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Numerical solution of nonlinear stationary magnetic field problems

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Abstract

This thesis deals with the solution of two-dimensional nonlinear stationary magnetic field problems. Starting point for this work is a project with the ACCM (Austrian Center of Competence in Mechatronics). The group in the ACCM uses a tool for the simulation of electric motors. Our task now is to present mathematical background of such magnetic field problems.
The starting point for deriving an appropriate mathematical model are the Maxwell equations. In the following we derive the variational formulation for our model and show the existence of a unique solution of the problem.
For a better understanding we also discuss a one-dimensional model problem. We discretize the problem with the FEM (Finite Element Method). The Newton procedure is used, to solve the nonlinear problem via a sequence of linearized problems. Finally, we present some numerical results for our one-dimensional model problem. These numerical results were obtained using a self-developed C^{++} program.

Zusammenfassung

Diese Bakkalaureatsarbeit behandelt die Lösung von zweidimensionalen nichtlinearen stationären Magnetfeldproblemen. Ausgangspunkt dafür war ein in diesem Zusammenhang stehendes Projekt mit dem ACCM (Austrian Center of Competence in Mechatronics). Dieses Team nutzt ein Tool zur Simulation von Elektromotoren. Unsere Aufgabe ist es nun, den mathematischen Hintergrund für solche Magnetfeldprobleme aufzubereiten.
Wir leiten im ersten Schritt ein mathematisches Modell aus den Maxwell'schen Gleichungen dafür her. Im Anschluss wird für das Modell die Variationsformulierung hergeleitet und es wird die Existenz einer eindeutigen Lösung des Problems gezeigt.
Zum besseren Verständnis wird weiters ein eindimensionales Modellproblem behandelt. Dieses Problem wird mit FEM (Finite Elemente Methode) diskretisiert. Mit Hilfe des Newtonverfahrens wird anschließend das nichtlineare Problem durch eine Folge von linearisierten Problemen gelöst. Zuletzt präsentieren wir noch einige numerische Resultate von unserem eindimensionalen Modellproblem. Diese Resultate erhalten wir durch Anwendung eines selbst entwickelten C^{++} Programms.

Acknowledgement

First of all, we would like to express our thanks to our supervisor Prof. Ulrich Langer for giving us the chance to write this thesis, for various discussions and for his time.
Special thanks go to Dipl.-Ing. Clemens Pechstein for help in programming and debugging.

This work has been carried out at the Institute of Computational Mathematics, JKU Linz, in cooperation with ACCM (Austrian Center of Competence in Mechatronics), a K2-Center of the COMET/K2 program of the Federal Ministry of Transport, Innovation, and Technology, and the Federal Ministry of Economics and Labour, Austria.

Last but not least we acknowledge the Institute of Computational Mathematics at the Johannes Kepler University of Linz for the technical support.

Contents

Chapter 1

Introduction

This work is closely related to a cooperation with the ACCM (Austrian Center of Competence in Mechatronics) within the Project Seminar *Numerik SS 2008*. The group in the ACCM uses a program called FEMAG for the analysis and simulation of electric machines. One of their applications is the shape optimization of electric motors by simulating for a large set of parameters, such as rotor diameters etc., and then choosing the paramter configuration with the best degree of efficiency. This method of operation works, but is very expensive. The main task now was to optimize this process by different methods of optimization. One way was discussed by Markus Kollmann, namely shape optimization by shape derivatives. Another method, the derivative free optimization, was described by Mykhaylo Yudytskiy. Gradient based optimization was discussed by Elisabeth Frank. For more details we refer to the final reports of these methods, which can be found at the Institute of Computational Mathematics, Johannes Kepler University.
The major task of our group was to formulate a concrete model of the behavior of an electric motor from the magnetic point of view. Furthermore the direct field simulation of underlying partial differential equations is discussed in this thesis. Another goal of the seminar was to get the idea of the simulation done by the ACCM with FEMAG.

Starting point of our discussion was a model problem of an electric motor sketched in Figure 1.1, which was provided by the group of the ACCM. The area A7 is the permanent magnet of the motor. Areas A1 up to A6 are the coils. Between the coils and the permanent magnet there is a little air gap. Area A8 is an iron layer.
From the famous Maxwell equations, which are used to describe electromagnetic phenomena, we derive the stationary magnetic field formulation. Under certain assumptions the system of nonlinear partial differential equations can be reduced to a single nonlinear differential equation in two dimensions.

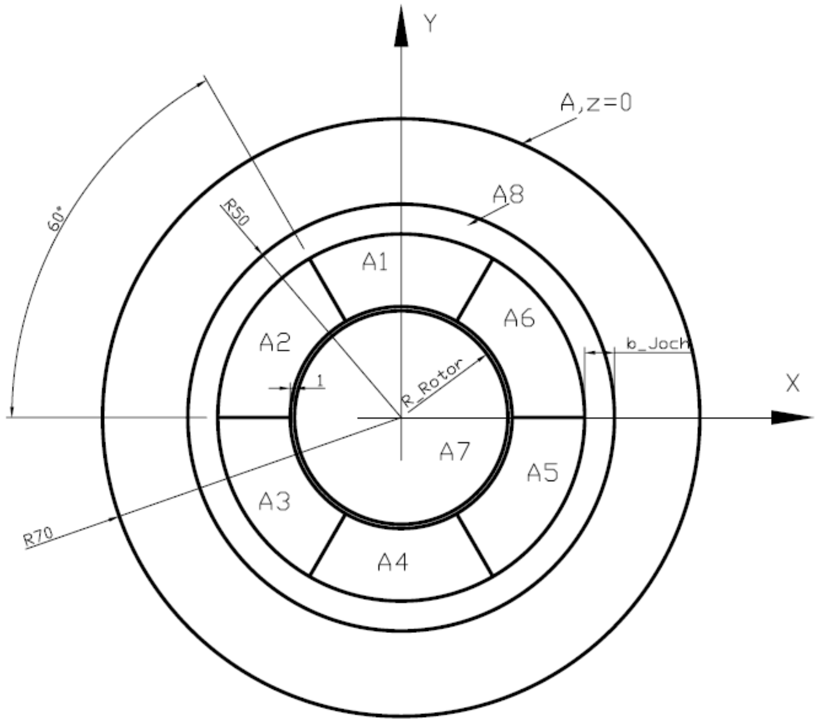

Figure 1.1: Motor

The nonlinearity occurs in a material relation between the magnetic field **H** and the magnetic induction **B**. The relation can be expressed by the so-called *B-H-curve*, or the closely related reluctivity ν. In general, none of these mappings are given analytically and must be approximated from measured data. With suitable boundary conditions, the magnetostatic 2D - problem can be formulated in a nonlinear variational formulation.
Existence and uniqueness is shown by the theorem of Zarantonello. Usually one uses Newton's method to obtain the solution of the nonlinear variational problem.
To demonstrate how such a simulation works in principle, we discuss a one-dimensional model problem and we develop a program to get the solution of this problem. The main part of this implementation was the Finite Element Method and the Newton‘s Method. Finally we take a look at the numerical results. We compare the results of our own 1D program and the results obtained by FEMAG and ParNFB, a program developed by Dipl.-Ing. Clemens Pechstein.

The organization of this thesis:

- Chapter 2 - Problem Formulation and Analysis:
  Starting from the physical model for magnetostatics, i.e. Maxwell's equations, the mathematical formulation is derived, namely the boundary value problem for a nonlinear partial differential equation (PDE). We discuss the solvability of this PDE under suitable conditions imposed on the *B-H-curve*.

- Chapter 3 - A 1D model problem and its discretization:
  For disretization we introduce a one-dimensional problem. After a short introduction on the Finite Element Method (cf. [2, 6]), FEM and Newton's method are applied to the variational formulation of the problem. At the end of this chapter we discuss the approximation of the *B-H-curve* (cf. [3, 4]).

- Chapter 4 - Computer implementation and numerical results:
  The theoretical results are used in numerical studies, tested on a 1D - model for a electric motor. Finally we compare the results of the two 2D- programs: FEMAG and ParNFB.

Since this thesis was written by a group of two people, we want to clarify which part of this work was mainly discussed by which person.

Alexander Lechner:

- The derivation of the variational formulation of the nonlinear partial differential equation. Here, a detailed reflection of different interfaces is a crucial point.
- The proof of existence and uniqueness of the solution.
- The simulation by FEMAG.

Stefan Mühlböck:

- The derivation of the mathematical model and of the nonlinear partial differential equation.
- The discussion of the conditions on the nonlinear parameter.
- The implementation of the program to solve the 1D-model problem based on FEM. Focus of discussion is the approximation of the B-H-Curve. Another important part was the interpretation of the results of this implementation.

Chapter 2

Problem Formulation and Analysis

2.1 Governing equations

2.1.1 Maxwell's equations

Maxwell's equations are used to describe electromagnetic phenomena (cf. [1, 3]):

$$\begin{aligned} curl\ \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, && (2.1) \\ curl\ \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, && (2.2) \\ div\ \mathbf{B} &= 0, && (2.3) \\ div\ \mathbf{D} &= \rho. && (2.4) \end{aligned}$$

The quantities involved are

H - magnetic field,
E - electric field,
B - magnetic flux,
D - electric induction,
J - electric current density,
ρ - electric charge density.

All these quantities depend on the position in the space $x = (x_1, x_2, x_3)$ and on the time t. The boldface letters are vector fields.
Via constitutive laws there exists a relation between the magnetic flux **B** and

the magnetic field $\mathbf{H}$:

$$\mathbf{B} = \mu_0\mu_r(\mathbf{H}+\mathbf{H}_0). \tag{2.5}$$

Here,

μ_0 denotes the permeability of the vacuum, $\mu_0 := 4\pi \cdot 10^{-7}\,\frac{Vs}{Am}$, and
μ_r denotes the relative permeability.

In the case of permanent magnetic materials, $-\mathbf{H}_0$ is called the magnetic field where the induction $\mathbf{B}$ disappears. For the quantities which are not permanent magnetic we assume that $\mathbf{H}_0 = 0$.

We neglect the effects of hysteresis, such that the relative permeability μ_r can be represented as a function of $|\mathbf{B}|$, such that

$$\mathbf{B} = \mu_0\mu_r(|\mathbf{B}|)\cdot(\mathbf{H}+\mathbf{H}_0). \tag{2.6}$$

Additionally we introduce another quantity, the so-called *relative reluctivity*

$$\nu(|\mathbf{B}|) := \frac{1}{\mu_0\mu_r(|\mathbf{B}|)}, \tag{2.7}$$

such that following relation holds:

$$\mathbf{H}+\mathbf{H}_0 = \nu(|\mathbf{B}|)\mathbf{B}. \tag{2.8}$$

Since $\mathbf{B}$ is divergence free, we can find a vector potential $\mathbf{A}$ such that

$$\mathbf{B} = \mathit{curl}\ \mathbf{A}. \tag{2.9}$$

Considering the low-frequency case of electromagnetism, displacement currents are negligible in comparison with the impressed currents, i.e.,

$$\left|\frac{\partial\mathbf{D}}{\partial t}\right| \ll |\mathbf{J}|.$$

So we are left with the following reduced set of equations, the stationary (no dependence on t) magnetostatic formulation:

$$\begin{aligned} \mathit{curl}\ \mathbf{H} &= \mathbf{J}, \\ \mathit{div}\ \mathbf{B} &= 0, \\ \mathbf{H}+\mathbf{H}_0 &= \nu(|\mathbf{B}|)\mathbf{B}. \end{aligned}$$

By writing **H** and **B** in terms of the vector potential **A** we arrive at the magnetostatic vector potential formulation,

$$curl\ (\ \nu(|\ curl\ \mathbf{A}|) \cdot curl\ \mathbf{A}\ ) \quad = \quad \mathbf{J} + curl\ \mathbf{H}_0. \tag{2.10}$$

Before we perform further simplifications to the model we first inspect the reluctivity ν.

2.1.2 Material laws

The material influence appears in form of the reluctivity ν which additionally depends on the position x. We can differentiate between three different kinds of materials: linear materials, where the reluctivity ν is a constant, permanent magnetic materials and nonlinear materials. Within one and the same material, ν is independent of the position.

Linear Materials:
The most famous linear material is vacuum. It is well known, that vacuum behaves linearly, i.e.,

$$\begin{aligned} \mu_r &\equiv 1, \\ \nu &\equiv \frac{1}{\mu_0} =: \nu_0, \end{aligned}$$

where we recall that $\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am}$ denotes the permeability of vacuum. For electromagnetic problems we can assume that air behaves just like vacuum.

Permanent magnetic Materials:
For Permanent magnetic materials we have the following material influence:

$$\mu_r \equiv 1.$$

From the previous subsection we know that for permanent magnetic materials we can assume

$$\mathbf{H}_0 \neq 0.$$

Nonlinear Materials:
In this thesis we neglect hysteresis effects. In this case there exists a bijective mapping $f : {\mathbb{R}_0}^+ \to {\mathbb{R}_0}^+$ such that

$$|\mathbf{B}| \quad = \quad f(|\mathbf{H}|). \tag{2.11}$$

The function f is called *B-H-curve* or *magnetization curve*. For large values of H the degree of amplification behaves again like in vacuum - in this case we say the material is saturated. These properties are summarized in the following assumption:

Assumption 2.1. *Any* B-H-curve

$$f : {\mathbb{R}_0}^+ \to {\mathbb{R}_0}^+$$

describing the relation $|\boldsymbol{B}| = f(|\boldsymbol{H}|)$ *fulfills the following conditions:*

1. f *is continuously differentiable,*
2. $f(0) = 0,$
3. $f'(s) \geq \mu_0, \, \forall s \geq 0,$
4. $f'(s) \overset{s\to\infty}{\longrightarrow} \mu_0.$

From (2.8), (2.11) and the assumption $\mathbf{H}_0 = 0$ for nonpermanent magnetic materials we know

$$\nu(|\mathbf{B}|) = \frac{|\mathbf{H}|}{|\mathbf{B}|} = \frac{f^{-1}(|\mathbf{B}|)}{|\mathbf{B}|}.$$

So the reluctivity ν is related to f via

$$\nu(s) \quad := \quad \frac{f^{-1}(s)}{s}. \tag{2.12}$$

We need several properties of f and ν that are summarized in the following lemma:

Lemma 2.2. *With* f *fulfilling Assumption* 2.1, *the following statements holds:*

1. *The function* $\nu(s) := \frac{f^{-1}(s)}{s}$ *is well-defined and continuous in* $[0, \infty)$ *with*

   $$\nu(0) \quad = \quad (f^{-1})'(0)$$

   and $\nu(s) \overset{s\to\infty}{\longrightarrow} \nu_0.$

2. *The function* $f^{-1}(s) = \nu(s)s$ *is Lipschitz continuous with Lipschitz constant* ν_0.

3. *The function* $f^{-1}(s) = \nu(s)s$ *is strongly monotone with monotonicity constant* $m > 0$*, i.e.*

$$(f^{-1}(s) - f^{-1}(t))(s-t) \;\; \geq \;\; m(s-t)^2, \quad \forall s, t \in \mathbb{R}_0^+,$$

where

$$m \;\; := \;\; \min_{s \geq 0} (f^{-1})'(s).$$

4. *The function* ν *is continuously differentiable on* $(0, \infty)$ *and* $\nu'(s) \stackrel{s\to\infty}{\longrightarrow} 0$.

Proof. 1. $\nu(s)$ is well-defined for $s > 0$. But for $s = 0$ we have a division by 0. In this special case we can conclude from Assumption 2.1 (2) and the definition of the derivative that

$$\lim_{s\to 0} \frac{f^{-1}(s)}{s} = \lim_{s\to 0} \frac{f^{-1}(s) - 0}{s - 0} = \lim_{s\to 0} \frac{f^{-1}(s) - f^{-1}(0)}{s - 0} = (f^{-1})'(0).$$

An application of de l'Hospital's theorem and Assumption 2.1 (4) yields

$$\lim_{s\to\infty} \nu(s) = \lim_{s\to\infty} \frac{f^{-1}(s)}{s} = \frac{\text{“}\infty\text{”}}{\infty} = \lim_{s\to\infty} (f^{-1})'(s) = \nu_0.$$

2. We show the Lipschitz continuity by using the mean value theorem:

$$\nu(s)s - \nu(t)t \;\; = \;\; f^{-1}(s) - f^{-1}(t) = (f^{-1})'(\xi)(s-t), \text{ for some } \xi \in (s,t).$$

From 1. we can conclude that

$$(f^{-1})'(\xi) \;\; \leq \;\; \nu_0.$$

Hence,

$$\nu(s)s - \nu(t)t \;\; \leq \;\; \nu_0(s-t).$$

3. Similarly to the proof of Lipschitz continuity, we gain strong monotonicity:

$$(\nu(s)s - \nu(t)t)(s-t) \;\; = \;\; (f^{-1}(s) - f^{-1}(t))(s-t) = \underbrace{(f^{-1})'(\xi)}_{\geq m}(s-t)^2$$

for some $\xi \in (s,t)$.

4. First, the derivative

$$\nu'(s) = \frac{(f^{-1})'(s)s - f^{-1}(s)}{s^2}$$

is well defined and continuous for $s > 0$. Secondly, we know from the definition of ν that

$$\nu'(s) = \frac{(f^{-1})'(s)s - f^{-1}(s)}{s^2} = \frac{(f^{-1})'(s)}{s} - \underbrace{\frac{f^{-1}(s)}{s^2}}_{=\frac{\nu(s)}{s}} = \frac{(f^{-1})'(s)}{s} - \frac{\nu(s)}{s}.$$

Since $(f^{-1})'(s)$ and $\nu(s)$ are bounded $\nu'(s) \xrightarrow{s\to\infty} 0$.

□

2.1.3 Reduction to 2D

We consider a magnetic field problem on the $x_1 - x_2-$ plane. It is requested, that the electric current density $\mathbf{J}$ is perpendicular to the magnetic field $\mathbf{H}$, which should lie on the $x_1 - x_2-$ plane and that both fields are independent of x_3, i.e.

$$\mathbf{J} = \begin{pmatrix} 0 \\ 0 \\ J_3(x_1, x_2) \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H_1(x_1, x_2) \\ H_2(x_1, x_2) \\ 0 \end{pmatrix}. \tag{2.13}$$

Additionally, we assume that

$$\mathbf{H}_0 = \begin{pmatrix} H_{01}(x_1, x_2) \\ H_{02}(x_1, x_2) \\ 0 \end{pmatrix}.$$

From this assumption we obtain

$$curl\ \mathbf{H}_0 = \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02} \end{pmatrix}. \tag{2.14}$$

From the B-H-relation (2.6), we immediately get that $\mathbf{B}$ has the form

$$\mathbf{B} = \begin{pmatrix} B_1(x_1, x_2) \\ B_2(x_1, x_2) \\ 0 \end{pmatrix}.$$

Since the third component vanishes and $\mathbf{B} = curl\ \mathbf{A}$, we find that

$$(curl\ \mathbf{A})_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = 0,$$

which leads to the following ansatz representing the vector potential

$$\mathbf{A} = \mathbf{A}(x_1, x_2) = \begin{pmatrix} 0 \\ 0 \\ A_3(x_1, x_2) \end{pmatrix}. \tag{2.15}$$

We conclude that

$$\mathbf{B} = curl\ \mathbf{A} = \begin{pmatrix} \frac{\partial A_3}{\partial x_2} \\ -\frac{\partial A_3}{\partial x_1} \\ 0 \end{pmatrix}. \tag{2.16}$$

Combining (2.10), (2.13), (2.14) and (2.15), we obtain

$$-div(\ \nu(\ |\nabla A_3|\ )\ \nabla A_3) \quad = \quad J_3 + (curl\ \mathbf{H}_0)_3. \tag{2.17}$$

From now the unknown A_3 will be identified by u:

$$u \quad := \quad A_3.$$

To summarize, our $2D$ reduction yields to the scalar partial differential equation

$$-div(\ \nu(\ |\nabla u|\ )\ \nabla u) \quad = \quad J_3 - \frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02}. \tag{2.18}$$

2.2 Variational Formulation

2.2.1 Sobolev Spaces

Before we derive the variational formulation of this problem, we define the Sobolev space $H^1(\Omega)$ (cf. [2, 6]). Therefore we introduce a generalized concept of a derivative.
Let Ω be a domain with boundary Γ. If for a function u there exists a continuous derivative $\frac{\partial u}{\partial x_i}$, we know with integration by parts that for every continuous differentiable function φ with $\varphi\Big|_\Gamma = 0$,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx \quad = \quad -\int_\Omega \frac{\partial u}{\partial x_i} \varphi dx.$$

With this formula we define a derivative of functions, which are not necessarily differentiable in the classical sense. Let u and w be functions, which are integrable such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx \;\; = \;\; -\int_\Omega w \varphi dx$$

for every differentiable function φ with $\varphi\Big|_\Gamma = 0$, then we call w the weak derivative of u.
With this new derivative it is possible to differentiate continuous, piecewise polynomial functions, which are used by the Finite Element Method.
A function u of the space $L_2(\Omega)$ is an element of the space $H^1(\Omega)$, if all its partial derivatives of order one are in $L_2(\Omega)$.

2.2.2 The Variational Formulation

First, for simplicity, we consider the classical formulation:
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with the boundary $\Gamma = \partial\Omega$. Find a function $u \in C^2(\Omega) \cap C(\Omega \cup \Gamma)$, such that the differential equation

$$-div(\nu(|\nabla u|)\nabla u) \;\; = \;\; J_3 - \frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02} \quad in \;\; \Omega \tag{2.19}$$

and the homogeneous Dirichlet boundary condition

$$u \;\; = \;\; 0 \quad on \;\; \Gamma \tag{2.20}$$

are satisfied.
We are looking for a classical solution:

$$u \;\; \in \;\; C^2(\Omega) \; \cap \; C(\Omega \cup \Gamma)$$

under classical assumptions imposed on the data

- $\nu \in C^1(\Omega)$
- $J_3 \in C(\Omega)$
- $H_{01}, H_{02} \in C^1(\Omega)$

Homogenous material

We derive now the variational formulation of our problem for the case, that the domain Ω consists only of one homogenous material. In the next section we derive the variational formulation for more than one material, because this is relevant for our motor.

Under appropriate differentiability and integrability conditions the following steps can be performed:

1. Choose the space of test functions:

$$V_0 = \{v \in V = H^1(\Omega) : v\Big|_{\Gamma} = 0\}$$

2. Multiply the differential equation with an arbitrary test function $v \in V_0$ and integrate over the computational domain Ω:

$$\int_\Omega \Big[- div(\nu(|\nabla u|)\nabla u)\Big] \; v \; dx \quad = \quad \int_\Omega \Big[J_3 - \frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02}\Big] \; v \; dx$$

3. Integration by parts in the principle part:

$$- \int_\Omega \Big[div(\nu(|\nabla u|)\nabla u)\Big] \; v \; dx \quad = \quad \int_\Omega \nu(|\nabla u|)\nabla u \cdot \nabla v dx - \int_\Gamma \nu(|\nabla u|)(\nabla u \cdot n) \; v \; ds$$

The test-functions $v \in V_0$ vanish on the boundary Γ, such that

$$\int_\Gamma \nu(|\nabla u|)(\nabla u \cdot n) \; v \; ds \quad = \quad 0.$$

4. Incorporate the natural boundary condition:
We have only essential (=Dirichlet) boundary conditions!

5. Define the linear manifold V_g of all admissible functions in which the solution u is looked for:

$$V_g \quad = \quad \{v \in V = H^1(\Omega) : v\Big|_{\Gamma} = 0\} = V_0$$

We can also apply integration by parts on the right hand side of our differential equation (2.19):

$$\int_\Omega \Big[J_3 - \frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02}\Big] \; v \; dx \quad = \quad \int_\Omega J_3 \; v \; dx - \int_\Omega \frac{\partial}{\partial x_2} H_{01} \; v \; dx + \int_\Omega \frac{\partial}{\partial x_1} H_{02} \; v \; dx$$

$$\begin{aligned}
\int_\Omega \frac{\partial}{\partial x_2} H_{01}\ v\ dx &= -\int_\Omega H_{01}\frac{\partial v}{\partial x_2} dx + \underbrace{\int_\Gamma (H_{01}\cdot n)\ v\ ds}_{=0} \\
\int_\Omega \frac{\partial}{\partial x_1} H_{02}\ v\ dx &= -\int_\Omega H_{02}\frac{\partial v}{\partial x_1} dx + \underbrace{\int_\Gamma (H_{02}\cdot n)\ v\ ds}_{=0}
\end{aligned}$$

The result is the following nonlinear variational formulation of the boundary value problem:

Find $u \in V_g$, such that

$$a(u,v) = \langle F, v\rangle \quad \forall v \in V_0, \tag{2.21}$$

where

$$\begin{aligned}
a(u,v) &= \int_\Omega \nu(|\nabla u|)\nabla u \cdot \nabla v dx, \\
\langle F, v\rangle &= \int_\Omega \Big[J_3\ v\ + H_{01}\frac{\partial v}{\partial x_2} - H_{02}\frac{\partial v}{\partial x_1}\Big] dx.
\end{aligned}$$

Here, $a(w,v)$ is only linear in the second argument v, but not necessarily in w. We equip the space V_0 with the norm $\|u\|_{V_0} := |u|_{H^1(\Omega)}$, which is indeed a norm due to Friedrichs' inequality (cf. [6]). It is relatively easy to show that $a(u,\cdot)$ is bounded for each fixed $u \in V_0$ and that F is a bounded linear form, in short $F \in V_0^*$. Then our problem can be rewritten as an operator equation in the dual space

$$A(u) \quad = \quad F \quad in\ V_0^* \quad , \tag{2.22}$$

with the nonlinear operator $A : V_0 \to V_0^*$ defined by the relation

$$\langle A(u), v\rangle \quad = \quad a(u,v), \tag{2.23}$$

so it follows

$$\langle A(u), v\rangle = \int_\Omega \nu(|\nabla u|)\nabla u \cdot \nabla v dx.$$

Heterogenous material

Without loss of generality we consider two different materials in our domain, such that we have an interface between the two materials. Like in Figure 2.1

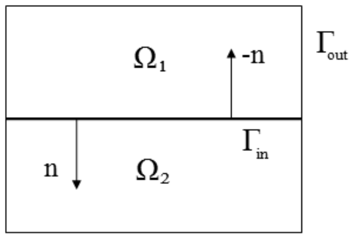

Figure 2.1: Domain Ω

we split our domain Ω into two nonoverlapping subdomains Ω_1 and Ω_2, with $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$. We set $\Gamma_{out} := \partial\Omega$ and define the interface $\Gamma_{in} := \partial\Omega_1 \cap \partial\Omega_2$. Let $n = n(x_1, x_2)$ denote the outer unit normal vector to Ω_1.
So we write our problem in the form

$$-div\Big(\nu_i(|\nabla u_i|)\nabla u_i\Big) = J_3^{(i)} - \frac{\partial}{\partial x_2}H_{01}^{(i)} + \frac{\partial}{\partial x_1}H_{02}^{(i)} \quad in\ \Omega_i,\ i \in \{1,2\}.$$

Let u_1 be the restriction of u to Ω_1, u_2 the restriction of u to Ω_2 and $H_0^{(1)}$, $H_0^{(2)}$ the restrictions of H_0 to Ω_1, Ω_2, respectively. Then the following interface conditions hold:

$$u_1 = u_2 \tag{2.24}$$

$$\nu_1(|\nabla u_1|)\nabla u_1 \cdot n + \begin{pmatrix} H_{02}^{(1)} \\ -H_{01}^{(1)} \end{pmatrix} \cdot n = \nu_2(|\nabla u_2|)\nabla u_2 \cdot n + \begin{pmatrix} H_{02}^{(2)} \\ -H_{01}^{(2)} \end{pmatrix} \cdot n \tag{2.25}$$

These can be derived from the three-dimensional (physical) interface condition

$$\mathbf{H}^{(1)} \times n = \mathbf{H}^{(2)} \times n.$$

With (2.8) we get

$$\nu_1(|\mathbf{B}^{(1)}|)\mathbf{B}^{(1)} \times n - \nu_2(|\mathbf{B}^{(2)}|)\mathbf{B}^{(2)} \times n - \mathbf{H}_0^{(1)} \times n + \mathbf{H}_0^{(2)} \times n = 0,$$
$$\nu_1(|\mathbf{B}^{(1)}|)\mathbf{B}^{(1)} \times n - \mathbf{H}_0^{(1)} \times n = \nu_2(|\mathbf{B}^{(2)}|)\mathbf{B}^{(2)} \times n - \mathbf{H}_0^{(2)} \times n.$$

With (2.14) and (2.16) the interface condition (2.25) follow.

We need to take these interface conditions into account, when deriving the variational formulation.

1. Choose the space of the test functions:

$$V_0 = \{v \in V = H^1(\Omega) : v\Big|_{\Gamma_{out}} = 0\}$$

2. Multiply the differential equation by an arbitrary test function $v \in V_0$ and integrate over the computational domain Ω:

$$\begin{aligned}\int_{\Omega_1} \Big[- div(\nu_1(|\nabla u_1|)\nabla u_1)\Big] \; v \; dx + \int_{\Omega_2} \Big[- div(\nu_2(|\nabla u_2|)\nabla u_2)\Big] \; v \; dx \;\; &= \\ = \int_{\Omega} \Big[J_3 - \frac{\partial}{\partial x_2} H_{01} + \frac{\partial}{\partial x_1} H_{02} \Big] \; v \; dx&\end{aligned}$$

3. Integration by parts in the principle part:

$$\begin{aligned}&\int_{\Omega_1} \Big[- div(\nu_1(|\nabla u_1|)\nabla u_1)\Big] \; v \; dx = \int_{\Omega_1} \nu_1(|\nabla u_1|)\nabla u_1 \cdot \nabla v dx - \\ &- \int_{\Gamma_{out}\cap\partial\Omega_1} \nu_1(|\nabla u_1|)(\nabla u_1 \cdot n) v ds - \int_{\Gamma_{in}} \nu_1(|\nabla u_1|)(\nabla u_1 \cdot n) v ds \\ &\int_{\Omega_2} \Big[- div(\nu_2(|\nabla u_2|)\nabla u_2)\Big] \; v \; dx = \int_{\Omega_2} \nu_2(|\nabla u_2|)\nabla u_2 \cdot \nabla v dx + \\ &+ \int_{\Gamma_{out}\cap\partial\Omega_2} \nu_2(|\nabla u_2|)(\nabla u_2 \cdot n) \; v \; ds + \int_{\Gamma_{in}} \nu_2(|\nabla u_2|)(\nabla u_2 \cdot n) v ds\end{aligned}$$

   By using the interface conditions (2.25) and $v\Big|_{\Gamma_{out}} = 0$ we get

$$\int_{\Omega_1} \Big[- div(\nu_1(|\nabla u_1|)\nabla u_1)\Big] v dx + \int_{\Omega_2} \Big[- div(\nu_2(|\nabla u_2|)\nabla u_2)\Big] v dx =$$

$$= \int_{\Omega} \nu(|\nabla u|)\nabla u \cdot \nabla v dx + \left(\begin{array}{c} H_{02}^{(1)} - H_{02}^{(2)} \\ -H_{01}^{(1)} + H_{01}^{(2)} \end{array} \right) \cdot n. \tag{2.26}$$

   The rightmost term will be put to the right hand side F.

4. Incorporate the natural boundary condition:
   We have only essential (=Dirichlet) boundary conditions!

5. Define the linear manifold V_g of all admissible functions in which the solution u is looked for:

$$V_g = \{v \in V = H^1(\Omega) : v\Big|_{\Gamma_{out}} = 0\} = V_0$$

Like in the previous case we can apply integration by parts also on the right hand side of our differential equation:

$$\int_{\Omega_1} \left[J_3^{(1)} - \frac{\partial}{\partial x_2} H_{01}^{(1)} + \frac{\partial}{\partial x_1} H_{02}^{(1)} \right] v \; dx =$$
$$= \int_{\Omega_1} J_3^{(1)} \; v \; dx - \int_{\Omega_1} \frac{\partial}{\partial x_2} H_{01}^{(1)} v dx + \int_{\Omega_1} \frac{\partial}{\partial x_1} H_{02}^{(1)} v dx =$$
$$= \int_{\Omega_1} J_3^{(1)} \; v \; dx + \int_{\Omega_1} H_{01}^{(1)} \frac{\partial v}{\partial x_2} dx - \int_{\Gamma_{in}} H_{01}^{(1)} n_2 v ds - \int_{\Omega_1} H_{02}^{(1)} \frac{\partial v}{\partial x_1} dx + \int_{\Gamma_{in}} H_{02}^{(1)} n_1 v ds$$

$$\int_{\Omega_2} \left[J_3^{(2)} - \frac{\partial}{\partial x_2} H_{01}^{(2)} + \frac{\partial}{\partial x_1} H_{02}^{(2)} \right] v \; dx =$$
$$= \int_{\Omega_2} J_3^{(2)} \; v \; dx - \int_{\Omega_2} \frac{\partial}{\partial x_2} H_{01}^{(2)} v dx + \int_{\Omega_2} \frac{\partial}{\partial x_1} H_{02}^{(2)} v dx =$$
$$= \int_{\Omega_2} J_3^{(2)} \; v \; dx + \int_{\Omega_2} H_{01}^{(2)} \frac{\partial v}{\partial x_2} dx + \int_{\Gamma_{in}} H_{01}^{(2)} n_2 v ds - \int_{\Omega_2} H_{02}^{(2)} \frac{\partial v}{\partial x_1} dx - \int_{\Gamma_{in}} H_{02}^{(2)} n_1 v ds$$

Incorporating the H_0-terms from (2.26) we obtain

$$\langle F, v \rangle = \int_{\Omega} \left[J_3 \; v \; + H_{01} \frac{\partial v}{\partial x_2} - H_{02} \frac{\partial v}{\partial x_1} \right] dx$$

Finally one can easily see that we have the same variational formulation for one material and for more than one material.

2.3 Existence and Uniqueness of the Solution

Existence and uniqueness of the solution to linear variational formulations is guaranteed by the Lax-Milgram theorem under suitable assumptions, in particular, ellipticity and boundedness of the bilinear form $a(\cdot, \cdot)$. In the nonlinear case, the operator equation (2.22) can be treated with a generalization of the Lax-Milgram theorem if the operator A is strongly monotone and Lipschitz continuous. Therefore, we first investigate these properties.

2.3.1 Conditions on the Reluctivity ν

Lemma 2.3. *If $\nu(\cdot)\cdot : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is strongly monotone with monotonicity constant $m > 0$, i.e.*

$$(\nu(t)t - \nu(s)s)(t - s) \;\; \geq \;\; m(t - s)^2 \; , \qquad \forall s, t \geq 0,$$

then the nonlinear operator A defined by (2.23) *is strongly monotone with monotonicity constant m, i.e.*

$$\langle A(u) - A(v), u - v\rangle \quad \geq \quad m\|u - v\|_{V_0}^2 \ , \qquad \forall u, v \in V_0.$$

Proof. First, we show that the monotonicity constant m is a lower bound for the reluctivity $\nu(\cdot)$. From the monotonicity of $\nu(\cdot)\cdot$ we get

$$\begin{aligned} \nu(|t|)|t|^2 = (\nu(|t|)|t| - \nu(0)0)(|t| - 0) \quad &\geq \quad m(|t| - 0)^2 = m|t|^2 \\ \Rightarrow \nu(|t|) \quad &\geq \quad m. \end{aligned}$$

Now we show that the mapping $\nu(|\cdot|)\cdot : \mathbb{R}^2 \to \mathbb{R}^2$ is strongly monotone. Let be $s, t \in \mathbb{R}^2$, then

$$\begin{aligned} &\Big[\nu(|t|)t - \nu(|s|)s\Big] \cdot (t - s) = \\ &\quad = \Big[\nu(|t|)t - \nu(|s|)s\Big] \cdot (t - s) + m|t - s|^2 - m|t - s|^2 = \\ &\quad = \Big[\nu(|t|)t - \nu(|s|)s - m(t - s)\Big] \cdot (t - s) + m|t - s|^2 = \\ &\quad = \Big[\nu(|t|)t - mt - \nu(|s|)s + ms\Big] \cdot (t - s) + m|t - s|^2 = \\ &\quad = \Big[(\nu(|t|) - m)t - (\nu(|s|) - m)s\Big] \cdot (t - s) + m|t - s|^2 = \\ &\quad = (\nu(|t|) - m)t \cdot (t - s) - (\nu(|s|) - m)s \cdot (t - s) + m|t - s|^2 = \\ &\quad = \underbrace{(\nu(|t|) - m)}_{\geq 0}\underbrace{(|t|^2 - s \cdot t)}_{\geq |t|(|t|-|s|)} - \underbrace{(\nu(|s|) - m)}_{\geq 0}\underbrace{(s \cdot t - |s|^2)}_{\leq |s|(|t|-|s|)} + m|t - s|^2 \geq \\ &\quad \geq \underbrace{\Big[\nu(|t|)|t| - \nu(|s|)|s|\Big](|t| - |s|)}_{\geq m(|t|-|s|)^2} - m(|t| - |s|)^2 + m|t - s|^2 \geq \\ &\quad \geq m(|t| - |s|)^2 - m(|t| - |s|)^2 + m|t - s|^2 = \\ &\quad \geq m|t - s|^2. \end{aligned}$$

By setting $t := \nabla u, s := \nabla v$ and integrating we get:

$$\begin{aligned} &\langle A(u) - A(v), u - v\rangle = \\ &\qquad = \int_\Omega \Big[\nu(|\nabla u|)\nabla u - \nu(|\nabla v|)\nabla v\Big] \cdot (\nabla u - \nabla v)dx \geq \\ &\qquad \geq \int_\Omega m|\nabla u - \nabla v|^2 dx = m\|u - v\|_{V_0}^2. \end{aligned}$$

$\square$

Lemma 2.4. *If $\nu(\cdot)\cdot : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is Lipschitz continuous with Lipschitz constant $L > 0$, i.e.*

$$|\nu(t)t - \nu(s)s| \;\leq\; L|t-s| \;, \qquad \forall s,t \geq 0$$

then the nonlinear operator A defined by (2.23) *is Lipschitz continuous with Lipschitz constant $3L$, i.e.*

$$\|A(u) - A(v)\|_{V_0^*} \;\leq\; 3L\|u-v\|_{V_0} \;, \qquad \forall u,v \in V_0.$$

Proof. First, we show that the Lipschitz constant L is an upper bound for the reluctivity $\nu(\cdot\,)$:

$$\nu(t)t = |\nu(t)t - \nu(0)0| \leq L|t-0| = Lt$$

$$\Longrightarrow \nu(t) \;\leq\; L$$

Let be $s,t \in \mathbb{R}^2$, then

$$\begin{aligned}
\Big|\nu(|t|)t - \nu(|s|)s\Big| &= \Big|\nu(|t|)t - \nu(|s|)s + \nu(|t|)s - \nu(|t|)s\Big| = \\
&= \Big|\nu(|t|)(t-s) + \Big(\nu(|t|) - \nu(|s|)\Big)s\Big| \leq \\
&\leq \nu(|t|)\Big|t-s\Big| + \Big|\Big(\nu(|t|) - \nu(|s|)\Big)|s|\Big| \leq \\
&\leq L\Big|t-s\Big| + \Big|\nu(|t|)|s| - \nu(|s|)|s| - \nu(|t|)|t| + \nu(|t|)|t|\Big| = \\
&= L\Big|t-s\Big| + \Big|\nu(|t|)(|s| - |t|) + \nu(|t|)|t| - \nu(|s|)|s|\Big| \leq \\
&\leq L\Big|t-s\Big| + \underbrace{\nu(|t|)}_{\leq L}\underbrace{\Big||s| - |t|\Big|}_{\leq|t-s|} + \underbrace{\Big|\nu(|t|)|t| - \nu(|s|)|s|\Big|}_{\leq L\big||t|-|s|\big|} \leq \\
&\leq 3L|t-s|
\end{aligned}$$

By setting $t := \nabla u, s := \nabla v$ and integrating we get

$$\begin{aligned}
|\langle A(u) - A(v), w\rangle| &= \int_\Omega \Big|[\nu(|\nabla u|)\nabla u - \nu(|\nabla v|)\nabla v] \cdot \nabla w\Big| dx \leq \\
&\leq \int_\Omega \Big|\nu(|\nabla u|)\nabla u - \nu(|\nabla v|)\nabla v\Big|\Big|\nabla w\Big| dx \leq \\
&\leq \int_\Omega 3L\Big|\nabla u - \nabla v\Big|\Big|\nabla w\Big| dx \leq \\
&\leq 3L\|u-v\|_{V_0}\|w\|_{V_0},
\end{aligned}$$

where in the last step we have used the Cauchy-Schwarz inequality. Hence

$$\|A(u) - A(v)\|_{V_0^*} = \sup_{w\neq 0} \frac{|\langle A(u) - A(v), w\rangle|}{\|w\|_{V_0}} \leq 3L\|u-v\|_{V_0} \quad .$$

□

2.3.2 Zarantonello's Theorem

For existence and uniqueness we use Zarantonello's Theorem, which is also known as the nonlinear Lax-Milgram theorem (cf. [6]).

Theorem 2.5. *(Zarantonello)*
Let $(V, (\cdot\,,\cdot)_V, \|\cdot\|_V)$ be a Hilbert space, $F \in V^*$ and $A : V \to V^*$ a nonlinear operator, fulfilling the following conditions:

1. *A is strongly monotone, i.e.*

$$\exists c_1 = const > 0 : \langle A(u) - A(v), u - v\rangle \geq c_1 \|u - v\|_V^2, \forall u, v \in V \tag{2.27}$$

2. *A is Lipschitz continuous, i.e.*

$$\exists c_2 = const > 0 : \|A(u) - A(v)\|_{V^*} \leq c_2 \|u - v\|_V, \forall u, v \in V \tag{2.28}$$

Then the operator equation

$$A(u) = F \quad in\ V^* \tag{2.29}$$

has a uniquely determined solution $u^* \in V$.

Proof. Like in the linear case, the proof is based on Banach's fixed point theorem (cf. [5]).
Let $\mathcal{J} : V^* \to V$ denote the Riesz isomorphism, such that

$$(\mathcal{J}w, v)_V = \langle w, v\rangle \quad for\ w \in V^*, v \in V.$$

With $\widetilde{A}(u) = \mathcal{J}A(u)$ and $\widetilde{f} = \mathcal{J}F$ we obtain the equivalent problem

$$\widetilde{A}(u) = \widetilde{f} \quad in\ V,$$

which can also be written in fixed point form

$$u = u - \tau\Big(\widetilde{A}(u) - \widetilde{f}\Big) =: K_\tau(u) \quad .$$

Now we search for some relaxation parameter $\tau > 0$ such that K_τ is a constraction:

$$\begin{aligned}
&\|K_\tau(u) - K_\tau(v)\|_V^2 = \\
&\quad = \|u - v\|_V^2 - 2\tau\Big(\widetilde{A}(u) - \widetilde{A}(v), u - v\Big)_V + \tau^2\|\widetilde{A}(u) - \widetilde{A}(v)\|_V^2 \leq \\
&\quad \leq \|u - v\|_V^2 - 2\tau\Big\langle A(u) - A(v), u - v\Big\rangle + \tau^2\|A(u) - A(v)\|_{V^*}^2
\end{aligned}$$

We use now (2.27) and (2.28), such that we get a Lipschitz constant of K_τ:

$$\|K_\tau(u) - K_\tau(v)\|_V^2 \leq \underbrace{(1 - 2\tau c_1 + \tau^2 c_2)}_{q(\tau)} \|u - v\|_V^2$$

We have

$$q(\tau) < 1 \Longleftrightarrow 0 < \tau < 2\frac{c_1}{(c_2)^2}.$$

The optimal value is $\tau_{opt} = \frac{c_1}{(c_2)^2}$ with $q_{opt} = \sqrt{1 - (\frac{c_1}{c_2})^2}$.

The existence and uniqueness follow from Banach's fixed point theorem. $\square$

Chapter 3

A 1D model problem and its discretization

For particular reasons we consider now a one dimensional problem. Most of the concepts given in this chapter can be generalized to two dimensions.

3.1 1D problem

Our nonlinear problem in classical formulation is given as follows.
Let $\Omega = (a, b)$ be a bounded domain with the boundary $\partial\Omega = \Gamma = \Gamma_D = \{a\} \cup \{b\}$. Find a function $u \in C^2(\Omega) \cap C(\Omega \cup \Gamma)$, such that the differential equation

$$-\Big[\underbrace{\frac{1}{\mu_0\mu_r(|u'(x)|)}}_{=:\ \nu(|u'(x)|)} u'(x)\Big]' = f(x) + \frac{dH}{dx} \quad \forall x \in (a, b) \tag{3.1}$$

and the Dirichlet boundary conditions

$$\begin{aligned} u(a) &= g_a \\ u(b) &= g_b \end{aligned}$$

are satisfied.

3.1.1 The Variational Formulation

For the case of a domain Ω, which consists of only one material we refer to the variational formulation of Section 2.2.2 for homogenous materials. Without loss of generality we derive the variational formulation for two materials

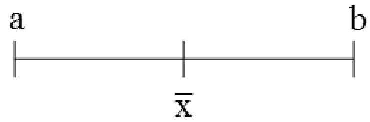

Figure 3.1: Domain Ω

according to Section 2.2.2 for heterogenous materials.
Let us consider the following situation:
The first material (domain $Omega_1$) lies between a and $\overline{x}$, whereas the second material (domain Ω_2) is located between $\overline{x}$ and b. So we write our problem in the form

$$-\Big[\nu_i(|u_i'(x)|)u_i'(x)\Big]' = f^{(i)}(x) + \frac{dH^{(i)}}{dx} \quad in\ \Omega_i,\ i \in \{1,2\}.$$

Let u_1 be a restriction of u to Ω_1, u_2 the restriction of u to Ω_2 and $H^{(1)}$, $H^{(2)}$ the restriction of H to Ω_1, Ω_2, respectively. Then following interface conditions hold:

$$-\nu_1(|u_1'(\overline{x})|)u_1'(\overline{x}) - H^{(1)}(\overline{x}) = -\nu_2(|u_2'(\overline{x})|)u_2'(\overline{x}) - H^{(2)}(\overline{x}) \quad (3.2)$$

$$u_1(\overline{x}) = u_2(\overline{x}) \quad (3.3)$$

To derive the variational formulation of our problem we can perform the following steps like in the two dimensional problem.

1. Choose the space of test functions:

$$V_0 = \{v \in V = H^1(a,b) : v\Big|_\Gamma = 0\}$$

2. Multiply the differential equations by an arbitrary test function $v \in V_0$ and integration over the computational domains $(a, \overline{x})$ and $(\overline{x}, b)$:

$$\int_a^{\overline{x}} -\Big[\nu_1(|u_1'(x)|)u_1'(x)\Big]' v(x)dx + \int_{\overline{x}}^b -\Big[\nu_2(|u_2'(x)|)u_2'(x)\Big]' v(x)dx$$

3. Integration by parts in the principle part:

$$\int_a^{\overline{x}} -\Big[\nu_1(|u_1'(x)|)u_1'(x)\Big]' v(x)dx =$$

$$= -\nu_1(|u_1'(x)|)u_1'(x)v(x)\Big|_a^{\overline{x}} + \int_a^{\overline{x}} \nu_1(|u_1'(x)|)u_1'(x)v'(x)dx =$$

$$= -\nu_1(|u_1'(\overline{x})|)u_1'(\overline{x})v(\overline{x}) + \nu_1(|u_1'(a)|)u_1'(a)\underbrace{v(a)}_{=0} + \int_a^{\overline{x}} \nu_1(|u_1'(x)|)u_1'(x)v'(x)dx$$

$$\int_{\overline{x}}^{b} -\Big[\nu_2(|u_2'(x)|)u_2'(x)\Big]' v(x)dx =$$

$$= -\nu_2(|u_2'(x)|)u_2'(x)v(x)\Big|_{\overline{x}}^{b} + \int_{\overline{x}}^{b} \nu_2(|u_2'(x)|)u_2'(x)v'(x)dx =$$

$$= -\nu_2(|u_2'(b)|)u_2'(b)\underbrace{v(b)}_{=0} + \nu_2(|u_2'(\overline{x})|)u_2'(\overline{x})v(\overline{x}) + \int_{\overline{x}}^{b} \nu_2(|u_2'(x)|)u_2'(x)v'(x)dx$$

Using the interface conditions (3.2) and (3.3) we get:

$$\int_{a}^{\overline{x}} \nu_1(|u_1'(x)|)u_1'(x)v'(x)dx + \int_{\overline{x}}^{b} \nu_2(|u_2'(x)|)u_2'(x)v'(x)dx =$$

$$= \int_{a}^{b} \nu(|u'(x)|)u'(x)v'(x)dx + H^{(2)}(\overline{x}) - H^{(1)}(\overline{x}) \tag{3.4}$$

The rightmost term will be put to the right hand side F.

4. Incorporate the natural boundary condition:
   We have only essential (=Dirichlet) boundary conditions!

5. Define the linear manifold V_g of all admissible functions in which the solution u is looked for:

$$V_g = \{v \in V = H^1(a,b) : v(a) = g_a \wedge v(b) = g_b\}$$

We can also apply integration by parts on the right hand side of our differential equation (3.1).

$$\int_{a}^{b} \Big[f(x) + \frac{d}{dx}H(x)\Big]\ v(x)dx =$$

$$= \int_{a}^{\overline{x}} f^{(1)}(x)v(x)dx + \int_{a}^{\overline{x}} \frac{d}{dx}H^{(1)}(x)v(x)dx +$$

$$+ \int_{\overline{x}}^{b} f^{(2)}(x)v(x)dx + \int_{\overline{x}}^{b} \frac{d}{dx}H^{(2)}(x)v(x)dx$$

$$= \int_{a}^{\overline{x}} f^{(1)}(x)v(x)dx + H^{(1)}(x)v(x)\Big|_{a}^{\overline{x}} - \int_{a}^{\overline{x}} H^{(1)}(x)v'(x)dx +$$

$$+ \int_{\overline{x}}^{b} f^{(2)}(x)v(x)dx + H^{(2)}(x)v(x)\Big|_{\overline{x}}^{b} - \int_{\overline{x}}^{b} H^{(2)}(x)v'(x)dx$$

$$= \int_{a}^{\overline{x}} f^{(1)}(x)v(x)dx + H^{(1)}(\overline{x})v(\overline{x}) - H^{(1)}(a)\underbrace{v(a)}_{=0} - \int_{a}^{\overline{x}} H^{(1)}(x)v'(x)dx +$$

$$+\int_{\overline{x}}^{b} f^{(2)}(x)v(x)dx + H^{(2)}(b)\underbrace{v(b)}_{=0} - H^{(2)}(\overline{x})v(\overline{x}) - \int_{\overline{x}}^{b} H^{(2)}(x)v'(x)dx$$

Incorporating the H-terms from (3.4) we obtain

$$\langle F, v\rangle = \int_a^b \Big[f(x)v(x) - H(x)v'(x)\Big]dx.$$

The result is the following nonlinear variational formulation of the boundary value problem:

Find $u \in V_g$, such that

$$a(u, v) = \langle F, v\rangle \quad \forall v \in V_0, \tag{3.5}$$

where

$$\begin{aligned} a(u, v) &= \int_a^b \nu(|u'(x)|)u'(x)v'(x)dx, \\ \langle F, v\rangle &= \int_a^b [f(x)v(x) - H(x)v'(x)]dx. \end{aligned}$$

3.2 Finite Element Method for the Boundary Value Problem

First of all we discretize the interval $[a, b]$ into N_h subintervals by introducing nodes $x_i, i = \overline{0, N_h}$, with

$$a = x_0 < x_1 < \ldots < x_{N_h} = b.$$

We obtain a subdivision $\mathcal{T}_h$ as a set of subintervals $T_k = (x_{k-1}, x_k)$ for $k = \overline{1, N_h}$. The meshsize h_k of each subinterval is given by

$$h_k \;=\; \mid x_{k-1} - x_k \mid.$$

Let P_k be the set of all polynomials of degree $\leq k$. V_h is the set of all continuous and piecewise affine linear functions on Ω.

$$V_h \;=\; \{v \in C(\bar{\Omega}) : v\Big|_T \in P_1 \;\; \forall T \in \mathcal{T}_h\}$$

The nodal basis for V_h reads as follows. For each node $x_i, i = \overline{0, N_h}$ we define the basis functions $\varphi_i \in V_h$ by

$$\varphi_i(x_j) \;=\; \delta_{ij} \qquad \forall i, j = \overline{0, N_h},$$

where δ_{ij} is the Kronecker symbol.
One immediately sees that $\{\varphi_i : i = \overline{0, N_h}\}$ is a basis of V_h, in particular each function $v_h \in V_h$ can be written in the form

$$v_h(x) = \sum_{i=0}^{N_h} v_i \varphi_i(x)$$

with $v_i = v_h(x_i)$.

• Test functions: $v_h\Big|_\Gamma = 0$:

$$V_{0h} = \{v_h \in V_h : v_h(a) = v_h(b) = 0\} = \{v_h \in V_h : v_h = \sum_{i=1}^{N_h-1} v_i \varphi_i\}$$

• Linear manifold for the solution: $v_h(a) = g_a \wedge v_h(b) = g_b$:

$$V_{gh} = \{v_h \in V_h : v_h(a) = g_a \wedge v_h(b) = g_b\} = \{v_h \in V_h : v_h = g_a\varphi_0 + \sum_{i=1}^{N_h-1} v_i \varphi_i + g_b \varphi_{N_h}\}$$

One easily sees that $V_{0h} \subset V_0$ and $V_{gh} = g_h + V_{0h} \subset V_g$ with $g_h = g_a\varphi_0 + g_b\varphi_{N_h}$.

In the next step we use the following ansatz for the approximate solution:

$$u_h \quad = \quad g_h + \sum_{j=1}^{N_h-1} u_j \varphi_j,$$

i.e. $u_h \in V_{gh}$. We require that this approximate solution satisfies the variational formulation (3.5) for all test functions $v \in V_{0h}$, in other words

$$a(g_h + \sum_{j=1}^{N_h-1} u_j \varphi_j, \sum_{i=1}^{N_h-1} v_i \varphi_i) \quad = \quad \langle F, \sum_{i=1}^{N_h-1} v_i \varphi_i \rangle$$

for all $v_i \in \mathbb{R}, i = \overline{1, N_h - 1}$.

Because of the linearity of $a(\cdot, \cdot)$ in the second argument, it suffices to test only with the basis functions φ_i.

$$a(g_h + \sum_{j=1}^{N_h-1} u_j \varphi_j, \varphi_i) \quad = \quad \langle F, \varphi_i \rangle$$

for all $i = \overline{1, N_h - 1}$.

Summarizing, we get a nonlinear system of equations:

Find $\underline{u}_h = [u_j]_{j=\overline{1,N_h-1}}$ such that the nonlinear system of equations

$$\underline{K}_h[\underline{u}_h] \quad = \quad \underline{f}_h$$

is fulfilled in $\mathbb{R}^{N_h-1}$ with the nonlinear map $\underline{K}_h : \mathbb{R}^{N_h-1} \to \mathbb{R}^{N_h-1}$ given by

$$\underline{K}_h[\underline{w}_h] = [\underline{K}_h[\underline{w}_h]]_i = [a(g_h + \sum_{j=1}^{N_h-1} w_j\varphi_j, \varphi_i)]_i$$

for all $i = \overline{1, N_h - 1}$ and

$$\underline{f}_h = [f_h]_i = \langle F, \varphi_i \rangle.$$

For any arbitrary u we define

$$a[u](w, v) \quad := \quad \int_a^b \nu(|u'(x)|)w'(x)v'(x)dx,$$

then, of course,

$$a(u, v) \quad = \quad a[u](u, v).$$

Obviously we have the relation

$$a[u_h](w_h, v_h) \quad = \quad (K_h[\underline{u}_h]\underline{w}_h, \underline{v}_h).$$

Therefore

$$\underline{K}_h[\underline{u}_h] \quad = \quad K_h[\underline{u}_h]\underline{u}_h,$$

where $K_h[\underline{u}_h]$ is a $(N_h - 1) \times (N_h - 1)-$ matrix, which can be generated element-wise like the stiffness matrix in the linear case.

3.2.1 Computation of the stiffness matrix

Like in the linear case, the stiffness matrix can be assembled element-wise

$$
\begin{aligned}
(K_h[\underline{u}_h]\underline{w}_h, \underline{v}_h) = a[u_h](w_h, v_h) &= \sum_{k=2}^{N_h-1} \int_{T_k} \nu(|u_h'(x)|) w_h'(x) v_h'(x) dx = \\
&= K_h^{(1)} w_1 v_1 + \sum_{k=2}^{N_h-1} \begin{pmatrix} \underline{v}_{k-1} \\ \underline{v}_k \end{pmatrix}^T K_h^{(k)} \begin{pmatrix} \underline{w}_{k-1} \\ \underline{w}_k \end{pmatrix} + K_h^{(N_h)} w_{N_h} v_{N_h},
\end{aligned}
$$

with the element stiffness matrices

$$
K_h^{(k)} = \begin{pmatrix} \int_{T_k} \nu(|u_h'|) \varphi_{k-1}'(x)^2 dx & \int_{T_k} \nu(|u_h'|) \varphi_{k-1}'(x) \varphi_k'(x) dx \\ \int_{T_k} \nu(|u_h'|) \varphi_k'(x) \varphi_{k-1}'(x) dx & \int_{T_k} \nu(|u_h'|) \varphi_k'(x)^2 dx \end{pmatrix}.
$$

Since u_h' is constant on T_k also $\nu(|u_h'|)$ is constant on T_k. So we are able to put the term $\nu(|u_h'|)$ considered on the elements T_k in front of the integrals, such that the following identity holds:

$$
\int_{T_k} \nu\Big(|u_h'|\Big) \psi(x) dx \quad = \quad \nu\Big(|u_h'\big|_{T_k}|\Big) \int_{T_k} \psi(x) dx
$$

Mapping each element T_k to the reference element $\hat{T} = (0, 1)$ one can show that

$$
K_h^{(k)} \quad = \quad \frac{1}{h_k} \, \nu\Big(|u_h'\big|_{T_k}|\Big) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
$$

From the entries of the element stiffness matrices we easily obtain the stiffness matrix

$$
K_h \quad = \quad \begin{pmatrix}
K_{11}^{(1)} + K_{00}^{(2)} & K_{01}^{(2)} & 0 & \dots & \dots & 0 \\
K_{10}^{(2)} & K_{11}^{(2)} + K_{00}^{(3)} & K_{01}^{(3)} & \ddots & & \vdots \\
0 & K_{10}^{(3)} & K_{11}^{(3)} + K_{00}^{(4)} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & \ddots & K_{01}^{(N_h-1)} \\
0 & \dots & \dots & 0 & K_{10}^{(N_h-1)} & K_{11}^{(N_h-1)} + K_{00}^{(N_h)}
\end{pmatrix}.
$$

For the special case of an equidistant subdivision and $\nu = const$ (like in vacuum) one obtains

$$K_h = \frac{1}{h}\,\nu \begin{pmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{pmatrix}.$$

3.2.2 Computation of the load vector

We use the same technique to compute the element load vector

$$f_h^{(k)} = \begin{pmatrix} \int_{T_k} \Big[f(x)\varphi_{k-1}(x) - H(x)\varphi'_{k-1}(x) \Big] dx \\ \int_{T_k} \Big[f(x)\varphi_k(x) - H(x)\varphi'_k(x) \Big] dx \end{pmatrix}.$$

These integrals are typically not computed exactly but approximatively with the help of a so-called quadrature rule, e.g.: the trapezoidal rule

$$\int_a^b \psi(x)dx \approx \frac{b-a}{2}\Big[\psi(a) + \psi(b)\Big].$$

Additionally we use the information that H is constant on the element T_k. So we get the element load vector

$$f_h^{(k)} = \begin{pmatrix} \frac{x_k - x_{k-1}}{2}\Big[f(x_{k-1})\Big] + H\Big|_{T_k} \\ \frac{x_k - x_{k-1}}{2}\Big[f(x_k)\Big] - H\Big|_{T_k} \end{pmatrix}.$$

With this element load vector we can assemble the load vector $\underline{f}_h$ similar to the stiffness matrix.

3.3 Newton Method

From the proof of Zarantonello's Theorem we obtain the following fixed point iteration:

$$u^{(n+1)} = u^{(n)} + \tau_n C_n^{-1}\Big(\underline{f} - K_h[\underline{u}^{(n)}]\underline{u}^{(n)}\Big),$$

where C_n^{-1} corresponds to the Riesz isomorphism or another suitable mapping from V^* to V. With the choice of

$$C_n = \underline{K}_h'[\underline{u}^{(n)}],$$

i.e. the Jacobian of the nonlinear mapping $\underline{K}_h : \mathbb{R}^{N_h-1} \to \mathbb{R}^{N_h-1}$ evaluated at $\underline{u}^{(n)}$, we get Newton's method:

1. Compute $\underline{r}^{(n)} = \underline{f} - K_h[\underline{u}^{(n)}]\underline{u}^{(n)}$
2. Solve $\underline{K}_h'[\underline{u}^{(n)}]\underline{w}^{(n)} = \underline{r}^{(n)}$
3. Compute $\underline{u}^{(n+1)} = \underline{u}^{(n)} + \tau_n \underline{w}^{(n)}$

The relaxation parameter τ_n is determined by a line search algorithm:
In every Newton step we set $\tau_n = 1$. We perform a new Newton step only if following condition is fulfilled:

$$\|\underline{r}^{(n+1)}\| \quad < \quad \|\underline{r}^{(n)}\| \tag{3.6}$$

But if the $\|\underline{r}^{(n+1)}\|$ is greater than the $\|\underline{r}^{(n)}\|$ we set

$$\tau_n \quad = \quad \frac{\tau_n}{2},$$

compute $\underline{r}^{(n+1)}$ with the new τ_n and look if our condition (3.6) is now fulfilled. When the condition is fulfilled we can perform the next Newton step, where the new relaxation parameter τ_{n+1} is determined by the same line search algorithm again, starting with $\tau_{n+1} = 1$.

3.3.1 Computation of the Jacobian matrix

In the following we show how to compute the Jacobian $\underline{K}_h'$.
Starting point:

$$\underline{K}_h'[\underline{u}]\underline{w} = \lim_{t\to 0} \frac{1}{t}\left(\underline{K}_h[\underline{u} + t\underline{w}] - \underline{K}_h[\underline{u}]\right)$$

From the discretization part we know that

$$[\underline{K}_h[\underline{u}+t\underline{w}]]_i = a(u+tw,\varphi_i)$$

and

$$[\underline{K}_h[\underline{u}]]_i = a(u,\varphi_i).$$

So for our one-dimensional model problem we obtain

$$a(u,w) = \int_a^b \nu(|u'(x)|)u'(x)w'(x)dx.$$

We try to compute the (Gâteaux-)derivative of $a[u]$ in a direction w:

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\Big(a(u+tw,\varphi)-a(u,\varphi)\Big)\\
&\quad=\lim_{t\to 0}\frac{1}{t}\int_a^b\Big[\nu(|(u(x)+tw(x))'|)(u(x)+tw(x))'-\nu(|u'(x)|)u'(x)\Big]\varphi'(x)dx\\
&\quad=\lim_{t\to 0}\frac{1}{t}\int_a^b\Big[\nu(|u'(x)+tw'(x)|)(u'(x)+tw'(x))-\nu(|u'(x)|)u'(x)\Big]\varphi'(x)dx\\
&\quad=\int_a^b\Big(\nu(|u'(x)|)+\nu'(|u'(x)|)|u'(x)|\Big)w'(x)\varphi'(x)dx\\
&\quad=:\langle A'(u)w,\varphi\rangle =: a'[u](w,\varphi)
\end{aligned}$$

Here, $A'(u)$ is the (Gâteaux-)derivative of the nonlinear operator $A: V\to V^*$ evaluated at $u: A'(u): V\to V^*$ with corresponding bilinear form $a'[u]$. Therefore, the Jacobian matrix $\underline{K}_h'[u]$ of the nonlinear function $\underline{K}_h:\mathbb{R}^{N_h-1}\to\mathbb{R}^{N_h-1}$ is obtained as the stiffness matrix associated to that bilinear form $a'[u]$,

$$\underline{K}_h'[u]_{ij} = a'[u](\varphi_j,\varphi_i).$$

Let us now define the factor $\vartheta(|u_h'|) := \nu(|u_h'|)+\nu'(|u_h'|)|u_h'|$. The element Jacobian matrix looks like the one before:

$$\underline{K}_h{}'^{(k)} = \begin{pmatrix} \int_{T_k}\vartheta(|u_h'|)\varphi_{k-1}'(x)^2dx & \int_{T_k}\vartheta(|u_h'|)\varphi_{k-1}'(x)\varphi_k'(x)dx \\ \int_{T_k}\vartheta(|u_h'|)\varphi_k'(x)\varphi_{k-1}'(x)dx & \int_{T_k}\vartheta(|u_h'|)\varphi_k'(x)^2dx \end{pmatrix}$$

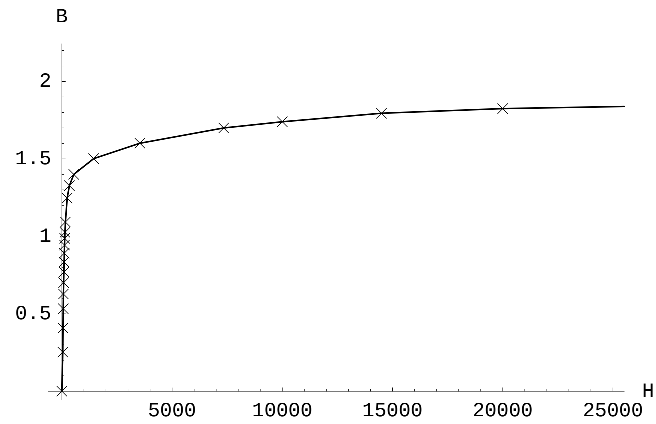

Figure 3.2: B-H-Curve from real-life measurements

Like before u_h' is constant on the element T_k, so also $\vartheta(|u_h'|)$ is constant on T_k. Like in the linear case we can assemble $\underline{K}_h'$ using the Jacobian matrices

$$\underline{K}_h'^{(k)} = \frac{1}{h_k}\vartheta(|u_h'\big|_{T_k}|) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

If ν is constant, the Jacobian matrix coincides with the stiffness matrix $K_h[u_h]$ for any u_h.

3.4 B-H-Curve Approximation

From Section 2.1.2 we know, that B-H-curves are needed for the modeling of ferromagnetic materials in connection with electromagnetic field computation. Due to the underlying physics, such curves are naturally monotone and must often be approximated from real-life measurements (see Figure 3.2). In practice such material curves are never given analytically. A technique for a approximation of such a B-H-curve based on the use of spline functions and a data depending smoothing functional is treated by [3, 4].

In this work we try to find a analytic function, which should approximate the inverse $f^{-1}(s)$ of the real B-H-curve and fulfills Assumption 2.1.

First we start with the analytic function

$$\widetilde{f}^{-1}(s) := \alpha s + e^{\beta s} + \gamma,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

For this function the assumption

$$\left(\widetilde{f}^{-1}(s)\right)' \stackrel{s\to\infty}{\longrightarrow} \frac{1}{\mu_0} := \nu_0$$

is not fulfilled. So we have to find a $s_0 > 0$, such that our function $\widetilde{f}^{-1}(s)$ can be continued at s_0 with a linear function

$$\widehat{f}^{-1}(s) \quad := \quad \eta s + \theta.$$

We choose s_0 such that $\widehat{f}^{-1}$ and the gradient of $\widetilde{f}^{-1}$ agree at s_0, i.e.

$$\left(\widetilde{f}^{-1}(s_0)\right)' \quad = \quad \left(\widehat{f}^{-1}(s_0)\right)'.$$

To guarantee that $f^{-1}(s)$ for $s \geq s_0$ have a gradient of ν_0, η has to be selected accordingly. The purpose of θ is to eliminate a jump in the function. So our function is continuously differentiable.

Finally we use the following analytic function (see Figure 3.3) in our program:

$$f^{-1}(s) \quad := \quad \begin{cases} \alpha s + e^{\beta s} + \gamma & for\ s < s_0 \\ \eta s + \theta & otherwise \end{cases}$$

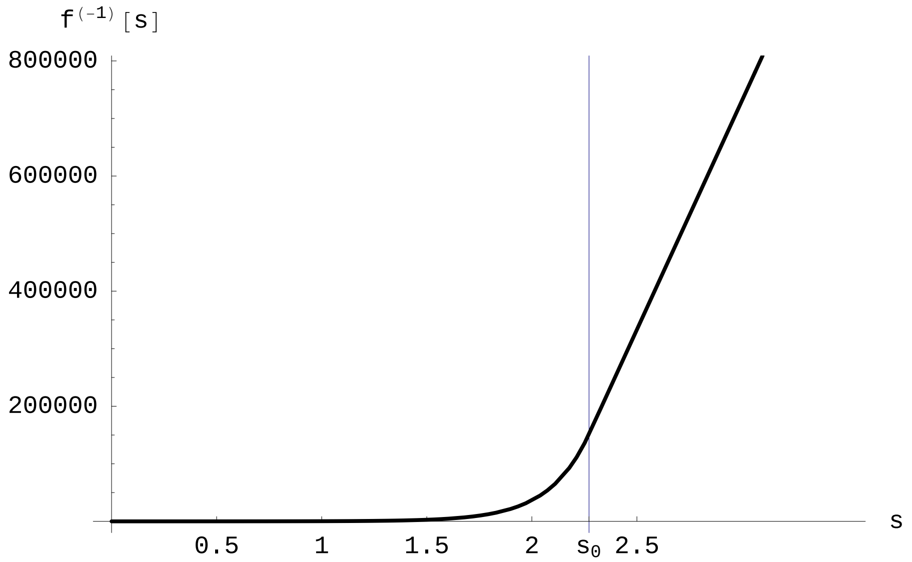

Figure 3.3: $f^{-1}(s)$

Chapter 4

Computer implementation and numerical results

4.1 1D- Program

For numerical studies of nonlinear magnetostatic problem, a C^{++} code was implemented. In this section, the Newton method is tested for a one-dimensional model of our motor. For the geometry see Figure 4.1.
We have the following regions:

I	iron
C^+	coil
A	air
M	permanent magnet
C^-	coil

For these regions we have the following assumptions:

$$\nu(\cdot) \quad := \quad \begin{cases} \nu_{iron}(\cdot), & \text{in } \Omega_{iron} \quad \text{(iron)} \\ \nu_0 = \text{const}, & \text{elsewhere} \quad \text{(coil,air,magnet)} \end{cases}$$

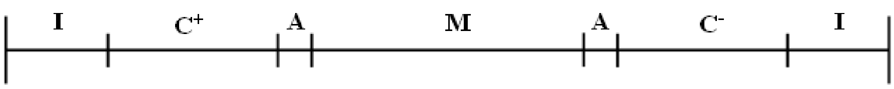

Figure 4.1: Geometry of the problem

$$f(x) := \begin{cases} J_3 = \text{const}, & \text{for } x \in \Omega_{C^+} \quad \text{(coil)} \\ -J_3, & \text{for } x \in \Omega_{C^-} \quad \text{(coil)} \\ 0 & \text{elsewhere} \quad \text{(iron,air,magnet)} \end{cases}$$

$$H(x) := \begin{cases} H_1 = \text{const}, & \text{for } x \in \Omega_M \quad \text{(magnet)} \\ 0 & \text{elsewhere} \quad \text{(iron,coil,air)} \end{cases}$$

For $\nu_{iron}(\cdot)$ we use the analytic function described in section 3.4. By varying the current density J_3 in the coils, the magnetic field in the permanent magnet and the number of unknowns we get the following result:

J_3	H	unknowns	iterations
10^4	10^5	100000	2
10^5	10^5	100000	7
10^6	10^5	100000	11
10^7	10^5	100000	9
10^8	10^5	100000	4
10^6	10^3	100000	33
10^6	10^6	100000	4
10^6	10^4	100000	30
10^6	10^4	10000	11
10^6	10^4	1000	5

In Figures 4.3 and 4.4, we see that the solution exhibits jumps in its derivative at the material interfaces, which goes along with the interface conditions (3.2) and (3.3).

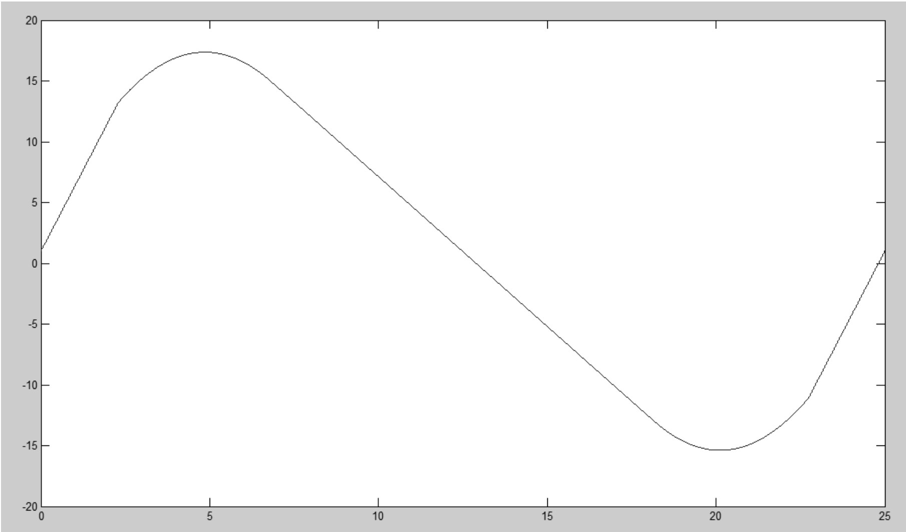

Figure 4.2: The solution u with $J_3 = 10^6$ and $H = 10^4$

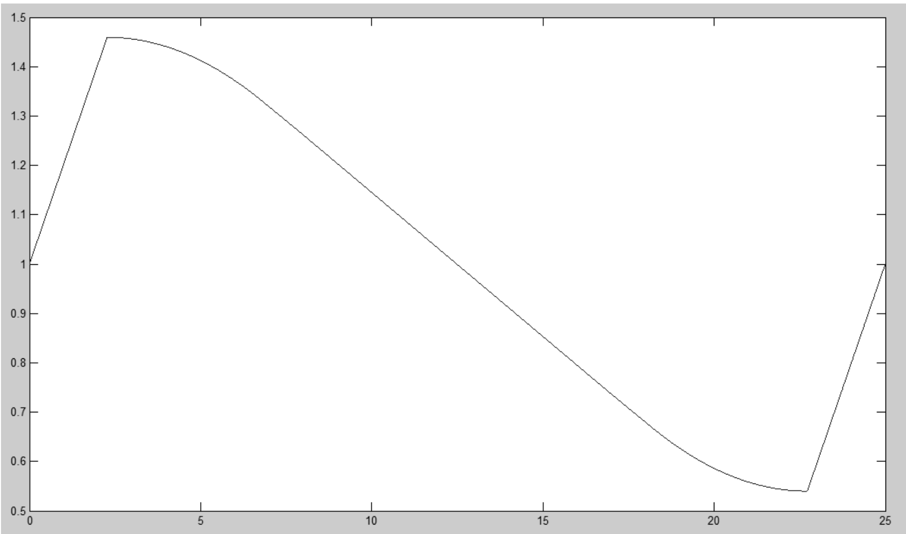

Figure 4.3: The solution u with $J_3 = 10^4$ and $H = 10^3$

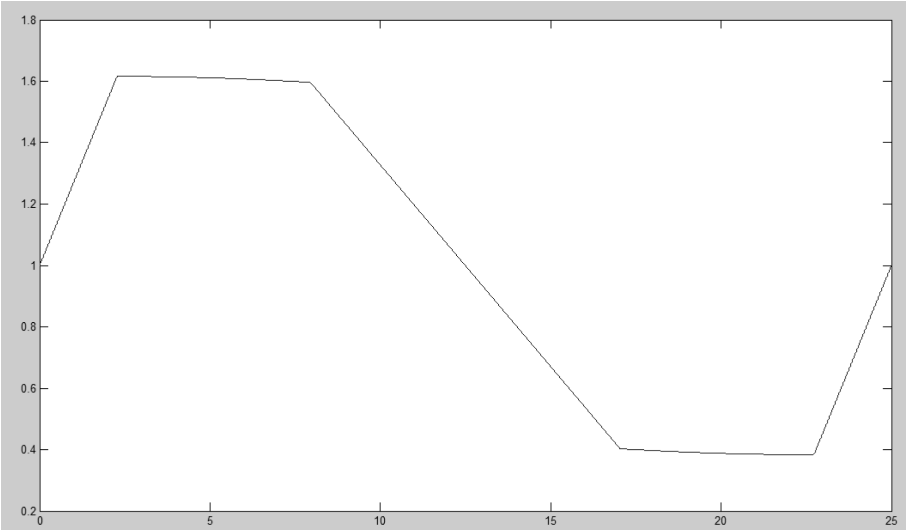

Figure 4.4: The solution u with $J_3 = 10^3$ and $H = 10^5$

4.2 FEMAG

In this section we briefly describe two-dimensional simulations using FEMAG. This program is used for calculations of

- two-dimensional magnetic fields ((x, y) or (r, φ) coordinates),
- rotationally symmetric magnetic fields ((r, z) coordinates).

and their characteristics (inductor, forces, losses,...) using the method of finite elements. In Figure 4.5 we see the user interface of FEMAG.

Figure 4.5: FEMAG: program

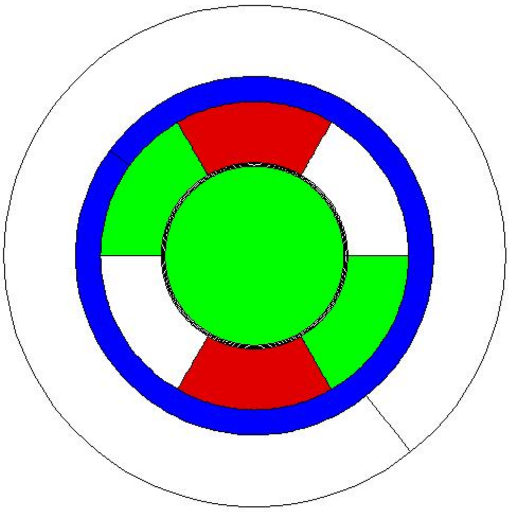

Figure 4.6: FEMAG: motor model

We got a Matlab function from ACCM, which does the simulation of our motor model. We start this function in Matlab with the command `simulate(R_rotor,b_yoke)`. If we start it without parameter values, the default parameters $R\_rotor = 25$ and $b\_yoke = 7$ will be used. The unit of these parameters is $[mm]$. In the process of this simulation a new directory is created, in which all the necessary data is stored. Then the geometry of the motor model is generated. Here the positions of the air gap, the windings, the iron layer,... are set. All this information is stored in a logfile. In the last step, FEMAG is started with this logfile and so the program is executed with the given information. The result of the function `simulate()` is a vector of different torques of the motor. This vector has 30 elements, which are the torques at the actual position. We get these positions, because the rotor is rotated in in steps of $2°$ in the interval of $60°$. The whole simulation for this model takes about two minutes on a standard PC.

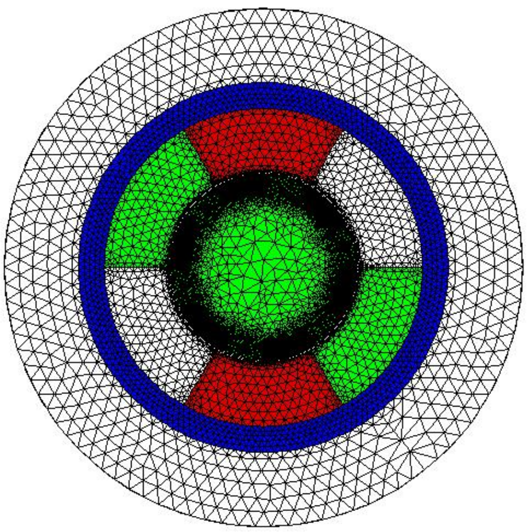

Figure 4.7: FEMAG: mesh

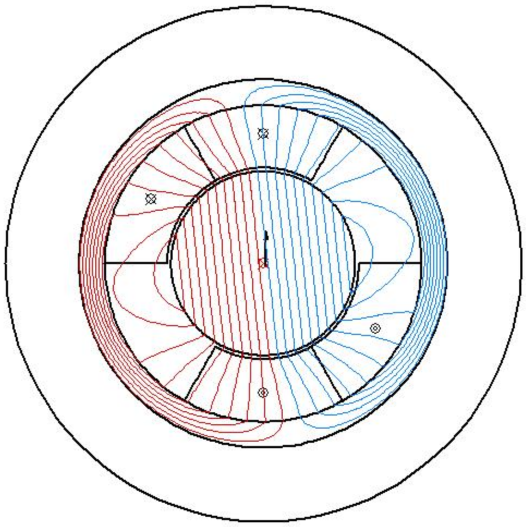

Figure 4.8: FEMAG: magnetic field lines

In Figure 4.6 we can see the motor model, that FEMAG has generated. Figure 4.7 shows the finite element mesh used for this motor. In this example the mesh has about 8500 nodes and about 17000 elements. We can also refine this mesh, such that there are more nodes and elements, but this gives only a small improvement and so we do not take refinement into account.
The magnetic field lines (the isolines of $|B|$) are displayed in Figure 4.8. As we can see, the field lines are parallel in the magnet, but they get diverted in the coils and go back through the iron layer.

4.3 ParNFB

Finally we compare the results of the simulation by FEMAG and the results attained by using ParNFB. ParNFB is a parallel solver for nonlinear coupled FEM/BEM (Boundary Element Method) systems. This program was developed by Dipl.-Ing. Clemens Pechstein at the Institute for Computational Mathematics at the Johannes Kepler University in Linz. ParNFB is applied to our problem sketched in Figure 1.1. In the air gap and in area A8 BEM was performed. For the rest we use interface concentrated FEM.
In Figure 4.9 we plot the mesh of our motor (cf. Figure 4.7). Figure 4.10 shows a similar distribution of the magnetic field lines like in Figure 4.8.

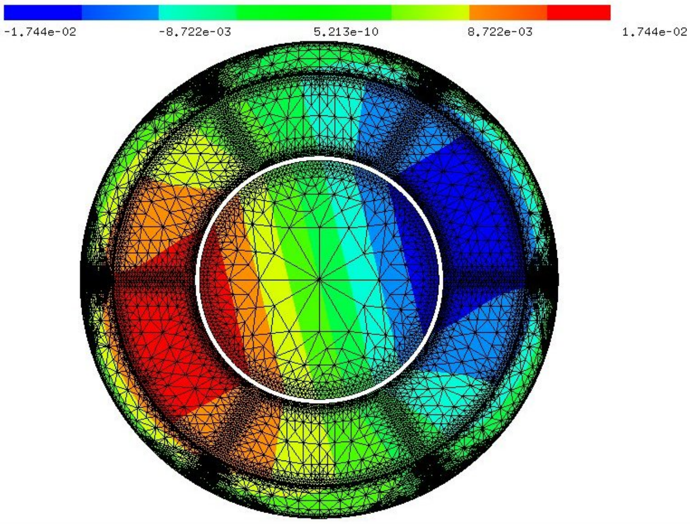

Figure 4.9: ParNFB: mesh

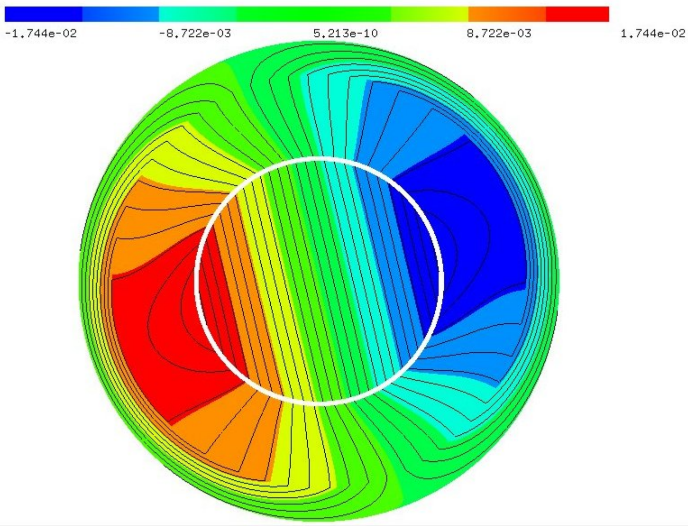

Figure 4.10: ParNFB: magnetic field

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