + t_k Dg(\bar{x})u + \frac{t_k^2}{2} g''(\bar{x}; u), C - t_k Dg(\bar{x})(u_k - u)) \\
&= \mathrm{d}(g(\bar{x}) + t_k Dg(\bar{x})u_k + \frac{t_k^2}{2} g''(\bar{x}; u), C) = o(t_k^2).
\end{aligned}$$

holds. Since all limits are uniform with respect to arbitrary sequences $(t_k) \downarrow 0$ and arbitrary directions $u \in \mathcal{C} \cap \mathcal{S}_X$, $u \to \{g''(\bar{x}; u)\}$ constitutes an inner second order approximation mapping. □

Note that the existence of some inner second order approximation mapping together with (3.8) imply that (3.10) holds uniformly for all $u \in \mathcal{C} \cap \mathcal{S}_X$. Hence, the main difference between the second order necessary and sufficient conditions for metric subregularity concerns the real ρ: Just as in the case of "No Gap" second order optimality conditions of optimization the difference is the change from an inequality to a strict inequality. This observation justifies our notion of "No Gap" second–order conditions for metric subregularity.

In the following proposition we discuss a possibility to simplify condition (3.9). We introduce the following set of multipliers

$$\Lambda^* := \{y^* \in \mathcal{S}_{Y^*} \mid \langle y^*, g(\bar{x})\rangle - \sigma_C(g(\bar{x})) = \|Dg(\bar{x})^* y^*\| = 0\}.$$

Proposition 3.6. *Assume that Assumptions 1-3 are fulfilled and that the contingent derivative $\overline{CG}$ is metrically subregular at $(0,0)$. Assume that there are reals $\xi \geq 0$, $\rho > 0$ and an inner second order approximation mapping $\mathcal{A} := \mathcal{A}_{C,g(\bar{x}),Dg(\bar{x}),\xi}$ for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and ξ, such that*

$$\frac{1}{2}(\langle y^*, g''(\bar{x}, u)\rangle - \sigma_{\mathcal{A}(u)}(y^*)) + \rho \|y^*{}_{|Y_1}\| \leq 0, \forall u \in \mathcal{C} \cap \mathcal{S}_X \forall y^* \in \Lambda^*. \tag{3.13}$$

Further assume that

$$\lim_k \mathrm{d}(y_k^*, \Lambda^*) = 0 \tag{3.14}$$

holds for every sequence $(y_k^*) \subset \mathcal{S}_{Y^*}$ with $\lim_k \langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$ and there is a real $\bar{\zeta} \geq 0$ such that

$$\sigma_{\mathcal{A}(u)}(y^*) = \sigma_{\mathcal{A}(u) \cap \bar{\zeta} B_Y}(y^*), \ \forall u \in \mathcal{C} \cap S_X \forall y^* \in \Lambda^*. \tag{3.15}$$

Then G is metrically subregular at $(\bar{x}, 0)$.

Proof. We shall show that all assumptions of Theorem 3.4 are fulfilled. Condition (3.8) follows from (3.15) and hence there remains (3.9) to show. This is done by contradiction. Assume on the contrary that there are sequences $(y_k^*) \subset \mathcal{S}_{Y^*}$, $(u_k) \subset \mathcal{C} \cap \mathcal{S}_X$ such that $\lim_k \langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$ and

$$\liminf_{k\to\infty} \{\frac{1}{2}(\langle y_k^*, g''(\bar{x}; u_k)\rangle - \sigma_{\mathcal{A}(u_k)}(y_k^*)) + \rho \|y^*_{k|Y_1}\|\} =: \epsilon > 0$$

Due to (3.14) we can find a sequence $\bar{y}_k^* \subset \Lambda^*$ satisfying $\lim_k \|y_k^* - \bar{y}_k^*\| = 0$. Using (3.15) we obtain $\sigma_{\mathcal{A}(u_k)}(y_k^*) \geq \sigma_{\mathcal{A}(u_k)}(\bar{y}_k^*) - \bar{\zeta}\|y_k^* - \bar{y}_k^*\|$ and by (3.7) we have $\|g''(\bar{x}; u_k)\| \leq 2\eta$. Therefore we obtain from (3.13)

$$\begin{aligned} 0 &\geq \limsup_{k\to\infty} \{\frac{1}{2}(\langle \bar{y}_k^*, g''(\bar{x}; u_k)\rangle - \sigma_{\mathcal{A}(u_k)}(\bar{y}_k^*)) + \rho \|\bar{y}^*_{k|Y_1}\|\} \\ &\geq \liminf_{k\to\infty} \{\frac{1}{2}(\langle y_k^*, g''(\bar{x}; u_k)\rangle - \sigma_{\mathcal{A}(u_k)}(y_k^*)) + \rho \|y^*_{k|Y_1}\| - \|y_k^* - \bar{y}_k^*\|(\eta + \frac{1}{2}\bar{\zeta} + \rho)\} = \epsilon > 0, \end{aligned}$$

a contradiction. □

By the following proposition we identify two important cases where the crucial condition (3.14) is fulfilled.

Proposition 3.7. *Assume that either*

1. *Y is finite dimensional, or*
2. *$Dg(\bar{x})$ has closed range and $N_C(g(\bar{x}))$ is a polyhedral cone of the form*

$$N_C(g(\bar{x})) = \{y^* \in Y^* \mid \langle y^*, y_i\rangle \leq 0, \ i = 1, \ldots, p\}.$$

Then for every sequence $(y_k^*) \subset \mathcal{S}_{Y^*}$ *with* $\lim_k \langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$ *we have*

$$\lim_k \mathrm{d}(y_k^*, \Lambda^*) = 0. \tag{3.16}$$

Proof. Consider a sequence $(y_k^*) \subset \mathcal{S}_{Y^*}$ with $\lim_k \langle y_k^*, g(\bar{x})\rangle - \sigma_C(y_k^*) = \lim_k \|Dg(\bar{x})^* y_k^*\| = 0$. In case that Y is finite dimensional the proof is done by contradiction. Assume on the contrary that $\limsup_k \mathrm{d}(y_k^*, \Lambda^*) =: \delta > 0$. By passing to a subsequence if necessary we can assume that $\lim_k \mathrm{d}(y_k^*, \Lambda^*) = \delta$ and y_k^* converges to some $\bar{y}^* \in \mathcal{S}_{Y^*}$. Then $Dg(\bar{x})^* \bar{y}^* = 0$ and, since $\sigma_C(\cdot)$ is weak-$*$ lower semi-continuous, $\sigma_C(\bar{y}^*) \le \limsup_k \sigma_C(y_k^*) = \limsup_k \langle y_k^*, g(\bar{x})\rangle = \langle \bar{y}^*, g(\bar{x})\rangle$. Since we also have $\sigma_C(\bar{y}^*) \ge \langle \bar{y}^*, g(\bar{x})\rangle$ due to $g(\bar{x}) \in C$, the property $\langle \bar{y}^*, g(\bar{x})\rangle = \sigma_C(\bar{y}_*)$ follows, showing $\bar{y}^* \in N_C(g(\bar{x})$ and consequently $\bar{y}^* \in \Lambda^*$. Hence $0 < \lim_k \mathrm{d}(y_k^*, \Lambda^*) \le \lim_k \|y_k^* - \bar{y}^*\| = 0$, a contradiction.

In the second case we conclude from the Closed Range Theorem that $Dg(\bar{x})^*$ has closed range and hence we can apply Hoffman's lemma [1, Thm. 2.200] to obtain a sequence $(\bar{y}_k^*) \subset \{y^* \in N_C(g(\bar{x})) \,|\, Dg(\bar{x})^* \bar{y}^* = 0\}$ and a constant γ such that

$$\|y_k^* - \bar{y}_k^*\| \le \gamma(\|Dg(\bar{x})^* y_k^*\| + \sum_{i=1}^p \max\{\langle y_k^*, y_i\rangle, 0\}).$$

Now consider the function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\zeta(t) := \max_{i=1,\ldots,p} \mathrm{d}(g(\bar{x}) + ty_i, C)$ Since $T_C(g(\bar{x})) = \{\sum_{i=1}^p \lambda_i y_i \,|\, \lambda_i \ge 0, i = 1, \ldots, p\}$ we have $y_i \in T_C(g(\bar{x}))$ and therefore $\zeta(t) = o(t)$. Thus there exist an increasing function $\tau : (0, \infty) \to (0, 1]$ such that for every $\delta > 0$ and every $t \in [0, \tau(\delta)]$ we have $\zeta(t) \le \frac{\delta}{2} t$. Defining $\tau(0)) = 0$ we obtain for each $i = 1, \ldots, p$ and every $y^* \in \mathcal{S}_{Y^*}$ the bound

$$\sigma_C(y^*) \ge \sup\{\langle y^*, g(\bar{x}) + ty_i\rangle - \zeta(t) \,|\, t \ge 0\} \ge \langle y^*, g(\bar{x})\rangle + \frac{\max\{\langle y^*, y_i\rangle, 0\}}{2} \tau(\max\{\langle y^*, y_i\rangle, 0\}).$$

Since $\lim_k \langle y_k^*), g(\bar{x})\rangle - \sigma_C(y_k^*) = 0$ we conclude that $\lim_k \max\{\langle y_k^*, y_i\rangle, 0\} = 0$ for each i and hence $\lim_k \|y_k^* - \bar{y}_k^*\| = 0$. Further $\lim_k \|\bar{y}_k^*\| = 1$ and, because of $\frac{\bar{y}_k^*}{\|\bar{y}_k^*\|} \in \Lambda^*$ we have $\lim_k \mathrm{d}(y_k^*, \Lambda^*) \le \lim_k \|y_k^* - \frac{\bar{y}_k^*}{\|\bar{y}_k^*\|}\| = 0$. This completes the proof. □

4. Application to constraint systems given by equality and inequality constraints. In this section we consider the following setting. We suppose that we are given Banach spaces X, $\hat{Y}_1$, $\hat{Y}_2$, natural numbers m_1, m_2, mappings $P : X \to \hat{Y}_1$ and $h_i : X \to \mathbb{R}$, $i = 1, \ldots, m_1$, a continuous linear operator $A \in \mathcal{L}(X, \hat{Y}_2)$, continuous linear functionals $x_i^* \in X^*$, $i = 1, \ldots, m_2$ and elements $b \in \hat{Y}_2$, $\zeta_i \in \mathbb{R}$, $i = 1, \ldots, m_2$. The constraint system under consideration is given by the following equalities and inequalities:

$$P(x) = 0, \tag{4.1}$$
$$h_i(x) \le 0,\ i = 1, \ldots, m_1 \tag{4.2}$$
$$Ax = b \tag{4.3}$$
$$\langle x_i^*, x\rangle \le \zeta_i,\ i = 1, \ldots, m_2 \tag{4.4}$$

Of course, (4.1)–(4.4) constitutes a constraint system of the form (1.2) with $Y = \hat{Y}_1 \times \mathbb{R}^{m_1} \times \hat{Y}_2 \times \mathbb{R}^{m_2}$, $g(x) = (P(x), h_1(x) \ldots, h_{m_2}(x), Ax - b, \langle x_1^*, x\rangle - \zeta_1, \ldots, \langle x_{m_2}^*, x\rangle - \zeta_{m_2})$ and $C = \{0_{\hat{Y}_1}\} \times R_-^{m_1} \times \{0_{\hat{Y}_2}\} \times R_-^{m_2}$. We denote by Ω the set of points fulfilling (4.1)–(4.4). Given a point $\bar{x} \in \Omega$, the multifunction G associated with (4.1)–(4.4) is metrically subregular at $(\bar{x}, 0)$ if there is some $\kappa > 0$ such that

$$\mathrm{d}(x, \Omega) \le \kappa(\|P(x)\| + \sum_{i=1}^{m_1} \max\{h_i(x), 0\} + \|Ax - b\| + \sum_{i=1}^{m_2} \max\{\langle x_i^*, x\rangle - \zeta_i, 0\})$$

holds for all x in a neighborhood U of $\bar{x}$.

To ease the notation we assume that all inequalities are active at the point $\bar{x} \in \Omega$ under consideration, i.e. $g(\bar{x}) = 0$. Hence $T_C(g(\bar{x}) = C$ and $N_C(g(\bar{x})) = \hat{Y}_1 \times \mathbb{R}^{m_1}_+ \times \hat{Y}_2 \times \mathbb{R}^{m_2}_+$. The derivative $Dg(\bar{x})$ is given by $Dg(\bar{x}) = (DP(\bar{x}), Dh_1(\bar{x}), \dots, Dh_{m_1}(\bar{x}), A, x_1^*, \dots, x_{m_2}^*)$. Therefore the cone of critical directions is

$$\mathcal{C} = \{u \in X \mid \begin{array}{l} DP(\bar{x})u = 0, \\ \langle Dh_i(\bar{x}), u\rangle \leq 0,\ i = 1, \dots, m_1, \\ Au = 0, \\ \langle x_i^*, u\rangle \leq 0,\ i = 1, \dots, m_2 \end{array} \}$$

and the set Λ^* of multipliers is given by

$$\Lambda^* = \{(y_1^*, \lambda^*, y_2^*, \mu^*) \in N_C(g(\bar{x})) \mid \begin{array}{l} \|y_1^*\| + \|\lambda^*\| + \|y_2^*\| + \|\mu^*\| = 1, \\ DP(\bar{x})^* y_1^* + \sum_{i=1}^{m_1} \lambda_i^* Dh_i(\bar{x}) + A^* y_2^* + \sum_{i=1}^{m_2} \mu_i^* x_i^* = 0 \end{array} \}.$$

THEOREM 4.1. *Assume that Assumptions 2, 3 are fulfilled for the mapping g associated with (4.1)–(4.4) and assume that both A and $(DP(\bar{x}), A)$ have closed range.*

1. *(First order sufficient conditions) Assume that either $\mathcal{C} \cap \mathcal{S}_X = \emptyset$ or*

$$\|y_1^*\| + \|\lambda^*\| = 0,\ \forall (y_1^*, \lambda^*, y_2^*, \mu^*) \in \Lambda^*.$$

Then the multifunction G associated with (4.1)–(4.4) is metrically subregular at $(\bar{x}, 0)$.

2. (a) *(Second order sufficient conditions) If there is some $\rho > 0$ such that for all $u \in \mathcal{C} \cap \mathcal{S}_X$ and all $(y_1^*, \lambda^*, y_2^*, \mu^*) \in \Lambda^*$ we have*

$$\frac{1}{2}(\langle y_1^*, P''(\bar{x}; u)\rangle + \sum_{i=1}^{m_1} \lambda_i^* h_i''(\bar{x}; u)) \leq -\rho(\|y_1^*\| + \|\lambda^*\|) \tag{4.5}$$

then the multifunction G associated with (4.1)–(4.4) is metrically subregular at $(\bar{x}, 0)$.

(b) *(Second order necessary conditions) If the multifunction G associated with (4.1)–(4.4) is metrically subregular at $(\bar{x}, 0)$, then for all $u \in \mathcal{C} \cap \mathcal{S}_X$ and all $(y_1^*, \lambda^*, y_2^*, \mu^*) \in \Lambda^*$ the following inequality holds:*

$$\langle y_1^*, P''(\bar{x}; u)\rangle + \sum_{i=1}^{m_1} \lambda_i^* h_i''(\bar{x}; u) \leq 0.$$

Proof. Since A has closed range we can apply Hoffman's lemma (cf. [1, Thm.2.200]) to fulfill the Assumption 1 with $Y_1 = \hat{Y}_1 \times \mathbb{R}^{m_1} \times \{0_{\hat{Y}_2}\} \times \{0_{\mathbb{R}^{m_2}}\}$ and $Y_2 = \{0_{\hat{Y}_1}\} \times \{0_{\mathbb{R}^{m_1}}\} \times \hat{Y}_2 \times \mathbb{R}^{m_2}$. Note that the first order sufficient conditions are implied by the second order sufficient conditions. Since for all $u \in \mathcal{C}$ and all $t \geq 0$ we have $g(\bar{x}) + tDg(\bar{x})u = tDg(\bar{x})u \in T_C(g(\bar{x})) = C$, the multifunction $\mathcal{A}(u) := \{0\}$, $\forall u \in \mathcal{C} \cap \mathcal{S}_X$ is an inner second order approximation mapping for C at $g(\bar{x})$ with respect to $Dg(\bar{x})$ and 0. We will now apply Proposition 3.6 for this inner second order approximation mapping to show the second order sufficient conditions. Since $(DP(\bar{x}), A)$ has closed range, the contingent derivative $\overline{CG}$ is metrically subregular at $(0, 0)$, again as a consequence of Hoffman's lemma. Taking into account that $\sigma_{\mathcal{A}(u)}(y^*) = \langle y^*, 0\rangle = 0$ we see that condition (3.13) is fulfilled due to condition (4.5)and that (3.15) is also fulfilled with $\bar{\zeta} = 0$. By the Closed Range Theorem we have that $(DP(\bar{x}), A)^*$ has closed range. Now the range of $Dg(\bar{x})^*$ is the sum of the range of $(DP(\bar{x}), A)^*$ and a finite dimensional space and therefore also closed. Again by using the Closed Range Theorem we conclude that $Dg(\bar{x})$ has closed range. Since $N_C(\bar{x})$ is a polyhedral cone we can use Proposition 3.7 to verify (3.14). Hence all assumptions of Proposition 3.6 are fulfilled and metric subregularity follows. To complete the proof of the theorem just note that the second order necessary conditions are a simple consequence of Lemma 3.5 and (3.6). ☐

Remark 4.2.

1. By invoking the results of §2 we see that the first order sufficient conditions of Theorem 4.1 remain valid without the Assumptions 2,3, when we either assume that Y has a Fréchet–smooth renorm or $\hat{Y}_2 = \{0\}$, $m_2 = 0$.

2. In case that not all inequalities are active at $\bar x$ Theorem 4.1 remains valid with

$$\mathcal{C} = \{u \in X \mid \begin{array}{l} DP(\bar x)u = 0, \\ \langle Dh_i(\bar x), u\rangle \le 0,\ i \in I_1(\bar x) \\ Au = 0, \\ \langle x_i^*, u\rangle \le 0,\ i \in I_2(\bar x) \end{array} \},$$

$$N_C(g(\bar x)) = \{(y_1^*, \lambda^*, y_2^*, \mu^*) \in \hat Y_1 \times \mathbb{R}^{m_1}_+ \times \hat Y_2 \times \mathbb{R}^{m_2}_+ \mid \begin{array}{l} \lambda_i^* = 0,\ i \in \{1,\dots,m_1\} \setminus I_1(\bar x) \\ \mu_i^* = 0,\ i \in \{1,\dots,m_2\} \setminus I_2(\bar x) \end{array} \},$$

where $I_1(\bar x) := \{i \in \{1,\dots,m_1\} \mid h_i(\bar x) = 0\}$ and $I_2(\bar x) := \{i \in \{1,\dots,m_2\} \mid \langle x_i^* \bar x\rangle = \zeta_i\}$ denote the index sets of active inequality constraints. Note that the change of $N_C(g(\bar x))$ causes also a change of the set Λ^*.

To illustrate our results we consider some examples:

Example 4.3. Consider the system

$$\begin{aligned} h_1(x) &:= -x_1 + x_1^2 - x_2^2 \le 0, \\ \langle x_1^*, x\rangle &:= -x_1 \le 0, \\ \langle x_2^*, x\rangle &:= x_1 \le 0 \end{aligned}$$

at $\bar x = (0,0)$. *Straightforward calculations show that* $\mathcal{C} = \mathbb{R}(0,1)$,

$$\Lambda^* = \{(\lambda_1^*, \mu_1^*, \mu_2^*) \in \mathbb{R}^3_+ \cap \mathcal{S}_{R^3} \mid -\lambda_1^* - \mu_1^* + \mu_2^* = 0\}$$

and hence the first order sufficient conditions are not fulfilled. However, for every $u \in \mathcal{C} \cap \mathcal{S}_{R^2}$ *and every* $(\lambda_1^*, \mu_1^*, \mu_2^*) \in \Lambda^*$ *we have*

$$\frac{1}{2}\lambda_1^* h_1''(\bar x; u) = -\lambda_1^* \le -|\lambda_1^*|,$$

showing that the multifunction associated with the system is metrically subregular at $(0,0)$ *owing to the second order sufficient conditions of Theorem 4.1.*

Example 4.4. Consider the system

$$\begin{aligned} h_1(x) &:= -x_1 + x_1^2 + x_2^2 \le 0, \\ \langle x_1^*, x\rangle &:= -x_1 \le 0, \\ \langle x_2^*, x\rangle &:= x_1 \le 0 \end{aligned}$$

at $\bar x = (0,0)$. $\mathcal{C}$ *and* Λ^* *are exactly the same as in the previous example, but for* $u = (0,1) \in \mathcal{C} \cap \mathcal{S}_{R^2}$, $(\lambda_1^*, \mu_1^*, \mu_2^*) = (1,0,1)/\sqrt{2} \in \Lambda^*$ *we have*

$$\lambda_1^* h_1''(\bar x; u) = \frac{2}{\sqrt{2}} > 0,$$

verifying that the multifunction associated with the system are not metrically subregular at $(0,0)$, *since the second order necessary conditions of Theorem 4.1 are violated.*

The last example demonstrates that the spaces Y_1 and Y_2 need not to be orthogonal:

Example 4.5. Consider the system

$$\begin{aligned} x_1 + x_2 + |x_1|^{\frac{3}{2}} + x_3^2 &= 0, \\ x_2 + |x_1|^{\frac{3}{2}} - x_2^2 &\le 0, \\ x_1 &\le 0. \end{aligned}$$

at $\bar x = (0,0,0)$. *With this system is associated the multifunction* $G(x) = g(x) - C$, *where* $C = \{0\} \times R^2_-$ *and*

$$g(x) = (x_1 + x_2 + |x_1|^{\frac{3}{2}} + x_3^2, x_2 + |x_1|^{\frac{3}{2}} - x_2^2, x_1).$$

Note that we cannot apply our second order theory immediately, since Assumptions 2, 3 are not fulfilled. However we obtain that

$$\mathcal{C} = \{(u_1, u_2, u_3)\,|\, \begin{array}{rcl} u_1 + u_2 & = & 0, \\ u_2 & \leq & 0, \\ u_1 & \leq & 0 \end{array} \} = \mathbb{R}(0,0,1),$$

$$\Lambda^* = \{(y_1^*, y_2^*, y_3^*) \in \mathcal{S}_{\mathbb{R}^3} \,|\, y_1^* + y_2^* = 0, y_1^* + y_3^* = 0, y_2^*, y_3^* \geq 0\} = (-1,1,1)/\sqrt{3}$$

By taking the decomposition $Y = Y_1 \oplus Y_2$, where

$$Y_1 = \mathbb{R}(1,1,0),\ Y_2 = \{0\} \times \mathbb{R}^2,$$

we have $y^*_{|_{Y_1}} = 0$, $\forall y^* \in \Lambda^*$. The projections p_1, p_2 are given by

$$p_1(y_1, y_2, y_3) = (y_1, y_1, 0),\ p_2(y_1, y_2, y_3) = (0, y_2 - y_1, y_3),$$

yielding $C_1 = p_1(C) = (0,0,0)$, $C_2 = p_2(C) = \{0\} \times R^2_-$ and $G_2(x) = (0, -x_1 - x_2^2 - x_3^2, x_1) - \{0\} \times \mathbb{R}^2_-$. Similar arguments as in Example 4.3 show that G_2 is metrically subregular at $(0,0)$. Hence Assumption 1 is fulfilled and since $\{(t,t,0)\,|\, t \in [-1,1]\} \subset (Dg(\bar{x})\mathcal{B}_{R^3} - C) \cap Y_1$ we can conclude from Theorem 2.1 and the discussion following the theorem that G is metrically subregular at $(0,0)$.

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